COMMENTS TO UNIVERSAL CLASSES
E54

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Abstract. We add improvements and give details on some points in [Sh:h].

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§1 [Sh 300a]

Verification of [Sh 300a, 1.13], Case 4:

$(*)_1$ $M \models \psi[\bar a'_\alpha, \bar b'_\beta]$ iff $M \models \varphi(a_{1+\alpha}, \bar b_{1+\beta}) \equiv \varphi[a_0, \bar b_{1+\beta}]$

$(*)_2$ $M \models \varphi[\bar a_0, \bar b_{1+\beta}]$ as $0 < 1+\beta$

$(*)_3$ $M \models \psi[\bar a'_\alpha, \bar b'_\beta]$ iff $M \models \varphi[\bar a_{1+\alpha}, \bar b_{1+\beta}]$ iff $1+\alpha < 1+\beta$ iff $\alpha < \beta$.

Comments on [Sh 300a, §1] end Here?:

1.1 Exercise: We call I a (λ,χ)-candidate when for some $\bar s$, the pair $(I,\bar s)$ is a (λ,χ)-candidate which means

(a) I is a linear order

(b) $\bar s = \langle s^\ell_\alpha : \alpha < \lambda, \ell < 3\rangle$ such that there is no repetition

(c) $s^0_\alpha <_I s^1_\alpha <_I s^2_\alpha$

(d) $s^0_\alpha, s^1_\alpha, s^2_\alpha$ induce the same cut of $\{s^\ell_\beta : \beta < \alpha, \ell < 3\}$

(e) in I there is no increasing sequence of length χ.

1) Assume $M = I, \varphi(x,y) = [x < y], \psi(x,\bar y) = [\varphi(x,y_1) \equiv \varphi(x,y_2)]$ letting $\bar y = \langle y_1, y_2\rangle$ and I is a (λ,χ)-candidate. Then

(α) M has the $(\varphi(x,y),\chi)$-non-order property

(β) M has the $(\psi(x,\bar y),\lambda)$-order property

(γ) $\psi(x,\bar y) \in \{\varphi(x,y)\}^{\rm es}$.

2) There is a candidate $(I,\bar s)$ as assumed in (1), in fact with no increasing ω-sequence.
[Hint: use the inverse of a well ordering of order type χ]
3) If there is a χ^+-Aronszajn tree then for Specker order I defined from it, not only is a (χ^+,χ^+)-candidate but in it there is no monotonic sequence of length $\theta := \chi^+$, so we can add in part (1)

(δ) M has the $(\{\varphi(x,y)\}^{i,r},\chi^+)$-non-order property.

4) Assume I^* is a linear order of cardinality λ with neither decreasing nor increasing sequence of length χ^+, e.g. has density $\leq \chi$ (an example is the order of the reals). Then there is a linear order I which is a (λ,χ^+)-candidate with no monotonic sequence of length χ^+ (so in part (1) we have also clause (δ)).

[Hint: use $I^* \times \{0,1,2\}$ ordered lexicographically.]

1.2 Exercise: In Lemma [Sh 300a, 2.9tex] we can replace $\leq^{\aleph_0}_{\mathrm{qf},\mu,\chi}$ by $\leq^{\aleph_0}_{\mathrm{qf},<\mu,\chi}$ and then get $\mathrm{LS}(\mathfrak{K} \leq \mu(=: 2^{2^\chi})$. For this we need other changes. [Saharon: more?]

By [Sh 300a, 1.15] we know that: $A \subseteq M, |A| \leq \mu \Rightarrow |\mathbf{S}^{<\kappa}_\Delta(A,M)| \leq \mu^{<\kappa} = \mu$. We try to choose $M_\alpha, \bar{c}_\alpha$ by induction on $\alpha < \mu^+$ such that:

⊛ (a) $A \subseteq M_\alpha \subseteq N$
(b) $\|M_\alpha\| = \mu$
(c) $\langle M_\beta : \beta \leq \alpha \rangle$ is $\subseteq$-increasing continuous
(d) $\bar{c}_\alpha \in {}^{\kappa >}M$ exemplifies $\neg(M_\alpha \leq^\kappa_{\Delta,\mu,\chi})$.

1.3 Question: 1) Is the cardinal bound in [Sh 300a, 5.1] optimal?
2) Similarly in [Sh 300a, 5.3=5.2tex].

§2 [Sh 300b]

<u>2.1 Question</u>: Can we allow $\langle A\rangle^{\rm gn}_M$ to be partial?

<u>Discussion</u>: 1) It seemed that if we check the proof in [Sh:h, II], we do not really use $\langle A\rangle^{\rm gn}_M$ is well defined for every $A \subseteq M$, but only under restricted circumstances, a first try is

- $(B0)$ if $B := \langle A\rangle^{\rm gn}_M$ is well defined then $A \subseteq B \subseteq M$
- $(B1)$ if $B = \langle A\rangle^{\rm gn}_M$ then $\langle B\rangle^{\rm gn}_M = B$
- $(B2)$ if $A \subseteq M \le_{\mathfrak s} N$ then $\langle A\rangle^{\rm gn}_M$ is well defined iff $\langle A\rangle^{\rm gn}_N$ is well defined and if so then they are equal
- $(B3)$ if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ then $\langle M_1 \cup M_2\rangle^{\rm gn}_{M_3}$ is well defined and $\le_{\mathfrak s} M_{\mathfrak s}$
- $(B4)$??.

2) Or should we use $\langle\{B_t : t \in I\}\rangle^{\rm gn}_N$ and it depends on the history?

2.2 Observation.: Ax(A3) follows from Ax(C1),(C3(a),(b)) and (A2).

Remark. This is [Sh 300b, 1.7=1.4.7tex](2).

Proof. Assume $M_0 \subseteq M_1$ and $M_\ell \le_{\mathfrak s} N$ for $\ell = 1, 2$. By Ax(C2) we can find $M^*_\ell(\ell \le 3)$ and f_1, f_2 such that:

- (a) $\mathrm{NF}_{\mathfrak s}(M^*_0, M^*_1, M^*_2, M^*_3)$
- (b) $M_0 = M^*_0$
- (c) f_1, f_2 is an isomorphism from N, M_0 onto M^*_1, M^*_2 respectively
- (d) $F_\ell \supseteq \mathrm{id}_{M_0}$.

By renaming $f_1 = \mathrm{id}_N$ so $M^*_2 = N$ (and of course $M^*_1 = M_0$) so $\mathrm{NF}_{\mathfrak s}(M_0, M_0, N, M^*_3)$.

By Ax(C3)(a) we have $\mathrm{NF}_{\mathfrak s}(M_0, M_0, M_0, M^*_3)$. Now $M_1 \le_{\mathfrak s} N \le_{\mathfrak s} M^*_3$ hence by Ax(A2) we have $M_1 \le_{\mathfrak s} M^*_3$ and of course $M_0 \cup M_0 \subseteq M_1$. Now apply Ax(C3)(c) with $M_0, M_0, M_0, M^*_3, M_1$ here standing for M_0, M_1, M_2, M_3, M^* there, its assumptions hold by the previous sentence. The conclusion of Ax(C3)(c) gives $\mathrm{NF}_{\mathfrak s}(M_0, M_0, M_0, M_1)$ which by Ax(C1) gives $M_0 \le_{\mathfrak s} M_1$, as required. $\square_{2.2}$

<u>2.3 Question</u>: In [Sh 300b, 2.3], use indiscernible sequence of cardinality $\mu = 2^{2^\chi}$ or χ^+, enough?

* * *

We can give more details on [Sh 300b, 2.3tex], the (D, x)-sequence-homogeneous. We may give details to uniqueness of (D, λ)-prime.

* * *

2.4 Discussion: Ax(D2) for [Sh 300b, 2.18=2.3Ctex]

Give details for:

- (a) for (D, x)-primary we have uniqueness,
- (b) for primes (nec?)

§3 On [Sh 300c]

We can give details of $(<\mu)$-stably constructible from [Sh 300c, §4] as in [Sh 300d, §5]. Saharon: prepare for quoting in [Sh 300f, §4,§5] where Ax(A4) we replaced by Ax(C2)$^+$, (A4)$^*_{<\theta}$.

In particular the uniqueness of "anti-prime".

3.1 Claim. *Assume* $\lambda \le |A| + \mathrm{LS}(\mathfrak{s})$ *and* $\lambda \ge \mu = c\ell(\mu) > \mathrm{LS}(\mathfrak{s})$. *There is an isomorphism from* $A^{\mathscr{A}_1}_{\ell g(\mathscr{A}_1)}$ *onto* $A^{\mathscr{A}_2}_{\ell g(\mathscr{A}_2)}$ *over* A *when for* $\ell = 1,2$ *we have:*

$\circledast_{\mathscr{A}_\ell}$ (a) $\mathscr{A}$ *is a* $(<\mu)$*-stable construction inside* N

(b) $A^{\mathscr{A}_\ell} = A$

(c) $B_* \le_{\mathfrak{s}} A$ *has cardinality* $<\mu$ *and* $u \subseteq \ell g(\mathscr{A}_\ell)$ *is closed of cardinality* $<\mu$ *and* $A^{\mathscr{A}_\ell}_u \cap A \subseteq B_*, B' = \langle B_u \cup B_* \rangle^{\mathrm{gn}}_N$ *so* $B' \le_{\mathfrak{s}} A^{\mathscr{A}_\ell}_{\ell g(\mathscr{A}_\ell)}$ *and* $B' \le_{\mathfrak{s}} B$ *and* B *is of cardinality* $<\mu$ *<u>then</u> for* λ*-oridnal* α *we have:*

(α) $\sup(u) < \alpha < \ell g(\mathscr{A}_\ell)$

(β) $w^{\mathscr{A}_\ell}_\alpha = u$

(γ) $B^{\mathscr{A}_\ell}_\alpha$ *is isomorphic to* B *over* B'.

§4 On [Sh 300d]

(4A) Details:

We give details on [Sh 300d, 2.12=2.9tex], [Sh 300d, 3.17=3.15tex]. See [Sh 300d, 2.9=2.6tex] + [Sh 300d, 2.11=2.8tex](2), expand? Refer to

(4B) On [Sh 300d] for quoting in [Sh 300e, 4.6]

4.1 Claim. *Assume $\langle M_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing continuous, $\langle N_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasin continuous $\alpha \leq \delta \Rightarrow M_\alpha \leq_{\mathfrak{s}} N_\alpha$ and $p \restriction M_\delta \in \mathscr{S}_c^{<\infty}(M_\delta)$.*
1) If $p \in \mathscr{S}^{<\alpha}(N_\delta), p \restriction N_\alpha$ does not fork over M_α for every $\alpha < \delta$, then p does not fork over M_δ.
2) If $M_\delta \leq_{\mathfrak{s}} M_{\delta+1}$ and $M_\alpha, N_\alpha, M_{\delta+1}$ are in stable amalgamation for $\alpha < \delta$ then $M_\delta, N_\delta, M_{\delta+1}$ are in stable amalgamation.

Proof. By [Sh 300c, 1.10](1)=1.0tex(1), [Sh 300d, 3.11](2), recalling Definition [Sh 300d, 3.3,3.5].

Remark. 1) Already exists?
2) Used in [Sh 300e, 4.6].

(4C) Comments On $\mathfrak{C}^{\rm eq}$

We give the model $\mathfrak{C}^{\rm eq}$ where equivalence classes can be represented as elements. It is good for superstable $\mathfrak{s}$, where each $p \in \mathscr{S}^1(M)$ has a canonical base consisting of a singleton, etc.
Generally, see remark [Sh 300d, 7.5] or below (?).

4.2 Definition. 1) Let

$$\mathbf{E}_\chi = \{\mathscr{E} : \mathscr{E} \text{ is an equivalence relation on } {}^\chi|\mathfrak{C}|, \text{ preserved by automorphism of } \mathfrak{C}\}.$$

2) For $\bar a \in {}^\chi|C|$, $E \in \mathscr{E}_\chi$ we say $\bar a/E$ is A-invariant where A is a subset of $\mathfrak{C}$ if every automorphism h of $\mathfrak{C}$, $h \restriction A = \text{id}_A$, maps $\bar a/E$ into itself.
3) We say $\bar a/E$ (where $\bar a \in {}^\chi|\mathfrak{C}|$, $E \in \mathscr{E}_\chi$) is finitary when:

if $M \leq_{\mathfrak{s}} \mathfrak{C}$, $\bar a/E$ is M-invariant and $M = \bigcup_{\alpha<\delta} M_\alpha$, $\langle M_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing then for some $\alpha < \delta$, $\bar a/E$ is M_α invariant.

4) We say $E \in \mathscr{E}_\chi$ is finitary <u>if</u> every $\bar a/E$ $(\bar a \in {}^\chi|\mathfrak{C}|)$ is finitary.
5) We say $\bar a/E$ has a base (a χ-base) <u>if</u> it is invariant over some A, $|A| < \|C\|$, $(|A| < \chi)$.
6) We say $E \in \mathscr{E}_\chi$ has base [μ-base] <u>if</u> every equivalence class has a base [μ-base].
7) Let $\mathscr{E}^*_\chi$ be the family of finitary $E \in \mathscr{E}_\chi$ which has a base.

4.3 Claim. *1) If $\bar a \in {}^{\omega>}\mathfrak{C}$ (or even $\bar a \in {}^{\chi\geq}\mathfrak{C}$), $\bar a/E$ has base and is finitary <u>then</u> it has a base $M <_{\mathfrak{s}} \mathfrak{C}$ such that $\|M\| \leq \chi$.*
2) The number of $E \in \mathscr{E}_\chi$ is $\leq 2^{2^{\chi+|\tau(\mathfrak{s})|}}$.
3) If $\chi \geq \chi_{\mathfrak{s}}$ then $E \in \mathscr{E}^*_\chi$ <u>iff</u> $E \in \mathscr{E}_\chi$ is finitary and has $\chi_{\mathfrak{s}}$-base.

4.4 Claim. *Suppose $\chi(0) < \chi(1)$ and $E_1 \in \mathscr{E}_{\chi(1)}$ and every $\bar a/E_1$ has a $\chi(0)$-base. <u>Then</u> we can find $E_0 \in \mathscr{E}_{\chi(0)}$ and functions h from the set of E_1-sequence classes onto the set of E_0-equivalence classes [of ordinals $< \|\mathfrak{C}\|^{\chi(1)}$] such that:*

(∗) *$\bar a/E_1$ has base A iff $h(\bar a/E_1)$ has base A.*

Proof. Fill.

4.5 Definition. 1) We let for any $M <_{\mathfrak{s}} \mathfrak{C}, M^{\text{eq}}$ be a model with universe

$$|M| \cup \left\{\bar a/E : a \in {}^{\chi(\mathfrak{s})>}|M|, E \in E^*_{\chi(\mathfrak{s})}\right\}$$

relations and functions:

those of $\mathfrak{C}$
$P_E = \{\bar a/E : a \in {}^{\chi(\mathfrak{s})\geq}M\}$
F_E the partial function $F(\bar a) = \bar a/E$

2) K^{eq} is the class of models isomorphic to some M^{eq} (using equivalence class $\mathfrak{C}$ as a class).
3) Next we define $\leq^{\text{eq}}$:

$$M^* \leq^{\text{eq}}_{\mathfrak{s}} N^* \text{ iff there are } M \leq_{\mathfrak{s}} N <_{\mathfrak{s}} \mathfrak{C}, (N^{\text{eq}}, M^{\text{eq}}) \cong (N^*, M^*).$$

4) NF^{eq} is the class of $(M^*_1, M^*_2, M^*_3, M^*_4)$ such that for some $M_\ell <_{\mathfrak{s}} \mathfrak{C}$ for $\ell \leq 3$ we have $M^*_\ell = M^{\text{eq}}_\ell$ for $\ell \leq 3$ and $\text{NF}(M_1, M_2, M_3, M_4)$.

§5 On [Sh 300e]

<u>(5A) Details on X</u>: [Sh 300e, 4.2=4.1.7tex]

Proof of [Sh 300e, 4.1.7](3). Check with [Sh 300, 5.3=5.3tex](6).

First, the implication $(a) \Rightarrow (b)$ is trivial.

Second, assume (b) and let $\bar{b} \in {}^{\beta}\mathfrak{C}$ such that $\mathbf{tp}(\bar{b}, A)$ does not fork over M. Let $\lambda = \|M\| + |\ell g(\bar{b})| + \chi_{\mathfrak{s}}$ and N be $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous such that $M \leq_{\mathfrak{s}} N$. Continue as in the proof of [Sh 300e, 4.8=4.6tex](2) below.

About (c) see xxxx.

Proof of [Sh 300e, 4.2]. 1) For $\underset{\rm wk}{\perp}$, i.e. Definition [Sh 300e, 4.1](1), [Sh 300d, 4.1] they say the same as in [Sh 300d, 4.1], we can find N_1, N_2 realizing p_1, p_2 respectively such that M, N_1, N_2 is in stable amalgamation.
2) For $\perp$, i.e. Definition [Sh 300e, 4.1](2), [Sh 300d, 4.3](2), the equivalence holds the definition of "stationarization" are compatible.
3) For $p \perp B$, i.e. Definition [Sh 300e, 4.1](4), [Sh 300d, 4.5](1), we are assuming $p \in \mathscr{S}^{<\infty}(N)$, again we use the equivalence of the definition of "stationarization" are compatible (and (b), i.e. the definitions of $\perp$ are compatible.
4) For $p \underset{\rm a}{\perp} M$ assume $M \leq_{\mathfrak{s}} N, p \in \mathscr{S}^{<\infty}_c(N)$, there seemingly is a difference: in [Sh 300d, 4.5](2), we demand $q \in \mathscr{S}^{<\infty}_c(M) \Rightarrow p \perp q$ and in [Sh 300e, 4.1](3) $q \in \mathscr{S}^{<\infty}(M) \Rightarrow p \perp q$, so in the second version the demand is seemingly strongly: we have more q. Butif the first version holds, let $q = \mathbf{tp}(\bar{a}, M) \in \mathscr{S}^{<\infty}(M)$, let $M \cup \bar{a} \subseteq M_1 <_{\mathfrak{s}} \mathfrak{C}$, and $\bar{c}$ list $M_1, \bar{a} \trianglelefteq \bar{c}$ so $q_1 := \mathbf{tp}(\bar{c}, M) \in \mathscr{S}^{<\infty}_c(M)$ hence $q_1 \perp p$. But if $N \leq_{\mathfrak{s}} N_1$ and $p_1 = \mathbf{tp}(\bar{b}, N_1)$ is a stationarization of p and $\mathbf{tp}(\bar{a}_1, N_1)$ is a stationarization of q then we can find $\bar{c}_1$ such that $\mathbf{tp}(\bar{c}_1, N_1)$ is a stationarization of q_1 and $\bar{a}_1 \trianglelefteq \bar{c}_1$, and we easily finish.

Remark. See 4.1, intended for quoting in [Sh 300e, 4.6].

<u>(5B) Details on x</u>: [Sh 300e, 4.8=4.6tex]

Proof of [Sh 300e, 4.8=4.6tex](2).

(Canibalize for [Sh 300e, 4.3](3)=4.1.7(3) revise) but see [Sh 300e, 5.3=5.3tex](6).
2) Let $M_\delta := \cup\{M_i : i < \delta\}$ and $N_\delta := \cup\{N_i : i < \delta\}$, hence $M_\delta \leq_{\mathfrak{s}} N_\delta <_{\mathfrak{s}} \mathfrak{C}$. Assume $\bar{b} \in {}^{\alpha}\mathfrak{C}$ and $\mathbf{tp}(\bar{b}, M_\delta \cup C)$ does not fork over M_δ, and we should prove

that it is weakly orthogonal to $\mathbf{tp}(N_\delta, M_\delta \cup C)$. For this it suffices to prove that $\mathbf{tp}(\bar{b}, N_\delta)$ does not fork over M_δ.

Let $M_{\delta+1}$ be such that $M_\delta \cup \bar{b} \subseteq M_{\delta+1} <_{\mathfrak{s}} \mathfrak{C}$ and let $\bar{b}^+$ list the members of $M_{\delta+1}$ such that $\bar{b} = \bar{b}^+ \restriction \alpha$. There is $\bar{b}'$ realizing $\mathbf{tp}(\bar{b}^+, M_\delta)$ such that $\mathbf{tp}(\bar{b}^+, N_\delta)$ does not fork over M_δ. So $\mathbf{tp}(\bar{b}' \restriction \alpha, M_\delta) = \mathbf{tp}(\bar{b}^+ \restriction \alpha, M_\delta) = \mathbf{tp}(\bar{b}, M_\delta)$ and $\mathbf{tp}(\bar{b}' \restriction \alpha, N_\delta)$ does not fork over $M_\delta$hence $\mathbf{tp}(\bar{b}' \restriction \alpha, M_\delta \cup C)$ does not fork over M_δ.

As also $\mathbf{tp}(\bar{b}, M_\delta \cup C)$ does not fork over M_δ and $\mathbf{tp}(\bar{b}, M_\delta) = \mathbf{tp}(\bar{b}' \restriction \alpha, M_\delta)$ is stationary so follows that $\mathbf{tp}(\bar{b}, M_\delta \cup C) = \mathbf{tp}(\bar{b}' \restriction \alpha, M_\delta \cup C)$.

Hence by [Sh 300e, 2.5](6) it suffices to prove that $\mathbf{tp}(\bar{b}', M_\delta \cup C)$ is weakly orthogonal to $\mathbf{tp}(N_\delta, M_\delta \cup C)$. So let $\bar{b}''$ realize $\mathbf{tp}(\bar{b}', M_\delta \cup C)$ and let $M''_{\delta+1} = \mathfrak{C} \restriction \text{Rang}(\bar{b}'')$. So $M_\delta \le_{\mathfrak{s}} M''_{\delta+1} <_{\mathfrak{s}} \mathfrak{C}$ and $\mathbf{tp}(M''_{\delta+1}, M_\delta \cup C)$ does not fork over M_δ and it suffices to prove that $\mathbf{tp}(M''_{\delta+1}, N_\delta)$ does not fork over M_δ.

By symmetry [Sh 300e, 2.10=2.9tex] we have $\mathbf{tp}(C, M''_{\delta+1})$ does not fork over M_δ. But $\mathbf{tp}(C, M_\delta)$ does not fork over M_0 hence by transitivity [Sh 300e, 2.5](4),2.4(2) we have $\mathbf{tp}(C, M''_{\delta+1})$ does not fork over M_0. For each $i < \delta, \mathbf{tp}(C, M''_{\delta+1})$ does not fork over M_i (by monotonicity) [Sh 300e, 2.5](1) but $\mathbf{tp}(N_i, M_i \cup C) \underset{a}{\perp} M_i$ hence $\mathbf{tp}(N_i, M''_{\delta+1})$ does not fork over M_i. By symmetry [Sh 300e, 2.5](4),2.4(2) we have $\mathbf{tp}(M''_{\delta=1}, N_i)$ does not fork over M_i hence by continuity ([Sh 300d, 3.11](2) recalling Definition [Sh 300d, 3.3,3.5] we have $\mathbf{tp}(M''_{\delta+1}, N_\delta)$ does not fork over M_δ, which as said above, suffice.

(5x) Everybody is nice

On nice types we can improve the result on being nice eliminating the superstability so this improves [Sh 300e, 6.3=6.3tex].

5.1 Claim. *If $M <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in {}^\alpha\mathfrak{C}$ and $\bar{c} \in {}^\alpha\mathfrak{C}$ then there are M^*, N^* such that*

(a) $M^* \le_{\mathfrak{s}} N^*$ *and* $\bar{c} \in {}^{\omega>}(N^*), M^* \le_{\mathfrak{s}} M$

(b) $\|N^*\| \le \lambda, \chi_{\mathfrak{s}} + |\ell g(\bar{c})|$

(c) $\mathbf{tp}(\bar{c}, M)$ *does not fork over* M^*

(d) $\mathbf{tp}(N^*, M^* \cup \bar{c})$ *is weakly orthogonal to* $\mathbf{tp}(M, M^* \cup \bar{c})$.

Proof. 1) We assume that such M^*, N^* does not exist and will eventually derive a contradiction. We choose M_i, $N_i(i < \lambda^+), f_i(i < \lambda^+)$ by induction on $i < \lambda^+$ such that:

$\boxdot$ (a) $M_i \le_{\mathfrak{s}} M$ is $\le_{\mathfrak{s}}$-increasing, $\mathbf{tp}(\bar{c}, M)$ does not fork over M_0

(b) $\bar{c} \in N_i, \|N_i\| \le \lambda$ and $j < i \Rightarrow N_j \le_{\mathfrak{s}} N_i$

(c) f_i is a $\le_{\mathfrak{s}}$-embedding of N_i into M_i increasing with i

(d) f_i is the identity on $M_0 \cup \bar{c}$

(e) $\mathbf{tp}(N_i, f_i(M_{i+1}))$ forks over M_i

(f) for i limit, $M_i = \bigcup_{j<i} M_j$, $N_i = \bigcup_{j<i} N_j$.

Construction:.

Case 1:. $i = 0$

Choose (as $\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based), $N_0 <_{\mathfrak{s}} \mathfrak{C}$ such that $\bar{c} \subseteq N_0$ and $N_0 \cap M$, N_0, M is in stable amalgamation and $\|N_0\| \leq \lambda$. Let $M_0 = N_0 \cap M$ and $f_0 = \mathrm{id}_{M_0}$.
Clearly clause (b) holds as well as "$M_0 \leq_{\mathfrak{s}} M$" from clause (a), clause (c) is trivial and the other conditions are inapplicable.

Case 2:. $i = j + 1$.

So N_j, M_j are defined (and are as required) and let g_j be an automorphism of $\mathfrak{C}_{g_j}$ extending f_j so $g_j \supseteq \mathrm{id}_{M_0 \cup \bar{c}}$. Consider $g_j(N_j), M_j$ as candidates for N^*, M^* in the conclusion of 5.1(1), so they should fail some demand. As $\|M_j\| \leq \|N_j\| \leq \lambda$, $M_j \leq_{\mathfrak{s}} M, M_j \leq_{\mathfrak{s}} g_j^{-1}(N_j) <_{\mathfrak{s}} \mathfrak{C}$ and $\bar{c} \in g^{-1}(N_j)$ necessarily $\mathbf{tp}(g_j^{-1}(N_j), M_j \cup \bar{c})$ is not weakly orthogonal to $\mathbf{tp}(M, M_j \cup \bar{c})$. So there is $N'_j <_{\mathfrak{s}} \mathfrak{C}$ isomorphic to $g_j^{-1}(N_j)$ over $M_j \cup \bar{c}$, say by the isomorphism h_j, such that:

$$\mathbf{tp}(N'_j, M) \text{ forks over } M_j.$$

Then we can find $N''_i <_{\mathfrak{s}} \mathfrak{C}, \|N''_i\| \leq \lambda$ such that $N'_j \subseteq N''_i$ and $N''_i \cap M, N''_i, M$ are in stable amalgamation (exists as $\mathfrak{s}$ is λ-based). We let $M_i =: M \cap N''_i$ and let h^+_j be an automorphism of $\mathfrak{C}$ extending h_j and satisfying $f^+_i = g_j \circ h^+_j, N_i = f^+_j(N''_i)$ and $f_i = f^+_j \restriction M_i$. Note $h^+_j \restriction (M_0 \cup \bar{c}) \subseteq h^+_j \restriction (M_j \cup \bar{c}) = \mathrm{id}_{M_j \cup \bar{c}}$ hence $h^+_j \restriction (M_0 \cup \bar{c}) = \mathrm{id}_{M_0 \cup \bar{c}}$ and $g_j \restriction (M_0 \cup \bar{c}) = f_j \restriction (M_0 \cup \bar{c}) = \mathrm{id}_{M_0 \cup \bar{c}}$ so together $f^+_i \restriction (M_0 \cup \bar{c}) = (g_j \circ h^+_j) \restriction (M_0 \cup \bar{c}) = \mathrm{id}_{M_0 \cup \bar{c}}$; i.e. clause (d) holds.

Recall $N_i := f^+_i(N'_i)$, now $M_i \leq_{\mathfrak{s}} N''_i$ hence $f_i(M_i) = f^+_i(M_i) \leq_{\mathfrak{s}} f^+_i(N''_i) = N_i$; so clause (c) holds, too; also $N'_j \leq_{\mathfrak{s}} N''_i$ hence $f^+_i(N'_j) \leq_{\mathfrak{s}} f^+_i(N''_i) = N_i$ but $f^+_i(N'_1) = g_j(h^+_j(N'_j)) = g^0_j(g_j^{-1}(N_j) = N_j$. Together $N_j \leq_{\mathfrak{s}} N_i$, i.e. clause (b) holds. Clause (a) holds trivially and clause (f) is irrelevant. Clause (e) holds as $\mathbf{tp}(N'_j, N''_i)$ forks over M_j by the choices of N'_j, N''_i and f^+_i preserves this.

So we are done with case 2.

Case 3. $i = \delta$ is a limit ordinal.

Let $M_\delta = \bigcup\limits_{\beta<\delta} M_\beta$ and $N_\delta = \bigcup\limits_{\beta<\delta} N_\beta$ and $f_\delta = \bigcup\limits_{\beta<\delta} f_\beta$.

So we have finished the construction, we can choose $M_{\lambda^+}, N_{\lambda^+}, \langle f_{\lambda^+,i} : i < \lambda^+\rangle$ such that the relevant demands in $\boxdot(a)-(f)$ hold. But then $\langle f_i(M_i), f(N_i) : i < \lambda^+\rangle$ contradict "$\mathfrak{s}$ is $\chi_{\mathfrak{s}}$-based" (see [Sh 300c, 2.8]).
2) Left to the reader (use [Sh 300e, 5.4=5.4tex](4)). $\square_{5.1}$

5.2 Remark. If $\bar{c} \subseteq N$ and $|\ell g(\bar{c})| = \lambda$, <u>then</u> $\mathbf{tp}(N, M \cup \bar{c})$ has character (= localness) $\leq \lambda + \chi_{\mathfrak{s}}$ as $\mathfrak{s}$ is $(\lambda + \chi_{\mathfrak{s}})$-based.

5.3 Conclusion. 1) Every $p \in \mathscr{S}^{\infty>}(N)$, (such that $N <_{\mathfrak{s}} \mathfrak{C}, m < \omega$) is prenice.
2) If $\lambda \geq \chi_{\mathfrak{s}}, M <_{\mathfrak{s}} \mathfrak{C}$ is $(\mathbb{D}_{\mathfrak{s}}, \lambda^+)$-homogeneous and $\bar{c} \in {}^{\lambda^+}\mathfrak{C}$ then $\mathbf{tp}(\bar{c}, M)$ is nice.
3) In [Sh 300e, §6,§7] we can waive "superstable" in all the claims except [Sh 300e, 7.12=7.9tex] and can weaken "regular $p \in \mathscr{S}^{<\omega}(M)$" to "regular $p \in \mathscr{S}^{<\infty}(M)$".

Proof. 1) By (2).
2) By 5.1.
3) Check. $\square_{5.3}$

§6 On [Sh 300f]

(A) On the n-place indiscernibility - FILL

(C) "Strengthening the order $\leq_{\mathfrak{s}}$" revisited

Concerning [Sh 300f, 3.2]

6.1 Claim. *Assume [Sh 300f, 3.1], i.e. fill.*
Then $\mathfrak{s}$ *is* $(\Lambda_{\mathfrak{s}}, \lambda)$*-stable when* $\chi \in [\chi_{\mathfrak{s}}, \theta^*), \lambda = \lambda^\chi = \beth_\ell(\chi)$ *when* $\ell = 2$*. Check.*

Proof. We combine the proofs of [Sh 300f, 2.10.7], [Sh 300a, 1.10]. Fill. (070523) What does $\ell = 2$ mean?

* * *

6.2 Question: Where is [Sh 300f], Ax(C10), rigidity, is used?

6.3 Question: Concerning [Sh 300f, 3.19=3.13tex], it is proved for $x = i$ (and $x = j$ is O.K.) what about $\lambda =$ nc?

The following answer Question **?**-6.3. That is, we try to eliminate the use of the
—> scite{f3.2F} undefined
rigidity axiom, paying a low price on cardinalities which does not affect the Main conclusion **?**, [Sh 300f, 3.32=3.15tex].
—> scite{3.15} undefined
First concerning [Sh 300f, 3.13=3.10tex].

We use freely

6.4 Definition.

$\circledast^{j,\lambda,\chi}_{\bar{N},\bar{M}}$ mean as in [Sh 300f, 3.11=3.8.21tex].

6.5 Claim. *Suppose* $x = i, \chi_{\mathfrak{s}} \leq \chi < \lambda = 2^\chi < \theta^*$*; if* $\mathrm{NF}^i_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ *then* $\langle M_1 \cup M_3\rangle^{\mathrm{gn}}_{\mathfrak{C}} \leq^x_{\chi,\chi} M_3$.

[Hint: We assume that this fails and to prove the $(\Lambda_\lambda, \beth_2(\lambda)^)$*-order property. First, without loss of generality* $\|M_\ell\| \leq \lambda$*. Second, let* $\alpha(*)$ *be an ordinal,* R *a two-place relation on* $\alpha(*)$ *such that* $\alpha R_\beta \Rightarrow$ *(*α *even* $\wedge \beta$ *odd). We now can define*

$M^{\alpha(*)}_R, M_{\{\alpha\}}(\alpha < \alpha(\delta)) M_{\{\alpha,\beta\}}$ *(for* $(\alpha,\beta) \in R$*) as in* **?** *with* M_0, M_1, M_2, M_3 *here*
−> *scite{2.12} undefined*
standing for $M_0, M^1_0, M^2_0, M^3_{0,0}$ *there. Now we like to prove them* $M_{\{\alpha\}} \le^x_{\chi,\chi} M^{\alpha(*)}_R$ *when* $\alpha < \alpha(*)$ *and* $M_{\{\alpha,\beta\}} \le^x_{\chi,\chi} M^{\alpha(*)}_R$ *when* $\alpha R \beta$ *and for* $\alpha < \beta$ *we have*

$$\langle M_{\{\alpha\}} \cup M_{\{\beta\}} \rangle^{\mathrm{gn}}_{M^{\alpha(*)}_R} \le^x_{\chi,\chi} M^{\alpha(*)}_R \Leftrightarrow \alpha R \beta.$$

Thus we prove first for the case $(\forall \alpha,\beta)[\alpha R \beta \Rightarrow \beta = \beta_t]$ *to which* **?** *apply. Then the*
−> *scite{3.13} undefined*
general case is done applying **?** *and the previous sentence.*
−> *scite{3.13} undefined*
Recall that **?** *does not depend on Ax(C10).]*
−> *scite{3.13} undefined*

For 6.8, instead of using §1 (the original idea) we use the following exercise. We get $\langle N_u : u \in [\lambda] \rangle$ *independent*$_2$ *by finding many independent realizations of* $tp(N_{\{i-j\}}, N_{\{i\}} \cup N_{\{j\}}$.

6.6 Claim. *Assume* $\chi_{\mathfrak{s}} \le \chi < \lambda = \lambda^\chi, \chi < \theta^*$. *Assume* $M_1 \le^j_{\lambda,\lambda} M_2$ *and* $\bar e \in {}^{\chi \ge}(M_2)$ *and for every* $N \le_{\mathfrak{s}} M_1$ *of cardinality* $\le 2^\chi$ *there is* $\bar e' \in {}^{\ell g(\bar e)}(M_1)$ *realizing* $tp_{\mathfrak{s},\Lambda_\chi}(\bar c, N)$ *such that* $M_2 \models (\exists \bar x)(\varphi(\bar x, \bar e', \bar c')$.

Then we can find N^*_ℓ *for* $\ell = 0,1,2,3$ *and*

- (a) $N^*_\ell \in K$ *has cardinality* $\le \lambda$
- (b) $N^*_0 \le^{\mathrm{nc}}_{\chi,\chi} N^*_1 \le^{\mathrm{nc}}_{\chi,\chi} M_3, N^*_3 \le^{\mathrm{nc}}_{\chi,\chi}$
- (c) $N^*_0 \le^j_{\chi,\chi} N^*_2 \le^{\mathrm{nc}}_{\chi,\chi} N^*_3 \le^{\mathrm{nc}}_{\chi,\chi} M_2$
- (d) $N^*_2 \le^j_{\chi,\chi} N^*_3$
- (e) π *is an isomorphism from* N^*_2 *onto* N^*_1 *over* N^*_0
- (f) $\bar c \subseteq N^*_2$
- (g) *if* $N^*_0 \le_{\mathfrak{s}} N^+_1 \le_{\mathfrak{s}^\chi} M_1$ *and* $\|N^+_1\| \le \lambda$ *then there is a* $\le_{\mathfrak{s}}$*-embedding (or even* $\le_{\mathfrak{s}}$*-embedding)* $\varkappa$ *of* N^*_2 *into* M_1 *over* N^*_0 *such that:*
  - (α) $\{\varkappa(N^*_2), N^*_2, N^+_1\}$ *is independent over* N^*_0 *inside* M_3
  - (β) $M_2 \models (\exists \bar x)(\bar x, \kappa(\bar c), \bar e)$.

6.7 Claim. *A relative of [Sh 300f, 1.6=1.4tex] but is*

- (A) *price: we assume no* $(\Lambda_{<*}, \bar\kappa)$*-order so we use, e.g.* $\mathfrak{s}^{\rm nc}_{<\theta^*,<\theta^+}$
- (B) *in the proof the* $N_{\{i,j\}}$ *part comes by having dim*$(\mathbf{tp}(N_{\{i,j\}}, \langle N_i \cup N_j\rangle^{\rm gn}_{\mathfrak{C}})$ *large*
- (C) *(by first larger submodels then shrink, i.e. using* $\le^{\rm nc}_{\lambda,\chi}$*-submodels (or* $\le^i_{\lambda,*}$*) so have the stronger result.*

Concerning [Sh 300f, 3.17=3.11tex]

6.8 Claim. *[Weak symmetry] Suppose* $x = j$ *and* $\mathrm{NF}^x_{\lambda,\lambda}(M_0, M_1, M_2, M_3)$ *and* $M_3 = \langle M_1 \cup M_2\rangle^{\rm gn}_{M_3}$ *then* $\mathrm{NF}^x_{\chi,\chi}(M_0, M_2, M_1, M_3)$ *when*

- (a) $\mathrm{NF}^j_{\lambda,\lambda}(M_0, M_1, M_2, M_3)$
- (b) $\chi_{\mathfrak{s}} \le \chi < \lambda = \beth_3(chi) < \theta^*$

Proof. Part (A):

Let $\chi_\ell = \beth_\ell(chi)$. Assume that the desired conclusion fails hence there is $\bar N$ such that $\circledast_{\bar N,\bar M}$ (Saharon: define) $\|N_\ell\| = \chi_\ell$ and there is no $\le_{\mathfrak{s}}$-embedding f of N_3 into M_0 over N_1 mapping N_2 into M_0. For the other direction there is a mapping so we can apply **?**.

→ scite{f3.9X} undefined

Part (B): Let $\bar a_\ell$ list N_ℓ for $\ell \le 3$, $\mathrm{Rang}(\bar a_\ell) \subseteq \mathrm{Rang}(a_\ell) \subseteq \mathrm{Rang}(\bar a_2)$ and $\varphi(\bar x_3, \bar x_2, \bar x_1, \bar x_0) = \varphi_N(\bar x_3, \bar x_2, \bar x_1, \bar x_0)$ so $M_3 \models \varphi_i(\bar a_3, \bar a_2, \bar a_2, \bar a_0)$.

Let $\bar N^1 = \langle N^1_\ell : \ell \le 3\rangle$ be such that $\circledast_{\bar N^1,\bar M}$ and $N_\ell \le_{\mathfrak{s}} N^\ell_\ell$ for $\ell \le 3$ and $N^1_\ell \subseteq^{\mathfrak{s}}_{\chi,\chi} M_\ell$ (or little more).

Part (C):

We use 6.7 instead of **?**.

→ scite{f3.9X} undefined

Concerning [\Sh:300f=3.11tex]

Claim. *Suppose* $\lambda = i, \chi_{\mathfrak{s}} \le \chi \le \lambda = 2^\chi$ *and rm* $NF^x_{\lambda,\chi}(M_0, M_1, M_2, M_3)$ *and* $M_0 \le^x_{\lambda,\lambda} \le M^*_0 \le^x_{\lambda,\lambda} M_1$ *where* $M^*_0 = \langle M^*_0 \cup M_2\rangle^{\rm gn}_{M_3}$. *Then* $\mathrm{NF}^x_{\chi,\chi}(M^*_0, M_1, M^*_2, M_3)$.

[Hint: We try to repeat the proof of **?**. *First, when we apply* **?** *there we apply part (1)*

→ *scite{3.11} undefined*

→ *scite{3.10} undefined*

here so $M^*_2 \le^x M_3$. *Second, the proof* $\mathrm{NF}^j_{\chi,\chi}(M^*_0, M_1, M^*_2, M_3)$ *causes no problem.*

Lastly, if $\neg \mathrm{NF}^j_{\chi,\chi}(M^*_0, M^*_2, M_1, M_3), f$ *then in addition to the asymmetry we have a strange situation: given* $\bar a \in {}^{\chi \geq}(M^*_2), \bar c \in {}^{\chi \geq}(M_3)$ *for some* $N_\ell\,(\ell \leq 3), N^*_0, N^*_2$*, of cardinality* $\leq \chi$ *all is natural and* $\bar c \subseteq N_3, \bar a \subseteq N_\ell$ *so we can "reflect"* N_3 *into* M^*_2 *over* N^*_2*, say for* ℓ *but not such that* $f(N_1) \subseteq M^*_0$*.*

(D) Revisiting: failure of $\mathrm{Ax}(\mathrm{A4})_{\aleph_0}$ implies non-structure.

Hypothesis. $\mathfrak{s}$ is an AxFr^-_1 and $\chi^*_{\mathfrak{s}}$ is well defined (or $\chi^{**}_{\mathfrak{s}}$?).

Discussion: Below we prefer to investigate AxFr^-_1, rather than rely on $\mathfrak{s} = \mathfrak{t}^+, \mathfrak{t}$ an AxFr.

6.9 Question: Give details to [Sh 300f, 4.5=4n.3.9](2), i.e. $(< \aleph_0)$-stable constructions; give details.

6.10 Question: Assume in Definition [Sh 300f, 3.19=3.13tex], $t \in I \Rightarrow M_t \leq_{\mathfrak{s}(+)} N$ but $\langle \bigcup_{t \in I} M_t \rangle^{\mathrm{gn}}_N \not\leq_{\mathfrak{s}} N$. Can we get a structure theory? Without loss of generality $|I|$ is minimal. $I = \kappa$, so without loss of generality κ is reular (putting blocks together). But this is §5, but maybe an easier case.

Was in the end of [Sh 300f, §4]:

6.11 Claim. *If* χ *and* $\bar N = \langle N_n : n < \omega \rangle$ *are as in [Sh 300f, 4.9=4f.8tex]'s conclusion (about* $\bar M$*) for the case* $\theta = \aleph_0$*, then for some* $\leq_{\mathfrak{s}(+)}$*-increasing sequence* $\bar M = \langle M_n : n < \omega \rangle$ *of members of* $K^{\mathfrak{s}(+)}_\chi$ *we have* $(\forall \alpha)(*)^\alpha_{\bar M}$ *from [Sh 300f, 4.7=4f.3tex](5).*

Remark. Proof copied January 2007 from [Sh 300f, 4.7tex], there is was moved to AP.

Proof. Let χ be as there and choose μ as 2^χ. So there is a sequence $\langle N_n : n < \omega \rangle$ be as there for μ and let $N = N_\omega := \cup\{N_n : n < \omega\}$. As $\neg(N_0 \leq_{\mathfrak{s}(+)} N)$, that is $\neg(N_0 \leq^i_{\chi,\chi} N)$ clearly we can find M_0, M such that

$(*)_1$ (a) $M_0 \leq_{\mathfrak{s}} M$ are from $K^{\mathfrak{s}}_\chi$
(b) $M_0 \leq_{\mathfrak{s}} N_0$ and $M \leq_{\mathfrak{s}} N$
(c) there is no $\leq_{\mathfrak{s}}$-embedding of M into N_0 over M_0.

By [Sh 300c, 3.7,3.8] without loss of generality

$(*)_n$ $M_\eta := M \cap N_n \le_{\mathfrak{s}} N_n$ for $n < \omega$.

Also

$(*)_3$ if $n < \omega$ then there is no $\le_{\mathfrak{s}}$-embedding of M into N_n over M_0.

[Why? Because if f is such a $\le_{\mathfrak{s}}$-embedding then applying the definition of $M_0 \le^i_{\mu,\chi} M_n$ to the pair of models $(M_0, f(M))$ getting an $\le_{\mathfrak{s}}$-embedding g of $f(M)$ into N_0 over M_0, so $g \circ f$ contradicts $(*)_1(c)$.]

Let $\bar{M} = \langle M_n : n < \omega\rangle$ and let $g_n = \mathrm{id}_{M_n}$.

Next

$(*)_4$ if $\alpha < \mu^+$ and $n < \omega$ then $\mathrm{rk}^{\mathrm{emb},\mu}_{\bar{M}}(g_n, N_n) \ge \alpha$ moreover[1] there is a canonical $(\mathfrak{s}, \mathrm{des}_\mu(\alpha))$-tree witnessing it (i.e. as in [Sh 300f, 4.7=4f.3](4)).

[Why $(*)_4$? We prove this by induction on $\alpha < \mu$ (for all $n < \omega$ simultaneously). For $\alpha = 0$ this is trivial. Arriving to α, fix $n < \omega$. We first note that by the induction hypothesis, for every $\beta < \alpha$ we have $\mathrm{rk}^{\mathrm{emb},\mu}_{\bar{M}}(g_{n+1}, N_{n+1}) \ge \beta$ hence by [Sh 300f, 4n.5.4tex] applied to $\mathfrak{s}$ there is a canonical tree $\langle N_{n+1,\beta}, N^{n+1,\beta}_\eta, f^{n+1}_\eta : \eta \in \mathrm{des}(\beta)\rangle$ for $\bar{M} \restriction [n+1, \omega)$ such that $f^{n+1,\beta}_{<>} = g_{n+1}$ and $N_{n+1,\beta} \le_{\mathfrak{s}} M_{n+1}$. Clearly there is $N^{n+1}_\alpha \le_{\mathfrak{s}} N_{n+1}$ of cardinality $\le \mu$ such that $\cup\{N_{n+1,\beta} : \beta < \alpha\} \subseteq N^{n+1}_\alpha$ (hence $N^{n+1,\beta}_\eta \subseteq N$ for $\beta < \alpha, \eta \in \mathrm{des}(\beta)$). As $N_n \le^i_{\mu,\mu} N_{n+1}$ there is a $\le_{\mathfrak{s}}$-embedding $h = h_{n,\alpha}$ of N^{n+1}_α into N_n over M_n.

Now we define $f^{\eta,\alpha}_\eta, N^{\eta,\alpha}_\eta$ for $\eta \in \mathrm{des}(\alpha)$ as follows $f^{\eta,\alpha}_{<>} = g_n, N^{n,\alpha}_{<>} = M_n$ and if $\eta = <\beta> \hat{}\, \nu, \beta < \alpha \cap \nu \in \mathrm{des}(\beta)$ then $f^{n,\alpha}_\eta = h \circ f^{n+1,\beta}_\nu$ (and $N^{n,\alpha}_\eta = h(N^{n+1,\beta}_\nu)$. So the "moreover" holds by [Sh 300f, 4.7=4f.3](4) (or directly) we can deduce that $\mathrm{rk}_{\bar{M}}(g_n, N_n) \ge \alpha$. So we have carried the induction proving $(*)_4$.]

Now by $(*)_4$ as $\|M_n\| = \chi$ and $\mu = 2^\chi = (2^\chi)^\chi = \mu^\chi$, by [Sh 300f, 4.7=4f.3tex](5) we get $(\forall \alpha \in \mathrm{Ord})[(*)^\alpha_{\bar{\mu}}]$, so we are done. $\square_{\mathbf{?}}$

—> scite{f4.3A} undefined

Remark. Saharon: 6.12 + **?** were copied from [Sh 300f], the question is: can we

—> scite{4f.6} undefined

prove them in weak framework rather than prove it in $\mathfrak{s}^+$ there, i.e.

[1] we can waive it here, but use trees as in [Sh 300f, 4.7=4f.3](4); however then we have to apply [Sh 300f, xxx-4n.5.4] proving $(*)_4$

6.12 Claim. *Assume* $\chi^*_{\mathfrak{s}}$ *is well defined and Ax(A6) holds (so* $\mathfrak{s}$ *is* μ*-based). If* $\bar M = \langle M_n : n < \omega\rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing, then we can find an independent* $(\mathfrak{s}, \text{des}(\alpha))$*-tree of models* $\mathbf{n}$ *for* $\bar M$ *with* $N_{\mathbf{n}} = N^*$ *and* $f^{\mathbf{n}}_{<>} = f$ *(hence by* **?***(2) = [Sh 300f,*
→ *scite{f4.3} undefined*
4f.3](2)) a related canonical tree in fact $\langle \bigcup_\eta N^{\mathbf{n}}_\eta\rangle^{\text{gn}}_{N^*} \leq_{\mathfrak{s}} N^*$*) provided that*

⊛) (a) $\bar M = \langle M_n : n < \omega\rangle$ *is* $\leq_{\mathfrak{s}}$*-increasing*
(b) $\lambda > \chi \geq \chi^*_{\mathfrak{s}} + \Sigma\{\|M_n\| : n < \omega\}$
(c) $N^+ \in K_{\mathfrak{s}}$
(d) f *is a* $\leq_{\mathfrak{s}}$*-embedding of* M_0 *into* N^*
(e) $\text{rk}^{\text{emb},\lambda}_{\bar M}(f, N^*; \mathfrak{s}) \geq \alpha$
(f) α *is an ordinal* $< \lambda^+$.

Proof. Let $\langle \eta_\gamma : \gamma < \gamma(*) \leq \lambda\rangle$ list des(α) such that $\eta_{\gamma_1} \triangleleft \eta_{\gamma_2} \Rightarrow \gamma_1 < \gamma_2$. Now we choose $\langle M^*_\gamma, f_{\eta_\gamma}\rangle$ by induction on $\gamma < \gamma(*)$ such that

$(*)_1$ (a) $M^*_\gamma \leq_{\mathfrak{s}} N^*$ is $\leq_{\mathfrak{s}}$-increasing continuous
(b) $\|M^*_\gamma\| \leq \chi + |\gamma|$
(c) f_{η_γ} is a $\leq_{\mathfrak{s}}$-embedding of $M_{\ell g(\eta_\gamma)}$ into N^*
(d) if $\beta < \gamma$ then $\text{Rang}(f_{\eta_\gamma}) \subseteq M^*_\gamma$
(e) if $\eta_\beta \triangleleft \eta_\beta$ then $f_{\eta_\gamma} \subseteq f_{\eta_\beta}$
(f) $f_{<>} = f$
(g) if $\gamma = \beta+1$ and $\eta_\beta = \eta_{\beta_1}\hat{}\langle\varepsilon\rangle$ then $\text{NF}_{\mathfrak{s}}(f_{\eta_{\beta_1}}(M_{\ell g(\eta_{\beta_1})}), M^*_\beta, f_{\eta_\beta}(M_{\ell g(\eta_\beta)}, N^*)$
(h) if $\gamma = \beta + 1, \eta_\beta = \eta_\beta\hat{}\langle\varepsilon\rangle$ then $\text{rk}^{\text{emb},\lambda}_{\bar M}(f_{\eta_\beta}, N^*) \geq \varepsilon$.

For $\gamma = 0$ let $f_{\eta_\gamma} = f$ and $M^*_0 = f_{\eta_0}(M_0)$. For γ limit use Ax(A6). The main point is to choose f_γ when $\eta_\gamma = \eta_\beta\hat{}\langle\varepsilon\rangle$ and $\gamma = \beta + 1$ and so $M^*_\gamma, f_{\eta_\beta}$ have already been chosen. Clearly $\text{rk}^{\text{emb},\lambda}_{\bar M}(f_{\eta_\beta}, N^*) > \varepsilon$ hence we can find a sequence $\bar f = \langle f_{\eta_\gamma,\zeta} : \zeta < \lambda\rangle$ such that

$(*)_2$ (a) $f_{\eta_\gamma,\zeta}$ is a $\leq_{\mathfrak{s}}$-embedding of M_{n+1} into N^*
(b) $f_{\eta_\gamma,\zeta}$ extends f_{η_β} and $\text{rk}^{\text{emb},\lambda}_{\bar M}(f_{\eta_\gamma,\zeta}, N^*) \geq \varepsilon$
(c) $\langle f_{\eta_\gamma,\zeta}(M_{n+1}) : \zeta < \lambda\rangle$ is independent over $f_{\eta_\beta}(M_n)$ inside N^*.

Hence it suffices to find one $\zeta < \lambda$ such that $\text{NF}_{\mathfrak{s}}(f_{\eta_\beta}(M_{\ell g(\eta_\beta)}), M^*_\gamma, f_{\eta_\gamma,\zeta}(M_{\ell g(\eta_\beta)+1}), N^*)$ and let $f_{\eta_\gamma} = f_{\eta_\gamma,\zeta}$. Such ζ exists by "$\mathfrak{s}$ is $(\chi + |\gamma|)$-based. $\square_{6.12}$

6.13 Claim. *Assume $\mathfrak{s}$ satisfies Ax(A6)$^+$ and $\chi^*_{\mathfrak{s}}$ is well defined, θ regular and Ax(A4)$^*_\theta$ fails.*

Then

(a) $\theta < \mathrm{cf}(\chi^*_{\mathfrak{s}})$

(b) *[possibly decrease θ?] failure is exemplified by models of cardinality $\le 2^{\chi^*_{\mathfrak{s}}}$, i.e. there is an $\le_{\mathfrak{s}}$-increasing continuous sequence $\langle M_i : i < \theta\rangle$ of members of $K_{\mathfrak{s}}$ of cardinality $\le 2^{\chi^*_{\mathfrak{s}}}$ such that $i < \theta \Rightarrow M_i \nleq_{\mathfrak{s}} M_\theta$ where $M_\theta := \cup\{M_i : i < \theta\}$.*

Proof. Let $\mu = 2^{\chi^*_{\mathfrak{s}}}$ by the definition of $\chi^*_{\mathfrak{s}}$ necessarily $\theta <$ cf$(\chi^*_{\mathfrak{s}})$. Now without loss of generality θ is minimal. Choose as counter example $\langle M_i : i < \theta\rangle\hat{\ }\langle M_\theta\rangle$ to Ax(A4)$^*_\theta$ with minimal $\lambda = \Sigma\{\|M_i\| : i < \theta\}$. If $\lambda \le \mu$ then we are done.

So assume $\lambda > \mu$. For $i < \theta$ let $\{a_{\alpha,i} : i < \lambda\}$ list the members of M_i. We choose by induction on $\alpha < \lambda, n < \omega$ for every $u \in [\lambda]^n$ a sequence $\langle M_{u,i} : i < \theta\rangle$ such that:

$\circledast$ (a) $M_{u,i} \le_{\mathfrak{s}} M_i$

(b) $\|M_{u,i}\| \le \mu$

(c) $M_{u,i}$ include $\cup\{M_{v,j} : v \subset u\hat{\ }j \le i$ or $v = u \wedge j < i\} \cup \{a_{\beta,i} : \beta \in u\}$.

By the definition of $\chi^*_{\mathfrak{s}}$ clearly $\mathfrak{s}$ satisfies LSP$_\mu$ hence we can carry the definition.

It is also clear that $u_1 \subseteq u_2 \in [\lambda]^{<\aleph_0} \wedge i_1 \le i_2 \Rightarrow M_{u_1,i_1} \le_{\mathfrak{s}} M_{u_2,i_2}$. Let $M_{u,\theta} = \cup\{M_{u,i} : i < \theta\}$. As λ is minimal clearly $u \in [\lambda]^{<\aleph_0} \wedge i < \theta \Rightarrow M_{u,i} \le_{\mathfrak{s}} M_{u,\theta}$ (so $M_{u,\theta} \in K_{\mathfrak{s}}$).

Now for $u \subset v \in [\lambda]^{<\aleph_0}$ by Ax(A4)$^*_{\ge\chi^*_{\mathfrak{s}}}$ applied to $\langle M_{u,i} : u \in [\lambda]^{<\aleph_0}, i < \theta\rangle, M_\theta$ we get that $M_{u,i} \le_{\mathfrak{s}} M_\theta$ so $M_\theta \in K_{\mathfrak{s}}$. By Ax(A6)$^+$ applied to $\langle M_{u,i} : u \in [\lambda]^{<\aleph_0}\rangle$ and M_θ we get $\cup\{M_{u,i} : u \in [\lambda]^{<\aleph_0}\} \le_{\mathfrak{s}} M_\theta$, i.e. $M_i \le_{\mathfrak{s}} M_\theta$.

6.14 Claim. *If χ and $\bar{N} = \langle N_n : n < \omega\rangle$ are as in **?**'s (or see [Sh 300f, §4])*
−> *scite{f4.5.3} undefined*
*conclusion for the case $\theta = \aleph_0$, then for some $\le_{\mathfrak{s}(+)}$-increasing sequence $\bar{M} = \langle M_n : n < \omega\rangle$ of members of $K^{\mathfrak{s}(+)}_\chi$ we have $(\forall\alpha)(*)^\alpha_{\bar{M}}$ from [Sh 300f, 4.7=4f.3tex](5). But the proof repeats **?**!*
−> *scite{f4.3A} undefined*

Remark. The proof repeats **?**?
−> scite{f4.3A} undefined

Proof. Let χ be as there and choose μ as 2^χ. So there is a sequence $\langle N_n : n < \omega\rangle$ be as there for μ and let $N = N_\omega := \cup\{N_n : n < \omega\}$. As $\neg(N_0 \le_{\mathfrak{s}(+)} N)$, that is $\neg(N_0 \le^i_{\chi,\chi} N)$ clearly we can find M_0, M such that

$(*)_1$ (a) $M_0 \le_{\mathfrak{s}} M$ are from $K^{\mathfrak{s}}_\chi$
(b) $M_0 \le_{\mathfrak{s}} N_0$ and $M \le_{\mathfrak{s}} N$
(c) there is no $\le_{\mathfrak{s}}$-embedding of M into N_0 over M_0.

By [Sh 300c, 3.7,3.8] without loss of generality

$(*)_n$ $M_\eta := M \cap N_n \le_{\mathfrak{s}} N_n$ for $n < \omega$.

Also

$(*)_3$ if $n < \omega$ <u>then</u> there is no $\le_{\mathfrak{s}}$-embedding of M into N_n over M_0.

[Why? Because if f is such a $\le_{\mathfrak{s}}$-embedding then applying the definition of $M_0 \le^i_{\mu,\chi} M_n$ to the pair of models $(M_0, f(M))$ getting an $\le_{\mathfrak{s}}$-embedding g of $f(M)$ into N_0 over M_0, so $g \circ f$ contradicts $(*)_1(c)$.]

Let $\bar M = \langle M_n : n < \omega\rangle$ and let $g_n = \mathrm{id}_{M_n}$.
Next

$(*)_4$ if $\alpha < \mu^+$ and $n < \omega$ <u>then</u> $\mathrm{rk}^{\mathrm{emb},\mu}_{\bar M}(g_n, N_n) \ge \alpha$ moreover[2] there is a canonical $(\mathfrak{s}, \mathrm{des}_\mu(\alpha))$-tree witnessing it (i.e. as in [Sh 300f, 4.7=4f.3tex](4)).

[Why $(*)_4$? We prove this by induction on $\alpha < \mu$ (for all $n < \omega$ simultaneously). For $\alpha = 0$ this is trivial. Arriving to α, fix $n < \omega$. We first note that by the induction hypothesis, for every $\beta < \alpha$ we have $\mathrm{rk}^{\mathrm{emb},\mu}_{\bar M}(g_{n+1}, N_{n+1}) \ge \beta$ hence by 6.12 applied to $\mathfrak{s}$ there is a canonical tree $\langle N_{n+1,\beta}, N^{n+1,\beta}_\eta, f^{n+1}_\eta : \eta \in \mathrm{des}(\beta)\rangle$ for $\bar M \restriction [n+1, \omega)$ such that $f^{n+1,\beta}_{<>} = g_{n+1}$ and $N_{n+1,\beta} \le_{\mathfrak{s}} M_{n+1}$. Clearly there is $N^{n+1}_\alpha \le_{\mathfrak{s}} N_{n+1}$ of cardinality $\le \mu$ such that $\cup\{N_{n+1,\beta} : \beta < \alpha\} \subseteq N^{n+1}_\alpha$ (hence $N^{n+1,\beta}_\eta \subseteq N$ for $\beta < \alpha, \eta \in \mathrm{des}(\beta)$). As $N_n \le^i_{\mu,\mu} N_{n+1}$ there is a $\le_{\mathfrak{s}}$-embedding $h = h_{n,\alpha}$ of N^{n+1}_α into N_n over M_n.

Now we define $f^{\eta,\alpha}_\eta, N^{\eta,\alpha}_\eta$ for $\eta \in \mathrm{des}(\alpha)$ as follows $f^{\eta,\alpha}_{<>} = g_n, N^{n,\alpha}_{<>} = M_n$ and if $\eta = <\beta> \hat{}\ \nu, \beta < \alpha \cap \nu \in \mathrm{des}(\beta)$ then $f^{n,\alpha}_\eta = h \circ f^{n+1,\beta}_\nu$ (and $N^{n,\alpha}_\eta = h(N^{n+1,\beta}_\nu)$. So the "moreover" holds by [Sh 300f, 4.3tex](4) (or directly) we can deduce that $\mathrm{rk}_{\bar M}(g_n, N_n) \ge \alpha$. So we have carried the induction proving $(*)_4$.]

Now by $(*)_4$ as $\|M_n\| = \chi$ and $\mu = 2^\chi = (2^\chi)^\chi = \mu^\chi$, by [Sh 300f, 4.3tex](5) we get $(\forall \alpha \in \mathrm{Ord})[(*)^\alpha_{\bar\mu}]$, so we are done. $\square_{6.14}$

[2]we can waive it here, but use trees as in [Sh 300f, 4.7=4f.3tex](4); however then we have to apply 6.12 proving $(*)_4$

End copying!

(E) Failure of $\mathrm{Ax(A4)}_\theta$ implies non-structure We now pay a Debt from [Sh 300f, §5]:

Giving details to the proof of [Sh 300f, 5.12=5f.5.29].

6.15 Hypothesis. $\mathfrak{s}$ satisfies AxFr_1^-.

We define $\mu_\theta(\mathfrak{s}), \theta(\mathfrak{s})$ as in [Sh 300f, 5.2=5.1tex] and $\mathbf{T}_\theta \leq_{\mathbf{T}_\theta}, \mathbf{T}_\theta^{\mathrm{nc}}, \mathbf{T}_\theta^\gamma$, see [Sh 300f, 5.4-5.9=5f.0-5f.3.7].

We can define $\mathbf{N}_\theta, \leq_{\mathbf{N}_\theta}$ as there, which rely on the choice of $\langle M_\varepsilon^* : \varepsilon < \theta\rangle$, a counterexample to $\mathrm{Ax(A4)}_\theta^*$. But what we prove here does not depend on this, so we prefer

6.16 Definition. [Revise!] 1) $\mathbf{T}_\theta$ is the class $\mathscr{T} = (\mathscr{T}, <)$ which satisfies:

- (a) $(\mathscr{T}, <)$ is a partial order with a minimal element
- (b) $(\mathscr{T}, <)$ is a normal well founded tree, that is: for every $t \in \mathscr{T}, \mathscr{T}_{<t} = \{s : s <_I t\}$ is well ordered (so in particular linearly ordered) and if it has no last element then x is its unique least upper bound in $\mathscr{T}$.
- (c) For $t \in \mathscr{T}$, $\mathrm{otp}\{s : s <_I t\}$ is $< \theta$ and we call it $\mathrm{lev}_{\mathscr{T}}(x)$ moreover
- (d) there is $<_T$-increasing sequence of length θ of members of $\mathscr{T}$.

2) $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} T_2$ (or T_2 extends $\mathscr{T}_1$) when $\mathscr{T}_1 \subseteq \mathscr{T}_2$ are from $\mathbf{T}_\theta$ and $s <_{\mathscr{T}_2} t \in \mathscr{T}_1 \Rightarrow s \in T_1$.
3) $\mathscr{T}_1 \leq_{\mathbf{T}_\theta}^{c\ell} \mathscr{T}_2$ or when $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2$ and if $t \in \mathscr{T}_2$ and $\mathrm{lev}_{I_2}(t)$ is a limit ordinal then $(\forall s)(s <_{I_2} t \rightarrow s \in T_1) \Rightarrow t \in I_1$.

6.17 Observation. [(1) copied [Sh 300f, 5f.4.8]] 1) $\leq_{\mathbf{N}_\theta^{\mathrm{gn}}}$ partially ordered $\mathbf{N}_\theta^{\mathrm{gn}}$.
2) Assume $\{M_t : t \in I\}$ is locally independent over M inside N. If we let $N' := \cup\{\langle \bigcup_{t \in J} M_t\rangle_N^{\mathrm{gn}} : J \subseteq I \text{ is finite}\}$ then $M, N', \langle M_t : t \in I\rangle$ are as in Definition [Sh 300f, 3.20=3.13Atex].

6.18 Claim. *1) If $\mathscr{T} \in \mathbf{T}_\theta^{\mathrm{nc}}$ then there is a canonical $\mathscr{T}$-tree $\mathbf{n}$ of models. Moreover, it is unique, i.e. if $\mathbf{n}_1, \mathbf{n}_2$ are $\mathscr{T}$-trees of models then there is an isomorphism f from $N_{\mathbf{n}_1}$ onto $N_{\mathbf{n}_2}$ such that $\eta \in \mathscr{T} \Rightarrow f \circ f_\eta^{\mathbf{n}_1} = f_\eta^{\mathbf{n}_2}$.*
2) If $\mathscr{T}_1 \leq_{\mathbf{T}_\theta} \mathscr{T}_2 \in \mathscr{T}_\theta^{\mathrm{nc}}$ and $\mathbf{m}$ is a $\mathscr{T}_1$-tree of models then there is $\mathbf{n} \in \mathbf{N}_\theta$ such that $\mathbf{m} \leq_{\mathbf{N}_\theta} \mathbf{n}$. Moreover, $\mathbf{n}$ is unique, i.e. if $\mathbf{n}_\ell$ are $\mathscr{T}_\ell$-trees of models and $\mathbf{m} \leq \mathbf{n}_\ell$

for $\ell = 1,2$ then there is an isomorphism f from $N_{\mathbf{n}_1}$ onto $N_{\mathbf{n}_2}$ over $N_{\mathbf{m}}$ such that $\eta \in \mathscr{T} \Rightarrow f \circ f^{\mathbf{n}_1}_\eta = f^{\mathbf{n}_2}$.

Remark. This just copies [Sh 300f, 5f.5.7tex].

6.19 Claim. *(Copied from [Sh 300f, 5f.5.29])*
Assume that $\mathscr{T}_* \in \mathbf{T}^{\rm nc}_\theta$ and $\mathbf{n}_*$ is a canonical $\mathscr{T}_*$-tree of models for $\bar M$.
1) If $\mathscr{T} \le_{\mathbf{T}_\theta} \mathscr{T}_*$ then for some canonical $\mathscr{T}$-tree $\mathbf{n}$ we have $\mathbf{n}_* \le_{\mathbf{N}_\theta} \mathbf{n}$.
2) In part (1), $\mathbf{n}$ is unique and $N_{\mathbf{n}} = \langle \cup\{N^{\mathbf{n}_*}_\eta : \eta \in \mathscr{T}\}\rangle^{\rm gn}_{N_{\mathbf{n}_*}}$.
3) Assume $\mathscr{T}_\ell \le_{\mathbf{T}_\theta} \mathscr{T}_*$ for $\ell = 0,1,2$ and $\mathscr{T}_1 \cap \mathscr{T}_2 = \mathscr{T}_0$ and $\mathbf{n}_\ell \le_{\mathbf{N}_\theta} \mathbf{n}_*$ is a canonical $\mathscr{T}_\ell$-tree for $\ell = 0,1,2$. Then $\mathrm{NF}_{\mathfrak{s}}(N_{\mathbf{n}_0}, N_{\mathbf{n}_1}, N_{\mathbf{n}_1}, N_{\mathbf{n}_*})$ and $\mathscr{T}_1 \cup \mathscr{T}_2 = \mathscr{T} \Rightarrow N_{\mathbf{n}_*} = \langle N_{\mathbf{n}_1} \cup N_{\mathbf{n}_2}\rangle^{\rm gn}_{N_{\mathbf{n}_*}}$.
4) If $\langle \mathscr{T}_\varepsilon : \varepsilon \le \alpha\rangle$ is $\le_{\mathbf{T}_\theta}$-increasing continuous and $\mathscr{T}_\alpha \le_{\mathbf{T}_\theta} \mathscr{T}_*$ and $\varepsilon \le \alpha \Rightarrow \mathbf{n}_\varepsilon = \mathbf{n} \restriction \mathscr{T}_\varepsilon$ then $\langle \mathbf{n}_\varepsilon : \varepsilon \le \alpha\rangle$ is $\le_{\mathbf{N}_\theta}$-continuous.
5) If $A \subseteq \mathscr{T}_*$ is a maximal set of pairwise $<_{\mathscr{T}_*}$-incomparable members of $\mathscr{T}_*$ and $\mathbf{n} = \mathbf{n}_* \restriction (\mathscr{T}_*)_{\le A}$ and $\mathbf{n}_\eta := \mathbf{n}_* \restriction (T^{[\eta]} \cup (\mathscr{T}_*)_{\le A})$ for $\eta \in A$ then $\langle N_{\mathbf{n}_\eta} : \eta \in A\rangle$ is independent in $N_{\mathbf{n}_*}$.

Remark. This copies [Sh 300f, 5f.5.29tex]. Recheck the proof.

Proof. We prove by induction on the ordinal γ that all parts of 6.18 holds when 6.18 $\mathscr{T}, \mathscr{T}_\ell \in \mathbf{T}^{\le\gamma}_\theta$ and all parts of **?** hold when $\mathscr{T}_* \in \mathbf{T}^\gamma_\theta$.
—> scite{f5.5.29} undefined

Case 1: $\gamma = 0$.
This is trivial as:

⊛ if $\mathscr{T}_1, \mathscr{T}_2 \le_{\mathbf{T}_\theta} \mathscr{T}_*$ then $\mathscr{T}_1 \le_{\mathbf{T}_\theta} T_2$ or $\mathscr{T}_2 \le_{\mathbf{T}_\theta} \mathscr{T}_1$.

Case 2: γ a limit ordinal.
Nothing to prove.

Case 3:
For $\eta \in A_*$ we let $\mathscr{T}^*_\eta = \mathscr{T}^{[\eta]}_* \cup (\mathscr{T}_*)_{\le A}$ then by the choice of $A_*, \mathscr{T}^*_\eta \in \mathbf{T}^{<\partial}_\theta$ and there is a canonical $\mathscr{T}^*_\eta$-tree $\mathbf{n}_\eta$ of models and a canonical $(\mathscr{T}_*)_{\le A}$-tree $\mathbf{n}_\emptyset$ of models such that $\mathbf{n}_\emptyset \le_{\mathbf{N}_\theta} \mathbf{n}_\eta \le_{\mathbf{N}_\theta} \mathbf{n}_*$ for $\eta \in A_*$ and $\langle N_{\mathbf{n}_\eta} : \eta \in A\rangle$ is independent over $N_{\mathbf{n}_\emptyset}$ in $N_{\mathbf{n}_*}$ and $N_{\mathbf{n}_*} = \langle \cup\{N_{\mathbf{n}_\eta} : \eta \in A_*\} \cup N_{\mathbf{n}_\emptyset}\rangle^{\rm gn}_{N_{\mathbf{n}_*}}$.

Now we prove each of the parts:

<u>Part (1) of ?</u>:

Without loss of generality assume $\mathscr{T} \leq_{\mathbf{T}_\theta} \mathscr{T}_*$ and let $\mathscr{T}'_\emptyset = \mathscr{T} \cap (\mathscr{T}_*)_{\leq A}$ and $\mathscr{T}'_\eta = \mathscr{T} \cap \mathscr{T}^*_\eta$ and $\mathscr{T}''_\eta = \mathscr{T}'_\eta \cup \mathscr{T}_\emptyset$.

As $\mathscr{T}'_\emptyset \in \mathbf{T}^{<\gamma}_\theta$ by the induction hypothesis there is a unique $\mathbf{n}'_\emptyset = \mathbf{n}_\emptyset \restriction \mathscr{T}'_\emptyset$ so $\mathbf{n}'_\emptyset \leq_{\mathbf{N}_\theta} \mathbf{n}_\emptyset$ such that $\mathscr{T}_{\mathbf{n}'_\emptyset} = \mathscr{T}'_\emptyset$.

As $\mathscr{T}_{\mathbf{n}_\varepsilon} = \mathbf{T}^*_\varepsilon \in \mathbf{T}^{<\gamma}_\theta$ by the induction hypothesis also $\mathbf{n}'_\varepsilon = \mathbf{n}_\varepsilon \restriction \mathscr{T}'_\varepsilon, \mathbf{n}''_\varepsilon \restriction \mathscr{T}''_\varepsilon$ are well defined as in $\mathscr{T}'_\varepsilon \cap \mathscr{T}'_\emptyset$ it follows that $\mathrm{NF}_{\mathfrak{s}}(N_{\mathbf{n}'_\emptyset}, N_{\mathbf{n}_\emptyset}, N_{\mathbf{n}'_\varepsilon}, N_{\mathbf{n}''_\varepsilon})$ holds.

By Ax(C2)$^+$ we know that there is $N^{**} \leq_{\mathfrak{s}} N_{\mathbf{n}_*}$ such that $N^{**} = \langle \cup\{N'_{\mathbf{n}''_\eta} : \eta \in A_*\}\rangle^{\mathrm{gn}}_{N_{\mathbf{n}_*}}$ and $\langle N_{\mathbf{n}''_\eta} : \eta \in A_*\rangle$ is independent over $N_{\mathbf{n}_\emptyset}$ inside N'' so $\mathbf{n}'' = \mathbf{n} \restriction (\cup\{\mathbf{T}''_\eta : \eta \in A_*\})$ is well defined. Easily $\langle N_{|boldn'_\eta} : \eta \in A_*\}\rangle\hat{}\langle N_{\mathbf{n}_\emptyset}\rangle$ is independent over $N_{\mathbf{n}'_\emptyset}$ inside N'' and $n'' = \langle \cup\{N_{\eta'_\eta} : \eta \in A_*\} \cup \{N_{\mathbf{n}_\emptyset}\}\rangle^{\mathrm{gn}}_{N''}$. So again by Ax(C2)$^-$ there is $N' \leq N'' = N_{\mathbf{n}''}$ such that $N' = \langle \cup\{N_{\mathbf{n}'_\eta} : \eta \in A_*\}\rangle^{\mathrm{gn}}_{N'}$ and so $\mathbf{n}' = \mathbf{n}_* \restriction (\cup\{\mathscr{T}'_\eta : \eta \in A_*\})$ is well defined and $N_{\mathbf{n}'} = N'$, but $\mathscr{T} = \cup\{\mathscr{T}'_\eta : \eta \in A_*\}$, as A_* is non-empty so we are done proving part (1) in Case 3.

<u>Part (2)</u>:

As $|N_{\mathbf{n}}|$ is necessarily $\langle \cup\{N^{\mathbf{n}_*}_\eta : \eta \in \mathscr{T}\rangle))^{\mathrm{gn}}_{N_{\mathbf{n}}}$.

<u>Part (3)</u>:

$(*)_1$ without loss of generality $(\mathscr{T}_*)_{\leq A_*} \cup \mathscr{T}_1 \cup \mathscr{T}_2 = \mathscr{T}_*$.

[Why? By part (1).]

$(*)_2$ without loss of generality $\mathscr{T}_1 \cup \mathscr{T}_2 = \mathscr{T}_*$.

[Why? As in the proof of part (1).]

$(*)_3$ if $(\mathscr{T}_*)_{\leq A} = \mathscr{T}_0$ the conclusion holds.

[Why? Let $\mathscr{T}^\ell_\eta = \mathscr{T}_\ell \cap \mathscr{T}^*_\eta$ for $\eta \in A_*$ for $\ell = 1, 2$. So $\mathbf{n}^\ell_\eta = \mathbf{n}_* \restriction \mathscr{T}^\ell_\eta$ is well defined and we apply Ax(C2)$^+(\alpha)$ to $\{N_{\mathbf{n}^\ell_\eta} : (\eta, \ell) \in A_* \times \{1, 2\}$ over $N_{\mathbf{n}_\emptyset}$ inside $N_{\mathbf{n}_*}$.]

$(*)_4$ without loss of generality $\mathscr{T}_0 \subseteq \mathscr{T}_\emptyset$.

[Why?]

$(*)_5$ without loss of generality $\mathscr{T}_0 = \mathscr{T}_\emptyset$.

[Why? We change the "heart" to be $\mathscr{T}_0$.]

Together we are done.

<u>Part (4)</u>:

<u>Version 1</u>: First deal $A \backslash (\mathscr{T})_{\leq A}$.

So without loss of generality $A \subseteq (\mathscr{T}_*)_{\le A}$ and easy.

Version 2: Let $\mathbf{n}'_\eta = \mathbf{n} \restriction (\mathscr{T}^{[\eta]}\ cup(\mathscr{T}_*)_{\le A}), \mathbf{n}'_\emptyset = \mathbf{n} \restriction (\mathscr{T}_*)_{\le A}$.
It is enough to prove that

(∗) for any $n < \omega$ and distinct $\eta_0, \dots, \eta_{n-1} \in A$, the sequence $\langle N_{\mathbf{n}'_{\eta_\ell}} : \ell < n\rangle$ is independent over $N_{\mathbf{n}'_\emptyset}$.

But (∗) can be proved easily by part (3) (compare with case ?).

Part (5):
Add $\mathscr{T}_\emptyset$ to $\mathbf{T}_{\mathbf{n}_\varepsilon}$, etc. See Case 4.

Part (1),(2) of 6.18:
Straight.

Case 4: $\alpha = \beta + 1, \beta$ a limit ordinal so $\mathrm{cf}(\delta) < \theta$; so without loss of generality $\delta < \theta$.
Let $\mathbf{n}^*_\varepsilon = \mathbf{n}_\varepsilon \restriction \mathscr{T}_\varepsilon$ for $\varepsilon < \delta$.

Part (1):
If $\mathscr{T} \subseteq \mathbf{T}_\varepsilon$ for some $\varepsilon < \delta$ this is obvious. In general, let $\mathscr{T}'_\varepsilon = \mathscr{T} \cap \mathscr{T}_0$, so $\mathbf{n}'_\varepsilon = \mathbf{n}_* \restriction \mathscr{T}_\varepsilon \le_{\mathbf{N}_\theta} \mathbf{n}_*$ is well defined and is $\le_{\mathbf{N}_\theta}$-increasing continuous.
Hence by $\mathrm{Ax(A4)}^*_{<\theta}$ the model $N'_\delta = \cup\{N_{\mathbf{n}'_\varepsilon} : \varepsilon < \delta\}$ belongs to $K_{\mathfrak{s}}$ and $\varepsilon < \delta \Rightarrow N_{\mathbf{n}'_\varepsilon} \le_{\mathfrak{s}} N'_\delta$. Clearly $\langle N_{\mathbf{n}_\varepsilon} : \varepsilon \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous, $\langle N_{\mathbf{n}'_\varepsilon} : \varepsilon < \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $\varepsilon < \zeta < \delta \Rightarrow ?$ and by $\mathrm{Ax(A4)}^*_{<\theta}$, as $\mathrm{cf}(\delta) < \theta$ also $\langle N_{\mathbf{n}'_\varepsilon} : \varepsilon < \delta\rangle\hat{\ }\langle N'_\delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.
Also $\varepsilon < \zeta < \delta \Rightarrow \mathrm{NF}_{\mathfrak{s}}(N_{\mathbf{n}'_\varepsilon}, N_{\mathbf{n}_\varepsilon}, N_{\mathbf{n}'_\zeta}, N_{\mathbf{n}_\zeta})$. As $\mathrm{Ax(A4)}^*_{<\theta}$ holds by [Sh 300b, 1.6=1.4tex] = [Sh:F822, 1b.5] we know that $N'_\delta \le_{\mathfrak{s}} N_{\mathbf{n}_*}$ and $\varepsilon < \delta \Rightarrow \mathrm{NF}(N_{\mathbf{n}'_\varepsilon}, N_{\mathbf{n}_\varepsilon}, N'_\delta, N_{\mathbf{n}_\delta})$.
Clearly we are done.

Part (2):
Should be clear.

Part (3):
By part (1) without loss of generality $\mathscr{T}_1 \cup \mathscr{T}_2 = \mathscr{T}_*$ and $\mathbf{n}_\ell := \mathbf{n} \restriction \mathbf{T}_\ell$ is well defined. For $\ell = 0, 1, 2$ let $\mathscr{T}^\ell_\varepsilon = \mathscr{T}'_\ell \cap \mathscr{T}^*_\varepsilon$ and $\mathbf{n}^\ell_\varepsilon = \mathbf{n}_* \restriction \mathscr{T}^\ell_\varepsilon$.
As in the proof of part (1) we have $\varepsilon < \zeta \le \delta \Rightarrow \mathrm{NF}_{\mathfrak{s}}(N_{\mathbf{n}^0_\varepsilon}, N_{\mathbf{n}^\ell_\varepsilon}, N_{\mathbf{n}^0_\varepsilon}, N_{\mathbf{n}^\ell_\zeta})$. For $\varepsilon \le \zeta \le \delta$ let $\mathbf{n}^\ell_{\varepsilon, z\eta} = \mathbf{n}_* \restriction ((\mathscr{T}_0 \cap \mathscr{T}^*_\zeta) \cup (\mathscr{T}_\ell \cap \mathscr{T}^*_\varepsilon))$.
Clearly for $\varepsilon < \zeta \le \delta$ we have $\mathbf{n}^\ell_{\varepsilon,\zeta} \le_{\mathbf{N}_\theta} \mathbf{n}_*$. Hence by [Sh 300c, 1.7=1.4Atex] = [Sh:F822, 1h.4A] we have $\langle N_{\mathbf{n}^\ell_{\varepsilon,\delta}} : \varepsilon \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous.
FILL.

Part (4):

For $\eta \in A$ let $\mathbf{n}'_\emptyset = \mathbf{n} \restriction (\mathscr{T}_*)_{\leq A}$ and $\mathbf{n}'_\eta = \mathbf{n}_* \restriction \mathscr{T}_*^{[\eta]}$, so $\mathbf{n}'_\emptyset \leq_{\mathbf{N}_\theta} \mathbf{n}_*$ and $\mathbf{n}'_\eta \leq \mathbf{n}_*$ and $\mathscr{T}_*^{[\eta]} \in \mathbf{T}^\gamma_\theta$. By Ax(C2)$^+(\alpha)$ it suffices to prove that:

(∗) for every $n < \omega$ and distinct $\eta_0, \ldots, \eta_{n-1} \in A$, $\langle N_{\mathbf{n}'_{\eta_\ell}} : \ell < n\rangle$ is independent over $N_{\mathbf{n}}$.

But this we can prove by induction on n by using part (3).

Part (5):
Let $\langle \mathscr{T}_\varepsilon : \varepsilon \leq \delta\rangle$ be gien (not necessary $\delta < \theta$!). So $\mathbf{n}_\varepsilon = \mathbf{n} \restriction \mathscr{T}_\varepsilon \leq_{\mathbf{N}_\theta} \mathbf{n}_*$ is well defined by part (1), so $N_{\mathbf{n}_\varepsilon} \leq_{\mathfrak{s}} N_{\mathbf{n}_*}$ and clearly by Ax(B) $\langle \mathbf{n}_\varepsilon : \varepsilon \leq \delta\rangle$ is $\subseteq$-increasing continuous. Hence it is $\leq_{\mathfrak{s}}$-increasing continuous so we are done.

Part (6),(7):
Should be clear. $\square_{6.18}, \square_{\mathbf{?}}$
—> scite{f5.5.29} undefined

Case 5: $\alpha = \beta + 1, \beta$ odd.
Easy.

Saharon: Also details for [Sh 300f, 5f.7].

§7 On [Sh 300g]

Concerning [Sh 300g, 1.4=1f.4tex]

7.1 Claim. *Assume $\mathfrak{s}_\alpha \in \mathfrak{S}$ is increasing for $\alpha < \delta$ and we define $\mathfrak{s}_\delta = \cup\{\mathfrak{s}_\alpha : \alpha < \delta\}$ as in [Sh 300g, 1.3=1f.3].*
1) $\mathfrak{s}_\delta$ belongs to $\mathfrak{S}$.
2) For each of the following axioms, if $\mathfrak{s}_\alpha$ satisfies it then so does $\mathfrak{s}_\delta$: (A4),(A4)$_*$,(A4)$_\theta$,(C3),(C4),(C6),(C7).
3) For each of the following sets of axioms, if $\mathfrak{s}_\alpha$ satisfies each member of the set then so does $\mathfrak{s}_\delta$

- (a) *(C2) + (C4); [also (C2)$'$ meaning in (C2) we add $M = \langle M_1^* \cup M_2^* \rangle^{\rm gn}_M$]*
- (b) *(C5) + (C4); [also strength (C5) as in [Sh 300c, §1]].*

Proof. Fill.

* * *

<u>Discussion</u>: Unfortunately in Theorem [Sh 300g, 1.7] we assume "the existence of stationary sets $\subseteq S^{\mu^+}_\theta$ non-reflecting in any $\delta \in S^{\mu^+}_{<{\rm cf}(\chi^*_{\mathfrak{s}}}$".

To avoid this we can try to develop "$\mathfrak{s}$ satisfied AxFr_1^- and $\chi^*_{\mathfrak{s}}$ well defined + (A4)$_*$

- (A) we have stable constructions
- (B) we can get non-structure from non-superstability (so it says $\langle M_i : i \le \theta + 1\rangle, a \in M_{\theta+1} \backslash M_\theta$, the type $\mathbf{tp}(a, M_\theta, M_{\theta+1})$ forks over M_i) for every $i < \theta$. Have to recheck everything.

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