

**CONSISTENCY OF “THE IDEAL OF NULL RESTRICTED TO
SOME A IS κ -COMPLETE NOT κ^+ -COMPLETE, κ WEAKLY
INACCESSIBLE AND $\mathbf{cov}(\text{meagre}) = \aleph_1$ ”**

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In this note we give an answer to the following question of Grinblat (Moti Gitik asked about it in the Oberwolfach meeting):

Grinblat’s Question 1. Is it consistent that

(**) for some set X , $\mathbf{cov}(\text{Null} \upharpoonright X) = \lambda$ is a weakly inaccessible cardinal (so X not null of course) while $\mathbf{cov}(\text{Meager})$ is small, say it is \aleph_1 .

A. THE FORCING:

Starting with a universe \mathbf{V} and a cardinal λ of cofinality $> \aleph_0$, regular for simplicity (otherwise the only difference is that J consists of “bounded subsets”), in fact weakly inaccessible for Grinblat’s question.

Let $\mathbb{P} = \mathbb{P}_\lambda$ be the result of FS iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \lambda \rangle$ with \mathbb{Q}_{2i} being the random real forcing, and \mathbb{Q}_{2i+1} being the Cohen forcing notion. Let \mathbb{R} be a \mathbb{P} -name for the forcing notion adding \aleph_1 random reals (i.e., forcing with the measure algebra of Borel subsets of ${}^{\omega_1}2$ of positive Lebesgue measure).

We claim that $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P} * \mathbb{R}}$ is as required.

Let $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}}$.

As the whole forcing satisfies the ccc, no cardinal is collapsed etc

B. WHY $\mathbf{cov}(\text{Meager}) = \aleph_1$?

As forcing by \mathbb{R} does it (well known).

C.

Let η_i be the \mathbb{Q}_i -generic real for $i < \lambda$. Clearly they are pairwise distinct. Let

$$X \stackrel{\text{def}}{=} \{\eta_{2i} : i < \lambda\}.$$

This is a set of cardinality λ . Let J be the ideal of subsets of X of cardinality $< \lambda$ (it is a λ -complete ideal on X).

It is enough to prove

(*) J is equal to the ideal of null subsets of X .

C1.

Now, for every $\alpha < \lambda$ the set $\{\eta_{2i} : i < \alpha\}$ is null in \mathbf{V}_2 . Why? Because $\eta_{2\alpha+1}$ is Cohen over $\mathbf{V}^{\mathbb{P}_{2\alpha+1}}$ the universe to which the above set belongs and is an inner model of \mathbf{V}_2 .

This is enough to show that every member of J is null.

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C2.

For the other direction, let \tilde{Y} be a $\mathbb{P} * \mathbb{R}$ name of an unbounded subset of λ . We shall prove that

$$\{\eta_{2i} : i \in \tilde{Y}\}$$

is forced to be non-null (this clearly suffices).

Let p be a condition in $\mathbb{P} * \mathbb{R}$ forcing the inverse, so for some $\mathbb{P} * \mathbb{R}$ -name \tilde{Z} of a null Borel subset of ${}^\omega 2$, we have

$$p \Vdash \text{“ } \{\eta_{2i} : i \in \tilde{Y}\} \subseteq \tilde{Z} \text{”}.$$

We can find $\alpha < \lambda$ such that, in $\mathbf{V}^{\mathbb{P}^\alpha}$, \tilde{Z} becomes an $\mathbb{R}^{\mathbf{V}^{\mathbb{P}^\alpha}}$ -name and p is a member of $\mathbb{R}^{\mathbf{V}^{\mathbb{P}^\alpha}}$.

Now for every i , if $\alpha < 2i < \lambda$ then η_{2i} is random over $\mathbf{V}^{\mathbb{P}^\alpha}$. Hence, by the Fubini theorem (i.e., random reals commute), it is also random over $(\mathbf{V}^{\mathbb{P}^\alpha})^{\mathbb{R}^{\mathbf{V}^{\mathbb{P}^\alpha}}}$. Consequently it does not belong to \tilde{Z} , so we are done.

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