GENERAL NON-STRUCTURE THEORY
E59

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Abstract. The theme of the first two sections, is to prepare the framework of how from a "complicated" family of index models $I \in K_1$ we build many and/or complicated structures in a class K_2. The index models are characteristically linear orders, trees with $\kappa+1$ levels (possibly with linear order on the set of successors of a member) and linearly ordered graph, for this we phrase relevant complicatedness properties (called bigness).

We say when $M \in K_2$ is represented in $I \in K_1$. We give sufficient conditions when $\{M_I : I \in K^1_\lambda\}$ is complicated where for each $I \in K^1_\lambda$ we build $M_I \in K^2$ (usually $\in K^2_\lambda$) represented in it and reflecting to some degree its structure (e.g. for I a linear order we can build a model of an unstable first order class reflecting the order). If we understand enough we can even build e.g. rigid members of K^2_λ.

Note that we mention "stable", "superstable", but in a self contained way, using an equivalent definition which is useful here and explicitly given. We also frame the use of generalizations of Ramsey and Erdös-Rado theorems to get models in which any I from the relevant K_1 is reflected. We give in some detail how this may apply to the class of separable reduced Abelian $\dot{p}$-group and how we get relevant models for ordered graphs (via forcing).

In the third section we show stronger results concerning linear orders. If for each linear order I of cardinality $\lambda > \aleph_0$ we can attach a model $M_I \in K_\lambda$ in which the linear order can be embedded such that for enough cuts of I, their being omitted is reflected in M_I, then there are 2^λ non-isomorphic cases.

But in the end of the second section we show how the results on trees with $\omega+1$ levels (on which concentrate [Sh:331] gives results on linear ordered (not covered by §3), on trees with $\omega+1$ levels see [Sh:331]. To get more we prove explicitly more on such trees. Those will be enough for results in model theory of Banach space of Shelah-Usvyatsov [ShUs:928].

Date: January 12, 2016.

2010 *Mathematics Subject Classification.* Primary: 03C55; Secondary: 03E05, 03E75, 03G05.

Key words and phrases. model theory, set theory, non-structure, number of non-isomorphic models, unstable theories, linear orders, EM-models.

The author thanks Alice Leonhardt for the beautiful typing. This is a revised version of [Sh:300, Ch.III,§1-§3], has existed (and somewhat revised) for many years. Was mostly ready in the early nineties, and public to some extent. For the sake of [LwSh:687] we add the part of §1 from 1.25. {1.22new}
For the sake of [ShUs:928] we add in the end of §2. Recently this work was used and continued in Farah-Shelah [FaSh:954]. This was written as Chapter III of the book [Sh:e], which hopefully will materialize some day, but in meanwhile it is [Sh:E59]. The intentions were: [Sh:E58] (revising [Sh:229]) for Ch.I, [Sh:421] for Ch.II, [Sh:E59] for Ch.III, [Sh:309] for Ch.IV, [Sh:363] for Ch.V, [Sh:331] for Ch.VI, [Sh:511] for Ch.VII, [Sh:E60], a revision of [Sh:128] for Ch.VIII, [Sh:E62] for the appendix, and probably [Sh:757], [Sh:384], [Sh:482], and [Sh:800]. References like [Sh:E62, 3.7=Lc2] means that c2 is the label of 3.7 in [Sh:E62], will only help the author if changes in the paper [Sh:E62] will change the number.

§ 0. Introduction

The main result presented in this paper is (in earlier proofs we have it only in "most" cases):

{0.1new} **Theorem 0.1.** *If $\psi \in \mathbb{L}_{\chi^+,\omega}, \varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\chi^+,\omega}, \ell g(\bar{x}) = \ell g(\bar{y}) = \partial$ and ψ has the*
{1.2} *$\varphi(\bar{x},\bar{y})$–order property (see Definition 1.2(5)) <u>then</u> $\dot{\mathbb{I}}(\lambda,\psi) = 2^\lambda$ provided that for example: $\lambda \ge \chi + \aleph_1, \partial < \aleph_0$ or $\lambda = \lambda^\partial + \chi + \partial^+ + \aleph_1$ or $\lambda > \chi + \partial^+$ or $\lambda^{\partial^+} < 2^\lambda, \lambda \ge \chi$.*

{3.14} *Proof.* By 3.25(2), clause (b) of 3.25(2) holds. When $\lambda \ge \chi + \aleph_1, \partial < \aleph_0$, by
{3.10} Theorem 3.20(3), $\dot{\mathbb{I}}(\lambda,\psi) = 2^\lambda$.

So we can assume that $\lambda \ge \chi$ and $\partial \ge \aleph_0$. When $\lambda^\partial = \lambda$ or $\lambda^{(\partial^+)} < 2^\lambda$ the
{3.11} conclusion holds by 3.22(a), 3.22(b), respectively, using $\kappa = \partial^+$ and the existence
{3.14} of such models follows from 1.18 they are as required by 3.8(4). When $\lambda > \chi + \partial^+$
{3.14} the conclusion holds by 3.25(1). So we are done. $\square_{0.1}$

Note that although some notions connected to stability appear, they are not used in any way which require knowing them: we define what we use and at most quote some results. In fact, the proof covered problems with no (previous) connection to stability. For understanding and/or checking, the reader does not need to know the works quoted below: they only help to see the background. Note that

For later chapters (please give specific numbers) §2 is essential to some of the later parts of non-structure (see [Sh:309], [Sh:331] [Sh:511]) them but not §1 or §3
{1.7} still but better read 1.1-1.9.

Generally the construction of many models (up to isomorphism in this paper) in K_λ $(=: \{M \in K : \|M\| = \lambda\})$ goes as follows. We are given a class K of models (with fix vocabulary), and we are trying to prove that K has many complicated members. To help us, we have a class K^1 of "index models" (this just indicates their role; supposedly they are well understood; they usually are linear orders or a class of trees). By the "non-structure property of K", for some formulas φ_ℓ (see below), for every $I \in K^1_\lambda$ there is $M_I \in K_\lambda$ and $\bar{a}_t \in M_I$ for $t \in I$, which satisfies (in M_I) some instances of $\pm\varphi_\ell$.

We may demand on M_I:

(0) nothing more (except the restriction on the cardinality),

{3.1} (1) $\langle \bar{a}_t : t \in I\rangle$ behaves nicely: like a skeleton (see 3.1(1)), <u>or</u> even

(2) M_I is "embedded" in a model built from I in a simple way (Δ–represented;
{2.2} see Definition 2.4(c)), <u>or</u>

(3) M_I is built from I in a simple way, an the extreme case being $EM_\tau\ (I,\Phi)$;
{1.6} see Definition 1.8 where $\tau = \tau(M_I)$ of course.

Now even for (0) we can have meaningful theorems (see [Sh:309, 1.1] and [Sh:309, 1.3]); but we cannot have all we would naturally like to have — see [Sh:309, 1.8] (i.e., we cannot prove much better results in this direction, as shown by a consistency proof).

Though it looks obvious by our formulation, experience shows that we must stress that the formulas φ_ℓ need not be first order, they just have to have the right vocabulary (but in results on "no M_i embeddable in M_j" this usually means embedding preserving $\pm\varphi_\ell$ (but see the proof of [Sh:331, 3.22(2)]. So they are just properties of sequences in the structures we are considering preserved by the morphism we have in mind.

Another point is that though it would be nice to prove

$$[I \ncong J \quad \Rightarrow \quad M_I \ncong M_J];$$

this does not seem realistic. What we do is to construct a family

$$\{I_\alpha : \alpha < 2^\lambda\} \subseteq K^1_\lambda$$

such that for $\alpha \neq \beta$, in a strong sense I_α is not isomorphic to (or not embeddable into) I_β (see 2.5, 3.8, [Sh:331, 1.1], [Sh:331, 1.4]), such that now we have {3.4}
$M_{I_\alpha}, M_{I_\beta}$ not isomorphic for $\alpha \neq \beta$. We are thus led to the task of constructing such I_α's, which, probably unfortunately, splits to cases according to properties of the cardinals involved. Sometimes we just prove $\{\alpha : M_\alpha \cong M_\beta\}$ is small for each β.

A point central to [Sh:E58], [Sh:421], [Sh:511],[Sh:384] and [Sh:482] but incidental here, is the construction of a model which is for example rigid or has few endomorphisms, etc. In particular in [Sh:511] we could use linear order for "the gluing".

The methods here can be combined with [Sh:220] or [Sh:188] to get non-isomorphic $\mathbb{L}_{\infty,\lambda}$–equivalent models of cardinality λ; Instead "$\mathbb{L}_{\infty,\lambda}$-equivalent non-isomorphic model of T" we can consider equivalence by stronger games, e.g. $\mathrm{EF}_{\alpha,\lambda}$-equivalence started in Hyttinen-Tuuri [HT91], and then Hyttinen-Shelah [HySh:474], [HySh:529], [HySh:602]; See Väänänen [Vaa95] or such games.

In the next few paragraphs we survey the results of this paper. In this survey we omit some parameters for at various defined notions. These parameters are essential for an accurate statement of the theorems. We suppress them here trying to make it easier reading while still communicating essential points.

In §1 we mainly represent E.M. models. This is how in a natural way we construct a model from an "index model". The proof of existence many times rely on partition theorems. We give definition, deal with the framework, quote important cases, and present general theorems for getting the E.M. models, i.e., the templates; we then, as an example, deal with random graphs for theories in $\mathbb{L}_{\kappa^+,\omega}$.

In §2 we discuss a method of "representability" (from [Sh:136]). This is a natural way to get for "a model gotten from an index model I" that "I is complicated" implies "M is complicated". We discuss applications (to separable reduced Abelian $\dot{p}$-groups and Boolean algebras), but the aim is to explain; full proofs of full results will appear later (see [Sh:331, §3], [Sh:511] respectively). We introduce two strongly contradictory notions, the Δ–representability of a structure M in the "free algebraq" (i.e., "polynomial algebra") of an index model (Definition 2.4) and the {2.2}
$\varphi(\bar{x}, \bar{y})$–unembeddability of one index model in another. Now, to show that a class K has many models it suffices if for some formula φ, one first shows that (a) an index class K_1 has many pairwise φ–unembeddable structures, second that (b) for each $I \in K_1$, there is a model M_I which is Δ–representable in the free algebra on I, and finally that (c) if $M_I \cong M_J$ and M_J is Δ–represented in the free algebras on J then I is φ–embeddable in J.

However, for building for example a rigid model of cardinality λ, it is advisable to use $\langle I_\alpha : \alpha < \lambda \rangle$ such that I_α is φ–unembeddable into $\sum_{\beta \neq \alpha} I_\beta$. (See 2.15, 2.16, {2.7}
more in [Sh:511]). Generally having suitable sequence of $I \in K_1$ is expressed by "K_1 has a suitable bigness property".

Now, §3 does not depend on §2. The point is that in this section our non-isomorphisms proofs are so strong that they do not need "representability", we use a much weaker property. In §3 we extend and simplify the argument showing that an unstable first order theory T has 2^λ models of cardinality λ if $\lambda \geq |T| + \aleph_1$. Rather than constructing Ehrenfeucht–Mostowski models we consider a weaker notion — that a linear order J indexes a weak (κ, φ)-skeleton like sequence in a model M. In this section, K_1 is the class of linear orders. The formula $\varphi(\bar{x}, \bar{y})$ need not be
{3.10} first order and after 3.20 may have infinitely many arguments. Most significantly we make no requirement on the means of definition of the class K of models (for example first order, $\mathbb{L}_{\infty,\infty}$, etc). We require only that for each linear order J there are an $M_J \in K$ and a sequence $\langle \bar{a}_s : s \in J \rangle$ which is weakly (κ, φ)–skeleton like in M_J.

{2.21} Note that having bigness properties for $K^\kappa_{\rm tr}$ implies the ones for $K_{\rm or}$ see 2.25, Ehrenfeucht and Mostowski [EM56] built what are here $EM_\tau(I, \Phi)$ for I a linear order and first order T where $\tau = \tau_T$. Ehrenfeucht [Ehr57], [Ehr58] (and Hodges in [Hod73] improved the set theoretic assumption) proved that if T has the property (E) then it has at least two non-isomorphic models (this property is a precursor of being unstable). Recall that the property (E) says that: some a formula $R(x_1, \ldots x_n)$ is asymmetric on some infinite subset of some model of T; note that (E) is not equivalent to being unstable as the theory of random graphs fail it. Morley [Mor65] prove that for well ordered I, the model is stable in appropriate cardinalities, to prove that non-totally transcendental countable theories are not categorical in any $\lambda > \aleph_0$. See more in [Sh:c, VII,VIII]; by it if $T \subseteq T_1$ are unstable, complete first order and $\lambda \geq |T_1| + \aleph_1$ then T_1 has 2^λ models of cardinality λ with pairwise non-isomorphic reducts to τ_T. On the cases for $\mathbb{L}_{\chi^+,\omega}, \lambda > \chi$, see Grossberg-Shelah [GrSh:222], [GrSh:259] which continue [Sh:11].

This paper is a revised version of sections §1,§2,§3 of chapter III of [Sh:300].

§ 1. Models from Indiscernibles

We survey here [Sh:a, Ch.VIII,§3], which was the starting point for the other works appearing or surveyed in this paper and [Sh:309], [Sh:363]. So we concentrate on building many models for first order theories, using E.M. models, i.e., in all respects taking the easy pass. Our aim there was

{1.1}

Theorem 1.1. *If T is a complete first order theory, unstable and $\lambda \geq |T| + \aleph_1$, then* $\dot{\mathbb{I}}(\lambda, T) = 2^\lambda$,

where

{1.2}

Definition 1.2. T is unstable when for some first order formula $\varphi(\bar{x}, \bar{y})$ ($n = \ell g(\bar{x}) = \ell g(\bar{y})$) in the vocabulary τ_T of T of course, for every λ there is a model M of T and $\bar{a}_i \in {}^n M$ for $i < \lambda$ such that

$$M \models \varphi[\bar{a}_i, \bar{a}_j] \text{ Iff } i < j \ (< \lambda).$$

{1.3new}

Definition 1.3. For a theory T and vocabulary $\tau \subseteq \tau_T$,
$\dot{\mathbb{I}}(\lambda, T) =$ the number of models of T of cardinality λ, up to isomorphism,
$\dot{\mathbb{I}}_\tau(\lambda, T) =$ the number of τ-reducts of models of T of cardinality λ,

up to isomorphism.

{1.4new}

Definition 1.4. 1) For a class K of models and set Δ of formulas:

$\dot{\mathbb{I}}(\lambda, K) =$ the number of models in K of cardinality λ up to isomorphism,

$\dot{\mathbb{I}}(K) =$ the number of models in K up to isomorphism,

$\dot{I}\dot{E}_\Delta(\lambda, K) = \sup\{\mu$: there are $M_i \in K_\lambda$, for $i < \mu$, such that for $i \neq j$ there is no Δ-embedding of M_i to $M_j\}$.

see part (2); and we may write τ instead $\Delta = \mathbb{L}(\tau_K)$, may omit Δ when it is $\mathbb{L}(\tau_M)$.
2) $f : M \longrightarrow N$ is a Δ-embedding (of M into N) iff (f is a function from $|M|$ into $|N|$ and) for every $\varphi(\bar{x}) \in \Delta$ and $\bar{a} \in {}^{\ell g(\bar{a})}|M|$, we have:

$$M \models \varphi[\bar{a}] \Rightarrow N \models \varphi[f(\bar{a})].$$

(so if $(x \neq y) \in \Delta$ then f is one to one).

{1.5new}

Definition 1.5. 1) A sentence $\psi \in \mathbb{L}_{\chi^+,\omega}$ is ∂-unstable iff there are $\alpha < \partial$ and a formula $\varphi(\bar{x}, \bar{y})$ from $\mathbb{L}_{\chi^+,\omega}$ with $\ell g(\bar{x}) = \ell g(\bar{y}) = \alpha$ such that ψ has the φ-order property, i.e., for every λ there is a model M_λ of ψ and a sequence $\bar{a}_\zeta$ of length α from M_λ such that for $\zeta, \xi < \lambda$ we have

$$M_\lambda \models \varphi[\bar{a}_\zeta, \bar{a}_\xi] \Leftrightarrow \zeta < \xi.$$

If $\partial = \aleph_0$ we may omit it.
2) For κ regular and T first order, we say $\kappa < \kappa(T)$ iff there are first order formulas $\varphi_i(\bar{x}, \bar{y}_i) \in \mathbb{L}(\tau_T)$ for $i < \kappa$ and for every λ there is a model M_λ of T and for $i \leq \kappa, \eta \in {}^i\lambda$ a sequence $\bar{a}_\eta$ from M_λ, with

$$i < \kappa \Rightarrow \ell g(\bar{a}_\eta) = \ell g(\bar{y}_i)$$

$$i = \kappa \Rightarrow \ell g(\bar{a}_\eta) = \ell g(\bar{x})$$

such that: if $\nu \in {}^i\lambda,\ \eta \in {}^\kappa\lambda,\ \nu \triangleleft \eta$ then $M_\lambda \models \varphi_{i+1}[\bar{a}_\eta, \bar{a}_{\nu\hat{}\langle\alpha\rangle}] \Leftrightarrow \eta(i) = \alpha$. [We shall not use this except in 1.11 below.] {1.8}
3) T, a first order theory, is unsuperstable if $\aleph_0 < \kappa(T)$ [but we shall use it only in 1.11]. {1.8}

* * *

{1.5}
Definition 1.6. 1) $\langle \bar{a}_t : t \in I\rangle$ is Δ-indiscernible (in M) iff

- (a) I is an index model (usually linear order or tree), i.e., it can be any model but its role will be as an index set,
- (b) Δ is a set of formulas in the vocabulary of M (i.e. in $\mathscr{L}_{\tau(M)}$ for some logic $\mathscr{L}$)
- (c) the Δ-type in M of $\bar{a}_{t_0}\hat{}\ldots\hat{}\bar{a}_{t_{n-1}}$ for any $n < \omega$ and $t_0, \ldots t_{n-1} \in I$) depends only on the quantifier free type of $\langle t_0, \ldots, t_{n-1}\rangle$ in I.

Recall that the Δ-type of $\bar{a}$ in M is $\{\varphi(\bar{x}) \in \Delta : M \models \varphi(\bar{a})\}$, where $\bar{a}, \bar{x}$ are indexed by the same set. So the length of $\bar{a}_t$ depend just on the quantifier free type which $\ell g(\bar{a}_t)$ realizes in I.

If we allow $\varphi(\bar{x}) \in \Delta, \kappa > \alpha = \ell g(\bar{x}) \geq \omega$ and we allow $\langle t_i : i < \alpha\rangle$ above, then we say (Δ, κ)-indiscernible.
2) For a logic $\mathscr{L}$, "$\mathscr{L}$-indiscernible" will mean Δ-indiscernible for the set of $\mathscr{L}$-formulas in the vocabulary of M. If $\Delta, \mathscr{L}$ are not mentioned we mean first order logic.
3) Notation: Remember that if $\bar{t} = \langle t_i : i < \alpha\rangle$ then $\bar{a}_{\bar{t}} = \bar{a}_{t_0}\hat{}\bar{a}_{t_1}\hat{}\ldots$.

Many of the following definitions are appropriate for counting the number of models in a pseudo elementary class. Thus, we work with a pair of vocabularies, $\tau \subseteq \tau_1$. Often τ_1 will contain Skolem functions for a theory T which is $\subseteq \mathscr{L}(\tau)$.

{1.5f}
Convention 1.7. For the rest of this section all predicates and function symbols have finite number of places (and similarly $\varphi(\bar{x})$ means $\ell g(\bar{x}) < \omega$).

{1.6}
Definition 1.8. 1) $M = \mathrm{EM}(I, \Phi)$ iff for some vocabulary $\tau = \tau_\Phi = \tau(\Phi)$ (called L_1^Φ in [Sh:a, Ch.VII]) and sequences $\bar{a}_t (t \in I)$ we have:

- (i) M is a τ_Φ-structure and is generated by $\{\bar{a}_t : t \in I\}$,
- (ii) $\langle \bar{a}_t : t \in I\rangle$ is quantifier free indiscernible in M,
- (iii) Φ is a function, taking (for $n < \omega$) the quantifier free type of $\bar{t} = \langle t_0, \ldots, t_{n-1}\rangle$ in I to the quantifier free type of $\bar{a}_{\bar{t}} = \bar{a}_{t_1}\hat{}\ldots\hat{}\bar{a}_{t_n}$ in M (so Φ determines τ_Φ uniquely).

2) A function Φ as above is called a template and we say it is proper for I if there is M such that $M = \mathrm{EM}(I, \Phi)$. We say Φ is proper for K if Φ is proper for every $I \in K$, and lastly Φ is proper for (K_1, K_2) if it is proper for K_1 and $\mathrm{EM}^1(I, \Phi) \in K_2$ for $I \in K_1$.
3) For a logic $\mathscr{L}$, or even a set $\mathscr{L}$ of formulas in the vocabulary of M, we say that Φ is almost $\mathscr{L}$-nice (for K) iff it is proper for K and:

(∗) for every $I \in K, \langle \bar{a}_t : t \in I\rangle$ is $\mathscr{L}$-indiscernible in $EM(I,\Phi)$.

4) In part (3), Φ is $\mathscr{L}$-nice iff it is almost $\mathscr{L}$-nice and

(∗∗) for $J \subseteq I$ from K we have $\mathrm{EM}(J,\Phi) \prec_{\mathscr{L}} \mathrm{EM}(I,\Phi)$.

5) In part (3) we say that Φ is $(\mathscr{L},\tau)$-nice when $\tau \subseteq \tau_\Phi$, it is almost $\mathbb{L}$-nice and (see 1.9(1)) {1.7}

(∗∗∗) for $I \subseteq J$ from K we have $\mathrm{EM}_\tau(J,\Phi) \prec_{\mathscr{L}} \mathrm{EM}_\tau(I,\Phi)$.

In the book [Sh:a], always $\mathbb{L}_{\omega,\omega}(\tau_\Phi)$-nice Φ were used and $\mathrm{EM}(I,\Phi), \mathrm{EM}_\tau(I,\Phi)$ here are $\mathrm{EM}^1(I,\Phi), \mathrm{EM}(I,\Phi)$ there.

{1.7}

Definition 1.9. 1) $\mathrm{EM}_\tau(I,\Phi) = \mathrm{EM}(I,\Phi){\restriction}\tau$, i.e., τ-reduct of $\mathrm{EM}(I,\Phi)$, where $\tau \subseteq \tau_\Phi$. We may omit τ when clear from the context and write $\mathrm{EM}(I,\Phi)$. Saying "an EM-model will mean "a model of the form $\mathrm{EM}_\tau(I,\Phi)$" where Φ, I, τ are understood from the context.
2) We identify $I \subseteq {}^{\kappa\geq}\lambda$ which is closed under initial segments, with the model $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$, where
$P_\alpha = I \cap {}^\alpha\lambda$,
$\rho = \eta \cap \nu$ if $\rho = \eta{\restriction}\alpha$ for the maximal α such that $\eta{\restriction}\alpha = \nu{\restriction}\alpha$,
$\triangleleft$= being initial segment of (including equality),
$<_{\mathrm{lx}}$= the lexicographic order.
3) Similarly to (2), for any linear order J, every $I \subseteq {}^{\kappa\geq}J$ which is closed under initial segments is identified with $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$ ($\leq_{\mathrm{lx}}$ is still well defined).
4) K^κ_{tr} is the class of such models, i.e., models isomorphic to such I, i.e., to $(I, P_\alpha, \cap, <_{\mathrm{lx}}, \triangleleft)_{\alpha\leq\kappa}$ for some $I \subseteq {}^{\kappa\geq}J$ which is closed under initial segments, J a linear order (tr stands for tree). We call I standard if J is an ordinal or at least well ordered.
5) K_{or} is the class of linear orders.

{1.7A}

Remark 1.10. The main case here is $\kappa = \aleph_0$. We need such trees for $\kappa > \aleph_0$, for example if we would like to build many κ-saturated models of T, $\kappa(T) > \kappa$, κ regular. If $\kappa(T) \leq \kappa$ there may be few κ–saturated models of T.

In [Sh:a, Ch.VIII] we have also proved:

{1.8}

Lemma 1.11. *1) If $T \subseteq T_1$ are complete first order theories, T is unstable as exemplified by $\varphi = \varphi(\bar{x},\bar{y})$, say $n = \ell g(\bar{x}) = \ell g(\bar{y})$, then for some template Φ proper for the class of linear orders and nice for first order logic, $|\tau_\Phi| = |T_1| + \aleph_0$ and for any linear order I and $s,t \in I$ we have*

$$\mathrm{EM}(I,\Phi) \vDash \varphi[\bar{a}_s, \bar{a}_t] \textit{ iff } I \vDash s < t.$$

2) If $T \subseteq T_1$ are complete first order theories and T is unsuperstable, then there are first order $\varphi_n(\bar{x},\bar{y}_n) \in \mathbb{L}(\tau_T)$ and a template Φ proper for every $I \subseteq {}^{\omega\geq}\lambda$ such that for any such I we have:

(a) *$\eta \in {}^\omega\lambda, \nu \in {}^n\lambda$ implies $\mathrm{EM}(I,\Phi) \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu]$ iff $\eta{\restriction}n = \nu$*
(b) *$\mathrm{EM}(I,\Phi) \models T_1$ and Φ is $\mathbb{L}_{\omega,\omega}(\tau_\Phi)$-nice, $|\tau_\Phi| = |T_1| + \aleph_0$ (note that for η_1,η_2 of the same length, $\eta_1 \neq \eta_2 \Rightarrow \bar{a}_{\eta_1} \neq \bar{a}_{\eta_2}$)*[1].

[1]In fact $EM^1(I,\Phi)$ is well defined for $I \in K^\omega_{\mathrm{tr}}$.

3) If $T \subseteq T_1$ are complete first order theories and $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$ <u>then</u>

- (a) *there is a sequence of first order formulas $\varphi_i(\bar{x}, \bar{y}_i)$ (for $i < \kappa$) witnessing $\kappa < \kappa(T)$ i.e. there are a model M of T and sequences $\bar{a}_\eta$ for $\eta \in {}^{\kappa \le}\lambda$ such that for $\eta \in {}^\kappa\lambda, \nu \in {}^i\lambda, i < \kappa, \alpha < \lambda$ we have $M \models \varphi_i[\bar{a}_\eta, \bar{a}_{\nu \char`^ \langle \alpha \rangle}]$ <u>iff</u> $\alpha = \eta(i)$*
- (b) *for any $\langle \varphi_i(\bar{x}, \bar{y}) : i < \kappa \rangle$ as in (a) there is a nice template Φ proper for K^κ_{tr} such that for any λ:*
  - (α) *if $\eta \in {}^\kappa\lambda$, $\nu \in {}^i\lambda, i < \kappa, \alpha < \lambda$ then*
  $$\mathrm{EM}({}^{\kappa \ge}\lambda, \Phi) \models \varphi_i[\bar{a}_\eta, \bar{a}_{\nu \char`^ \langle \alpha \rangle}] \text{ iff } \alpha = \eta(i);$$
  - (β) $\mathrm{EM}(I, \Phi) \models T_1$,
  - (γ) Φ *is* $\mathbb{L}_{\omega,\omega}(\tau_\Phi)$*-nice,*
  - (δ) $|\tau_\Phi| = |T_1| + \aleph_0$.

Proof. See [Sh:a, Ch.VII,§3], but here we can consider the conclusion as the definition of unstable or unsuperstable and of $\kappa < \kappa(T)$, respectively. $\square_{1.11}$

{1.12} *Remark* 1.12. On K^ω_{tr} for $\mathbb{L}_{\lambda^+,\omega}$ we need the Ramsey property defined below, see
{1.14} 1.19 (and 1.20+ 1.21).

In [Sh:a, Ch.VIII,§2] we actually proved:

{1.9}
{1.8} **Theorem 1.13.** *1) If $\lambda > |\tau_\Phi|$, and $\Phi, \tau_\Phi, \langle \varphi_n : n < \omega \rangle$ are as in Lemma 1.11(2) (and Φ is almost $\mathbb{L}_{\omega,\omega}$-nice) <u>then</u>: we can find $I_\alpha \subseteq {}^{\omega \ge}\lambda$ (for $\alpha < 2^\lambda$), $|I_\alpha| = \lambda$ such that for $\alpha \ne \beta$ there is no one-to-one function from $\mathrm{EM}(I_\alpha, \Phi)$ onto $\mathrm{EM}(I_\beta, \Phi)$ preserving the $\pm\varphi_n$ for $n < \omega$.*
2) If λ is regular, also for $\alpha \ne \beta$ there is no one-to-one function from $\mathrm{EM}(I_\alpha, \Phi)$ into $\mathrm{EM}(I_\beta, \Phi)$ preserving the $\pm\varphi_n$ for $n < \omega$.
3) The φ_n's do not need to be first order, just their vocabularies should be $\subseteq \tau_\Phi$. But instead "Φ is almost $\mathbb{L}_{\omega,\omega}(\tau_\Phi)$-nice" we need just "$\Phi$ is almost $\{\pm\varphi_n(\ldots, \sigma_\ell(\bar{x}_\ell), \ldots)_{\ell < \ell(n)} : n < \omega, \sigma_\ell$ terms of $\tau_\Phi\}$-nice" and we should still demand (as in all this section)

- $(*)$ *the $\bar{a}_\eta$ are finite (and we are assuming that the functions are finitary).*

{1.8} *4) So if as in Lemma 1.11, $\varphi_n \in \mathscr{L}(\tau)$ <u>then</u> $\{M_\alpha {\restriction} \tau : \alpha < 2^\lambda\}$ are 2^λ non-isomorphic models of T of cardinality λ.*

Proof. This is proved in [Sh:a, §2 of Ch.VIII] (though it is not explicitly claimed, it was used elsewhere and there is no need to change the proofs). Also we shall later
{1.9} (in [Sh:331, 3.1] we prove better theorems, mainly getting 1.13(2) also for singular λ. $\square_{1.13}$

{1.9A}
{1.9} *Remark* 1.14. 1) Applying 1.13, we usually look at the τ-reducts of the models $\mathrm{EM}^1(I, \Phi)$ as the objects we are interested in, where the φ_n's are in the vocabulary τ. E.g., for $T \subseteq T_1$ first order, T unsuperstable, we use $\varphi_n \in \mathbb{L}(T)$.
2) The case $\lambda = |\tau_\Phi|$ is harder. In [Sh:a, Ch.VIII,§2,§3], the existence of many models in λ is proved for T unstable, $\lambda = |\tau_\Phi| + \aleph_1$ and there (in some cases) "T_1, T first order" is used.

* * *

{1.8} How do we find templates Φ as required in 1.11 and parallel situations?

Quite often in model theory, partition theorems (from finite or infinite combinatorics) together with a compactness argument (or a substitute) are used to build models. Here we phrase this generally. Note that the size of the vocabulary (μ in the "(μ,λ)-large")) is a variant of the number of colours, whereas λ is usually μ; it becomes larger if our logic is complicated.

{1.10}

Definition 1.15. Fix a class K (of index models) and a logic (or logic fragment) $\mathscr{L}$.

1) An index model $I \in K$ is called (μ,λ,χ)-Ramsey for $\mathscr{L} i$ when:

- (a) the cardinality of I is $\le \chi$ and every qf (= quantifier free) type p (in $\tau(K)$) which is realized in some $J \in K$ is realized in I,
- (b) for every vocabulary τ_1 of cardinality $\le \mu$, a τ_1-model M_1 and an indexed set $\langle \bar b_t : t \in I\rangle$ of finite sequences from $|M_1|$ with $\ell g(\bar b_t)$ determined by the quantifier free type which t realizes in I there is a template Φ, which is proper for K, with $|\tau_\Phi| \le \lambda$ such that ($\tau_1 \subseteq \tau_\Phi$ and):
  - $(*)$ for any $\tau(K)$-quantifier free type $p, I_1 \in K$ and $s_0,\dots,s_{n-1} \in I_1$ for which $\langle s_0,\dots,s_{n-1}\rangle$ realizes p in I_1 and for any formula
  $$\varphi = \varphi(x_0,\dots,x_{m-1}) \in \mathscr{L}(\tau_1)$$
  and τ_1-terms $\sigma_\ell(\bar y_0,\dots,\bar y_{n-1})$ for $\ell = 0,\dots,m-1$ we have
  - $(**)$ if for every $t_0,\dots,t_{n-1} \in I$ such that $\langle t_0,\dots,t_{n-1}\rangle$ realizes p in I we have $M_1 \models \varphi[\sigma_0(\bar b_{t_0},\dots,\bar b_{t_{n-1}}),\sigma_1(\bar b_{t_0},\dots,\bar b_{t_{n-1}}),\dots,\sigma_{m-1}(\bar b_{t_0},\dots,\bar b_{t_{n-1}})]$ then $\mathrm{EM}(I_1,\Phi) \models \varphi[\sigma_0(\bar a_{s_0},\dots,\bar a_{s_{n-1}}),\sigma_1(\bar a_{s_0},\dots,\bar a_{s_{n-1}}),\dots,\sigma_{m-1}(\bar a_{s_0},\dots,\bar a_{s_{n-1}})]$.

2) The class K of index models is called explicitly (μ,λ,χ)-Ramsey for $\mathscr{L}$ iff some $I \in K$ of cardinality $\le \chi$ is (μ,λ)-Ramsey for $\mathscr{L}$. A class $K' \subseteq K$ of index models is called (μ,λ,i,χ)-Ramsey (inside K, which is usually understood from context), iff

- (a) every member of K' has cardinality $\le \chi$ and every quantifier free type p in $\tau(K')$ realized in some $J \in K$ is realized in some $I \in K'$,
- (b) for every vocabulary τ_1 of cardinality $\le \mu$ and τ_1-models M_I for $I \in K'$, and $\bar b_{I,t} \in {}^{k(I,t)}(M_I)$, where $k(I,t) < \omega$ depends just on $\mathrm{tp}_{\mathrm{qf}}(\langle t\rangle,\emptyset,I)$ there is a template Φ proper for K with $|\tau_\Phi| \le \lambda$ such that $\tau^1 \subseteq \tau_\Phi$ we have $(*)$ only in $(**)$ we should also say "every $I \in K'$". Let "(μ,χ)-Ramsey" mean "(μ,μ,χ)-Ramsey". Let "μ-Ramsey" mean "(μ,χ)-Ramsey for some χ".

3) In all parts of 1.15, 1.16, 1.17, if $\mathscr{L}$ is first order logic, we may omit it. {13Anew}{1.10}

4) For f : Card $\longrightarrow$ Card, K is f-Ramsey iff it is $(\mu,f(\mu))$-Ramsey for $\mathscr{L}$ for every (infinite) cardinal μ. We say K is Ramsey for $\mathscr{L}$ if it is (μ,μ)-Ramsey for $\mathscr{L}$ for every μ.

5) We say K is $*$-Ramsey for $\mathscr{L}$ if it is f-Ramsey for $\mathscr{L}$ for some f : Card $\longrightarrow$ Card.

{13Anew}

Definition 1.16. Let K be a class of (index) models and $\mathscr{L}$ a logic.

1) We say $I \in K$ is (almost) $\mathscr{L}$-nicely (μ,λ,χ)-Ramsey for K iff 1.15(1) holds, and Φ is (almost) $\mathscr{L}$-nice. Similarly replacing I by a set $K' \subseteq K$. {1.10}

2) The class K is called explicitly (almost) $\mathscr{L}$-nice (μ, λ, χ)-Ramsey <u>iff</u> some $I \in K$ is (almost) $\mathscr{L}$-nicely (μ, λ, χ)-Ramsey.
3) For f : Card $\longrightarrow$ Card, we say K is (almost) $\mathscr{L}$-nicely f-Ramsey <u>iff</u> for every μ we have: K is (almost) $\mathscr{L}$-nicely $(\mu, f(\mu))$-Ramsey for every (infinite) cardinal μ. We omit f for the identity function.
4) We say K is (almost) $\mathscr{L}$-nicely $*$-Ramsey <u>iff</u> for some f, it is (almost) $\mathscr{L}$-nicely f-Ramsey.

{1.10A}
{1$Ant0} **Definition 1.17.** In 1.15, 1.16 we add "strongly" if we strengthen 1.15(1) by asking in $(*)$ in addition that for any $\tau(K)$-quantifier free type p and $s_0, \ldots, s_{n-1} \in I_1$ such that $\langle s_0, \ldots, s_{n-1} \rangle$ realizes p in I_1) we can find some $t_0, \ldots, t_{n-1}$ suitable for all $\varphi, \sigma_0, \ldots$ simultaneously (this helps for omitting types).

{1.11}
Theorem 1.18. *1) For $\mathbb{L}_{\omega,\omega}$, the class of linear orders is nicely Ramsey, moreover every infinite order is (μ, λ)-Ramsey for any $\mu \leq \lambda$.*
2) For $\mathbb{L}_{\omega_1,\omega}$ the class of linear orders is nicely $*$-Ramsey. In fact nicely f-Ramsey for the functions $f(\mu) = \beth_{(2^\mu)^+}$.
3) For any fragment of $\mathbb{L}_{\lambda^+,\omega}$ or of $\Delta(\mathbb{L}_{\lambda^+,\omega})$ (see, e.g. [Mak85]*) of cardinality λ, the class of linear orders is nicely f-Ramsey when $f(\mu) = \beth_{(2^\mu)^+}$, even strongly; moreover is strongly nicely f-Ramsey.*
{1.7} *4) $K^\omega_{\rm tr}$ (and even $K^\kappa_{\rm tr}$) is Ramsey for $\mathbb{L}_{\omega,\omega}$. For definitions of $K^\omega_{\rm tr}$ see 1.9 above.*
5) The class $K_{\rm org}$ of linear ordered graphs is explicitly nicely Ramsey. The class $K_{{\rm or},n}$ of linear orders expanded by an n-place relation is explicitly nicely Ramsey.

Proof. 1) This is the content of the Ehrenfeucht-Mostowski proof that E.M. models
{1.8} exist and it use the finitary Ramsey theorem as used in the proof of 1.11(1). see [Sh:c, Ch.VII].
2) By repeating the proof of Morley's omitting type theorem which use the Erdös-Rado theorem, see [Sh:c, Ch.VII,§5]; the to uncountably vocabulary (and many types) is a generalization noted by Chang.
{1.11} 3) Like 1.18(2); see [Sh:16, Theorem 2.5], and more in [GrSh:222], [GrSh:259].
4) By [Sh:c, Ch.VII,§3] (we use the compactness of $\mathbb{L}_{\omega,\omega}$ and partition properties of trees).
5) By the Nessetril-Rodl theorem (see e.g. [GRS90]). $\square_{1.18}$

By Grossberg-Shelah [GrSh:238] (improving [Sh:a, Ch.VII], where compactness of the logic $\mathscr{L}$ was used, but no large cardinals):

{1.12}
Theorem 1.19. *$K^\omega_{\rm tr}$ is the nicely $*$-Ramsey for $\mathbb{L}_{\lambda^+,\omega}$ <u>iff</u> for example there are arbitrarily large measurable cardinals (in fact, large enough cardinals consistent with the axiom $\mathbf{V} = \mathbf{L}$ suffice).*

We shall not repeat the proof.

{1.13}
Lemma 1.20. *Suppose K_1, K_2, K_3 are classes of models, Φ is proper template for $(K_1, K_2), \Psi$ proper template for (K_2, K_3) <u>then</u> there is a unique template Θ that is proper for (K_1, K_3) and for $I \in K_1$*

$$\mathrm{EM}(I, \Theta) = \mathrm{EM}(\mathrm{EM}(I, \Phi), \Psi)).$$

We write Θ as $\Psi \circ \Phi$.

Proof. Straightforward. $\square_{1.20}$

{1.14}

Lemma 1.21. *Suppose K is a class of index models, $\tau = \tau(K)$ and*

- $(*)$ *there is a template Ψ proper for K such that $|\tau_\Psi| = |\tau_K| + \aleph_0$ and for $I \in K$: $\mathrm{EM}_\tau(I,\Psi) \in K$ and $J =: \mathrm{EM}_\tau(I,\Psi)$ is strongly $(\aleph_0, \mathrm{qf})$-homogeneous over I, i.e., if $\bar t = \langle t_1,\dots,t_n\rangle, \bar s = \langle s_1,\dots,s_n\rangle$ realize the same quantifier free type in I, then some automorphism of J takes $\bar a_{\bar t}$ to $\bar a_{\bar s}$.*

We conclude that: if K is (μ,λ,χ)-Ramsey for $\mathscr{L}$ and $|\tau_\Psi| \le \mu$ <u>then</u> K is almost $\mathscr{L}$-nicely (μ,λ,χ)-Ramsey for $\mathscr{L}$.

Proof. Just chase the definitions. $\square_{1.21}$

{1.14A}

Remark 1.22. 1) E.g. for $\mathscr{L} \subseteq \mathbb{L}_{\omega_1,\omega}$ we get in 1.21 even $\mathscr{L}$-nice. {1.14}
2) The assumption $(*)$ of 1.21(1) holds for $K_{\mathrm{or}}, K^\omega_{\mathrm{tr}}, K^\kappa_{\mathrm{tr}}$ (as well as the other K's from [Sh:331]). {1.14}

{1.15}

Conclusion 1.23. *Assume that*

- (a) *K_{or} is (μ,λ)-Ramsey for $\mathscr{L}$,*
- (b) *T is an $\mathscr{L}$-theory (in the vocabulary $\tau(T)$), $|\tau(T)| \le \mu$,*
- (c) *$\varphi_\ell(\bar R_\ell, \bar x, \bar y) \in \mathscr{L}(\tau(T) \cup \{\bar R_\ell\})$ for $\ell = 1,2$ (and $\bar R_\ell$ is disjoint from $\tau(T)$ and from $\bar R_{3-\ell}$), and $T \cup \{\varphi_1(\bar R_1,\bar x,\bar y), \varphi_2(\bar R_2,\bar x,\bar y)\}$ has no model,*
- (d) *for every $I \in K_{\mathrm{or}}$ there is a model M_I of T, and $\bar a_t \in {}^{\omega>}M$ for $t \in I$ such that:*

$$t < s \Rightarrow M \models (\exists \bar R_1)\varphi_1(\bar R_1, \bar a_t, \bar a_s)$$

and

$$s < t \Rightarrow M \models (\exists \bar R_2)\varphi_2(\bar R_2, \bar a_s, \bar a_t).$$

<u>Then</u> for $\lambda \ge \mu + \aleph_1, \dot{\mathbb{I}}(\lambda, T) = 2^\lambda$.

Proof. Obvious by now (mainly 1.18(3) and 3.20(3) below). $\square_{1.23}$ {3.10} {1.16}

Conclusion 1.24. *The parallel of 1.23 for K^ω_{tr} instead K_{or} holds if $\lambda > \mu$.* {1.15}

Proof. By 1.13 (or use [Sh:331]). $\square_{1.24}$ {1.9}

* * *

{1.22new}

Discussion 1.25. We return to the general Ramsey properties for other classes (not just linear orders and trees). For compact logics, finitary generalization of Ramsey theorem suffices. More generally, certainly it is nice to have them for $\mathscr{L} = \mathbb{L}_{\lambda^+,\omega}$, and even $\Delta(\mathbb{L}_{\lambda^+,\omega})$, so we need a partition theorem generalizing Erdös-Rado theorem, i.e., the case with infinitely many colours. We may for example look at ordered graph as index models, quite natural one. It consistently holds ([Sh:289]) though unfortunately it does not necessarily hold (Hajnal-Komjath [HK97]). However, our main point is that this is enough when the consistency is by forcing with e.g. complete enough forcing notion. So the consistency result in [Sh:289] yields a "real", ZFC theorem here. The following is an abstract version of the omitting type theorem.

{1.23new}

Claim 1.26. *Assume that*

(a) K is a definition of a class of models with vocabulary τ (the "index models"); where τ and the parameters in the definition belongs to $\mathscr{H}(\chi^+)$,

(b) $\mathscr{L}$ is a definition of a logic or logic fragment, the parameters of the definition belong to $\mathscr{H}(\chi^+)$ and $\lambda \geq \chi$,

{1.6} (c) in the definition of "Φ is (almost) $\mathscr{L}$-nice" for Φ proper for K with $|\tau_\Phi| < \chi$ (see 1.8(3), (4); so without loss of generality $\Phi \in \mathscr{H}(\chi)$) it suffices to restrict ourselves to I of cardinality $< \chi$,

(d) $\mathbb{P}$ is a forcing notion not adding subsets to λ, and preserving clauses (a), (b) and (c) (i.e., the definitions of K and $\mathscr{L}$ have these properties) and no new quantifier free complete n-types are realized in $I \in K$,

(e) in $\mathbf{V}^{\mathbb{P}}$, there is a member I^* of K, which is (χ, λ)-Ramsey for $\mathscr{L}$ (or an almost $\mathscr{L}$-nicely (χ, λ)-large) [or an $\mathscr{L}$-nicely (χ, λ)-Ramsey] or such a subset K' of K. For $I \in K$ let $\mathbf{P}^n_I = \{p : p$ is complete quantifier-free τ_K-type realized by some $\bar{t} \in {}^n I\}$. Let $\mathbf{P}_n$ be $\mathbf{P}^n_{I^*}$ or $\cup\{\mathbf{P}^n_I : I \in K\}$ according to the case above; if $q \in \mathbf{P}^n_I$ as exemplified by $\bar{t} \in {}^n I$ let $\mathrm{proj}_\ell(q)$ be the quantifier-free type which t_ℓ realizes in I

(f) $\tau_0 \in \mathscr{H}(\chi^+)$ is a vocabulary, $q_* \in \mathbf{P}_1$ and $\langle \Omega_q : q \in \mathbf{P}_n$ for some $n < \omega\rangle$ are such that for every $q \in \mathbf{P}_n$ we have: $\Omega_q \subseteq \{p(\bar{x}_0, \ldots, \bar{x}_{n-1}) : p$ an $\mathscr{L}(\tau_0)$-type in the variables $\bar{x}_0, \ldots, \bar{x}_{n-1}$ where $\bar{x}^\ell = \langle x_{\ell,i} : i < \alpha_{\mathrm{proj}_\ell(q)}\rangle \in \mathscr{H}(\chi^+)$ for some $n < \omega\}$, and in $\mathbf{V}^{\mathbb{P}}$, for every $I \in K$ (in the $\mathbf{V}^{\mathbb{P}}$'s sense) or just $I = I^*$ [or just $I \in K'$, according to the case in clause (e)], there is a τ_0-model M_I and $\bar{b}^I_t \in {}^{\alpha_t}(M_I)$ for $t \in I$ such that:
  - (α) $\alpha_t = \alpha_q$ if q is the quantifier free-τ_0-1-type which t realizes in I,
  - (β) for no $t_0, \ldots, t_{n-1} \in I$, does $\langle t_0, \ldots, t_{n-1}\rangle$ realize in I the complete quantifier free $\tau_\kappa - n$-type q and $p = p(\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \Omega_q$, does $\bar{b}^I_{t_0} {}^\frown \bar{b}^I_{t_1} {}^\frown \ldots {}^\frown \bar{b}^I_{t_{n-1}}$ realizes p and $\alpha_{t_\ell} = \ell g(\bar{x}_\ell)$.

<u>Then</u> we can conclude that there is a Φ such that:

ℵ Φ is an (almost) $\mathscr{L}$-nice template Φ, proper for K,

ℶ $\Phi \in \mathscr{H}(\lambda^+)$ hence also $\tau_\Phi \in \mathscr{H}(\lambda^+)$

ℷ if $M = \mathrm{EM}(I, \Phi)$, and $t_0, \ldots, t_{n-1} \in I$, and $\bar{t} = \langle t_0, \ldots, t_{n-1}\rangle$ realizes the complete quantifier free $\tau_\kappa - n$-type q <u>then</u> $\bar{a}_{\bar{t}}$ does not realize in M any $p \in \Omega_q$.

Proof. Straightforward. $\square_{??}$

{1.24new}

Claim 1.27. *Assume that*

(a) K is a class of (index) models,

(b) κ is a cardinal, for $\alpha < (2^\kappa)^+$ the structure $I_\alpha \in K$ realizes all quantifier free τ_K-types (in $< \omega$ variables) realized in some $I \in K$, and their number is $\leq \kappa$,

(c) if $n < \omega, \alpha < \beta < (2^\kappa)^+, N$ is a model, $\tau(N) \leq \kappa, \alpha^*_r < \kappa^+$ for a complete quantifier free $\tau_K - 1$-type r realized in $I_\beta, \bar{b}_r \in {}^{\alpha^*_r}N$, <u>then</u> we can find $I'_\alpha \subseteq I_\beta$ isomorphic to I_α such that

(∗) *if* $\bar t, \bar s \in {}^m(I'_\alpha), m \le n$ *and they realize the same quantifier free type in* I'_α *then* $\bar b_{\bar t} = \langle \bar b_{t_\ell} : \ell < m\rangle$ *and* $\bar b_{\bar s} = \langle \bar b_{\bar s_\ell} : \ell < m\rangle$ *realizes the same quantifiers free type in* N*,*

(*d*) τ *is a vocabulary,* $|\tau| \le \kappa, \psi \in \mathbb{L}_{\kappa^+,\omega}(\tau)$ *and* $\alpha^*_p < \kappa^+$ *for* p *a complete quantifier free* $\tau_K - 1$*-type realized in every* $I_\alpha, \mathscr{L} \subseteq \mathbb{L}_{\kappa^+,\omega}(\tau)$ *is a fragment of cardinality* κ *to which* ψ *belongs,*

(*e*) *for every* $\alpha < (2^\kappa)^+$*, there is a model* N_α *of* ψ *with* $\bar b^\alpha_t \in {}^{\alpha^*_t}(N_\alpha)$ *for* $t \in I_\alpha$*, where* $\alpha^*_t = \alpha^*_{\mathrm{tp}_{\mathrm{qf}}(t,\emptyset,I_\alpha)}$.

<u>Then</u> *there is a* $\mathscr{L}$*-nice template* Φ*, such that:*

⊗ *for* $I \in K, m < \omega$ *and* $\bar t \in {}^m I$ *we have: the* $\mathscr{L}$*-type which is* $\bar a_{\bar t}$*-realized in* $\mathrm{EM}(I,\Phi)$ *is realized in some* N_α *by some* $\bar b_{\bar s}$*, where* $\mathrm{tp}_{\mathrm{qf}}(\bar s,\emptyset,I_\alpha) = \mathrm{tp}_{\mathrm{qf}}(\bar t,\emptyset,I)$.

In other words, $\{I_\alpha : \alpha < (2^\kappa)^+\}$ *is* κ*-Ramsey for* $\mathscr{L}$*.*

Proof. We can expand N_α by giving names to all formulas in $\mathscr{L}$ and adding Skolem functions (to all first order formulas in the new vocabulary), so we have a τ^+-model $N^+_\alpha, \tau^+ \supseteq \tau = \tau(\psi), |\tau^+| \le \kappa$, correspondingly we extend $\mathscr{L}$ to a fragment $\mathscr{L}^+$ of $\mathbb{L}_{\kappa^+,\omega}(\tau^+)$ of cardinality κ.

By induction on $n < \omega$ we choose A_n, f_n, $\langle I^n_\alpha : \alpha \in A_n\rangle$ such that:

(*i*) A_n is an unbounded subset of $(2^\kappa)^+$,

(*ii*) f_n is an increasing function from $(2^\kappa)^+$ onto A_n such that $\alpha < f_n(\alpha)$,

(*iii*) I^n_α is a submodel of I_α isomorphic to $I_{f_n^{-1}(\alpha)}$,

(*iv*) if $n > m > 0, \alpha_1, \alpha_2 < (2^\kappa)^+, \bar t^1 \in {}^m(I^n_{f_n(\alpha_1)}), \bar t^2 \in {}^m(I^m_{f(\alpha_2)}), \mathrm{tp}_{\mathrm{qf}}(\bar t^1,\emptyset,I_{f_n(\alpha_1)}) = \mathrm{tp}_{\mathrm{qf}}(\bar t^2,\emptyset,I_{f_n(\alpha_2)})$, then the quantifier free type of $\bar b_{\bar t^1}$ in $N_{f_n(\alpha_1)}$ is equal to the quantifier free type of $\bar b_{\bar t^2}$ in $N_{f_n(\alpha_2)}$,

(*v*) $A_{n+1} \subseteq A_n$ and $\alpha \in A_{n+1} Rightarrow I^{n+1}_\alpha \subseteq I^{n+1}_\alpha$.

For $n = 0$ let $A_0 = (2^\kappa)^+$ and $I^0_\alpha = I_\alpha$.

For $n+1$, for each α we apply assumption (c) to $N_{f_n(\alpha+n+1)}, I^n_{f_n(\alpha+n+1)}, \langle \bar b^\alpha_t : t \in I^n_{f_n(\alpha+n+1)}\rangle$, getting $J^n_{f_n(\alpha+n+1)}$. We define an equivalence relation E_n on $(2^\kappa)^+$: $\alpha\ E_n\ \beta$ if and only if $\mathrm{tp}(\bar b^{f_n(\alpha+n+1)}_{\bar s},\emptyset,N_{f_n(\alpha+n+1)}) = \mathrm{tp}(\bar b^{f_n(\alpha+n+1)}_{\bar t},\emptyset,N_{f_n(\beta+n+1)})$, whenever $m < \omega, \bar s \in {}^m(J^n_{f_n(\alpha+n+1)}), \bar t \in {}^m(J^n_{f_n(\beta+n+1)})$ and $\mathrm{tp}_{\mathrm{qf}}(\bar s,\emptyset,I_{f_n(\alpha+n+1)}) = \mathrm{tp}_{\mathrm{qf}}(\bar t,\emptyset,I_{f_n(\beta+n+1)})$.

Clearly E_n has $\le 2^\kappa$ equivalence classes, so some equivalence class B is unbounded in $(2^\kappa)^+$. Let

$$A_{n+1} = \{f_n(\alpha+n+1) : \alpha \in B\}, \quad f_{n+1}(\alpha) = f_n(\min(B \setminus \alpha) + n + 1),$$

and $I^{n+1}_{f_n(\alpha+n+1)} = J^n_{f_n(\alpha+n+1)}$ for $\alpha \in B$.

Having completed the induction, clearly we have gotten Φ, as the limit. $\square_{1.27}$

Conclusion 1.28. *Assume that* {1.25new}

(*a*) $\mathscr{L}$ *a fragment of* $\mathbb{L}_{\kappa^+,\omega}$*,* T *is theory in* $\mathscr{L}(\tau)$*, and* $\theta \ge \kappa + |T| + |\tau| + |\mathscr{L}|$*,*

(*b*) $\varphi_\alpha = \varphi_\alpha(x_0,\dots,x_{k_\alpha-1}) \in \mathscr{L}(\tau)$ *for* $\alpha < \alpha(*)$ *(where* $\alpha(*) < \kappa^+$ *may be finite),* (*c*) *for some* $\mu > \theta$, *in any forcing extension of* $\mathbf{V}$ *by a* μ*-complete forcing notion the following holds for any* λ:

if R_α *is a subset of* $[\lambda]^{k_\alpha}$ *for* $\alpha < \alpha(*)$ *then for some model* M *of* T *and* $a_\alpha \in M$ *for* $\alpha < \lambda$ *we have: if* $\alpha < \alpha(*), \gamma_0 < \ldots < \gamma_{k_\alpha - 1} < \lambda$, *then* $M \models \varphi_\alpha[a_{\gamma_0}, \ldots, a_{\gamma_{k_\alpha - 1}}] \Leftrightarrow \{\gamma_0, \ldots, \gamma_{k_\alpha - 1}\} \in R_\alpha$

(*d*) *Let* K *be the class of* $(I, <, R_0, \ldots, R_\alpha, \ldots)_{\alpha < \alpha(*)}, (I, <)$ *linear order,* R_α *a symmetric irreflexive* k_α*-place relation on* I.

Then we can find a complete $T_1 \supseteq T$ *with Skolem functions, and a template* Ψ *proper for* K *and nice, such that:*

(*α*) $\tau \subseteq \tau_\Psi$ *(even* τ_Ψ *extends* τ*), and* $|\tau_\Psi| \leq \theta$ *and* $|T_1| \leq \theta$,

(*β*) Ψ *is nice for* $\mathscr{L}$ *and* $\mathrm{EM}^1(I, \Psi) \vDash T_1$ *for* $I \in K$,

(*γ*) *if* $\alpha < \alpha(*)$, *and* $I \models t_0 < \ldots < t_{k_\alpha - 1}$ *then:*

$$\mathrm{EM}(I, \Psi) \models \varphi_\alpha[a_{t_0}, \ldots, a_{t_{k_\alpha - 1}}] \text{ iff } I \models R_\alpha(t_0, \ldots, t_{k_\alpha - 1}).$$

{1.24new} *Proof.* We would like to apply 1.27, e.g., with $I_\alpha \in K$ being of cardinality $\beth_{\omega\alpha+1}(\theta)$, and being $\beth_{\omega\alpha}(\theta)^+$-saturated for quantifier free types in the natural sense (such N_α
{1.24new} exists by the compactness theorem). However why does assumption (c) of 1.27 hold? By [Sh:289] there is a θ^+-complete forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$ this will hold; it would not make a real difference if we replace $\beth_{\omega\alpha+1}(\theta)$ by other suitable cardinal.
{1.23new} {1.25d} But by 1.26 this suffices (as our assumptions are absolute enough). $\square_{1.28}$

Remark 1.29. For first order T, this help in Laskowski-Shelah [LwSh:687].

{1.26new} **Conclusion 1.30.** *If* T *is first order countable with the OTOP (see* [Sh:c, Ch.XII], *the omitting type order property) then for some sequence* $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}, \bar{z}) : i < i(*) \rangle$ *of first order formulas in* $\mathbb{L}_{\omega,\omega}(\tau_T)$ *and template* Φ *proper for linear orders we have:*

(*α*) $\tau_T \subseteq \tau_\Phi$, $|\tau_\Phi| = |\tau_T| + \aleph_0$,

(*β*) $\mathrm{EM}_{\tau(T)}(I, \Phi) \models T$ *for* $I \in K_{\mathrm{org}}$,

(*γ*) *if* $I \in K_{\mathrm{org}}$ *and* $s, t \in I$ *then*

$$\mathrm{EM}_{\tau(T)}(I, \Phi) \models (\exists \bar{x}) \bigwedge_{i < i(*)} \varphi_i(\bar{x}, \bar{a}_s, \bar{a}_t) \text{ iff } I \models sRt.$$

Proof. Similarly: OTOP is defined in [Sh:c, Ch.XII,4.1,p.608], in a way giving
{1.24new} clause (e) of 1.27 above directly, but we need to know that it is absolute (or just preserved by λ-complete forcing), which holds by [Sh:c, Ch.XII,4.3,p.609]. $\square_{1.30}$

{1.28}
{1.24new} **Conclusion 1.31.** *Claim 1.27 applies to the class of trees with* ω *levels.*

Proof. By the proof in [Sh:c, Ch.VII,§3], i.e., looking at what we use and applying the Erdös-Rado theorem. $\square_{1.31}$

§ 2. Models Represented in Free Algebras and Applications

This section presents a framework, which tries to separate the model theory and combinatorics of [Sh:c, Ch.VIII] and improve it. We shall prove the combinatorics in [Sh:309] and [Sh:331]; here we try to show how to apply it. More applications and combinatorics are in [Sh:511].

Discussion 2.1. We sometimes need τ_Φ with function symbols with infinitely many places and deal with logics $\mathscr{L}$ with formulas with infinitely many variables. Why? {2.1}

Example 2.2. We would like to build complete Boolean algebras without non-trivial one-to-one endomorphisms. How do we get completeness? We build a Boolean algebra, B_0 and take its completion. Even when $\mathbf{B}_0$ satisfies the c.c.c. we need the term $\bigcup\limits_{n<\omega} x_n$ to represent elements of the Boolean algebra from the "generators" $\{\bar{a}_t : t \in I\}$. {2.1A}

Discussion 2.3. We also sometimes would like to rely on a well ordered construction, i.e., on the universe of $\mathscr{M}_{\mu,\kappa}$ there is a well ordering which is involved in the definition of indiscernibility (see 2.4). This means that we have in addition an arbitrary well-order relation. E.g., we would like to build many non-isomorphic $\aleph_1$–saturated models for a stable not superstable first order theory, with the DOP (dimensional order property, see [Sh:c, Ch.X]) so for some $\varphi(\bar{x}, \bar{y})$ (not first order), for any cardinal λ for some model M of T, we have a family $\{\bar{a}_\alpha : \alpha < \lambda\}$ of sequences of length $\leq |T|$ in M with $M \models \varphi[\bar{a}_\alpha, \bar{a}_\beta]$ <u>iff</u> $\alpha < \beta$. The formula φ says: there are z_α ($\alpha < |T|^+$) such that $\bar{x}{}^\frown\bar{y}{}^\frown\langle z_\alpha : \alpha < |T|^+\rangle$ realizes a type p. So there is a template Φ proper for $K_{\rm or}$ such that for $I \in K_{\rm or}$ and $s, t \in I$ we have {2.1B} {2.2}

$$\mathrm{EM}_{\tau(T)}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t] \text{ iff } I \models s < t$$

($<$ a relevant order), but we need to make them $\aleph_1$-saturated. Ultrapowers may well destroy the order. The natural thing is to make M_I $\aleph_1$-constructible over $EM_{\tau(T)}(I, \Phi)$, that is it's set of elements is $\{b_\alpha : \alpha < \alpha\}$, b_α realizing over $\mathrm{EM}_\tau(I, \Phi) \cup \{b_\beta : \beta < \alpha\}$ in M_I a complete type which is $\aleph_1$-isolated. So not only are the $\bar{a}_t$ infinite and the construction involves infinitary functions, but *a priori* the quite arbitrary order of the constructions may play a role.

With some work we can eliminate the well order of the construction for this example (using symmetry, the non-forking calculus) but there is no guarantee generally and certainly it is not convenient, for example see the constructions in [Sh:136, §3]. Moreover, it is better to delete the requirement that the universe of the model is so well defined.

This motivates the following definition. {2.2}

Definition 2.4.

(*a*) $\tau(\mu, \kappa) = \tau_{\mu,\kappa}$ is the vocabulary with function symbols

$$\{F_{i,j} : i < \mu,\ j < \kappa\},$$

where $F_{i,j}$ is a j-place function symbol and κ is $\aleph_0$ or an uncountable regular cardinal

(*b*) $\mathscr{M}_{\mu,\kappa}(I)$ is the free τ-algebra generated by I for $\tau = \tau_{\mu,\kappa}$.

We use the following notation in the remainder of this definition.

Let $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$. For $\bar{a} = \langle a_i : i < \alpha\rangle \in {}^{\alpha}M$ let for $i < \alpha$, $f(a_i) = \sigma_i(\bar{t}_i)$, where $\bar{t}_i$ is a sequence of length $< \kappa$ from I and σ_i is a term from $\tau_{\mu,\kappa}$.

Now if $\alpha < \kappa$ then there is one sequence $\bar{t}$ of members of I of length $< \kappa$ such that

$$\bigwedge_i \mathrm{Rang}(\bar{t}_i) \subseteq \mathrm{Rang}(\bar{t});$$

so we can find terms σ'_i satisfying $f(a_i) = \sigma'_i(\bar{t})$, so without loss of generality $\bar{t}_i = \bar{t}$, we let $\bar{\sigma} = \langle \sigma_i : i < \alpha\rangle$ and $\bar{\sigma}(\bar{t})$ be $\langle \sigma_i(\bar{t}) : i < \alpha\rangle$, so $f(\bar{a}) = \bar{\sigma}(\bar{t})$.

- (c) We say that M is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ iff there is a function $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$ such that the Δ-type of $\bar{a} \in {}^{\kappa>}M$ (i.e., $\mathrm{tp}_\Delta(\bar{a}, \emptyset, M)$) can be calculated from the sequence of terms $\langle \sigma_i : i < \alpha\rangle$ and $\mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}_i : i < \alpha\rangle, \emptyset, I)$ where $f(\bar{a}) = \langle \sigma_i(\bar{t}_i) : i < \alpha\rangle$ (from (b), so if $f(\bar{a}) = \bar{\sigma}(\bar{t})$ from then can be calculated $\bar{\sigma}$ and $\mathrm{tp}_{\mathrm{qf}}(\bar{t}, \emptyset, I)$). We may say "$M$ is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ by f"; similarly below.
- (d) We say that M is weakly Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$ iff for some function $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$, there is a well-ordering $<$ of the universe of $\mathscr{M}_{\mu,\kappa}(I)$ such that for $\bar{a} \in {}^{\alpha}M$ the Δ-type of $\bar{a}$ can be computed from the information described in (c) and the order $<$ restricted to the family of subterms of the terms $\langle \sigma_i(\bar{t}_i) : i < \alpha\rangle$.

[We introduce weak representability to deal with the dependence on the order of a
{2.1B} construction, (cf. 2.3)].

- (e) For $i = 1, 2$ if $\bar{a}_i = \langle \sigma^i_j(\bar{t}^i_j) : j < \alpha\rangle$, $\sigma^1_j = \sigma^2_j$ and

$$\mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}^1_j : j < \alpha\rangle, \emptyset, I) = \mathrm{tp}_{\mathrm{qf}}(\langle \bar{t}^2_j : j < \alpha\rangle, \emptyset, I)$$

  we write $\bar{a}^1 \sim \bar{a}^2 \mod \mathscr{M}_{\mu,\kappa}(I)$ and may say $\bar{a}^1, \bar{a}^2$ are similar in $\mathscr{M}_{\mu,\kappa}(J)$. For the case of weak representability we write $\bar{a}^1 \sim \bar{a}^2 \mod (\mathscr{M}_{\mu,\kappa}(I), <)$ and may say $\bar{a}^1, \bar{a}^2$ are similar in $(\mathscr{M}_{\mu,\kappa}(J), <)$ when in addition the mapping

$$\{\langle \sigma(\bar{t}^1_i), \sigma(\bar{t}^2_i)\rangle : i < \alpha,\ \sigma \text{ is a subterm of } \sigma^1_i = \sigma^2_i\}$$

  is a $<$-isomorphism (and both sides are linear orders). We write $\bar{a}^1 \sim_A \bar{a}^2 \mod \ldots$ if $\bar{a}^1{}^\frown\bar{b} \sim \bar{a}^2{}^\frown\bar{b} \mod \ldots$ whenever $\bar{b} \in {}^{\kappa>}A$ where $A \subseteq \mathscr{M}_{\mu,\kappa}(I)$. (This latter is especially important when we work over a set of parameters). We might, for instance, insist that $\bar{t}^1_i$ and $\bar{t}^1_j$ realize the same Dedekind cut over $I_0 \subseteq I$. (So "M is Δ-represented in $\mathscr{M}_{\mu,\kappa}(I)$" means: $f(\bar{a}^1)$ similar to $f(\bar{a}^2) \mod \mathscr{M}_{\mu,\kappa}$ implies $\bar{a}^1$ and $\bar{a}^2$ realize the same Δ-type in M.)
- (f) We say the representation is full when

$$c_1 \sim c_2 \mod \mathscr{M}_{\mu,\kappa}(I)) \text{ implies } [c_1 \in \mathrm{Rang}(f) \Leftrightarrow c_2 \in \mathrm{Rang}(f)].$$

  We say the weak representation is full if we replace $\mathscr{M}_{\mu,\kappa}(I)$ by $(\mathscr{M}_{\mu,\kappa}(I), <)$, where $<$ is a given well ordering from clause (d).
- (g) If Δ is the family of quantifier free formulas it may be omitted.

(h) For $f : M \longrightarrow \mathscr{M}_{\mu,\kappa}(I)$, let $\bar{a} \sim \bar{b} \mod (f, \mathscr{M}_{\mu,\kappa}(I))$ means

$$f(\bar{a}) \sim f(\bar{b}) \mod \mathscr{M}_{\mu,\kappa}(I).$$

Similarly, $\bar{a} \sim \bar{b} \mod (f, \mathscr{M}_{\mu,\kappa}(I), <)$ means

$$f(\bar{a}) \sim f(\bar{b}) \mod (\mathscr{M}_{\mu,\kappa}(I), <).$$

(i) There is no harm in allowing f (in clauses (c),(d)) to be multi-valued, but we shall mention explicitly when we allow multi-valued functions.

(j) We may restrict ourselves to well orderings $<$ of $\mathscr{M}_{\mu,\kappa}(I)$ which respect subterms; this means that if $\sigma_1(\bar{t}_1)$ is a subterm of $\sigma_2(\bar{t}_2)$ then $\sigma_1(\bar{t}_1) \leq \sigma_2(\bar{t}_2)$.

Now we define a very strong negation (when φ is "right") to even weak representability. {2.3}

Definition 2.5. 1) I is strongly $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into J iff for every $f : I \longrightarrow \mathscr{M}_{\mu,\kappa}(J)$ and well ordering $<$ (of $\mathscr{M}_{\mu,\kappa}(J)$) there are sequences $\bar{x}, \bar{y}$ of members of I such that $I \models \varphi[\bar{x}, \bar{y}]$ and $\bar{x}, \bar{y}$ have "similar" (2.4(e)) images in $\mathscr{M}_{\mu,\kappa}(J), <)$. If we delete the well ordering, we get only "I is $\varphi(\bar{x}, \bar{y})$-unembeddable". If φ clear from the context we may omit it. Note that the formula $\varphi(\bar{x}, \bar{y})$ should be in the vocabulary τ_I; here almost always we have $\tau_J = \tau_I$ but this is not really necessary. {2.2}

2) K has the [strong] $(\chi, \lambda, \mu, \kappa)$-bigness property for $\varphi(\bar{x}, \bar{y})$ iff there are $I_\alpha \in K_\lambda$ for $\alpha < \chi$ such that for $\alpha \neq \beta$ we have I_α is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into I_β.

3) K has the full [strong] $(\chi, \lambda, \mu, \kappa)$-bigness property for $\varphi(\bar{x}, \bar{y})$ iff there are $I_\alpha \in K_\lambda$ for $\alpha < \chi$ such that, for $\alpha < \chi$, I_α is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable for $\tau(\mu, \kappa)$ into $\sum\limits_{\beta<\chi, \beta\neq\alpha} I_\beta$ (where $\sum\limits_{\beta\in u} I_\beta$, when all the I_β are τ-models for some fixed vocabulary τ, is a τ-model I with universe $\bigcup\limits_{\beta\in u} |I_\beta|$; if those universes are not pairwise disjoint we use $\bigcup\limits_{\beta\in u} (\{\beta\} \times (I_\beta))$; for a predicate $P \in \tau$, $P^I = \bigcup\limits_{\beta\in u} P^{I_B}$, for every function symbol $F \in \tau$, F^I is the (partial) function $\bigcup\limits_{\beta\in u} F^{I_\beta}$).

4) Saying "I is [strongly] $\varphi(\bar{x}, \bar{y})$-unembeddable into J for function f satisfying Pr" means we restrict ourselves (in 2.5(1)) to function f from I to $\mathscr{M}_{\mu,\kappa}(J)$ satisfying Pr. {2.3}

5) The most popular restriction is "f finitary on some P" which means that for every $\eta \in P^I$ for some $n < \omega$, $\tau_{\mu,\kappa}$-term σ and $\eta_0, \ldots, \eta_{n-1} \in J$ we have $f(\eta) = \sigma(\eta_0, \ldots, \eta_{n-1})$. We say f is strongly finitary if in addition σ has only finitely many subterms.

6) Clearly (4) induces parallel variants of 2.5(2), 2.5(3). {2.3} {2.3A}

Remark 2.6. 1) This definition is used in proving that the model constructed from I is not isomorphic to (or not embeddable into) the model constructed from J. For existence see [Sh:331, 2.15(2)] (which we deduce from [Sh:331, 1.7(2)]).

2) We may in 2.5(1) and the other variants, add: moreover, given $A \subseteq J$ of cardinality $< \kappa$ we demand that $\bar{x}, \bar{y}$ are similar over A. This does not make a real difference so far. {2.3}

3) About the connection to $\dot{I}\dot{E}(\lambda, T_1, T)$ see [Sh:331].

Claim 2.7. *If Φ is proper for I and $\mu = |\tau_\Phi|$ then* $\mathrm{EM}(I,\Phi)$ *can be represented in* $\mathscr{M}_{\mu,\aleph_0}$. {2.3B}

Proof. Easy. $\square_{2.7}$

* * *

{2.4}

Discussion 2.8. The following example illustrates the application of this method. {2.14A} We first fix $K^\omega_{\rm tr}$ (see 1.9) as the class of index models and fix a formula $\varphi_{\rm tr}$ (see 2.9 below); note that we shall prove later that for many pairs $I,J \in K^\omega_{\rm tr}, I$ is $\varphi_{\rm tr}(\bar x,\bar y)$-unembeddable in J. In 2.12 below we choose for each $I \in K^\omega_{\rm tr}$ a reduced separable Abelian $\dot p$-group $\mathbb{G}_I$ which is representable in $\mathscr{M}_{\omega,\omega}(I)$. In 2.13 below we show that: [I is $\varphi_{\rm tr}$-unembeddable in J implies $\mathbb{G}_I \not\cong \mathbb{G}_J$]; thus the number of reduced separable Abelian $\dot p$-groups of cardinality λ is at least as great as the number of trees in $K^\omega_{\rm tr}$ with cardinality λ which are pairwise $\varphi_{\rm tr}$-unembeddable. We showed in [Sh:136] that this number is 2^λ for regular λ and many singulars. But as said in 1.13 for every uncountable λ we get 2^λ pairwise non-isomorphic such groups in λ, using $\mathbb{G}_I$ as below. {2.5A} {2.5B} {1.9}

We may like to strengthen "$\mathbb{G}_I \not\cong \mathbb{G}_J$" to "$\mathbb{G}_I$ not embeddable in $\mathbb{G}_J$". This depends on the exact notion of embeddability we use (we shall return to this in [Sh:331, 3.22]).

{2.4A}

Example 2.9. For the class of $I \in K^\omega_{\rm tr}$

$$\begin{aligned}\varphi_{\rm tr}(x_0,x_1:y_0,y_1) := \quad & [x_0 = y_0] \text{ and } P_\omega(x_0) \text{ and} \\ & \bigvee_{n<\omega} [P_n(x_1) \text{ and } P_n(y_1) \text{ and } P_{n-1}(x_1 \cap y_1)] \text{ and} \\ & [x_1 \triangleleft x_0 \wedge y_1 \ntriangleleft y_0] \text{ and } y_1 <_{\rm lx} x_1]\end{aligned}$$

in other words, when for transparency we restrict ourselves to standard $I \subseteq {}^{\omega\ge}\lambda$: $x_0 = y_0 \in {}^\omega\lambda$, and for some $n<\omega$ and $\alpha<\beta<\lambda$ we have

$$x_1 = (x_0 \restriction n)\hat{}\langle\alpha\rangle \triangleleft x_0$$

and

$$y_1 = (x_0 \restriction n)\hat{}\langle\beta\rangle$$

{2.3} The connection of the bigness properties from 2.5 to the results on $\dot I\dot E(\lambda,T_1,T)$ is done by:

{2.4B}

Claim 2.10. *Assume that*

{1.8} (*a*) Φ,φ_n *are as in the conclusion of 1.11(1),* $\mu = |\tau_\Phi|$,

(*b*) $I,J \in K^\omega_{\rm tr}$, *$I$ is strongly $\varphi_{\rm tr}$-unembeddable into J for a* $\tau_{\mu,\aleph_0}$,

(*c*) $\tau_0 \subseteq \tau_\Phi$ *is a vocabulary including that of the φ_n's.*

Then $\mathrm{EM}_{\tau_0}(I,\Phi)$ *cannot be elementarily embedded into* $\mathrm{EM}_{\tau_0}(J,\Phi)$. *Moreover, no function from* $\mathrm{EM}(I,\Phi)$ *into* $\mathrm{EM}(J,\Phi)$ *preserves the formulas* $\pm\varphi_n$ *(for $n<\omega$).*

Proof. Straightforward, reread the definitions. $\square_{2.10}$

{2.5}
Subexample 2.11. Separable reduced Abelian $\dot{p}$-groups.
(See more in [Sh:331, §3]; as p denote types we use $\dot{p}$ for prime numbers.)

Definition 2.12. 1) A separable reduced Abelian $\dot{p}$-group $\mathbb{G}$ is a group $\mathbb{G}$ which satisfies (we use additive notation): {2.5A}

- (a) $\mathbb{G}$ is commutative (that is "Abelian"),
- (b) for every $x \in \mathbb{G}$ for some n, x has order $\dot{p}^n$ (i.e., $\dot{p}^n x$ is the zero and n is minimal),
- (c) $\mathbb{G}$ has no divisible non-trivial subgroup (= reduced),
- (d) every $x \in \mathbb{G}$ belongs to some 1-generated subgroup which is a direct summand of $\mathbb{G}$ (= separable).

2) Any such group is a normed space:

$$\|x\| = \inf\{2^{-n} : (\exists y \in \mathbb{G})\dot{p}^n y = x\}.$$

3) For a tree $I \in K^\omega_{\rm tr}$ we define the $\dot{p}$-group $\mathbb{G}_I$ as follows, $\mathbb{G}_I$ is generated (as an Abelian group) by

$$\{x_\eta : \eta \in \bigcup_{n<\omega} P^I_n\} \cup \{y^n_\eta : \eta \in P^I_\omega \text{ and } n < \omega\},$$

freely except for the relations:

$$\dot{p}^{n+1} x_\eta = 0 \text{ for } \eta \in P^I_n;$$

and

$$\dot{p} y^{n+1}_\eta - y^n_\eta = x_{\eta \restriction n} \text{ and } \dot{p}^{n+1} y^n_\eta = 0 \text{ for } \eta \in P^I_\omega.$$

4) It is well known that $\mathbb{G}_I$ is a reduced separable Abelian $\dot{p}$-group. Also note that we have essentially say

$$y^n_\eta = \sum\{\dot{p}^{\ell-n} x_{\nu_\ell} : \ell \text{ satisfies } n \le \ell < \omega, \nu_\ell \in P^I_\ell \text{ and } \nu_\ell \triangleleft \eta\}$$

(the infinitary sum may be well defined as $\mathbb{G}_I$ is a normed space).

It is easy to see that

Fact 2.13. $\mathbb{G}_I$ is a reduced separable Abelian $\dot{p}$-group which is represented in $\mathscr{M}_{\omega,\omega}(I)$. {2.5B}

We shall prove now

Fact 2.14. If I is $\varphi_{\rm tr}$-unembeddable into J <u>then</u> $\mathbb{G}_I \not\cong \mathbb{G}_J$. {2.5C}

Proof. Let $g : \mathbb{G}_I \cong \mathbb{G}_J \longrightarrow_h g$ be an isomorphism from $\mathbb{G}_I$ onto $\mathbb{G}_J$ and $h : \mathbb{G}_J \longrightarrow \mathscr{M}_{\omega,\omega}(J)$, where h witnesses that $\mathbb{G}_J$ is representable in $\mathscr{M}_{\omega,\omega}(J)$.

Let $f : I \longrightarrow \mathbb{G}_I$ be:

$$f(\eta) = \begin{cases} \sum\limits_{1 \le \ell \le \ell g(\eta)} \dot{p}^{\ell-1} x_{\eta \restriction \ell} & \text{if} \quad \eta \in \bigcup\limits_{n<\omega} P^I_n, \\ y^1_\eta & \text{if} \quad \eta \in P^I_\omega. \end{cases}$$

So $(h \circ g \circ f) : I \longrightarrow \mathscr{M}_{\omega,\omega}(J)$. Now we use the fact that I is $\varphi_{\rm tr}$-unembeddable into J.

So suppose

$$I \models \varphi_{\rm tr}[\eta_0, \nu_0; \eta_1, \nu_1] \text{ and } h \circ g \circ f(\eta_0, \nu_0) \sim h \circ g \circ f(\eta_1, \nu_1).$$

Invoking the definition of $\varphi_{\rm tr}$: for some $\eta := \eta_0 = \eta_1 \in P^I_\omega$ and for some n,

$$\nu_1 \lhd \eta_1, \nu_1 \in P^I_n, \nu_0 \in P^I_n,$$

$$\nu_1 \restriction (n-1) = \nu_0 \restriction (n-1), \nu_0(n-1) < \nu_1(n-1).$$

For $i = 0, 1$ let

$$z_{\nu_i} = \sum \{\dot{p}^{\ell-1} x_\nu : \nu \lhd \nu_i, \nu \in P^I_\ell \text{ and } 1 \le \ell \le n\}.$$

Now $\mathbb{G}_I \models$ "$\dot{p}^n$ divides $(y^1_\eta - z_{\nu_0})$", hence, as g is an isomorphism, $\mathbb{G}_J \models$ "$\dot{p}^n$ divides $(g(y^1_\eta) - g(z_{\nu_0}))$", which means $\mathbb{G}_J \models$ "$\dot{p}^n$ divides $(g \circ f(\eta) - g \circ f(\nu_0))$".

Similarly, $\mathbb{G}_J \models$ "$\dot{p}^n$ does not divide $(g \circ f(\eta) - g \circ f(\nu_1))$", but

$$h \circ g \circ f(\langle \eta_0, \nu_0 \rangle) \sim h \circ g \circ f(\langle \eta_1, \nu_1 \rangle) \quad \mathrm{mod}\ \mathscr{M}_{\omega,\omega}(J),$$

{2.5C} a contradiction, proving 2.14. □

* * *

{2.6}

Discussion 2.15. We still can get considerable amounts of information by the general theory. When we try to construct many models of K (no one embeddable into the others) we need

(*) there are 2^λ index models I of cardinality λ each $\varphi_K(\bar{x}, \bar{y})$-unembeddable into any other.

But when you intend to construct rigid, indecomposable, etc., you need:

(**) there are $\{I_\alpha \in K : \alpha < \lambda\}, I_\alpha, \varphi_K$-unembeddable into $\sum\limits_{\beta \ne \alpha} I_\beta$ (and I_α has cardinality λ).

Why?

{2.7}

Example 2.16. Constructing Rigid Boolean Algebras. (See more, and for more details, in [Sh:511, §2].) For $I \in K^\omega_{\rm tr}$ let ${\rm BA}_{\rm tr}(I)$ be the Boolean Algebra freely generated by $\{a_\eta : \eta \in I\}$ except the relations

$$a_\eta \le a_\nu \text{ when } \nu \in P^I_\omega, n < \omega, \eta = \nu \restriction n.$$

We shall choose a sequence $\langle \mathbf{B}_i, a_j : i \le \lambda, j < \lambda \rangle$ such that $\mathbf{B}_i$ is a Boolean algebra, $\subseteq$-increasing with $i, a_i \in \mathbf{B}_i$ and if $i < \lambda$ and $a \in \mathbf{B}_i$ then $a = a_j$ for some $j \in [i, \lambda)$. Start with $\mathbf{B}_0 = {\rm BA}_{\rm tr}(I_0)$, successively for some $a_i \in \mathbf{B}_i, 0 < a_i < 1$, take

$$\mathbf{B}_{i+1} = (\mathbf{B}_i \restriction (1 - a_i)) + ((\mathbf{B}_i \restriction a_i) * {\rm BA}_{\rm tr}(I_i)),$$

$$\mathbf{B}_\lambda = \bigcup_{i<\lambda} \mathbf{B}_i = \{a_i : i < \lambda\}, |I_\alpha| = \lambda.$$

(In such situations we say that $\mathbf{B}_{i+1}$ is a result of the $\mathrm{BA}_{\mathrm{tr}}(I_i)$-surgery of $\mathbf{B}_i$ at a_i that is, below $1 - a_i$ we add nothing and below a_i we use the free product of $\mathbf{B}_i \restriction a_i$ and $\mathrm{BA}_{\mathrm{tr}}(I_i)$.)
Of course, we choose $\{I_\alpha : \alpha < \lambda\}$ such that I_α is φ_{tr}-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$.
The point is that each $a \in \mathbf{B}_\lambda \setminus \{0,1\}$ was "marked" by some I_α, (the α such that $a_\alpha = a$). Now $\mathrm{BA}_{\mathrm{tr}}(I_\alpha)$ is embeddable into $\mathbf{B}_\lambda \restriction a_\alpha$; but $\mathbf{B}_\lambda \restriction (1 - a_\alpha)$ is weakly $\mathbb{L}_{\omega,\omega}$-represented in $\mathscr{M}_{\omega,\omega}(\sum_{\beta \neq \alpha} I_\beta)$. So for no automorphism f of $\mathbf{B}_\lambda$ do we have, $f(a_\alpha) \leq 1 - a_\alpha$, which suffices to get "$\mathbf{B}_\lambda$ is rigid"; in fact, it has no one-to-one endomorphism. If we are trying to get stronger rigidity and/or $\mathbf{B}_\lambda \models$ c.c.c., and/or $\mathbf{B}_\lambda$ is complete, we may have to change K^ω_{tr} and/or φ_{tr}.

This illustrates the need for some of the complications in definition 2.1. E.g., the weak representation and the uncountable κ (for complete Boolean Algebras). {2.1}

The definition below (variants of closure under sums) are satisfied by the cases we shall deal with and enable us to translate results e.g. from the full (strong) $(\lambda, \lambda, \mu, \kappa)$-bigness to the (strong) $(2^\lambda, \lambda, \mu, \kappa)$-bigness.
Of course:

{2.15}

Definition 2.17. We say that the class K of τ-structures; with τ a relational vocabulary for transparency, is closed under sums <u>when</u> for every sequence $\langle I_s : s \in S\rangle$ of members of K, pairwise disjoint for simplicity, also I belongs to K where I is the τ- structure which is the union of $\langle I_s : s \in S\rangle$; that is the set of elements of I is the union of the sets of elements of I_s for $s \in S$ and $P^I = \cup\{P^{I_s} : s \in S\}$ for every predicate P from τ.

But in many cases which interest us, this is only almost true, hence we define:

{2.16}

Definition 2.18. 1) We say that K is almost (μ, κ)-closed under sums for λ and ψ where $\psi = \psi(\bar{x}, \bar{y}), \ell g(\bar{x}) = \ell g(\bar{y})$, <u>iff</u> for every $I_\alpha \in K$ (for $\alpha < \alpha_0 \leq \lambda$), I_α of cardinality $\leq \lambda$, there are $J, g, h_\alpha (\alpha < \alpha_0)$ such that:

- (a) $J \in K, |J| \leq \lambda$,
- (b) $h_\alpha : I_\alpha \longrightarrow J$, and for any $x_0, \dots, y_0, \dots \in I_\alpha, I_\alpha \models \psi[\langle x_0, \dots\rangle, \langle y_0, \dots\rangle]$ implies $J \models \psi[\langle h_\alpha(x_0), \dots\rangle, \langle h_\alpha(y_0), \dots\rangle]$,
- (c) $g : J \longrightarrow \sum\limits_{\alpha<\alpha_0} \mathscr{M}_{\mu,\kappa}(I_\alpha)$ satisfies, for any $\gamma < \kappa, \bar{x}, \bar{y} \in {}^\gamma J$ and $A \subseteq J$ of cardinality $< \kappa$,
  $\boxdot_0$ if $g(\bar{x}) \approx g(\bar{y}) \mod \mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha<\alpha_0} I_\alpha)$ then $\bar{x} \approx \bar{y} \mod \mathscr{M}_{\mu,\kappa}(J)$.

2) We replace "almost" by "semi", if in clause (c) above we weaken $\boxdot_0$ to:

$\boxdot_1$ if $g(\bar{x}) \approx g(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha<\alpha_0} I_\alpha), R)$ then $\bar{x} \approx \bar{y} \mod \mathscr{M}_{\mu,\kappa}(J)$, where we define
$$R = \{\langle\langle \eta, i\rangle, \langle \nu, j\rangle\rangle : \eta \in I_i, \nu \in I_j \text{ and } i < j\} \subseteq (\sum_{\alpha<\alpha_0} I_\alpha) \times (\sum_{\alpha<\alpha_0} I_\alpha).$$

3) We add "strongly" to close in part (1) if we strengthen clause (c) to:

(c)$^+$ $g : J \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha)$ such that for any well ordering $<_0$ of $\mathscr{M}_{\mu,\kappa}(J)$ (as in 2.4(d)), there is a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha)$ such that: for any $\gamma < \kappa$ and $\bar{x}, \bar{y} \in {}^\gamma J$ and $A \subseteq J$ of cardinality $< \kappa$, {2.2}

$\boxdot_2$ if $g(\bar{x}) \approx g(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha), <_1)$, then $\bar{x} \approx \bar{y} \mod (\mathscr{M}_{\mu,\kappa}(J), <_0)$.

4) We add strongly in part (2) iff we strengthen (c) to (c)$^+$, only using $(\mathscr{M}_{\mu,\kappa}(\sum_{\alpha<\alpha_0} I_\alpha), <_1, R)$.
5) We may omit "(μ, κ)" above if $\mathrm{Rang}(g) \subseteq J$.
6) We say that K is essentially closed under sums for λ iff in part (1) in addition, $\mathrm{Rang}(h_\alpha)$, $\mathrm{Rang}(g)$ are unions of equivalence classes of (R is from part (2))

$$\approx \mod J, \qquad \approx \mod (\sum_{\alpha<\alpha_0} I_\alpha, R), \qquad \text{respectively.}$$

Remark 2.19. We could have made, for example $h_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(J)$, or in the definition of sum expand by R, without serious changes in the paper.

{2.8}
Claim 2.20. *0) "K is closed under sums" implies "K is essentially closed under sums", which implies "K is almost closed under sums", which implies "K is almost (μ, κ)-closed under sums". If $\mu_1 \le \mu_2, \kappa_1 \le \kappa_2$ then "K is almost (μ_1, κ_1)-closed under sums" implies "K is (μ_2, κ_2)-closed under sums".*

In all above implications we can add "strongly" to both sides (when relevant, related).

1) If K is closed under sums, then the full (strong) $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property implies the (strong) $(\chi_1, \lambda, \mu, \kappa) - \psi$-bigness property, where $\chi_1 = \min\{2^\chi, 2^\lambda\}$.
2) In (1), instead of "K closed under sums" it is enough to assume that K is (strongly) almost closed under sums for λ, ψ.
{1.7} *3) The classes defined in 1.9 above $K^\kappa_{\rm tr}, K_{\rm or}$ are almost closed under sums and almost strongly closed under sums.*
{2.3} *4) The relations defined in 2.5(2), (3), (6) have obvious monotonicity properties in χ, μ, κ; and for all our K, for λ too. For example*

$$\chi \le \chi \Rightarrow [(\chi', \lambda, \mu, \kappa)\text{-}bigness \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-}bigness]$$

$$\mu \le \mu' \& \kappa \le \kappa' \Rightarrow [(\chi, \lambda, \mu', \kappa')\text{-}bigness \Rightarrow (\chi, \lambda, \mu, \kappa)\text{-}bigness].$$

Proof. 0) Obvious.
1) So we assume K has the full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property. Without loss of generality $\langle I_\alpha : \alpha < \chi \rangle$ are pairwise disjoint.

As K has the [strong] full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property, there are $I_\alpha \in K$ (for $\alpha < \chi$), each of cardinality λ, such that I_α is ψ-unembeddable intao $\sum_{\beta \ne \alpha} I_\beta$.

Case 1: $\chi \le \lambda$.

For $U \subseteq \chi$ let $J_U = \sum_{\alpha \in U} I_\alpha$. Let $\mathscr{P}$ be a collection of subsets of χ such that $|\mathscr{P}| = 2^\chi$ and $U \ne V \in \mathscr{P} \Rightarrow U \nsubseteq V$. Suppose $U, V \in \mathscr{P}, f : J_U \longrightarrow M(J_V)$.

Choose $\alpha \in U \setminus V$. Thus $f \restriction I_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum\limits_{\beta \neq \alpha} I_\beta)$ and the desired conclusion follows.

Case 2: $\lambda < \chi$.

Take a family $\mathscr{W}$ of subsets of λ, each of cardinality λ, such that

$$U \neq V \in H \Rightarrow U \nsubseteq V$$

and proceed as in Case 1.

2) As K has the [strong] full $(\chi, \lambda, \mu, \kappa) - \psi$-bigness property, there are $I_\alpha \in K$ (for $\alpha < \chi$), each of cardinality λ, such that I_α is ψ-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$. By the assumption of (2) (that K is almost (strongly) closed under sums) for every $U \subseteq \chi, |U| \leq \lambda$ let J_U, g_U, h^U_α $(\alpha \in U)$ satisfy clauses (a), (b), (c) of Definition 2.18(1) for $\sum\limits_{\alpha \in U} I_\alpha$. As in the proof of (1), it suffices to show: {2.16}

(∗) if $U, V \subseteq \chi, |U| \leq \lambda, |V| \leq \lambda, U \setminus V \neq \emptyset$ and $f : J_U \longrightarrow \mathscr{M}_{\mu,\kappa}(J_V)$, then for some $\bar{a}, \bar{b} \in {}^{\ell g(\bar{x})}(J_U), J_U \models \psi[\bar{a}, \bar{b}]$ and $f(\bar{a}) \approx_A f(\bar{b}) \mod \mathscr{M}_{\mu,\kappa}(J_V)$; or mod $(\mathscr{M}_{\mu,\kappa}(J_V), <)$ for the strong version.

Choose $\alpha \in U \setminus V$.

In the strong case let $<_0$ be a well ordering of $\mathscr{M}_{\mu,\kappa}(J_V)$ (as in 2.4(d),2.18(3)); choose a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(\sum\limits_{\alpha < \alpha_0} I_\alpha)$ as guaranteed by Definition 2.18(3); in the non-strong case let $<_0$, $<_1$ be the empty relations. {2.16} {2.16}

Now define

$$g^*_V : \mathscr{M}_{\mu,\kappa}(J_V) \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum_{i \in V} I_i)$$

by

$$g^*_V(\tau(x_0, \ldots)) = \tau(g_V(x_0), \ldots).$$

Consider the sequence of mappings:

$$I_\alpha \underset{h^U_\alpha}{\longrightarrow} J_U \underset{f}{\longrightarrow} \mathscr{M}_{\mu,\kappa}(J_V) \underset{g^*_V}{\longrightarrow} \mathscr{M}_{\mu,\kappa}\big(\sum_{i \in V} I_i\big).$$

So $g^*_V \circ f \circ h^U_\alpha : I_\alpha \longrightarrow \mathscr{M}_{\mu,\kappa}(\sum\limits_{i \in V} I_i)$. As $\sum\limits_{i \in V} I_i$ is a submodel of $\sum\limits_{i \neq \alpha} I_i$, also without loss of generality $\mathscr{M}_{\mu,\kappa}(\sum\limits_{i \in V} I_i)$ is a submodel of $\mathscr{M}_{\mu,\kappa}(\sum\limits_{i \neq \alpha} I_i)$. But we know that I_α is ψ-unembeddable into $\sum\limits_{i \neq \alpha} I_i$. Hence there are $\bar{x}, \bar{y} \in I_\alpha$ such that:

(i) $I_\alpha \models \psi[\bar{x}, \bar{y}]$,

(ii) $g^*_V \circ f \circ h^U_\alpha(\bar{x}) \approx g^*_V \circ f \circ h^U_\alpha(\bar{y}) \mod (\mathscr{M}_{\mu,\kappa}(\sum\limits_{i \in V} I_i), <_1)$.

By (i) and clause (b) from 2.18(1), {2.16}

(iii) $J_U \models \psi[\bar{x}', \bar{y}']$, where $\bar{x}' = h^U_\alpha(\bar{x}), \bar{y}' = h^U_\alpha(\bar{y})$.

By (ii) and the definition of $\bar{x}', \bar{y}'$,

$$(iv)\ \ g^*_V(f(\bar{x}')) \approx g^*_V(f(\bar{y}'))\ \ \text{mod}\ (\mathscr{M}_{\mu,\kappa}(\sum_{i\in V} I_i), <_1).$$

{2.16} By (iv), clause (c) of 2.18(1) or clause $(c)^+$ of 2.18(3), the definition of $\mathscr{M}_{\mu,\kappa}(\sum_{i\in V} I_i)$, and of g^*_V,

$$(v)\ \ f(\bar{x}') \approx f(\bar{y}')\ \ \text{mod}\ (\mathscr{M}_{\mu,\kappa}(J_V), <_0).$$

So we have proved $(*)$ (by (iii) and (v)), which suffices.
3)-6) Left to the reader. $\square_{2.20}$

{2.17} **Claim 2.21.** *The following classes are almost (and also semi) (μ,κ)-closed under sums for λ*

(a) $K_{\rm or}$ *(the class linear orders)*
(b) $K^\omega_{\rm tr}$ *(trees with $\omega+1$ levels)*
(c) $K^\kappa_{\rm tr}$ *(trees with $\kappa+1$ levels)*
(d) $K_{\rm org}$ *(ordered graphs).*

Proof. Case (a)
If $\langle I_\alpha : \alpha < \alpha_0\rangle$ is a sequence of linear orders then we let:

(i) $J = \cup\{\{\alpha\}\times I_\alpha : \alpha < \alpha_0\}$
(ii) $(\alpha_1,t_1) <_J (\alpha_2,t_2)$ <u>if and only if</u> $\alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2\ \&\ t_1 <_{I_{\alpha_1}} t_2)$
(iii) $h_\alpha : I_\alpha \to J$ is $h_\alpha(t) = (\alpha, t)$
(iv) $g : J \to \sum_{\alpha<\alpha_0} I_\alpha$ is the identity.

Now check

Case (b):
Given $\langle I_\alpha : \alpha < \alpha_0\rangle$ the unique we identify the member of $P^{J_\alpha}_0$ for $\alpha < \alpha_0$ but make then otherwise disjoint and take the union.

Case (c):
Similar to case (b).

Case (d):
Similar to case (a). $\square_{2.21}$

Another way to present those matters is to do it around the following definition and claim.

{2.18n} **Definition 2.22.** We say that J_2 does (μ,κ)-dominate J_1 <u>when</u> there is a function g from $\mathscr{M}_{\mu,\kappa}(J_1)$ into $\mathscr{M}_{\mu,\kappa}(J_2)$ such that: if $\rho\varphi\xi < \kappa$ and $\bar{a},\bar{b} \in {}^\xi(\mathscr{M}_{\mu,\kappa}(J_1))$ and $g(\bar{a}) \cong g(\bar{b})$ mod $\mathscr{M}_{\mu,\kappa}(J_2)$ then $\bar{a} \cong \bar{b}$ mod $\mathscr{M}_{\mu,\kappa}(J_1)$.

We say that J_2 strongly (μ,κ)-dominate J_1 <u>when</u> there is a function g from $\mathscr{M}_{\mu,\kappa}(J_1)$ into $\mathscr{M}_{\mu,\kappa}(J_2)$ such that: if $\xi < \kappa$ and $\bar{a},\bar{b} \in {}^\xi(\mathscr{M}_{\mu,\kappa}(J_1))$ and $g(\bar{a}) \cong g(\bar{b})$ mod $\mathscr{M}_{\mu,\kappa}(J_2)$ and $<_2$ is a well ordering of $(\mathscr{M}_{\mu,\kappa}(J_2), <_2)$ then there is a well ordering $<_1$ of $\mathscr{M}_{\mu,\kappa}(J_1)$ such that $\bar{a} \cong \bar{b}$ mod $(\mathscr{M}_{\mu,\kappa}(J_1, <_1))$.

We say J_1, J_2 are [strongly] (μ,κ)-equivalent when J_2 [strongly] dominate J_1 and vice versa.

{2.19n}

Claim 2.23. *If I is [strongly] $\varphi(\bar{x},\bar{y})$-unembeddable into J_2 and J_2 [strongly] (μ,κ)-dominate J_1 then I is [strongly] $\varphi(\bar{x},\bar{y})$-unembeddable into J_2.*

* * *

As we have remarked in the introduction to this paper, results on trees can be translated to results on linear orders; this is done seriously in [Sh:363]. Originally this was neglected as the results on unsuperstable T (and trees with $\omega+1$ levels) give the results on unstable theories (and linear orders). Anyhow, now we deal with the simplest case parallel to [Sh:c, Ch.2.1].

Definition 2.24. 1) For any $I \in K^\kappa_{\rm tr}$ we define $\mathbf{or}(I)$ as the following linear order (See Def 1.11(4)). {2.20} {1.8}

set of elements is chosen as $\{(t,\ell) : \ell \in \{1,-1\}, t \in I\}$

the order is defined by $(t_1,\ell_1) < (t_2,\ell_2)$ if and only if $t_1 \triangleleft t_2 \wedge \ell_1 = 1$ or $t_2 \triangleleft t_1 \wedge \ell_2 = -1$ or $t_1 = t_2 \wedge \ell_1 = -1 \wedge \ell_2 = 1$ or $t_1 <_{\rm lx} t_2 \wedge (t_1,t_2$ are $\triangleleft$-incomparable.

2) Let $\varphi_{\rm or} = \varphi_{\rm or}(x_0,x_1;y_0,y_1)$ be the formula $x_0 < x_1 \wedge y_1 < y_0$.

3) Let $\varphi^\kappa_{\rm tr} = \varphi^\kappa_{\rm tr}(x_0,x_1;y_0,y_1)$ be (this is for $K^\kappa_{\rm tr}$, for $\kappa = \aleph_0$ see example 2.9) {2.4A}

$$\begin{aligned}\varphi_{\rm tr}(x_0,x_1 : y_0,y_1) := \quad & [x_0 = y_0] \text{ and } P_\kappa(x_0) \wedge \bigvee_{\epsilon<\kappa} [P_{\epsilon+1}(x_1) \wedge P_{\epsilon+1}(y_1) \wedge \\ & P_\epsilon(x_1 \cap y_1)] \wedge [x_1 \triangleleft x_0 \wedge \neg(y_1 \triangleleft y_0)] \text{ and } y_1 <_{\rm lx} x_1].\end{aligned}$$

Claim 2.25. *1) Assume that $I, J \in K^\kappa_{\rm tr}$* {2.21}

- (a) *If I is strongly $\varphi^\kappa_{\rm tr}$-unembeddable for $\tau_{\mu,\kappa}$ into J then $\mathbf{or}(I)$ is strongly $\varphi^\kappa_{\rm tr}$-unembeddable for $\tau_{\mu,\kappa}$ into $\mathbf{or}(J)$*
- (b) *similarly without "strongly".*

2) If $K^\kappa_{\rm tr}$ has the strong $(\chi,\lambda,\mu,\kappa)$-bigness property then $K_{\rm or}$ has the strong $(\chi,\lambda,\mu,\kappa)$-bigness property.

3) In part (2) we may add "full" and/or omit "strong" in the assumption and the conclusion.

Proof. The main point is that:

$(*)$ if $I \models \varphi^\kappa_{\rm tr}(x_0,x_1;y_0,y_1)$ then $\mathbf{or} \models \varphi((x_0,1),(x_1,1);(y_0,1),(y_1,1))$.

$\square_{2.25}$

Remark 2.26. 1) We deal mainly with $K^\omega_{\rm tr}$, see [Sh:331, 3.1], so by it we know that $K^\omega_{\rm or}$ has the full strong $(\lambda,\lambda,\mu,\aleph_0)$-bigness property when $\mu < \lambda$. {2.22}

2) For κ regular uncountable, there are parallel results, noting that obviously $K^\kappa_{\rm or}$ have the full strong $(\chi,\lambda,\mu,\kappa)$ when λ is regular $> |\alpha|^{<\kappa} + \mu$ for every $\alpha < \lambda$ and $\lambda \le \chi$.

It seems reasonable to conjecture that the parallel of [Sh:331, 3.1(2)] holds, but we have not tried to work on it, see part (3) of the remark.

3) The results below (on $\varphi_{{\rm or},\alpha,\beta,\pi}$) seem to me a natural step but have actually set down to phrase and prove them for Usvyatsov-Shelah [ShUs:928].

4) Even for $\kappa = \aleph_0$ we do not deal with λ singular below, it seems reasonable that this, i.e., the parallel of [Sh:331, §1] holds, but the results below are more

than sufficient for its purpose, as for $\chi > \mu$ singular we can use the result here for $(\chi, \lambda, \mu, \kappa)$ for any regular $\lambda \in (\mu, \chi)$.
5) In 2.17 we use α, β well orders. {2.15}

It seems reasonable that we can say more for a more general case but again this was not required.
{2.23} {2.23} 6) We use freely the obvious observation 2.27.

Observation 2.27. *1) $K_{\rm or}$ is essentially closed under sums for λ and $\varphi_{\rm or}$, recalling*
{2.16} {2.18} *Definitions 2.18, 2.21.*
2) Similar for $\varphi_{{\rm or},\alpha,\beta,\pi}$ defined below.

{2.24} **Definition 2.28.** We define the following (quantifier free infinitary) formulas for the vocabulary $\{<\}$. For any ordinal α, β and a one-to-one function π from α onto β, and we let $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar{x}, \bar{y})$ where $\bar{x} = \bar{x}^\alpha = \langle x_i : i < \alpha \rangle$ and $\bar{y} = \bar{y}^\alpha = \langle y_i : i < \alpha \rangle$, be

$$\bigwedge \{x_i < x_j : i < j < \alpha\} \text{ and } \bigwedge \{y_i < y_j : i, j < \alpha \text{ and } \pi(i) < \pi(j)\}.$$

{2.25} **Claim 2.29.** *Assume $\chi \geq \lambda = {\rm cf}(\lambda) > \mu^{<\kappa}, \kappa = {\rm cf}(\kappa)$ and $\gamma < \lambda \Rightarrow |\gamma|^{<\kappa} < \lambda$.*
{2.24} *1) For (α, β, π) as in Definition 2.28, such that $\alpha, \beta \leq \lambda$, the class $K_{\rm or}$ has the full strong $(\lambda, \chi, \mu, \kappa)$-bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}(\bar{x}, \bar{y})$.*
{2.24} *2) For (α, β, π) as in Definition 2.28 such that $\alpha, \beta \leq \lambda$, the class $K_{\rm or}$ has the strong $(2^\lambda, \chi, \mu, \kappa)$ bigness property for $\varphi_{{\rm or},\alpha,\beta,\pi}$.*
3) In fact in both part (1) and (2) we can find examples which satisfies the conclusion for all triples (α, β, π) as there simultaneously.

{2.26} *Proof.* 1) By 2.30 below because there are λ pairwise disjoint stationary sets $S \subseteq S^\lambda_{\aleph_0}$.
{2.22} {2.28} 2) By part (1) and 2.27(1) and 2.20(1).
3) Check the proof. $\square_{2.29}$

{2.26} **Claim 2.30.** *Assume $\kappa = {\rm cf}(\kappa) \leq \mu, \mu^{<\kappa} < \lambda = {\rm cf}(\lambda) \leq \lambda_1, \kappa \leq \partial = {\rm cf}(\partial) < \lambda$ and $\gamma < \lambda \Rightarrow |\gamma|^{<\kappa} < \lambda$.*

If $I, J \in K^\kappa_{\rm or}$ satisfies $\circledast$ below and $\alpha_*, \beta_* \leq \lambda$ and π is a one-to-one function from
{2.20} *α_* onto β_* __then__ (recalling Definition 2.24) **or**(I) is strongly $\varphi_{{\rm or},\alpha_*,\beta_*,\pi}(\bar{x}^{\alpha_*}, \bar{y}^{\alpha_*})$-unembeddable for (μ, κ) into **or**(J) where*

- $\circledast$ (a) *$S_1, S_2 \subseteq S^\lambda_\partial$ such that $S_1 \setminus S_2$ is a stationary subset of λ*
  - (b) *$\bar{\eta} = \langle \eta_\delta : \delta \in S_1 \cup S_2 \rangle$ where η_δ is an increasing sequence of ordinals $< \delta$ with limit δ of length ∂*
  - (c) *for every $\alpha < \lambda$ the set $\{\eta_\delta \restriction i : \delta \in S, i < \partial$ and $\sup {\rm Rang}(\eta_\delta \restriction i) \leq \alpha\}$ has cardinality $< \lambda$*
  - (d) *$I \in K^\kappa_{\rm tr}$ is $\{\eta_\delta \restriction i : i \leq \partial, \delta \in S_1\} \cup \{\langle \alpha \rangle : \alpha < \lambda_1\}$*
  - (e) *$J \in K^\kappa_{\rm tr}$ is $\{\eta_\delta \restriction i : i \leq \partial, \delta \in S_1\} \cup \{\langle \alpha \rangle : \alpha < \lambda_1\}$.*

Proof. So let f be a function from **or**(I) into $\mathscr{M}_{\mu,\kappa}(\mathbf{or}(J))$ so actually a function from $I \times \{1, -1\}$ into $\mathscr{M}_{\mu,\kappa}(J \times \{1, -1\})$, and $<_*$ a well ordering of $\mathscr{M}_{\mu,\kappa}(J)$ but we "forget" to deal with it, as there are no problems, and let χ be large enough. Let $\bar{N} = \langle N_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in)$ such that $I, J, \lambda, \bar{\eta}, \mathscr{M}_{\mu,\kappa}(J), f, <_*$ belong to N_0 and $N_\alpha \cap \lambda \in \lambda, \bar{N} \restriction (\alpha +$

$1) \in N_{\alpha+1}$ for every $\alpha < \lambda$; as it happens "$\alpha_*, \beta_*, \pi \in N_0$" is not needed. So $E := \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$ is club of λ hence we can choose $\delta \in E \cap S_1 \setminus S_2$.

For any $\eta \in I$, clearly $f((\eta, 1))$ is well defined and $\in \mathscr{M}_{\mu,\kappa}(J)$ so let $f((\eta, 1)) = \sigma_\eta(\bar{\nu}_\eta), \bar{\nu}_\eta = \langle(\nu_{\eta,\epsilon}, \iota_{\eta,\epsilon}) : \epsilon < \epsilon_\eta\rangle, \nu_{\eta,i} \in J$ and $\iota_{\eta,\epsilon} \in \{1, -1\}, \epsilon_\eta < \kappa$.

Let $\epsilon_* = \epsilon_{\eta_\delta}, \iota_\epsilon = \iota_{\eta_\delta,\epsilon}, i^*_\epsilon = \lg(\nu_{\eta_\delta,\epsilon})$, so $i^*_\varepsilon \leq \partial$ for $\epsilon < \epsilon_*$ and let $j^*_\epsilon = \sup\{j \leq i^*_\epsilon : \sup \mathrm{Rang}(\nu_{\eta_\delta,\epsilon} \restriction j) < \delta\}$. By our assumption $j^*_\epsilon = \partial$ implies that $i_\epsilon = \partial$ hence as $\delta \notin S_2$ it follows that $\sup \mathrm{Rang}(\nu_{\eta_\delta,\epsilon}) < \delta$ hence by clause (c) of the assumption $\nu_{\eta_\delta,\epsilon} \in N_\delta$. Also $\alpha < \delta \Rightarrow J \cap {}^{\kappa>}\alpha \subseteq N_{\alpha+1}$ because it has cardinality $< \lambda$ and it belongs to $N_{\alpha+1}$; also let $\nu^*_\epsilon = \nu_{\eta_\delta,\epsilon} \restriction j^*_\epsilon$, it too belongs to N_δ.

So $\{\nu^*_\epsilon : \epsilon < \epsilon_*\} \subseteq N_\delta$, and it has cardinality $< \kappa$ as $\alpha < \lambda \to |\alpha|^{<\kappa} < \lambda$ and $\mathrm{cf}(\delta) = \partial \geq \kappa$ it follows that $\bar{\nu}^* = \langle \nu^*_\epsilon : \epsilon < \epsilon_*\rangle \in N_\delta$.

Let $u_* = \{\epsilon < \epsilon_* : j^*_\epsilon < i^*_\epsilon\}$. For $\epsilon \in u_*$ let $\alpha^*_\epsilon = \min(N_\delta \cap (\lambda + 1) \setminus \nu_{\eta_\delta,\epsilon}(j^*_\epsilon))$, so also $\bar{\alpha}^* := \langle \alpha_\epsilon : \epsilon \in u_*\rangle$ belongs to N_δ.

Now for $\eta \in {}^{\partial>}\lambda$ we define $\mathscr{U}_\eta$ as the set of $\beta \in S_1$ such that:

$(*)_{\eta,\beta}$ (a) $\eta \triangleleft \eta_\beta$

(b) $\sigma_{\eta_\beta} = \sigma_*$ so $\epsilon_{\eta_\beta} = \epsilon_*$

(c) $\lg(\nu_{\eta_\beta,\epsilon}) = i^*_\epsilon$ for $\epsilon < \epsilon_*$

(d) $\nu_{\eta_\beta,\epsilon} \restriction j^*_\epsilon = \nu^*_\epsilon$ for $\epsilon < \epsilon_*$

(e) $\iota_{\eta_\beta,\epsilon} = \iota_\epsilon$ for $\epsilon < \epsilon_*$

Note

⊛ if $\eta \triangleleft \eta_\delta$ then

(a) $\delta \in \mathscr{U}_\eta$ and $\mathscr{U}_\eta \in N_\delta$

(b) $\mathrm{cf}(\alpha^*_\epsilon) = \lambda$ for $\epsilon \in u_*$

(c) if $\bar{\alpha} \in \prod_{\epsilon \in u_*} \alpha^*_\epsilon$ <u>then</u> for arbitrarily large $\beta \in \mathscr{U}_\eta$ we have $\epsilon \in u_* \Rightarrow \nu_{\eta_\beta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon)$

(d) $\mathscr{U}_\eta$ is an unbounded subset of S_1.

[Why? Clause (a) directly. Why clause (d)? Otherwise $\sup(\mathscr{U}_\eta)$ is $< \lambda$ and it belongs to N_δ because $\mathscr{U}_\eta \in N_\delta$, hence $\sup(\mathscr{U}_\eta) \in N_\delta \cap \delta$ so $\sup(\mathscr{U}_\eta) < \delta$ contradicting clause (a). The other clauses follows by them.]

Next let Λ be the set of $\eta \in {}^{\partial>}\lambda$ such that

$\odot_\eta$ for every $\bar{\alpha} \in \prod_{\epsilon \in u_*} \alpha^*_\epsilon$ there is $\beta \in \mathscr{U}_\eta$ such that $\epsilon \in u_* \Rightarrow \nu_{\eta_\beta,\epsilon}(j^*_\epsilon) \in (\alpha_\epsilon, \alpha^*_\epsilon)$.

So

$(*)_1$ $\eta_1 \triangleleft \eta_2 \in \Lambda \Rightarrow \eta_1 \in \Lambda$

$(*)_2$ $\epsilon < \kappa \Rightarrow \eta_\delta \restriction \epsilon \in \Lambda$.

Hence

$(*)_3$ for some $\eta_* \in \Lambda$ the set $\mathscr{W} = \{\gamma < \lambda : \eta_* \hat{\ } \langle\gamma\rangle \in \Lambda\}$ is an unbounded subset of λ.

Let $\langle \gamma_\zeta : \zeta < \lambda \rangle$ list $\mathscr{W}$ in increasing order, and let $\alpha, \beta \le \lambda$ and π be a one-to-one function from α onto β.

Now first we choose $\delta(1,\zeta) \in S_1$ by induction on $\zeta < \alpha$ such that

$(*)_4$ (a) $\delta(1,\zeta) \in \mathscr{U}_{\eta_* \hat{} \langle \gamma_\zeta \rangle}$ i.e. $\gamma_\zeta \in \mathscr{W}$

(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\delta(1,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \mathrm{sub}\{\nu_{\eta_{\delta(1,\xi)},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

This is easy.

Second we choose $\delta(2,\zeta) \in S_1$ by induction on $\zeta < \beta$ such that:

$(*)_5$ (a) $\delta(2,\zeta) \in \mathscr{U}_{\eta_* \hat{} \langle \gamma_\xi \rangle}$ when $\pi(\xi) = \zeta$

(b) if $\epsilon \in u_*$ then $\nu_{\eta_{\delta(2,\zeta)},\epsilon}(j^*_\epsilon)$ is $< \alpha^*_\epsilon$ but is $> \sup\{\nu_{\eta_{\delta(2,\xi)},\epsilon}(j^*_\epsilon) : \xi < \zeta\}$.

Let $\bar{a} = \langle a_\zeta : \zeta < \alpha \rangle$, $\bar{b} = \langle b_\zeta : \zeta < \alpha \rangle$ from ${}^\alpha I$ be chosen as follows: $a_\zeta = (\eta_{\delta(1,\zeta)}, 1), b_\zeta = (\eta_{\delta(1,\pi(\zeta))}, 1)$ for $\zeta < \alpha$.

Now check, e.g.:

$(*)_6$ $a_{\zeta(1)} <_{\mathrm{or}(I)} a_{\zeta(2)}$ <u>iff</u> $\gamma_{\zeta(1)} < \gamma_{\zeta(2)}$ <u>iff</u> $\zeta(1) < \zeta(2)$

$(*)_7$ $b_{\zeta(1)} <_{\mathrm{or}(I)} b_{\zeta(2)}$ <u>iff</u> $\gamma_{\pi(\zeta)(1)} < \gamma_{\pi(\zeta)(2)}$ <u>iff</u> $\pi(\zeta)(1) < \pi(\zeta)(2)$.

$\square_{2.30}$

{2.27}
{2.26} **Conclusion 2.31.** *For $(\kappa, \mu, \lambda, \lambda_1, \alpha_*, \beta_*, \pi)$ as in 2.30, the class K_{or} has the full strong $(\lambda, \lambda_1, \mu, \kappa) - \varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property and the strong $(2^\lambda, \lambda_1, \mu, \kappa) - \varphi_{\mathrm{or},\alpha_*,\beta_*,\pi}$-bigness property.*

{2.26} *Proof.* By 2.30. $\square_{2.31}$

§ 3. Order Implies Many Non-Isomorphic Models

In this section (in a self contained way) we prove that not only the old result that any unstable (first order) T has in any $\lambda \geq |T| + \aleph_1$, the maximal number (2^λ) of pairwise non-isomorphic models holds, but for example that for any template Φ proper for linear orders, if the formula $\varphi(\bar{x}, \bar{y})$ with vocabulary τ, linearly orders $\{\bar{a}_s : s \in I\}$ in $\mathrm{EM}_\tau(I, \Phi)$ (Ehrenfeucht-Mostowski model, see §1) for every I, then the number of non-isomorphic models of the form $\mathrm{EM}_\tau(I, \Phi)$ of cardinality λ up to isomorphism is 2^λ when $\lambda \geq |\tau_\Phi| + \aleph_1$.

Dealing with this problem previously, the author (in the first attempt [Sh:12]) excluded some of the cardinals λ which satisfy $\lambda = |\tau_\Phi| + \aleph_1$ and in the second [Sh:a, Ch.VIII§3], replaced the $\mathrm{EM}_\tau(I, \Phi)$ with some kind of restricted ultrapower (of itself). Subsequently ([Sh:100]) we proved that for some unsuperstable first order complete theory T, and a first order theory T_1 extending T, $|T_1| = \aleph_1$, $|T| = \aleph_0$ the class

$$\mathrm{PC}(T_1, T) = \{M{\restriction}\tau(T) : M \models T_1\}$$

may be categorical in $\aleph_1$, "may be categorical" mean that some forcing extension this holds for some T, T_1; in fact if the original universe $\mathbf{V}$ satisfies CH, we may choose T, T_1 in $\mathbf{V}$.

We also prove there for $T =$ the theory of dense linear order, that we may, i.e. in some forcing extension, have a universal model in $\aleph_1$ even though CH fails. We then thought that the use of ultrapower in [Sh:a, Ch.VIII,§3] was necessary. This is not true. (We thank Rami Grossberg for a stimulating discussion which directed me to this problem again).

By the present theorem we can get the theorem also for the number of models of $\psi \in \mathbb{L}_{\lambda^+, \aleph_0}$ in λ $(> \aleph_0)$ when ψ is unstable. Incidentally the proof is considerably easier.

Note that we do not need to demand $\varphi(\bar{x}, \bar{y})$ to be first-order; a formula in any logic is O.K.; it is enough to demand $\varphi(\bar{x}, \bar{y})$ to have a suitable vocabulary. This is because an isomorphism from N onto M preserves satisfaction of such φ and its negation. However, the length of $\bar{x}$ (and $\bar{y}$) is crucial. Naturally we first concentrate on the finite case (in 3.1–3.20). But when we are not assuming this, we can, "almost always" save the result. In first reading, it may be advisable to concentrate on the case "λ is regular". {3.10}

For this section, the notion "$\langle \bar{a}_t : t \in I \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y})$-skeleton like inside M" is central and in Definition 3.1 the reader can concentrate on it. {3.1} {3.1}

Definition 3.1. Let M be a model, I an index model; for $s \in I$, $\bar{a}_s$ is a sequence from M, the length of $\bar{a}_s$ depends on the quantifier-free type of s over $\emptyset$ in I only; Λ is a set of formulas of the form $\varphi(\bar{x}, \bar{a})$, $\bar{a}$ from M, φ has a vocabulary contained in $\tau(M)$.

1) We say that $\langle \bar{a}_s : s \in I \rangle$ is weakly κ-skeleton like inside M for[2] Λ when: for every $\varphi(\bar{x}, \bar{a}) \in \Lambda$, there is $J \subseteq I$, $|J| < \kappa$ such that:

(∗) if $s, t \in I$ and $\mathrm{tp}_{\mathrm{qf}}(t, J, I) = \mathrm{tp}_{\mathrm{qf}}(s, J, I)$ then
$$M \models \text{“}\varphi[\bar{a}_s, \bar{a}] \equiv \varphi[\bar{a}_t, \bar{a}]\text{”}.$$

[2]The simplest example is: Λ the set of first order formulas with parameters from M.

2) If $\Lambda = \{\varphi(\bar{x}, \bar{a}) : \varphi(\bar{x}, \bar{y}_\varphi) \in \Delta,$
$bara \in \dot{\mathbf{J}}\}$ we may write $(\Delta, \dot{\mathbf{J}})$ instead of Λ; if $\Delta = \{\varphi(\bar{x}, \bar{y})\}$ we write $\varphi(\bar{x}, \bar{y})$ instead of Δ. If

$$\dot{\mathbf{J}} = \{\bar{a} : \bar{a} \text{ from } A, \text{ and for some } \varphi(\bar{x}, \bar{y}) \in \Delta, \ \ell g(\bar{a}) = \ell g(\bar{y})\}$$

we write A instead of Λ. If $|M| = A$ we write M instead A, and we omit it if clear from the context.
3) Supposing $\psi(\bar{x}, \bar{y}) =: \varphi(\bar{y}, \bar{x})$, I a linear order, we say $\langle \bar{a}_s : s \in I \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M for $\dot{\mathbf{J}}$ iff: $\varphi(\bar{x}, \bar{y})$ is asymmetric (at least in M) with vocabulary contained in $\tau(M)$, $\ell g(\bar{a}_s) = \ell g(\bar{x}) = \ell g(\bar{y}), \langle \bar{a}_s : s \in I \rangle$ is weakly κ-skeleton like inside M for $(\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}, \dot{\mathbf{J}})$ and for $s, t \in I$ we have:

$$M \models \varphi[\bar{a}_s, \bar{a}_t] \text{ iff } I \models s < t.$$

4) In (1), (3), if M is clear from the context then we may omit "inside M". In part (3), if $\dot{\mathbf{J}} = {}^\alpha|M|, \alpha = \ell g(\bar{x}) = \ell g(\bar{y})$ then we may omit it.

{3.1A}
{3.1} **Discussion 3.2.** Note that Definition 3.1 requires considerably more than "the $\bar{a}_s$ are ordered by φ" and even than "the $\bar{a}_s$ are order indiscernibles ordered by φ", but much less than "$M = \mathrm{EM}_\tau(I, \Phi)$".

We now would like to assign invariants to linear orders. We prove that there are enough linear orders with well defined pairwise distinct invariants. This is related to proofs from the Appendix to [Sh:a]=[Sh:c], where different terminology was employed. Speaking very roughly, we discussed there only $\mathrm{inv}^\alpha_\kappa$ where $\kappa = \aleph_0$. The assertion in the appendix of [Sh:c] that two linear orders are contradictory corresponds to the assertion here that the invariants are defined and different.

{3.1B} *Notation* 3.3. In the following, for any regular cardinal $\mu > \aleph_0, D_\mu$ denotes the filter on μ generated by the closed unbounded sets.
2) If D is a filter on μ and $X \subseteq \mu$ intersects each member of D, then $D + X$ denotes the filter generated by $D \cup \{X\}$.
3) For a linear order $I = (I, <_I)$ the cofinality $\mathrm{cf}(I)$ of I is

$$\mathrm{Min}\{|J| : J \subseteq I \text{ and } (\forall s \in I)(\exists t \in J) I \models s < t\}.$$

4) I^* is the inverse linear order and $\mathrm{cf}^*(I)$ is the cofinality of I^*.
5) For a linear order I and a cardinal κ, let

$$D = \mathscr{D}(\kappa, I) := \mathscr{D}_{\mathrm{cf}(I)} + \{\delta < \mathrm{cf}(I) : \kappa \leq \mathrm{cf}(\delta)\}.$$

6) Two functions f and g from $\mathrm{cf}(I)$ to some set X, are equivalent $\mod D$ if $\{\delta : f(\delta) = g(\delta)\} \in D$.
7) We write f/D for the equivalence class of f for this equivalence relations.

{3.2} **Definition 3.4.** 1) For a regular cardinal κ (for example $\aleph_0$) and an ordinal α we define $\mathrm{inv}^\alpha_\kappa(I)$ for linear orders I (sometimes undefined), by induction on α, by cases:

$\alpha = 0, \mathrm{inv}^\alpha_\kappa(I)$ is the cofinality of I if $\mathrm{cf}(I)$ is $\geq \kappa$, and is undefined otherwise.
$\alpha = \beta + 1$

Let $I = \bigcup_{i<\mathrm{cf}(I)} I_i$, where I_i is increasing and continuous in i and I_i is a proper initial segment of I. For $\delta < \mathrm{cf}(I)$ let $J_\delta = (I \setminus I_\delta)^*$ (where X^* denotes the inverse order of X). recalling 3.3(4).

{3.1B} If $\mathrm{cf}(I) > \kappa$ and for some club $\mathscr{C}$ of $\mathrm{cf}(I)$:

$(*)_{\mathscr{C}}$ $[\delta \in \mathscr{C}$ and $\mathrm{cf}(\delta) \geq \kappa] \Rightarrow \mathrm{inv}^\beta_\kappa(J_\delta)$ is defined,

<u>then</u> we let

$$\mathrm{inv}^\alpha_\kappa(I) = \langle \mathrm{inv}^\beta_\kappa(J_\delta) : \mathrm{cf}(\delta) \geq \kappa, \delta < \mathrm{cf}(I)\rangle / \mathscr{D}(\kappa, I).$$

Otherwise (i.e., there is no such $\mathscr{C}$ or $\mathrm{cf}(I) \leq \kappa$) $\mathrm{inv}^\alpha_\kappa(I)$ is not defined.

α is limit

$\mathrm{inv}^\alpha_\kappa(I) = \langle \mathrm{inv}^\beta_\kappa(I) : \beta < \alpha\rangle$.

2) If $\mathbf{d} = \mathrm{inv}^\alpha_\kappa(I)$ <u>then</u> "the cofinality of $\mathbf{d}$" means $\mathrm{cf}(I)$, clearly well defined. {3.2A}

Remark 3.5. 1) Really just $\alpha = 0, 1, 2$ are used. For regular λ, $\alpha = 1$ suffices, but for singular λ, $\alpha = 2$ is used (see 3.8). {3.4}

2) To understand the aim of 3.7 below, think of J as a linear order such that for some linear order U, and $\langle \bar{c}_t : t \in U\rangle$ we have $\bar{c}_t \in {}^{\ell g(\bar{x})}M$ and $\langle \bar{a}_s : s \in I\rangle\hat{\ }\langle \bar{c}_t : t \in U\rangle$ and $\langle \bar{b}_t : t \in U\rangle\hat{\ }\langle \bar{c}_t : t \in U\rangle$ are both weakly $(\kappa, \varphi(x, y))$-skeleton like in M and $\mathrm{cf}(U^*) \geq \kappa$. {3.3}

3) We can omit assumption (c) in 3.7, so the conclusion will tell us that if one of $\mathrm{inv}^\alpha_\kappa(I)$, $\mathrm{inv}^\alpha_\kappa(J)$ is well defined then both are, but presently there is no real gain. {3.3}

4) The following lemma will be helpful as we will try to deal with cases of inv inside models and try to prove that it is quite independent of a (relevant) choice of representatives. {3.2B}

Observation 3.6. *1) If $\beta \leq \alpha$ and $\mathrm{inv}^\alpha_\kappa(I) = \mathrm{inv}^\beta_\kappa(J)$, and both are well defined <u>then</u> $\mathrm{inv}^\beta_\kappa(I), \mathrm{inv}^\beta_\kappa(J)$ are well defined and equal.*

2) If I, J are linear orders, $\mathrm{inv} = \mathrm{inv}^\alpha_\kappa(I)$ is well defined, $\mathbf{E}$ is a convex equivalence relation on J, $f : J \overset{\text{onto}}{\longrightarrow} I$ preserves $\leq$, and $(f(x) = f(y)) \equiv (x\mathbf{E}y)$, <u>then</u> $\mathbf{d} = \mathrm{inv}^\alpha_\kappa(J)$.

3) Assume that $\psi(\bar{x}, \bar{y}) = \varphi(\bar{y}, \bar{x})$ and $\varphi_\ell(\bar{x}, \bar{y}) \in \{\varphi(\bar{x}, \bar{y}), \neg\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y}), \neg\psi(\bar{x}, \bar{y})\}$ for $\ell = 1, 2$. <u>Then</u> $\langle \bar{a}_s : s \in I\rangle$ is weakly $(\kappa, \varphi_1(\bar{x}, \bar{y}))$-skeleton like in M if and only if $\langle \bar{a}_s : s \in I^*\rangle$ is weakly $(\kappa, \varphi_2(\bar{x}, \bar{y}))$-skeleton like in M; also in M we have $\varphi(\bar{x}, \bar{y}) \vdash \neg\psi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y}) \vdash \neg\varphi(\bar{x}, \bar{y})$. {3.3}

Lemma 3.7. *Suppose that κ is a regular cardinal, I, J are linear orders, and $\bar{a}_s$ (for $s \in I$), $\bar{b}_t$ (for $t \in J$) are from M, and $\varphi(\bar{x}, \bar{y})$ is a $\tau(M)$-formula ($\kappa > \ell g(\bar{x}) = \ell g(\bar{y}) = \ell g(\bar{a}_s) = \ell g(\bar{b}_t)$), and $\psi(\bar{x}, \bar{y}) := \varphi(\bar{y}, \bar{x})$.*

Assume:

- *(a) (α) for every $s \in I$ for every large enough $t \in J, M \models \varphi[\bar{a}_s, \bar{b}_t]$,*
  - *(β) for every $t \in J$ for every large enough $s \in I, M \models \neg\varphi[\bar{a}_s, \bar{b}_t]$,*
- *(b) (α) $\langle \bar{a}_s : s \in I\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M,*
  - *(β) $\langle \bar{b}_t : t \in J\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M,*
- *(c) $\mathrm{inv}^\alpha_\kappa(I)$, $\mathrm{inv}^\alpha_\kappa(J)$ are defined.*

<u>*Then*</u> $\mathrm{inv}^\alpha_\kappa(I) = \mathrm{inv}^\alpha_\kappa(J)$.

Proof. By induction on α.

First Case: $\alpha = 0$

Assume not, so $\mathrm{inv}^0_\kappa(I) \neq \mathrm{inv}^0_\kappa(J)$. Then $\mathrm{cf}(I), \mathrm{cf}(J)$ are distinct (and $\geq \kappa$). By symmetry, without loss of generality $\mathrm{cf}(I) > \mathrm{cf}(J)$, so $\mathrm{cf}(I) > \kappa$.

Let $\langle t_\zeta : \zeta < \mathrm{cf}(J)\rangle$ be increasing unbounded in J. For each $\zeta < \mathrm{cf}(J)$ (by clause
{§32B} (a)(β) of 3.7 and 3.6) there is $s_\zeta \in I$ such that:

$$s_\zeta \leq s \in I \Rightarrow M \models \neg\varphi[\bar{a}_s, b_{t_\zeta}].$$

As $\mathrm{cf}(I) > \mathrm{cf}(J)$ there is $s \in I$ such that $\bigwedge_\zeta s_\zeta < s$. Now, the set

$$\{t \in J : M \models \neg\varphi[\bar{a}_s, \bar{b}_t]\}$$

includes each t_ζ (as $s_\zeta < s \in I$), and hence it is unbounded in J, contradicting
{3.3} clause (a)(α) of 3.7.

Second Case: $\alpha = \beta + 1$

{3.2B} By the first case and Observation 3.6, $\mathrm{cf}(I) = \mathrm{cf}(J) \geq \kappa$. Let $\lambda = \mathrm{cf}(I) = \mathrm{cf}(J)$; let

$$I = \bigcup_{i<\lambda} I_i,$$

where I_i is increasing continuous in i, I_i a proper initial segment of I and $[i \neq j \Rightarrow I_i \neq I_j]$.

Similarly let

$$J = \bigcup_{i<\lambda} J_i.$$

Choose $s_i \in I_{i+1} \setminus I_i$ and $t_i \in J_{j+1} \setminus J_j$. By assumption (a), for every $i < \lambda$ there is $j_i < \lambda$ such that:

$(\alpha)'$ if $t \in J \setminus J_{j_i}$ then $M \models \varphi[\bar{a}_{s_i}, \bar{b}_t]$,

$(\beta)'$ if $s \in I \setminus I_{j_i}$ then $M \models \neg\varphi[\bar{a}_s, \bar{b}_{t_i}]$.

Let

$$\mathscr{C} = \{\delta < \lambda : \delta \text{ is a limit ordinal and } i < \delta \Rightarrow j_i < \delta\};$$

{3.2} it is a club of λ. For $\delta \in \mathscr{C}$ let $I^\delta = (I \setminus I_\delta)^*$ and let $J^\delta = (J \setminus J_\delta)^*$. By Definition 3.4 above it suffices to prove, for $\delta \in \mathscr{C}$ satisfying $\mathrm{cf}(\delta) \geq \kappa$ such that $\mathrm{inv}^\beta_\kappa(I^\delta), \mathrm{inv}^\beta_\kappa(J^\delta)$ are defined, that:

$(*)_\delta$ $\mathrm{inv}^\beta_\kappa(I^\delta) = \mathrm{inv}^\beta_\kappa(J^\delta)$.

For this we use the induction hypothesis, but we have to check that the assumptions (a), (b), (c) hold for this case.

Now clause (c) is part of the assumption of $(*)_\delta$, and clause (b) is inherited from the same property of $\langle \bar{a}_s : s \in I\rangle$, $\langle \bar{b}_t : t \in J\rangle$; lastly clause (a) follows from $(\alpha)' + (\beta)'$ above as $\delta \in \mathscr{C}$. In detail, if $t \in J^\delta$ then $J \models$ "$t_j < t$" for $j < \delta$. Hence, for $i < \delta, M \models \varphi[\bar{a}_{s_i}, \bar{b}_t]$ (by clause $(\alpha)'$ above). So by clause (b)(β) from the assumptions, for every large enough $s \in I^\delta$ we have $M \models \varphi[\bar{a}_s, \bar{b}_t]$, which means that $\langle \bar{a}_s : s \in I^\delta \rangle$, $\langle \bar{a}_t : t \in J^\delta \rangle$ satisfy clause (a)(α). Similarly clause (a)(β) holds.

Third Case: α is limit

Immediate by Definition 3.4. $\square_{3.7}$ {3.2}

{3.4}

Lemma 3.8. *1) If λ, κ are regular, $\lambda > \kappa$, then there are 2^λ linear orders I_α (for $\alpha < 2^\lambda$), each of cardinality λ, with pairwise distinct $\mathrm{inv}^1_\kappa(I_\alpha)$ (for $\alpha < 2^\lambda$), each well defined.*
2) If $\lambda > \kappa$, κ is regular, then there are linear orders I_α (for $\alpha < 2^\lambda$), each of cardinality λ with pairwise distinct $\mathrm{inv}^2_\kappa(I_\alpha)$ (for $\alpha < 2^\lambda$), each well defined.
3) If in (2) we have $\lambda \geq \theta = \mathrm{cf}(\theta) > \kappa$, then we can have $\mathrm{cf}(I_\alpha) = \theta$ if we use inv^3_α. Similarly, if in part (1) we have $\lambda \geq \theta = \mathrm{cf}(\theta) > \kappa$, then we can have $\mathrm{cf}(I_\alpha) = \theta$ if we use inv^2_κ; of course can use $\mathrm{inv}^\alpha_\kappa$ for $\alpha \geq 2$ (similarly elsewhere).
4) Assume Φ is an almost $\mathscr{L}$-nice template proper for linear orders (see Definition 1.8). Then for any linear order I, the sequence $\langle \bar{a}_t : t \in I \rangle$ is $\aleph_0$-skeleton like for {1.6}
$\mathscr{L}$ inside $\mathrm{EM}(I, \Phi)$; $\mathscr{L}$ can be any set of formulas in the vocabulary τ_Φ.
5) In part (4), if I is $\aleph_0$-homogeneous (i.e., for any $n < \omega$ and $t_0 <_I \ldots <_I t_{n-1}, s_0 <_I \ldots <_I s_{n-1}$, there is an automorphism of I mapping t_ℓ to s_ℓ for $\ell < n$), then we can omit "almost $\mathscr{L}$-nice".

{3.4d}

Remark 3.9. 1) The construction of the linear orders is "hinted" by the proof 3.7, {3.3}
and by the properties of stationary sets. Alternatively see the inductive construction in Claims 3.7, 3.8 of the Appendix of [Sh:a] or see [Sh:12] where $\mathrm{inv}^\alpha_\kappa(1), \alpha < \lambda^+, \lambda = |I|$ are used.
2) Note that part (4) says that being skeleton-like really is a property of the skeleton of EM-models.
3) Note that 3.8(4) apply to $\mathrm{EM}_\tau(I, \Phi)$ whenever $\tau \subseteq \tau_\Phi$. {3.4}

Proof. 1) So $\lambda > \kappa$ are regular. The set $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary and hence we can find a partition $\langle S_\epsilon : \epsilon < \lambda \rangle$ of S into pairwise disjoint stationary subsets (well known, see Solovay theorem). For $u \subseteq \lambda$ we define I_u as the set

$$\{(\alpha, \beta) : \alpha < \lambda \text{ and } \alpha \in \bigcup_{\epsilon \in u} S_\epsilon \Rightarrow \beta < \kappa^+ \text{ and } \alpha \in \lambda \setminus \bigcup_{\epsilon \in u} S_\epsilon \Rightarrow \beta < \kappa\}$$

linearly ordered by

$$(\alpha_1, \beta_1) <_I (\alpha_2, \beta_2) \text{ iff } \alpha_1 < \alpha_2 \vee (\alpha_1 = \alpha_2 \text{ and } \beta_1 > \beta_2).$$

By the proof of 3.7 above clearly $\langle I_u : u \subseteq \lambda \rangle$ is as required. {3.3}
2) So we have $\lambda > \kappa$, $\kappa = \mathrm{cf}(\kappa)$.
Let $\lambda = \sum\limits_{i < \mathrm{cf}(\lambda)} \lambda_i$, λ_i increasing continuous $> \kappa$, let $\theta = \mathrm{cf}(\lambda) + \kappa^+$, or just $\kappa^+ + \mathrm{cf}(\lambda) \leq \theta = \mathrm{cf}(\theta) \leq \lambda$. Let $h : \theta \longrightarrow \mathrm{cf}(\lambda)$ be such that for any $i < \mathrm{cf}(\lambda)$ the set $\{\delta < \theta : \mathrm{cf}(\delta) = \kappa \text{ and } h(\delta) = i\}$ is stationary.

For each i, let $\langle I_{i,\epsilon} : \epsilon < 2^{\lambda_i^+} \rangle$ be as in the proof of (1) (for λ_i^+). For any $\nu \in \prod\limits_{i < \mathrm{cf}(\lambda)} 2^{\lambda_i^+}$ let $J_\nu = \sum\limits_{\alpha < \theta} J^*_{\nu,\alpha}$ with $J_{\nu,\alpha} \cong I_{h(\alpha),\nu(\alpha)}$.

3) Let $\langle I_\epsilon : \epsilon < 2^\lambda \rangle$ be as guaranteed in part (2) (or part (1) if λ is regular). For each $\epsilon < 2^\lambda$, let $J_\epsilon = \sum\limits_{i<\theta} J^*_{\epsilon,i}$ where $J_{\epsilon,i} \cong I_\epsilon$; now the sequence $\langle I_\epsilon : \epsilon < 2^\lambda \rangle$ is as required.
4) Let $\varphi = \varphi(\bar{x}, \bar{b}) \in \mathscr{L}(\tau_\Phi)$, so for some finite sequence $\bar{t}$ from I and a sequence $\bar{\sigma}$ of τ_Φ-terms we have $\bar{b} = \bar{\sigma}(\bar{t})$. So if s_1, s_2 realize the same quantifier free type over $\bar{t}$ in I, by indiscernibility (i.e., almost $\mathscr{L}$-niceness) $\mathrm{EM}(I, \Psi) \vDash$ "$\varphi[\bar{a}_{s_1}, \bar{b}] = \varphi[\bar{a}_{s_2}, \bar{b}]$". So $\mathrm{rang}(\bar{t})$ is as required.
5) Should be clear. $\square_{3.8}$

* * *

Now we would like to attach the invariants of a linear order I to a model M
{3.5} which has a skeleton-like sequence indexed by I. In (α) (in Definition 3.10 below) we define what it means for a sequence indexed by I to (κ, θ)-represent the (φ, ψ)-type of $\bar{c}$ over A.

{3.5}

Definition 3.10. Let $A \subseteq M, \bar{c} \in M$ and $\varphi(\bar{x}, \bar{y})$ be an asymmetric formula with vocabulary contained in $\tau(M)$ and $\psi(\bar{x}, \bar{y}) =: \varphi(\bar{y}, \bar{x})$

- (α) We say that $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represents $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ <u>iff</u>: I is a linear order, $\mathrm{cf}(I) \geq \kappa$ and for some linear order J of cofinality $\theta \geq \kappa$ disjoint to I, there are $\bar{a}_t \in {}^{\ell g(\bar{x})}A$ for $t \in J$, such that:
  - (i) for every large enough $t \in I$, $\bar{a}_t$ realizes $\mathrm{tp}_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\}}(\bar{c}, A, M)$, and
  - (ii) $\langle \bar{a}_s : s \in J + (I)^* \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M (I^* denotes the inverse of I).
- (β) We say that $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ has a (κ, θ, α)-invariant <u>when</u>:
  - (i) if for $\ell = 1, 2$, $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does (κ, θ)-represents $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\mathrm{inv}^\alpha_\kappa(I_\ell)$ are defined[3] for $\ell = 1, 2$ <u>then</u> $\mathrm{inv}^\alpha_\kappa(I_1) = \mathrm{inv}^\alpha_\kappa(I_2)$,
  - (ii) some $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\mathrm{inv}^\alpha_\kappa(I)$ is well defined.
- (γ) Let $\mathrm{INV}^\alpha_{\kappa,\theta}(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ be $\mathrm{inv}^\alpha_\kappa(I)$ when $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ has (κ, θ, α)-invariant and $\langle \bar{a}_s : s \in I \rangle$ does (κ, θ)-represent it
- (δ) Let "(κ, α)-invariant" means "(κ, θ, α)-invariant for some regular $\theta \geq \kappa$".
{3.5A} Similarly for "κ-represents" and $\mathrm{INV}^\alpha_\kappa(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ (justified by Fact 3.11 below).

{3.5A}

Fact 3.11. Suppose that for $\ell = 1, 2$, the sequence $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does (κ, θ_ℓ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$. <u>Then</u> $\theta_1 = \theta_2$.

Proof. So let for $\ell = 1, 2$ the sequence $\langle \bar{a}^\ell_s : s \in J_\ell \rangle$ witness that $\langle \bar{a}^\ell_s : s \in I_\ell \rangle$ does
{3.5} (κ, θ_ℓ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$, i.e., they are as in (α) of 3.10. Assume toward contradiction that $\theta_1 \neq \theta_2$ and by symmetry without loss of generality $\theta_1 < \theta_2$. Let $\langle s_\ell(\alpha) : \alpha < \theta_\ell \rangle$ be an increasing unbounded sequence of members of J_ℓ for $\ell = 1, 2$. So for each $\alpha < \theta_1$ we have

$$t \in I_1 \quad \Rightarrow \quad M \vDash \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^1_t]$$

{3.8A} [3]but see 3.18(2)

{3.5} and hence by clause (i) of (α) of Definition 3.10 we have $M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{c}]$ recalling $\bar{a}^1_{s_1(\alpha)} \subseteq A$, so for every large enough $t \in I_2, M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_t]$. But $\langle \bar{a}^2_t : t \in J_2 + (I_2)^* \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M, hence for some $\beta_\alpha < \theta_2$ we have

$$s_2(\beta_\alpha) \leq t \in J_2 \Rightarrow M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_t]$$

and so $\beta(*) = \sup\{\beta_\alpha + 1 : \alpha < \theta_1\} < \theta_2$ (as $\theta_1 < \theta_2 = \mathrm{cf}(\theta_2)$). So $M \models \varphi[\bar{a}^1_{s_1(\alpha)}, \bar{a}^2_{s_2(\beta(*))}]$ for $\alpha < \theta_1$.

But $t \in I_2 \Rightarrow M \models \neg\varphi[\bar{a}^2_t, \bar{a}^2_{s_2(\beta)}]$ and hence $M \models \neg\varphi[\bar{c}, \bar{a}^2_{s_2(\beta)}]$. Therefore, for every large enough $t \in I_1, M \models \neg\varphi[\bar{a}^1_t, \bar{a}^2_{s_2(\beta)}]$ and hence for every large enough $t \in J_1$, $M \models \neg\varphi[\bar{a}^1_t, \bar{a}^2_{s_2(\beta)}]$. Hence this holds for $t = s_1(\alpha)$, α large enough, a contradiction to the previous paragraph. $\square_{3.10}$

{3.6y}

Discussion 3.12. Each of Definition 3.13, Lemmas 3.15 and 3.17, and the proof of Theorem 3.19 have 3 cases. In the easiest case $\lambda = \|M\|$ is regular. When λ is singular the computation of $\mathrm{inv}^\alpha_\kappa(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ is easier when $\mathrm{cf}(\lambda) > \kappa$ (second case). The third case arises when $\lambda > \kappa > \mathrm{cf}(\lambda)$. {3.8} {3.9}

The relative easiness of the regular case is caused by the fact that any two increasing representations of a model with cardinality λ must "agree" on a club. In the second case we are able to restrict the first argument to a cofinal sequence of M. For the third case we must construct a "dual argument", noticing that much of a long sequence must concentrate on one member of the representation.

{3.6}

Definition 3.13. Let $\varphi(\bar{x}, \bar{y})$ be an asymmetric formula with vocabulary $\subseteq \tau(M)$ (where $\ell g(\bar{x}) = \ell g(\bar{y})$ is finite), and let M be a model of cardinality $\lambda, \lambda > \kappa, \kappa$ regular, α be an ordinal.

0) A representation of the model M is an increasing continuous sequence $\bar{M} = \langle M_i : i < \mathrm{cf}(\lambda) \rangle$ such that $\|M_i\| < \lambda$, and $M = \bigcup\limits_{i<\mathrm{cf}(\lambda)} M_i$.

Similarly for sets.

1) For a regular cardinal λ:

$\mathbf{INV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y})) = \{\mathbf{d} :$ for every representation $\langle A_i : i < \lambda \rangle$ of $|M|$, there are $\delta < \lambda$ and $\bar{c} \in M$ (of course, $\ell g(\bar{c}) = \ell g(\bar{x})$ such that $\mathrm{cf}(\delta) \geq \kappa$ and $\mathbf{d} = \mathrm{INV}^\alpha_\kappa(\bar{c}, A_\delta, M, \varphi(\bar{x}, \bar{y}))$ (in particular so the latter is well defined) $\}$.

2) For regular cardinals $\theta > \kappa$ such that $\lambda > \mathrm{cf}(\lambda) = \theta$ we let

$$\mathscr{D}_{\theta,\kappa} = \mathscr{D}_\theta + \{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$$

and

bfINV$^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y})) = \{\langle \mathbf{d}_i : i < \theta \rangle / \mathscr{D}_{\theta,\kappa} :$ for every representation $\langle A_i : i < \theta \rangle$ of $|M|$, there is $S \in \mathscr{D}_{\theta,\kappa}$ satisfying: for every $\delta \in S$ there is $\bar{c}_\delta \in M$ such that $\mathbf{d}_\delta = \mathrm{INV}^\alpha_\kappa(\bar{c}_\delta, A_\delta, M, \varphi(\bar{x}, \bar{y}))$ so is well defined and the cofinality of $\mathbf{d}_\delta$ is $> |A_\delta|\}$.

3) For regular cardinals $\kappa > \theta$, $\lambda > \theta > \kappa + \mathrm{cf}(\lambda)$ and a function h with domain a stationary subset of $\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$ and range a set of regular cardinals $< \lambda$, we let

$$\mathscr{D}_{\theta,h} = \mathscr{D}_\theta + \{\{\delta < \theta : h(\delta) \geq \mu \quad (\text{hence } \delta \in \mathrm{Dom}(h))\} : \mu < \lambda\},$$

and assuming that $\mathscr{D}_{h,\lambda}$ is a proper filter we let:

$\mathrm{bfINV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(x,y)) = \{\langle \mathbf{d}_i : i < \theta\rangle / \mathscr{D}_{\theta,h}$: for every representation $\langle A_i : i < \mathrm{cf}(\lambda)\rangle$ of $|M|$, there are $\gamma < \mathrm{cf}(\lambda)$ and $S \in \mathscr{D}_{h,\lambda}, S \subseteq \mathrm{Dom}(h)$, satisfying the following for each $\delta \in S$, if $h(\delta) > |A_\gamma|$ <u>then</u> for some $\bar{c}_\delta$ we have $\mathbf{d}_\delta = \mathrm{INV}^\alpha_\kappa(\bar{c}_\delta, A_\gamma, M, \varphi(\bar{x}, \bar{y}))$ so is well defined and the cofinality of e_δ is $> |A_\gamma|\}$.

{3.6A} {3.6} *Remark* 3.14. 1) Of course, also in 3.13(1) we could have used $\langle \mathbf{d}_i : i < \lambda\rangle / \mathscr{D}_\lambda$ as the invariant.

{3.6} {3.7} 2) In 3.13(3), we may demand "$\mathrm{cf}(\mathbf{d}_\delta) > |A_\delta|$".

Lemma 3.15. *Suppose $\varphi(\bar{x}, \bar{y})$ is a formula in the vocabulary of M, $\ell g(\bar{x}) = \ell g(\bar{y}) < \omega$.*

1) If $\lambda > \aleph_0$ is regular, M a model of cardinality λ, κ regular $< \lambda$, <u>then</u> $\mathrm{bfINV}^\alpha_\kappa(M, \varphi(\bar{x}, \bar{y}))$ has cardinality ! $\leq \lambda$.

2) If λ is singular, $\theta = \mathrm{cf}(\lambda) > \kappa$, <u>then</u> $\mathrm{bfINV}^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$ almost has cardinality $\leq \lambda$, which means: there are no $\mathbf{d}^\zeta_i$ (for $i < \theta$, $\zeta < \lambda^+$) such that:

(i) for $\zeta < \lambda^+, \langle \mathbf{d}^\zeta_i : i < \theta\rangle / \mathscr{D}_{\theta,\kappa} \in \mathrm{bfINV}^\alpha_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$,

(ii) for $i < \theta, \zeta < \xi < \lambda^+$, we have $\mathbf{d}^\zeta_i \neq \mathbf{d}^\xi_i$.

3) If λ is singular, θ, κ are regular, $\kappa + \mathrm{cf}(\lambda) < \theta < \lambda$, h is a function from some stationary subset of $\{i < \theta : \mathrm{cf}(i) \geq \kappa\}$ into

$$\{\mu < \lambda : \mu \text{ is a regular cardinal }\}$$

such that $\mathscr{D}_{\theta,h}$ is a proper filter, <u>then</u> $\mathrm{bfINV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$ almost has cardinality $\leq \lambda$, which means: there are no $\mathbf{d}^\zeta_i (i < \theta, \zeta < \lambda^+)$ such that:

(i) for $\zeta < \lambda^+, \langle \mathbf{d}^\zeta_i : i < \theta\rangle / \mathscr{D}_{\theta,h} \in \mathrm{bfINV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$,

(ii) for $i < \theta, \zeta < \xi < \lambda^+$, we have $\mathbf{d}^\zeta_i \neq \mathbf{d}^\xi_i$.

Proof. Straightforward. $\square_{3.15}$

* * *

We now show that (for example for the case λ regular) if $|I| \leq \lambda$ and $\mathrm{inv}^\alpha_\kappa(I)$ is well defined then there is a linear order J such that: if a model M has a weakly (κ, φ)-skeleton like sequence inside M of order-type J then $\mathrm{inv}^\alpha_\kappa(I) \in \mathrm{bfINV}^\alpha_\kappa(M, \varphi)$.

Again, the proof splits into three cases depending on the cofinality of λ. The following result provides a detail needed for the proof.

{3.7A}

Claim 3.16. *Suppose that κ is a regular cardinal and $\langle \bar{a}_t : t \in J\rangle$ is a weakly (κ, φ)-skeleton like inside M and $I \subseteq J$. If for each $s \in J \setminus I$ either $\{t \in I : t < s\}$ or the inverse order on $\{t \in I : t > s\}$ has cofinality less than κ (for example 1) then $\langle \bar{a}_t : t \in I\rangle$ is weakly (κ, φ)-skeleton like for M.*

Proof. As usual let $\psi(\bar{x}, \bar{y}) = \varphi(\bar{y}, \bar{x})$. We must show that for every $\bar{a} \in {}^{\ell g(\bar{x})}M$ there is an $I_{\bar{a}} \subseteq I$ with $|I_{\bar{a}}| < \kappa$ such that: if $s, t \in I$ and $\mathrm{tp}_{\mathrm{qf}}(s, I_{\bar{a}}, I) = \mathrm{tp}_{\mathrm{qf}}(t, I_{\bar{a}}, I)$ then

$$M \models \text{“}\varphi(\bar{a}_s, \bar{a}) \equiv \varphi(\bar{a}_t, \bar{a})\text{” and } M \models \text{“}\psi(\bar{a}_s, \bar{a}) \equiv \psi(\bar{a}_t, \bar{a})\text{”}.$$

We know that there is such a set $J_{\bar{a}}$ for J and $\bar{a}$ and for each $s \in J_{\bar{a}}$ choose a set X_s of $< \kappa$ elements of I such that X_s tends to s, i.e., to the cut that s induces in I (either from above or below). (So if $s \in I$, $X_s = \{s\}$; otherwise use the assumption). Let $I_{\bar{a}} = \bigcup_{s \in J_{\bar{a}}} X_s$; as κ is regular, $|X_s| < \kappa$ for $s \in J_{\bar{a}}$ and $|J_{\bar{a}}| < \kappa$ clearlly $I_{\bar{a}}$ has cardinality $< \kappa$; also trivially $J_{\bar{a}} \subseteq I$.

Now it is easy to see that if t_1 and $t_2 \in I$ have the same quantifier free type over $I_{\bar{a}}$, then they have the same quantifier free type over $J_{\bar{a}}$, and the claim follows. $\square_{3.16}$

Lemma 3.17. *Assume $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$ and $\varphi = \varphi(\bar{x}, \bar{y})$.* {3.8}
1) Let $\lambda > \aleph_0$ be regular. If I is a linear order of cardinality $\leq \lambda$, and $\mathrm{inv}^{\alpha}_{\kappa}(I)$ is well defined, then for some linear order J of cardinality λ the following holds:

- $(*)$ *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(x)}M, \langle \bar{a}_s : s \in J\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M (hence $\varphi(\bar{x}, \bar{y})$ is asymmetric), then $\mathrm{inv}^{\alpha}_{\kappa}(I) \in \mathrm{bfINV}^{\alpha}_{\kappa}(M, \varphi(\bar{x}, \bar{y}))$.*

2) Let λ be singular, $\theta = \mathrm{cf}(\lambda) > \kappa$, $\lambda = \sum_{i<\theta} \lambda_i$, where the sequence $\langle \lambda_i : i < \theta\rangle$ is increasing continuous. Suppose that for $i < \theta$, I_i is a linear order of cofinality $> \lambda_i$ and cardinality $\leq \lambda$ such that $\mathrm{inv}^{\alpha}_{\kappa}(I_i)$ is well defined. Then for some linear order J of cardinality λ the following holds:

- $(**)$ *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(x)}M$ for $s \in J$, $\langle \bar{a}_s : s \in J\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton inside M, (so $\varphi(\bar{x}, \bar{y})$ asymmetric), then $\langle \mathrm{inv}^{\alpha}_{\kappa}(I_i) : i < \theta\rangle / \mathscr{D}_{\theta,\kappa}$ belongs to $\mathrm{bfINV}^{\alpha}_{\kappa}(M, \varphi(\bar{x}, \bar{y}))$.*

3) Let λ be singular, θ, κ be regular, $\lambda > \theta > (\mathrm{cf}(\lambda)+\kappa)$, $\lambda = \sum_{i<\mathrm{cf}(\lambda)} \lambda_i$, λ_i increasing continuous. If, for $i < \theta$, I_i is a linear order such that $\mathrm{inv}^{\alpha}_{\kappa}(I_i)$ is well defined, then for some linear order J of cardinality λ the following holds:

- $(***)$ *if M is a model of cardinality λ, $\bar{a}_s \in {}^{\ell g(x)}M$ for $s \in J$, $\langle \bar{a}_s : s \in J\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M, (so $\varphi(\bar{x}, \bar{y})$) asymmetric), h is a function from a stationary subset of $\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\}$ with range a set of regular cardinals $< \lambda$ but $> \theta$ such that $\mathrm{cf}(I_i) \geq h(i)$ and $\mathscr{D}_{\theta,h}$ is a proper filter then $\langle \mathrm{inv}^{\alpha}_{\kappa}(I_i) : i < \theta\rangle / \mathscr{D}_{\theta,h}$ belongs to $\mathrm{bfINV}^{\alpha,h}_{\kappa,\theta}(M, \varphi(\bar{x}, \bar{y}))$.*

Proof. 1 We must choose a linear order J of cardinality λ such that: if J indexes a weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like sequence inside M, a model of cardinality λ, then

$$\mathrm{inv}^{\alpha}_{\kappa}(I)\in \mathrm{bfINV}^{\alpha}_{\kappa}(M,\varphi(\bar{x},\bar{y})).$$

For this, for any continuous increasing decomposition $\bar{A}$ of $|M|$, we must find a sequence $\bar{c}\in M$ and an ordinal δ with

$$\mathrm{INV}^{\alpha}_{\kappa}(\bar{c},A_\delta,M,\varphi(\bar{x},\bar{y}))=\mathrm{inv}^{\alpha}_{\kappa}(I).$$

To obtain $\bar{c}$, we shall use a function from λ to J. Let I_α for $\alpha<\lambda$ be pairwise disjoint linear orders isomorphic to I.

Let $J=\sum\limits_{\alpha<\lambda} I^*_\alpha$ (where I^* means we use the inverse of I as an ordered set). Suppose $\langle \bar{a}_s : s\in J\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like inside M, (hence $\varphi(\bar{x},\bar{y})$) is asymmetric), M has cardinality λ. For $\alpha<\lambda$ let $s(\alpha)\in I_\alpha$ and let $\langle A_\alpha:\alpha<\lambda\rangle$ be an increasing continuous sequence such that $M=\bigcup\limits_{\alpha<\lambda} A_\alpha$, $|A_\alpha|<\lambda$. By the
{3.1} definition of weak $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like (Definition 3.1(1)), for every $\bar{a}\in {}^{\ell g(\bar{x})}M$, here is a subset $J_{\bar{a}}$ of J of cardinaltiy $<\kappa$ such that: if $s,t\in J\setminus J_{\bar{a}}$ induces the same Dedekind cut on $J_{\bar{a}}$, then $M\models$ "$\varphi[\bar{a}_s,\bar{a}]\equiv\varphi[\bar{a}_t,\bar{a}]$" and $M\models$ "$\varphi[\bar{a},\bar{a}_s]\equiv\varphi[\bar{a},\bar{a}_t]$". Since λ is regular, for some closed unbounded subset $\mathscr{C}^*$ of λ, for every $\delta\in\mathscr{C}$ we have:

$(*)$ (i) $\bar{a}_{s(\alpha)}\in {}^{\ell g(\bar{x}}(A_\delta)$ for $\alpha<\delta$,

(ii) $J_{\bar{a}}\subseteq \sum\limits_{\beta<\delta} I^*_\beta$ for $\bar{a}\in {}^{\ell g(\bar{x}}(A_\delta)$.

So it is enough to prove that for any $\delta\in\mathscr{C}^*$ of cofinality $\le\kappa$ we have

$$\mathrm{inv}^{\alpha}_{\kappa}(I)=\mathrm{INV}^{\alpha}_{\kappa}(\bar{a}_{s(\delta)},A_\delta,M,\varphi(\bar{x},\bar{y})).$$

Let $\mathscr{C}\subseteq\delta$ be closed unbound of order types $\mathrm{cf}(\delta)$. It is easy to see that $\langle\bar{a}_s:s\in I_\delta\rangle$
{3.5} does κ-represents $(\bar{a}_{s(\delta)},A_\delta,M,\varphi(\bar{x},\bar{y}))$ as: the required θ and J in Definition
{3.7A} 3.10(α) are $\mathrm{cf}(\delta)$ and $\langle \bar{a}_{s(\beta)}:\beta\in\mathscr{C}\rangle$, and now use claim 3.16 with $J,\{s(\beta):\beta\in\mathscr{C}\}\cup I^*_\delta$ here standing for J,I there.
{3.5} So (see Definition 3.10(γ)) it is enough to show that $(\bar{a}_{s(\delta)},A_\delta,M,\varphi(\bar{x},\bar{y}))$ has a
{3.5} (κ,α)-invariant. Now in Definition 3.10(β), part (ii) is obvious by the above; so it remains to prove (i).

Let $\theta=:\mathrm{cf}(\delta)$. So assume that for $\ell=1,2$,

$$\langle\bar{a}^{\ell}_s:s\in I^{\ell}\rangle \text{ weakly } (\kappa,\theta)\text{-represents } (\bar{a}_{s(\delta)},A_\delta,M,\varphi(\bar{x},\bar{y})).$$

Let J^ℓ, $\langle a^\ell_t:t\in J^\ell\rangle$ exemplify this (so each $\bar{a}^\ell_t$ belongs to A_δ) and let $J^*_\ell=J^\ell+(I^\ell)^*$ and assume $\mathrm{inv}^{\alpha}_{\kappa}(I^\ell)$ are well defined. We have to prove that $\mathrm{inv}^{\alpha}_{\kappa}(I^1)=\mathrm{inv}^{\alpha}_{\kappa}(I^2)$.
{3.8A} This follows by 3.18(2) below. $\square_{3.17}$
{3.8A}

Fact 3.18. 1) Suppose $\langle\bar{a}_s:s\in J+I^*\rangle$ is weakly $(\kappa,\varphi(\bar{x},\bar{y}))$-skeleton like inside M and both J and I have cofinality $\ge\kappa$. Then for every $\bar{b}\in M$ there exist $s_0\in J$ and $s_1\in I^*$ such that if $s_0<t_\ell<s_1$ (in $J+I^*$) for $\ell=0,1$, then$M\models$ "$\varphi(\bar{a}_{t_0},\bar{b})\equiv\varphi(\bar{a}_{t_1},\bar{b})$", $M\models$ "$\psi(\bar{a}_{t_0},\bar{b})\equiv\psi(\bar{a}_{t_1},\bar{b})$".

2) Suppose that, for $\ell = 1, 2$, $\langle \bar{a}^\ell_s : s \in I^\ell \rangle$ does (κ, θ)-represent $(\bar{c}, A, M, \varphi(\bar{x}, \bar{y}))$ and $\langle \bar{a}^\ell_s : s \in J^\ell \rangle$ witnesses this. Then $\mathrm{inv}^\alpha_\kappa(I^1) = \mathrm{inv}^\alpha_\kappa(I^2)$.

Proof. 1) Easy.
2) As we can replace I^ℓ by any end segment, without loss of generality

(∗) for $\ell = 1, 2$ for every $t \in I^\ell, \bar{a}_t$ realizes $\mathrm{tp}_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})\}}(\bar{c}, A, M)$.

We shall use Lemma 3.7 (with I^1, I^2 here standing for I, J there and ψ for φ). Conditions (b),(c) from 3.7 are met trivially, for (b) using 3.6 and by similar arguments in condition (a) it is enough to prove clause (α). {3.3} {3.2B}

Let us prove (a)(α) from 3.7. So suppose it fails, i.e., $s \in I^1$ but for arbitrarily large $t \in I^2$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$. {3.3}

Since $\langle \bar{a}^2_t : t \in J^2 + (I^2)^* \rangle$ is weakly (κ, φ)-skeleton like inside M, the preceding Fact 3.18(1) yields that for arbitrarily large $t \in J^2$, $M \models \neg\varphi[\bar{a}^1_s, \bar{a}^2_t]$. Since $\bar{a}^1_s$ and $\bar{c}$ realize the same $\{\varphi, \psi\}$-type over A_δ (see definition 3.10(α) and (*) above), and as $\bar{a}^2_t \subseteq A_\delta$ for $t \in J^2$, this implies $M \models \neg\varphi[\bar{c}, \bar{a}^2_t]$, so this holds for arbitrarily large $t \in J^2$. Choose such $t_0 \in J^2$, this quickly contradicts the choice of J^2 and I^2. For, it implies that for every $t \in I^2$ (as $\bar{c}, \bar{a}^2_t$ realize the same $\{\varphi, \psi\}$-type over A_δ) we have {3.8A} {3.5}

$$M \models \neg\varphi[\bar{a}^2_t, \bar{a}^2_{t_0}],$$

which is impossible as $\langle \bar{a}_s : s \in J^2 + (I^2)^* \rangle$ is weakly (κ, φ)-skeleton like (see Definition 3.1(3) the last phrase). {3.1}

Continuing the proof of 3.17(2),(3): Left to the reader (or see the proof of case (d) and formulation of case (e) in Theorem 3.22). Take $J = \sum_{i<\theta} (I_i)^*$ where $I_i \cong I$ are pairwise disjoint. $\square_{3.18}$ {3.11}

{3.9}

Theorem 3.19. *Suppose that $\lambda > \kappa$, K_λ is a family of τ-models, each of cardinality $\lambda, \varphi(\bar{x}, \bar{y})$ is an asymmetric formula with vocabulary $\subseteq \tau$, and $\ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0$. Further, suppose that for every linear order J of cardinality λ there are $M \in K_\lambda$ and $\bar{a}_s \in M$ for $s \in J$ such that $\langle \bar{a}_s : s \in J \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like in M.*

Then*, in K_λ, there are 2^λ pairwise non-isomorphic models.*

Proof. First let $\lambda > \aleph_0$ be regular.

By 3.8(1) there are linear order I_ζ (for $\zeta < 2^\lambda$) each of cardinality λ, such that $\mathrm{inv}^1_\kappa(I_\zeta)$ are well defined and distinct. Let J_ζ relate to I_ζ as guarantee by 3.17(1). Let $M_\zeta \in K_\lambda$ be such that there are $\bar{a}^\zeta_s \in M_\zeta$ for $s \in J_\zeta$ such that $\langle \bar{a}^\zeta_s : s \in J_\zeta \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M_ζ (exists by assumption). By 3.17(1), that is our choice of J_ζ, we have {3.4} {3.8} {3.8}

$$\mathrm{inv}^1_\kappa(I_\zeta) \in \mathrm{bfINV}^1_\kappa(M_\zeta, \varphi(\bar{x}, \bar{y})).$$

Clearly,

$$M_\zeta \cong M_\xi \Rightarrow \mathrm{bfINV}^1_\kappa(M_\zeta, \varphi(\bar{x}, \bar{y})) = \mathrm{bfINV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y})),$$

and hence

$$M_\zeta \cong M_\xi \quad \Rightarrow \quad \mathrm{inv}^1_\kappa(I_\zeta) \in \mathrm{bfINV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y})).$$

So if for some $\xi < 2^\lambda$, the number of $\zeta < 2^\lambda$ for which $M_\zeta \cong M_\xi$ is $> \lambda$, then $\mathrm{bfINV}^1_\kappa(M_\xi, \varphi(\bar{x}, \bar{y}))$ has cardinality $> \lambda$ (remember $\mathrm{inv}^1_\kappa(I_\zeta)$ were pairwise distinct
{3.7} for $\zeta < 2^\lambda$). But this contradicts 3.15(1).

So

$$\{(\zeta, \xi) : \zeta, \xi < 2^\lambda \text{ and } M_\zeta \cong M_\xi\},$$

which is an equivalence relation on 2^λ, satisfies: each equivalence class has cardinality $\leq \lambda$. Hence there are 2^λ equivalence classes and we finish.

For λ singular the proof is similar. If $\mathrm{cf}(\lambda) > \kappa$, we can choose $\theta = \mathrm{cf}(\lambda)$ and
{3.8} use $\mathrm{INV}^2_{\kappa,\theta}$, 3.8(2), 3.17(2), 3.15(2) instead of $\mathrm{bfINV}^1_{\kappa,\theta}$, 3.8(1), 3.17(1), 3.15(1) respectively.

If $\mathrm{cf}(\lambda) \leq \kappa$, let $\theta = \kappa^+$ so $\lambda > \theta > \kappa + \mathrm{cf}(\lambda)$. Hence we can find a mapping

$$h : \{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa\} \longrightarrow \{\mu : \mu = \mathrm{cf}(\mu) < \lambda\}$$

such that for each $\mu = \mathrm{cf}(\mu) < \lambda$ the set

$$\{\delta < \theta : \mathrm{cf}(\delta) \geq \kappa \text{ and } h(\delta) \geq \mu\}$$

{3.8} is stationary. Now we can use $\mathrm{bfINV}^{2,h}_{\kappa,\theta}$, 3.8(2), 3.17(3), 3.15(3) instead bfINV^1_κ,
{3.8} 3.8(1), 3.17(1), 3.15(1) respectively.
{3.16} Alternatively, for singular λ see the proof of 3.28 and 3.22 case (d) below. $\square_{3.19}$
{3.10}

Conclusion 3.20. *1) If T_1 is a first order $T \subseteq T_1$, T is unstable and complete, $\lambda \geq |T_1| + \aleph_1$, then there are 2^λ pairwise non-isomorphic models of T of cardinality λ which are reducts of models of T_1.*
2) If $T \subseteq T_1$ are as above, $\lambda \geq |T_1| + \kappa^+$, $\lambda = \lambda^{<\kappa}$, κ is regular, then there are 2^λ pairwise non-isomorphic models of T of cardinalty λ which are reducts of models M^1_i of T_1 such that M_i, M^1_i are κ-compact and κ–homogeneous. [Really we can get strongly homogeneous; see [Sh:363, §1]*].*
3) Assume that $\psi \in \mathbb{L}_{\kappa^+,\omega}(\tau_1)$, $\tau \subseteq \tau^1$, ψ has the order property for $\mathbb{L}_{\kappa^+,\omega}(\tau)$ [i.e., for some formula $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}_{\kappa^+,\omega}(\tau)$ for arbitrarily large μ there is a model M of ψ and $\bar{a}_i \in M$ for $i < \mu$ such that

$$M \models \varphi[\bar{a}_i, \bar{a}_j] \textit{ iff } [i < j \textit{ and } \ell g(\bar{x}) = \ell g(\bar{y}) < \aleph_0].$$

Then for $\lambda \geq \kappa + \aleph_1$, ψ has 2^λ models of cardinality λ, with pairwise non-isomorphic τ-reducts.

Proof. 1) Let $\varphi = \varphi(\bar{x}, \bar{y})$ be a first order formula exemplifying "T is unstable"
{1.2} (see Definition 1.2). By 1.11(1) there is a template Φ proper for linear orders such that $|\tau_\Phi| = |\tau_1|$ and for any linear order I, $EM(I, \Phi)$ is a model of T_1 satisfying $\varphi[\bar{a}_s, \bar{a}_t]$ if and only if $I \models s < t$. Clearly $\mathrm{EM}_{\tau(T_1)}(I, \Phi)$ has cardinality $\geq |I|$ but $\leq |\tau_\Phi| + |I| + \aleph_0$. So for every $\lambda \geq |T_1| + \aleph_0 = |\tau_\Phi| + \aleph_0$ and linear order I of cardinality λ the model $M = EM_\tau(I, \Phi)$ is a τ–model, a reduct of a model of

{3.4} T_1, hence M is a model of T of cardinality exactly λ, and by 3.8(4) the sequence
{3.9} $\langle \bar{a}_t : t \in I \rangle$ is weakly κ-skeleton like. So we have the assumption of 3.19, hence its conclusion as required.
2) By [Sh:c, Ch.VII 3.1], or case II of the proof of Theorem 3.2 (there) we have the assumption of 3.19; but [Sh:363, §1] supersedes upon this. {3.9}
3) See 1.18(3) and Definition 1.15 why the assumption of 3.19 holds. $\square_{3.20}$ {3.10} {3.10A}

Remark 3.21. Also 1.23 is a similar result. {1.15}

* * *

Now we turn our attention to the case in which the sequences on which $\varphi(\bar{x}, \bar{y})$ speaks are infinite. {3.11}

Theorem 3.22. *Suppose $\partial < \kappa < \lambda$ are cardinals, κ regular. Assume K is a class of τ-models, $\varphi = \varphi(\bar{x}, \bar{y})$ is a formula with vocabulary $\subseteq \tau$, and $\partial = \lg(\bar{x}) = \lg(\bar{y})$, and*

- $(*)$ *$K = K_\lambda$ and for every linear order I of cardinality λ there are $M_I \in K_\lambda$ and a sequence $\langle \bar{a}_t : t \in I \rangle$ which is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M_I.*

We can conclude that $\dot{\mathbb{I}}(K) = 2^\lambda$ iff at least one of the following conditions holds:

- (a) $\lambda = \lambda^\partial$
- (b) $\lambda^\kappa < 2^\lambda$
- (c) *We replace the assumption $(*)$ by:*
  - $(*)_0$ $K = K_\lambda$,
  - $(*)_1$ $\lambda^\partial < 2^\lambda$, $\mathrm{cf}(\lambda) > \partial$,
  - $(*)_2$ *for every linear order J of cardinality λ there are $M_J \in K_\lambda$ and a weakly $(\kappa, < \lambda, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M_J sequence $\langle \bar{a}_s : s \in J \rangle$ (where $\bar{a}_s \in {}^\partial|M_J|$), see Definition 3.23 below.* {3.12}
- (d) *We replace the assumption $(*)$ by: for some $\lambda(0) \leq \lambda(1) \leq \lambda \leq \lambda(3) < 2^\lambda$, $\mu(0) \leq mu(1) \leq 2^\lambda$ with $\lambda(1)$ and $\mu(1)$ are regular, we have:*
  - $(*)_0$ $K = K_{\lambda(3)}$,
  - $(*)_1$ $\lambda^\partial < 2^\lambda$,
  - $(*)_2$ *for every linear order J of cardinality λ there is $M_J \in K_{\lambda(3)}$ (of cardinality $\lambda(3)$) and $\langle \bar{a}_s : s \in J \rangle$ (where $\bar{a}_s \in {}^\partial|M_J|$) which is weakly $(\kappa, \lambda(0), < \lambda(1), \varphi(\bar{x}, \bar{y}))$-skeleton like inside M_J (see Definition 3.23 below),* {3.12}

$(*)_{3,\mu(0),\lambda(0)}$ *for $J \in K^{\mathrm{or}}_\lambda (= (K_{\mathrm{or}})_\lambda)$ and a set $A \subseteq M_J$ (M_J is from $(*)_2$) if $|A| < \lambda(0)$ then:*

- (i) $\mu(0) > |\mathbb{S}^\partial_{\{\varphi,\psi\}}(A, M_J)|$, *or at least*
- (ii) $\mu(0) > |\{\mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar{b}_i : i < \kappa \rangle, A, M_J : \bar{b}_i \in A$ *for* $i < \kappa$, *the average is well defined and is realized in* $M\}|$, *where*

$$\mathrm{Av}_\Delta(\langle b_i : i < \kappa \rangle, A, M_J) := \{\varphi(\bar{x}, \bar{a})^{\mathbf{t}} : \varphi(\bar{x}, \bar{y}) \in \Delta, \mathbf{t} \text{ a truth value, } \bar{a} \in A \text{ and for all but a bounded set of } i < \kappa,\ M_J \models \varphi[\bar{b}_i, \bar{a}]^{\mathbf{t}}\},$$

$(*)_{4,\lambda,\mu(1),\mu(0),\lambda(0)}$ *if* $\dot{\mathbf{I}}_i \subseteq {}^{\partial}\lambda(3)$ *and* $|\dot{\mathbf{I}}_i| = \lambda$ *for* $i < \mu(1)$, *then for some* $B \subseteq \lambda(3)$ *we have:*

$$|B| < \lambda(0) \text{ and } |\{i : |dot\mathbf{I}_i \cap {}^{\partial}B| \geq \kappa\}| \geq \mu(0).$$

(e) *We replace assumption* $(*)$ *by: for some* $\lambda_{0,\epsilon} \leq \lambda_{1,\epsilon} \leq \lambda \leq \lambda_3, \mu_{0,\epsilon} \leq \mu_1 \leq 2^\lambda$, *for* $\epsilon < \epsilon(*), \mu_1$ *is regular and:*

$(*)_0$ $K = K_{\lambda_3}$,

$(*)_1$ $\lambda^\partial < 2^\lambda$,

$(*)_2$ *for every linear order* J *of cardinality* λ *there is* $M_J \in K_{\lambda(3)}$ *and* $\langle \bar{a}_s : s \in J\rangle$ *(where* $\bar{a}_s \in {}^{\partial}|M_J|$*) which for each* $\epsilon < \epsilon(*)$ *is weakly* $(\kappa_1, < \lambda_{0,i}, < \lambda_{1,i}, \varphi(\bar{x}, \bar{y}))$*-skeleton like inside* M_J,

$(*)_{3,\mu_{0,\epsilon},\lambda_{0,\epsilon}}$ *if* $\epsilon < \epsilon(*)$ *and* $J \in K^{\rm or}_\lambda (= (K_{\rm or})_\lambda)$ *and a set* $A \subseteq M_J$ *(*M_J *is from* $(*)_2$*) if* $|A| < \lambda_{0,\epsilon}$ *then:*

(i) $\mu_{0,\epsilon} > |\mathbf{S}^\partial_{\{\varphi,\psi\}}(A, M_J)|$ *or at least*

(ii) $\mu_{0,\epsilon} > |\{\mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar{b}_i : i < \kappa\rangle, A, M_J) : \bar{b}_i \in A$ *for* $i < \kappa$, *the average is well defined and is realized in* $M\}|$, *where*

$$\mathrm{Av}_\Delta(\langle b_i : i < \kappa\rangle, A, M_J) := \quad \{\varphi(\bar{x}, \bar{a})^{\mathbf{t}} : \varphi(\bar{x}, \bar{y}) \in \Delta, \ \mathbf{t} \text{ a truth value,}$$
$$\bar{a} \in A \text{ and for all but a bounded se}$$

$(*)_4$ *there are* $h_\alpha : \lambda \longrightarrow \{\theta : \theta$ *regular,* $\kappa \leq$ *theta* $\leq \lambda\}$ *for* $\alpha < 2^\lambda$ *such that: if* $S \subseteq 2^\lambda$, $|S| \geq \mu(1)$ *and* $f_\alpha : \lambda \longrightarrow {}^{\partial}(\lambda_3)$ *for* $\alpha \in S$, *then we can find* $\epsilon < \epsilon(*), B \subseteq \lambda_3$ *satisfying:* $|B| < \lambda_{0,\epsilon}$ *and the set* $\{\alpha :$ *the closure of* $\{\zeta < \lambda : f_\alpha(\zeta) \subseteq B\}$ *has a member* δ *of cofinality* κ *such that* $h_\alpha(\delta) \geq \lambda_{1,\epsilon}\}$ *has* $\geq \mu_{0,\epsilon}$ *members. [Note:* $\mathrm{cf}(\delta) = \kappa' \geq \kappa$ *can be allowed if* $(*)_{3,\mu_{0,\epsilon},\lambda_{0,\epsilon}}$ *is changed accordingly].*

(f) *For some* $\mu < \lambda$, *there is a linear order of cardinality* μ *with* $\geq \lambda$ *Dedekind cuts each with upper and lower cofinality* $\geq \kappa$ *and* $2^{\mu+\partial} < 2^\lambda$.

(g) *there is* $\mathscr{P} \subseteq [\lambda^\partial]^\kappa$ *of cardinality* $< 2^\lambda$ *such that every* $X \subseteq \lambda^\partial$ *of cardinality* λ *contains at least one of them (and* $(*)$*); (can use similar considerations in other places).*

{3.12}

Definition 3.23. We say $\langle \bar{a}_s : s \in I\rangle$ is weakly $(\kappa, \mu, < \lambda, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M; if $\mu = \lambda$ we may omit μ; iff:

(i) for $s, t \in I$ we have

$$M \models \varphi[\bar{a}_s, \bar{a}_t] \text{ if and only if } I \models s < t,$$

{3.1} (ii) for every $\bar{c} \in {}^{\ell g(\bar{a}_s)}M$ for some $J \subseteq I$, $|J| < \kappa$ and $(*)$ of 3.1(1) holds, and

(iii) moreover, for each $A \subseteq M$, $|A| < \mu$, there is $J \subseteq I$, $|J| < \lambda$ such that for
{3.1} every $\bar{c} \in {}^{\ell g(\bar{x})}A$,the statement $(*)$ of 3.1 holds for J.

Proof. Case (a):

{3.5} In Definition 3.10 we can replace A by $\dot{\mathbf{J}}$, a set of sequences of length ∂ from M,
{3.5} which means that clause (i) in (α) of 3.10 now becomes (i)' for every large enough $t \in I$, for every $I \in \dot{\mathbf{J}}$ we have $M \models \varphi[\bar{a}, \bar{b}] = \varphi[\bar{a}_t, \bar{b}]$ and $M \models \psi[\bar{c}, \bar{b}] \equiv \varphi[\bar{a}_t, \bar{b}]$.

{3.6} Thus in Definition 3.13, replace $\langle A_i : i < \lambda \rangle$ by $\langle \dot{\mathbf{J}}_i : i < \mathrm{cf}(\lambda)) \rangle, {}^{\partial}|M| = \bigcup_i \dot{\mathbf{J}}_i, |\dot{\mathbf{J}}_i| < \lambda$, $\dot{\mathbf{J}}_i$ increasing continuous. No further changes in 3.1-3.19 is needed.

{3.9} Alternatively, we can define $N = F_\partial(M)$ as the model with universe $|M| \cup {}^{\partial}|M|$, assuming of course $|M|$ is disjoint to ${}^{\partial}|M|$,

$$\tau(N) = \tau(M) \cup \{F_i : i < \partial\},$$

$$R^N = R^M \text{ for } R \in \tau(M),$$

$$G^N(x_1, \ldots, x_n) = \begin{cases} G^M(x_1, \ldots, x_n) & \text{if } x_1, \ldots, x_n \in |M|, \\ x_1 & \text{otherwise} \end{cases}.$$

for function symbol $G \in \tau(M)$ which has n-places and

$$F_i^N(x) = \begin{cases} x(i) & \text{if } x \in {}^{\partial}M, \\ x & \text{if } x \in M \end{cases}$$

for $i < \partial$, so F_i is a new, unary function symbol for $i < \partial$.

Note that [$M_1 \cong M_2$ if and only if $F_\partial(M_1) \cong F_\partial(M_2)$], and $\|F_\partial(M)\| = \|M\|^\partial$, etc. So we can apply 3.19 to the class $\{F_\partial(M) : M \in K_\lambda\}$ and we can get the {3.9} desired conclusion.

Case (b): We use weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like sequences $\langle \bar{a}_s : s \in \kappa + (I_\zeta)^* \rangle$ in $M_\zeta \in K_\lambda$ for $\zeta < 2^\lambda$, with $\langle \mathrm{inv}^2_\kappa(I_\zeta) : \zeta < 2^\lambda \rangle$ pairwise distinct, and count the number of models $(M_\zeta, \langle \bar{a}_s : s \in \kappa \rangle)$ up to isomorphism. Then "forget the $\bar{a}_s$, $s \in \kappa$", i.e., use 3.24 below. {3.13}

Case (c): We revise 3.10–3.20; we use this opportunity to present another reasonable choice in clause (α) of 3.10. {3.10} {3.5}
Change 1: In 3.10(α) we replace (i), (ii) by {3.5}

- $(i)'$ for every formula $\vartheta(\bar{x}, \bar{d}) \in \mathrm{tp}_{\{\varphi(\bar{x},\bar{y}),\psi(\bar{x},\bar{y})}(\bar{c}, A, M)$, for every large enough $t \in I$ we have $M \models \vartheta[\bar{c}, \bar{d}] \equiv \vartheta[\bar{a}_t, \bar{d}]$,
- $(ii)'$ $\langle \bar{a}_s : s \in J + (J)^* \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M,
- $(iii)'$ $\theta > \mathrm{cf}(J)$ (actually $\theta \neq \mathrm{cf}(J)$ would suffice, but no real need) (not actually needed, but natural).

Of course, the meaning of Definition 3.10(β)-(δ) changes, and the reader can check {3.5} that, e.g., the proof of the Fact is still valid.

Change 2: In Definition 3.13(1), inside the definition of $\mathrm{bfINV}^\alpha_\kappa$, we demand $\mathrm{cf}(\mathbf{d}) =$ {3.6} λ recalling λ is regular.

Change 3: In Definition 3.13(2), inside the definition of $\mathrm{INV}^\alpha_{\kappa,\theta}$ add $\mathrm{cf}(\mathbf{d}_\delta) > \mathrm{cf}(\delta)$ {3.6} (necessitate by change 1, actually $\mathrm{cf}(\mathbf{d}_\delta) \neq \mathrm{cf}(\delta)$ suffices).

Change 4: In Definition 3.13(3) demand $\mathrm{cf}(\lambda) > \partial$. {3.6}

Change 5: In 3.15, in all cases the "cardinality $\leq \lambda$" is replaced by "cardinality {3.7}
$\leq \lambda^\partial$" and part (2) becomes like part (3).

Change 6: We replace "$(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton likeq" by $(\kappa, <\lambda, \varphi(\bar{x}, \bar{y}))$-skeleton like.
{3.8} In 3.17(3) add the demand $\mathrm{cf}(\lambda) > \partial$, $h(i) > \mathrm{cf}(i)$.

{3.8} Change 7: Inside the proof of 3.17(1), now not for every $\bar{a} \in {}^{\ell g(\bar{x})}M$ we define $J_{\bar{a}}$,
{3.12} but for every $A \subseteq M$ of cardinality $< \lambda$ we choose $J_A \subseteq J$, $|J_A| < \lambda$ by Definition 3.23, and in $(*)(ii)$ in the proof there we demand

$$(\forall \alpha < \delta)(\exists \beta < \delta)[\bigcup_{s \in J_{A_\alpha}} \bar{a}_s \subseteq A\beta].$$

{3.8} Change 8: In the proof of 3.17(2) let $\langle I_i : i < \theta \rangle$ be as in the statement of 3.17(2), and let $J = \sum_{i<\theta} I^*_i$, and assume $\langle \bar{a}_s : s \in J \rangle$ is $(\kappa, <\lambda, \varphi(\bar{x}, \bar{y}))$-skeleton like inside $M \in K_\lambda$. So let $\langle A_i : i < \theta \rangle$ be a representation of M, and for each $i < \theta$ let
{3.12} $J_{A_i} \subseteq J$, $|J_{A_i}| < \lambda$ be as in Definition 3.23.

Define

$$\mathscr{C} = \{\delta < \theta : \quad \delta \text{ is a limit ordinal such that for every } \alpha < \delta \text{ the cardinality of } J_{A_i} \text{ is } < \lambda_\delta\}.$$

So let $\delta \in C, \mathrm{cf}(\delta) \geq \kappa$. Recall that $\mathrm{cf}(I_\delta) > \lambda_\delta$ so clearly we can find $s(\delta) \in I_\delta$ such that

$$I_\delta \models s(\delta) \leq s \Rightarrow s \notin \bigcup_{i<\delta} J_{A_i}.$$

Now $(\bar{c}_{s(\delta)}, A_\delta, M, \varphi(\bar{x}, \bar{y}))$ is as required.

{3.12} Change 9: In the proof of 3.23(3) let $J = \sum_{\alpha<\theta} I^*_\alpha$ and M, $\langle \bar{a}_s : s \in J \rangle$, $\langle A_i : i < \mathrm{cf}(\lambda) \rangle$, $J_{A_i} \subseteq J$ be as above, and let $s(\alpha) \in I_\alpha$. As $\mathrm{cf}(\lambda) > \partial$ by $(*)_1$ of the assumption, for each $s \in J$ for some $i(s) < \mathrm{cf}(\lambda)$ we have $\bar{c} \subseteq A_{i(s)}$, but $\theta = \mathrm{cf}(\theta) > \mathrm{cf}(\lambda)$ hence for some $i(*) < \mathrm{cf}(\lambda)$ the set $W = \{\alpha < \theta : i(\alpha) \leq i(*)\}$ is unbounded in θ. Let $\mathscr{C} = \{\delta < \theta : \delta = \sup(\delta \cap W)\}$. We can choose $\delta \in \mathscr{C}$ of cofinality $\geq \kappa$ such that $h(\delta) > |J_{A_{i(*)}}|$, and continue as in the previous case.

{3.8A} Change 10: Proof of 3.18(2) (necessitated by change 1)
{3.3} We shall use Lemma 3.7 (with I^1, I^2 here standing for I, J there and ψ for φ).
{3.3} Conditions (b), (c) from 3.7 are met trivially and by similar arguments in condition (a) it is enough to prove clause (α).

{3.3} Let us prove (a)(α) from 3.7. Let $I^\ell_* \subseteq I^\ell$ be unbounded of order type $\mathrm{cf}(I^\ell) = \theta$ and let $J^\ell_* \subseteq J^\ell$ be unbounded of order type $\mathrm{cf}(J^\ell)$, which is $\neq \theta$. Possibly shrinking those sets the truth values of $\varphi[\bar{a}^1_s, \bar{a}^2_t]$ when $s \in I^1_*, y \in J^2 \wedge (\exists t')(t' \in J^2_*$ and $t' <_{J^2} t)$ is constant. We can continue as before.

Note that if $\mathrm{cf}(\lambda) > \kappa$ this follows from case (d). If λ is regular, choose $\lambda(0) = \lambda(1) = \lambda(3) = \lambda$ and $\mu(0) = \mu(1) = (\lambda^\partial)^+$ and now the assumptions hold. If λ is singular, let $\epsilon(*) = \mathrm{cf}(\lambda), \chi = (\mathrm{cf}(\lambda) + \kappa)^+ \leq \lambda$, $\mu_0 = \mu_{1,\epsilon} = (\lambda^\partial)^+$ and let

$\{(\lambda_{0,\epsilon},\lambda_{1,\epsilon}) : \epsilon < \epsilon(*)\}$ list $\{(\lambda_i^+,\lambda_j^+) : i < j < \mathrm{cf}(\lambda)\}$ and choose $h_\lambda = h : \lambda \longrightarrow \{\theta : \theta \text{ regular}, \kappa \le \theta \le \lambda\}$ such that $\epsilon < \epsilon(*) = \mathrm{cf}(\lambda)$ implies $\{\delta < \chi : \mathrm{cf}(\delta) = \kappa$ and $h(\delta) = \epsilon\}$ is stationary. Now we can apply case (e).

Case (d): Let $\langle I_\alpha : \alpha < 2^\lambda\rangle$ be a sequence of linear orders of cofinality $\mathrm{cf}(\lambda(1)) = \lambda(1)$, each of cardinality λ, with pairwise distinct $\mathrm{inv}^2_\kappa(I_\alpha)$ if λ is regular, $\mathrm{inv}^3_\kappa(I_\alpha)$ if λ is singular exists by 3.8. Let $J_\alpha = \sum_{\zeta\le\lambda} I^*_{\alpha,\zeta}$, where $I_{\alpha,\zeta}$ are pairwise disjoint, {3.4}
$I_{\alpha,\zeta} \cong I_\alpha$. Let M_{J_α} be a model as guaranteed in $(*)_2$ with $\langle \bar a_s : s \in J_\alpha\rangle$ as there. Suppose $\{M_{J_\alpha}/\cong\ : \alpha < 2^\lambda\}$ has cardinality $< 2^\lambda$, then without loss of generality $M_{J_\alpha} = M_{J_0}$ for $\alpha < \mu(1)$ and without loss of generality M_{J_0} has universe $\lambda(3)$. Let $s(\alpha,\zeta) \in I_{\alpha,\zeta}$, so

$$\dot{\mathbf{I}}_\alpha := \{\bar a_{s(\alpha,\zeta)} : \zeta < \lambda\}$$

is a subset of ${}^\partial(\lambda(3))$ of cardinality λ. By $(*)_{4,\lambda,\mu(1),\mu(0),\lambda(0)}$ there is $B \subseteq \lambda(3)$, $|B| < \lambda(0)$ such that

$$S =: \{\alpha < \mu(1) : |\dot{\mathbf{I}}_\alpha \cap {}^\partial B| \ge \kappa\}$$

has cardinality $\ge \mu(0)$. Choose for each $\alpha \in S$ a set

$$S_\alpha \subseteq \{\zeta : \bar a_{s(\alpha,\zeta)} \subseteq B\},$$

which has order type κ, and let

$$\delta_\alpha =: \sup(S_\alpha).$$

Clearly $\delta_\alpha \le \lambda$, hence I_{α,δ_α} is well defined. For each $\alpha \in S$, as $\langle \bar a_s : s \in J_\alpha\rangle$ is $(\kappa,\lambda(0),<\lambda(1),\varphi(\bar x,\bar y))$-skeleton like and $|B| < \lambda(0)$, there is a subset $J_{\alpha,B}$ of J_α as in Definition 3.23. But I_{α,δ_α} has cofinality $\lambda(1) > |B|$, hence for all large enough {3.12}
$t \in I_{\alpha,\delta_\alpha}$, the type $\mathrm{tp}_{\{\varphi,\psi\}}(\bar a_t, B, M_{J_0})$ is the same; choose such t_α. Clearly (for $\alpha \in S$)

$$\mathrm{tp}_{\{\varphi,\psi\}}(\bar a_{t_\alpha}, B, M_{J_0}) = \mathrm{Av}_{\{\varphi,\psi\}}(\langle \bar a_{s(\alpha,\zeta)} : \zeta \in S_\alpha\rangle, B, M_{J_0}),$$

so by $(*)_{3,\mu(0),\lambda(0)}$ from the assumption of case (d) without loss of generality for some $\alpha \ne \beta$ we get the same type. But I_α, I_β have different (and well defined) inv^2_κ (or inv^3_κ), contradicting 3.18(2). {3.8A}

Case (e):
Similar proof (to (d)).

Case (f):
By 3.24 below. {3.13}

Case (g):
Similar to case (b). $\square_{3.23}$ {3.13}

Fact 3.24. If $\tau_2 = \tau_1 \cup \{c_i : i \in I\}$, c_i are individual constants, K_ℓ is a class of τ_ℓ-models (for $\ell = 1,2$), $M \in K_2 \ \Rightarrow\ M{\restriction}\tau_1 \in K_1$, and $\mu = \dot{\mathbb{I}}(\lambda, K_2) > \lambda^{|I|}$, then $\dot{\mathbb{I}}(\lambda, K_1) \ge \mu$ (so if $\mu = 2^{\lambda+|\tau_1|}$, equality holds).

Proof. Straight (or see [Sh:a, Ch.VIII,1.3]). $\square_{3.24}$

{3.10} In 3.20-3.22 above we do not get anything when $\lambda^\partial = 2^\lambda$, however if we assume that M_J has a clearer structure , e.g., is an EM-model, we can get better results as
{3.14} done below.

Conclusion 3.25. *1) Suppose $\psi \in \mathbb{L}_{\chi^+,\omega}(\tau_1)$, $\tau \subseteq \tau_1$, $\varphi(\bar{x},\bar{y}) \in \mathbb{L}_{\chi^+,\omega}(\tau)$, $\ell g(\bar{x}) = \ell g(\bar{y}) = \partial \leq \chi$, and ψ has the $\varphi(\bar{x},\bar{y})$-order property that is for every μ for some model M of ψ there are $\bar{a}_i \in {}^\partial M$ (for $i < \mu$) such that*

$$M \models \varphi[\bar{a}_i, \bar{a}_j] \quad \textit{iff} \quad i < j.$$

Then for every λ such that $\lambda > \chi^\partial$ or $\lambda > \chi$ and $2^\lambda > \lambda^\partial$, ψ has 2^λ models of cardinality λ with pairwise non-isomorphic τ-reducts.
2) Suppose $\psi \in \mathbb{L}_{\chi^+,\omega}(\tau)$, $\varphi_\ell(\bar{x},\bar{y}) \in \mathbb{L}_{\chi^+,\omega}(\tau_\ell)$, for $\ell = 1,2$, $\ell g(\bar{x}) = \ell g(\bar{y}) = \partial$, $\tau_0 = \tau_1 \cap \tau_2 = \tau_1 \cap \tau = \tau_2 \cap \tau$, $\{\psi, \varphi_1(\bar{x},\bar{y}), \varphi_2(\bar{x},\bar{y})\}$ has no model and ψ has the (φ_1,φ_2)-order property, which means that

- $(*)$ *for every α there is a τ_0-model M and $\bar{a}_\beta \in {}^\partial|M|$ for $\beta < \alpha$, such that: if $\beta < \gamma < \alpha$ then*
  - (i) *for some expansion M' of M, $M' \models \varphi_1[\bar{a}_\beta, \bar{a}_\gamma]$,*
  - (ii) *for some expansion M' of M, $M' \models \varphi_2[\bar{a}_\gamma, \bar{a}_\beta]$.*

Let $\varphi(\bar{x},\bar{y}) = (\exists \ldots, R, \ldots)_{R \in \tau_1 \setminus \tau_0} \varphi_1(\bar{x},\bar{y})$; it is a formula in the vocabulary τ_0 (but of second order). Then

- (a) *for λ such that $\lambda > \chi^\partial$ or $\lambda > \chi$ and $2^\lambda > \lambda^\partial, \dot{\mathbb{I}}_\tau(\lambda,\psi) = 2^\lambda$ i.e., there are 2^λ non-isomorphic τ-models of ψ of cardinality λ, in fact even their τ_0–reducts are not isomorphic;*
- (b) *for $\lambda \geq \chi$ there are $\langle M_J : J \in (K_{\rm or})_\lambda \rangle$, M_J a model of ψ of cardinality λ with a weakly (∂^+,φ)-skeleton like $\langle \bar{a}_s : s \in J \rangle$, $\bar{a}_s \in {}^\partial|M_J|$, fully repre-*
{2.2} *sented in $\mathscr{M}_{\chi,\aleph_0}$ and $\bar{a}_s = \bar{\sigma}(s)$ for some sequence $\bar{\sigma}$ of term of $\tau_{\chi,\aleph_0}$ see 2.4, or even $\bar{a}_s = \langle F_{1,i}(s) : i < \partial \rangle$.*

Proof. 1) Follows from (2), by taking $\varphi(\bar{x},\bar{y}) = \varphi_1(\bar{x},\bar{y}) = \varphi_2(\bar{y},\bar{x})$.
{1.16} 2) By 1.18(3), 1.23 there is Φ, proper for the class of linear orders (see Definition 1.8) such that for every linear order $I, {\rm EM}_\tau(I,\Phi)$ is a model of ψ of cardinality $\chi + |I|$, for $t \in I, \bar{a}_t$ is a sequence of length ∂ of members of ${\rm EM}_\tau(I,\Phi)$, in fact is $\bar{\sigma}(t)$ for a fixed $\bar{\sigma}$, such that for $s,t \in I$:

$$\begin{aligned} {\rm EM}_\tau(I,\Phi) \models \varphi[\bar{a}_s,\bar{a}_t] \quad & \text{iff } s < t \\ & \text{iff } {\rm EM}_t au(I,\Phi) \vDash \neg(\exists \ldots, R, \ldots)_{R \in \tau_2 \setminus \tau_1} \varphi_2[\bar{a}_t,\bar{a}_t]. \end{aligned}$$

{3.11} By 3.2 $\langle \bar{a}_s : s \in I \rangle$ is weakly (∂^+,φ)-skeleton like (see Definition 3.1). Clearly
{3.14} ${\rm EM}_t au(I,\Phi)$ is represented in $\mathscr{M}_{\chi,\aleph_0}$. So the clause (b) of 3.25(2) holds. To prove
{3c.16} clause (a) we can use 3.28, Case A (as $\theta = \aleph_0$) below. $\square_{3.25}$

* * *

{3.10} We may like in, for example, 3.20 to get not just non-isomorphic models, but non-isomorphic because of some nice invariant is different. The following definition serves

{3.15}

Definition 3.26. 1) Let μ be a regular uncountable cardinal, h_0, h_1 be functions from some stationary $S \subseteq \mu$ to a set of regular cardinals $\leq \lambda$ satisfying $(\forall \delta \in S)(h_0(\delta) \leq h_1(\delta)), \bar{h} = (h_0, h_1)$. Let M be a τ-model, $\varphi(\bar{x}, \bar{y})$ a formula in the vocabulary τ such that $\ell g(\bar{x}) = \ell g(\bar{y}) = \partial$.

Now, we say that M κ-obeys $(\bar{h}, \varphi)$, or (h_0, h_1, φ), if the following holds:

$(*)_0$ there is a function $\mathbf{H}$ from ${}^{\mu>}([M]^{<\mu})$ to $[M]^{<\mu}$ such that: if $\langle A_i : i < \mu \rangle$ is an increasing continuous sequence of subsets of M, $|A_i| < \mu$, and $\mathbf{H}(\langle A_i : i \leq j \rangle) \subseteq A_{j+1}$ for every $j < \mu$, then for some club $\mathscr{C} \subseteq \mu$, for every $\delta \in \mathscr{C} \cap S$ of cofinality $\geq \kappa$ the following holds:

$\oplus$ if for each $i < \mathrm{cf}(\delta)$, $\bar{a}_i \subseteq A_{\alpha_i}$ for some $\alpha_i < \delta$, $\langle \bar{a}_i : i < \mathrm{cf}(\delta) \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M (so $\ell g(\bar{a}_i) = \partial$), for each $\alpha < \delta$ the sequence

$$\langle \mathrm{tp}_{\{\varphi,\psi\}}(\bar{a}_i, A_\alpha) : i < \mathrm{cf}(\delta) \rangle$$

is eventually constant then:

$(*)_1 = (*)^1_{h_0(\delta),h_1(\delta)}$ if every $B \subseteq |M|$ of cardinality $< h_0(\delta)$ belongs to $\mathscr{P}_1$, then every $B \subseteq |M|$ of cardinality $< h_1(\delta)$ belongs[4] to $\mathscr{P}_1$, where

$(*)_2$ $\mathscr{P}_0 = \{B \subseteq M : B \subseteq M \text{ and } p^* {\restriction} B \text{ is realized in } M\}$, see on p^* below,

$$\mathscr{P}_1 = \{B \in M : B \subseteq M \text{ and } B \cup A_\delta \in \mathscr{P}_0\},$$

where

$(*)_3$ $p^* = p^*_{M,\langle \bar{a}_i : i < \mathrm{cf}(\delta)\rangle} =: \big\{\vartheta(\bar{x}, \bar{c}) : \bar{c} \subseteq M$, and for every $i < \mathrm{cf}(\delta)$ large enough $M \models \vartheta[\bar{a}_i, \bar{c}]$
and $\vartheta(\bar{x}, \bar{y}) \in \{\varphi(\bar{x}, \bar{y}), \neg\varphi(\bar{x}, \bar{y}), \varphi(\bar{y}, \bar{x}), \neg\varphi(\bar{y}, \bar{x})\}\big\}$.

2) In (1), we say that M obeys $(\bar{h}, \varphi(\bar{x}, \bar{y}))$ exactly, when in $(*)$, for $\delta \in \mathscr{C} \cap S$, the statement $\oplus$ fails for $h_1(\delta)^+$ (i.e., for some $\langle \bar{a}_i : i < \mathrm{cf}(\delta) \rangle$, p, p^* as there, $|p| = h(\delta)$, p is not realized in M.)

3) We say that M weakly κ-obeys $(\bar{h}, \varphi)$ when the following variant of $(*)$ of part (1) holds: we replace $(*)^1_{h_0(\delta),h_1(\delta)}$ by

$(*)_0 = (*)^0_{h_0(\delta),h_1(\delta)}$ if every $B \subseteq M$ of cardinality $< h_0(\delta)$ belongs to $\mathscr{P}_1$ then every $B \subseteq M$ of cardinality $< h_1(\delta)$ belongs to $\mathscr{P}_0$

4) We say that M weakly obeys $(h_0, h_1, \varphi(\bar{x}, \bar{y}))$ exactly iff in $(*)$ of part (3), for $\delta \in \mathscr{C} \cap S$, the statement $(*)^0_{h_0(\delta),h_1(\delta)^+}$ fails.

5) We add in the definition above the adjective "semi" to κ-obeys iff we change $(*)$ to

$(*)'$ given $\bar{b}_\alpha \in {}^\partial M$ for $\alpha < \mu$ and $\langle \bar{b}_\alpha : \alpha < \mu \rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like, there are an unbounded $Y \subseteq \mu$ and a function $\mathbf{H}$ from $\mu([M]^{<\mu})$ to $[M]^{<\mu}$ such that: if $\langle A_i : i < \mu \rangle$ is an increasing continuous sequence of subsets of M, $|A_i| < \mu$ and $i > \mu \Rightarrow \mathbf{H}(\langle A_i : i \leq j \rangle) \subseteq A_{j+1}$ then for some club $\mathscr{C}$ of μ, for every $\delta \in \mathscr{C} \cap S$ of cofinality $\geq \kappa$, the following holds:

[4] so if $h_0(\delta) = h_1(\delta)$ this is an empty requirement

$\oplus$ there are sequences $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$, $\langle \beta_i : i < \mathrm{cf}(\delta)\rangle$ both increasing with limit $\delta, \beta_i \in S$, and we let $\bar{a}_i = \bar{b}_{\beta_i} \subseteq A_{\alpha_i}$ (not necessarily $\langle \bar{a}_i : i < \mathrm{cf}(\delta)\rangle$ is weakly $(\kappa, \varphi(\bar{x}, \bar{y}))$-skeleton like inside M) and for each $\alpha < \delta$ the sequence $\langle \mathrm{tp}_{\{\varphi,\psi\}}(\bar{a}_i, A_\alpha) : i < \mathrm{cf}(\delta)\rangle$ is eventually constant <u>then</u> $(*)_0$ or $(*)_1$ etc.

6) We say "exactly semi κ-obeys (h_0, h_1, φ)" <u>iff</u> M semi κ-obeys (h_0, h_1, φ) and if $\bigwedge_{\delta \in S} h_1(\delta) \leq h_1^+(\delta)$ and $(\exists^{\mathrm{stat}} \delta \in S)(h_1(\delta) < h_1^+(\delta))$, then M does not semi κ-obeys (h_0, h_1^+, φ). We write (h, φ) if in (h_0, h_1, φ), $h_1 = h$ and h_0 is constantly κ.

{3c.15d}
{3.15} *Remark* 3.27. 1) In 3.26(5), (6) we can avoid $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ with small changes.
2) Note that assuming below $\lambda < \chi^{<\theta}$ is very reasonable as $\chi^{<\theta}$ is the number of distinct terms, and we have no information on a representation in $\mathscr{M}_{\chi,\theta}(I)$ using every term only once. Also $\lambda < \partial^+$ seems reasonable.

{3c.16}
Theorem 3.28. *Assume that $\varphi(\bar{x}, \bar{y})$ is an asymmetric $\tau(K)$-formula, $\partial = \ell g(\bar{x}) = \mathrm{lg}(\bar{y})$. Suppose that for every $I \in K_\lambda^{\mathrm{or}}$ there is a τ-model $M_I \in K_\lambda$, weakly full $\varphi(\bar{x}, \bar{y})$-represented in $\mathscr{M}_{\chi,\theta}(I)$, by the identity function for notational simplicity*
{2.2} *(see Definition 2.4), where $\lambda > \chi^{<\theta} + \partial^+$ and for $s \in I$, $\bar{a}_s = \langle F_{i,1}(s) : i < \partial\rangle \in {}^{\partial}|M_I|$ and $M_I \models \varphi[\bar{a}_s, \bar{a}_t]$ <u>if and only if</u> $s < t$ (for $s, t \in I$) (where $F_{i,1} \in \tau_{\chi,\theta}$ is a one place function symbol for $i < \partial$).*

<u>*Then*</u>

(a) $\dot{\mathbb{I}}(\lambda, K_\lambda) = 2^\lambda$ *if:* $\lambda \geq \chi^{<\theta} + \chi^\partial$ *and* : $\lambda > \chi^\theta + \chi^\partial$ *or* $\lambda^\partial < 2^\lambda$ *and* $\mathrm{cf}(\lambda) > \partial$ *or* $\lambda^\partial < 2^\lambda$ *and* $\theta = \aleph_0$ *or there is a linear order I with $\geq \lambda$ Dedekind cuts of cofinality $\geq \kappa$ with $2^{|I|} < 2^\lambda$,*

{3.15} (b) *the cardinal invariants from Definition 3.26(5), suffice to distinguish 2^λ models in K_λ if $\lambda > \chi^{<\theta} + \chi^\partial$.*

Remark 3.29. 1) In the cases $M_I = EM_\tau(I, \Phi), |\tau_\Phi| \leq \chi, \ell g(\bar{a}_s) = \partial$, clearly M_I is weakly full $\varphi(\bar{x}, \bar{y})$-represented in $\mathscr{M}_{\chi,\theta}$ by some f, $f(\bar{a}_s) = \langle F_{i,1}(s) : i < \partial\rangle$ for $\theta = \aleph_0$, $\chi = |\tau_\Phi| + \aleph_0$.
{2.2} 2) On "weakly full $\varphi(\bar{x}, \bar{y})$-represented" see Definition 2.4 clauses (d)+(f).

Proof. Note that, letting $\kappa := \partial^+ + \theta$, (so it is a regular cardinal):

$(*)$ in M_I, $\langle \bar{a}_s : s \in I\rangle$ is weakly $(\kappa, < \mu, \varphi(\bar{x}, \bar{y}))$-skeleton like in M_I, see
{3.12} Definition 3.23 whenever $\mu \geq \kappa$. So in particular $(*)$ Definition 3.22 holds.

[Why? Assume $A \subseteq M_I$ and $|A| < \mu$, so for each $a \in A$ let $a = \sigma_a(\bar{t}_a), \bar{t}_a \in {}^{\theta>}I$ and let $J = \cup\{\bar{t}_a : a \in A\}$ so $J \subseteq I$ is of cardinality $< \mu$ such that $A \subseteq \{\sigma(\bar{t}) : \bar{t} \in {}^{\theta>}J$ and σ a $\tau_{\chi,\theta}$-term$\}$. Clearly J is as required].

$(**)$ $\lambda > \chi^{<\theta} + \partial^+ \geq \kappa = \mathrm{cf}(\kappa)$,

by the assumption

$(***)$ $\chi \geq \partial$ and of course $\lambda > \chi^{<\theta} + \partial^+$ hence

{3.11} We shall use $(*), (**), (***)$, freely. Let us see why the cases below and 3.22 cover all the possibilities.

<u>Why does clause (a) hold?</u>

First, if $\lambda > \chi^{<\theta} + \chi^{\partial}$ then clause (b) proved below suffices, so without loss of generality $\lambda \le \chi^{<\theta} + \chi^{\partial}$, but $\lambda \le \chi^{<\theta} + \chi^{\partial}$ so $\lambda = \chi^{<\theta} + \chi^{\partial}$.

{3.11} If $\lambda^{\partial} < 2^{\lambda}$ and $\mathrm{cf}(\lambda) > \partial$ then we can apply claim 3.22 clause (c); so we have to
{3.11} check the assumptions there. The general assumption of 3.22, holds trivially. Now
{3c.16} $(*)_0$ there holds by the general assumption of 3.28 and $(*)_1$ there holds by the case of (a) we are dealing with and $(*)_3$ holds by (*) above.

Second, assume $\lambda^{\partial} < 2^{\lambda}$ and $\lambda > \chi < \theta = \aleph_0$, so as without loss of generality the previous case does not holds, we have $\mathrm{cf}(\lambda) \le \partial$.

Third, let $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ be strictly increasing with limit λ, $\lambda_i = \mathrm{cf}(\lambda_i) > \chi^{<\theta} + \partial^+$, and without loss of generality $\langle 2^{\lambda_i} : i < \mathrm{cf}(\lambda)\rangle$ is constant (so is constantly 2^{λ}) or is strictly increasing (still $2^{\lambda} = \prod\limits_{i<\mathrm{cf}(\lambda)} 2^{\lambda_i}$). In the former case by Fact 3.31 {3c.18}
below we can reduce the problem to any λ_i, so assume that $\langle 2^{\mu_i} : i < \mathrm{cf}(\lambda)\rangle$ is strictly increasing. As we are assuming $\chi^{<\theta} < \lambda \le \chi^{\partial}$, clearly λ is not strong limit, So without loss of generality $2^{\lambda_i} \ge \lambda$, and hence $2^{\lambda_i} \ge \lambda^{\partial}$, so without loss of generality $2^{\lambda_1} > \lambda^{\partial}$.

Fourth, note that if there is a linear order I with $\ge \lambda$ Dedekind cuts with both cofinalities $\ge \kappa$ and $2^{|I|} < 2^{\lambda}$ then we are done as in claim 3.22 clause (f). But {3.11}
as $\langle 2^{\mu_i} : i < \mathrm{cf}(\lambda)\rangle$ is strictly increasing there is such linear order, see [Sh:E62, 3.7=Lc2].

<u>Clause (b)</u>:

If λ is regular $> \kappa^+$, we apply case (C) or case (F). If $\lambda = \kappa^+$ we apply case (D) (case (G) is empty) and if λ is singular we apply case (E) or (H).

<u>Case A</u>: $\lambda^{\partial} = \lambda$ or $\lambda^{\kappa} < 2^{\lambda}$.

As $\kappa =: \partial^+ + \theta < \lambda$ by (*) above we can apply 3.22 case (a) or case (b) and get {3.11}
$\dot{\mathbb{I}}(\lambda, K_\lambda) = 2^{\lambda}$.

<u>Case B</u>: $\lambda^{\partial} < 2^{\lambda}$ and $\mathrm{cf}(\lambda) > \partial$ and we get $\dot{\mathbb{I}}(\lambda, K_\lambda) = 2^{\lambda}$.

By 3.22 case (c) (and $(*)$ above). {3.11}

<u>Case C</u>: λ is regular, $(\forall \mu < \lambda)[\mu^{<\kappa} < \lambda], \lambda \ge \kappa^{++}$.

Let $S_0 = \{\delta < \lambda : \mathrm{cf}(\delta) \ge \kappa\}$ and let h_0 be the function with domain S_0 and constant value $\chi^{<\theta}$. Let $J^{[\kappa]}$ be a linear order of cardinality κ such that $\alpha < \kappa \Rightarrow J^{[\kappa]} \times (\alpha+1) \cong J^{[\kappa]} \cong J^{[\kappa]} \times ((\alpha+1)^*)$. (e.g. let J be a κ-dense strongly κ-homogeneous linear order, hence $\alpha \le \kappa \Rightarrow J \times (\alpha+1) \cong J = J \times ((\alpha+1)^*)$, and by the Löwenheim-Skolem argument there is a dense $J' \subseteq J$ of cardinality κ with this property; alternatively use [Sh:E62, 2.21=Lc73]).

For a function

$$h : S_0 \longrightarrow \{\mu : \mu \text{ is a regular cardinal}, \kappa \le \mu < \lambda\}$$

let I_h be the linear order with the set of elements

$$\{(\alpha,\beta,t) : \quad \begin{array}{l} \alpha < \lambda + \kappa,\ t \in J^{[\kappa]} \text{ and} \\ \beta < h(\alpha) \text{ if } \alpha \in S_0, \text{ and } \beta < \kappa \text{ otherwise}\}. \end{array}$$

The order is:

$$(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \text{ if and only if } \begin{array}{l} \alpha_1 < \alpha_2, \text{ or} \\ \alpha_1 = \alpha_2 \text{ and } \beta_1 \geq \beta_2, \text{ or} \\ \alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2 \text{ and } t_1 <_{J^*} t_2. \end{array}$$

Now

{3.15} $\boxdot$ M_{I_h} semi κ-obeys the pair $(h, (\varphi(\bar{x}, \bar{y}))$ exactly (see Definition 3.26).

{3.15} First we prove "obey". So (see Definition 3.26(5) with $\mu = \lambda$) let $\bar{b}_\alpha \in {}^{\partial}(M_I)$ for $\alpha < \lambda$. So for some sequence $\bar{\sigma}^\alpha$ of $\bar{\sigma}$-terms $\bar{b}_\alpha = \bar{\sigma}^\alpha(\bar{t}^\alpha)$ with $\bar{t}^\alpha \in {}^{\kappa >}(I_n)\zeta^* < \kappa, u \subseteq \zeta^*$, and for some stationary set $Y \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ and term $\bar{\sigma}^*$ we have

- $\circledast_1$ $\alpha \in Y \Rightarrow \bar{\sigma}^\alpha = \bar{\sigma}^*, \ell g(\bar{t}) = \zeta^*$, order type of $I \restriction \bar{t}^\alpha$ is constant, and $\bar{t}^\alpha \restriction u = \bar{t}^*$ and
- $\circledast_2$ $\epsilon \in \zeta^* \setminus u \Rightarrow$ the sequence $\langle t^\alpha_\epsilon : \alpha \in Y\rangle$ is $<_I$-increasing
- $\circledast_3$ the truth value of $t^{\alpha_1}_{\epsilon_1} <_{I_n} t^{|\alpha_2}_{\epsilon_2}$ for $\alpha_1, \alpha_2 \in Y$ and $\epsilon_1, \epsilon_2 < \zeta^*$ depend just on the truth values of $\alpha_1 < \alpha_2, \alpha_2 < \alpha_1$ and the values of ϵ_1, ϵ_2.

We define a function $\mathbf{H}$ from ${}^{\lambda >}([M_I]^{<\lambda})$ to $[M_I]^{<\lambda}$ by: given $\langle A_j : j < i\rangle$, with $A_j \subseteq M_{I_h}$ increasing, $|A_j| < \mu$ let

$$\gamma = \gamma_{A_i} = \mathrm{Min}\{\gamma : \begin{array}{l} A_j \subseteq \{\sigma^*(\bar{t}) : \bar{t} \in {}^{\kappa >}(\gamma \times \mu \times J^{[\kappa]} \cap I_h)\} \text{ and} \\ (\forall j \leq i)\bar{t}^j \subseteq \gamma \times \mu \cap I_h\}. \end{array}$$

Let $A_i \in [M_{I_h}]^{<\lambda}$ be increasing continuous, $\mathbf{H}(\langle A_j : j \leq i\rangle) \subseteq A_{j+1}$, and let

$$\mathscr{C} = \{\delta < \lambda : \begin{array}{l} (\forall \alpha, \beta)(\alpha < \delta \cap (\alpha, \beta) \in I_h \Rightarrow \beta < \delta) \text{ and} \\ (\forall i)(\gamma_i < \delta \equiv i < \delta), \text{ and} \\ \alpha < \delta \text{ and } i \in Y \setminus \delta \text{ and } j \in Y \setminus \delta \Rightarrow \\ \mathrm{tp}_{\{\varphi,\psi\}}(\bar{b}_i, A_\alpha, M_{I_h}) = \mathrm{tp}_{\{\varphi,\psi\}}(\bar{b}_\delta, A_\alpha, M_{I_n}), \text{ and} \\ \delta = \sup(\delta \cap Y) \text{ and} \\ \epsilon \in \sigma \setminus u \Rightarrow (\forall i)(t^i_\epsilon \in \delta \times \mu \times J^{[\kappa]} \equiv i < \delta)\}. \end{array}$$

Clearly $\mathscr{C}$ is a club of λ. Now let $\delta \in S \cap \mathscr{C}$. We can choose $\beta(i) \in Y$ for $i < \mathrm{cf}(\delta)$ increasing with limit δ. By the definition of representable clearly $\langle \bar{b}_{\beta(i)} : i < \mathrm{cf}(\delta)\rangle$ as required from $\langle \bar{a}_i : i < \mathrm{cf}(\delta)\rangle$ in Definition 3.26(1), and so $p^* = p^*_{M_{I_h}, \langle \bar{b}_{\beta(i)} : i < \mathrm{cf}(\delta)\rangle}$ {3.15}
is well defined.

Now

$(*)_0$ if $B \in [M]^{<h_1(\delta)}$ then $p^* \restriction (A_\delta \cup B)$ is realized in N.

[Why? Let $I^* \in [I]^{<h_1(\delta)}$ be such that

$$B \subseteq \{\sigma(\bar{t}) : \sigma \text{ is a } \tau_{\chi,\theta}\text{-term and } \bar{t} \in {}^{\kappa >}(I^*)\}.$$

We can find $\beta^* < h_1(\delta)$ such that

$$(\alpha', \beta', t') \in I' \setminus (\delta \times \delta \times J^{[\kappa]}) \Rightarrow \beta' < \beta^*.$$

Now we can choose $\bar{t}^\otimes \in {}^{\kappa >}I$ such that $\bar{t}^\otimes \restriction u = \bar{t}^* \restriction u$, and

$$\epsilon \in \partial \setminus u \Rightarrow t^\otimes_\epsilon \in \{\delta\} \times \{\beta^*\} \times J^{[\kappa]}$$

and

$$epsilon, \zeta < \partial \Rightarrow [t^\otimes_\epsilon < t^\otimes_\zeta \equiv t^*_\epsilon < t^*_\zeta],$$

possible by the choice of $J^{[\kappa]}$. By "represented" and the definition of p^*, clearly $\bar\sigma^*(\bar t^\otimes)$ realizes $p^* \restriction (A_\delta \cup B)$, so $(*)_0$ holds.]

Now $(*)$ tells us that M_{I_n} semi κ-obeys $(0, h_1, \varphi(\bar x, \bar y))$. As for the "exactly", it is enough to find $\langle \bar b_\alpha : \alpha < \mu\rangle$ exemplifying that, i.e. that for every unbounded $S \subseteq \mu, \langle \bar b_\alpha : \alpha \in S\rangle$ fulfill the demand there more then needed it follows by Fact 3.30 below. {3c.17} $\square_{??}$

{3c.17}

Fact 3.30. Assume

(a) μ is regular $\leq \lambda$, and $(\forall \alpha < \mu)(\kappa + \chi + |\alpha|^{<\theta} < \mu)$,
(b) $I \in K^{\rm or}_\lambda$,
(c) $\langle t_\alpha : \alpha < \mu\rangle$ is $<_I$–increasing,
(d) $S = \{\delta < \mu : {\rm cf}(\delta) > \kappa\}$ and h is the function with domain S defined by $h(\delta) = {\rm cf}(I^* \restriction \{t : (\forall i < \delta) t_i <_I t\})$.

Then there is a function $\mathbf{H}$ from ${}^{\mu>}([M]^{<\mu})$ to $[M]^{<\mu}$ satisfying $\bigcup\{\bar a_{t_j} : j < i\} \subseteq \mathbf{H}(\langle A_j : j < i\rangle\}$ and such that: if $A_i \in [M]^{<M}$ is increasing continuous, $H(\langle A_i : j < i\rangle) \subseteq A_{i+1}$ and

$$\mathscr{C} = \{\delta < \mu : \delta \text{ a limit ordinal such that } (\forall i < \mu)(\bar a_{t_i} \subseteq A_\delta \Leftrightarrow i < \delta\},$$

then

(α) $\mathscr{C}$ is a club of μ,
(β) there is an increasing continuous sequence $\langle I_\alpha : \alpha < \mu\rangle$, $I_\alpha \subseteq I$, $|I_\alpha| < \mu$ such that
  (i) $A_\alpha \subseteq \{\sigma(\bar t) : \sigma \text{ an } \tau_{\chi,\theta}\text{-term}, \bar t \in {}^{\theta>}(I_{\alpha+1})\} \subseteq A_{\alpha+1}$,
  (ii) $t_\alpha \in I_{\alpha+1}$,
  (iii) $\mathscr{C}_1 = \{\delta \in \mathscr{C}$: if $t_\alpha \in I_\delta$ and $(\exists \beta)(t <_I t_\beta) \Rightarrow (\exists \beta < \delta)(t <_I t_\beta)\}$ is a club of μ,
  (iv) $\bar a_{t_\alpha} \in A_{\alpha+1}$
(γ) if $\delta \in \mathscr{C} \cap S$ there are $\langle \alpha_\epsilon : \epsilon < {\rm cf}(\delta)\rangle$, $\langle \beta(\epsilon) : \epsilon < {\rm cf}(\delta)\rangle$ increasing with limit δ, such that $\bar a_{t_{\beta(\epsilon)}} \subseteq A_{\alpha_\epsilon}$,
(δ) if $\delta, \langle \alpha_\epsilon, \beta(\epsilon) : \epsilon < {\rm cf}(\delta)\rangle$ are as in clause (β) then for each $\alpha < \delta$ the sequence $\langle {\rm tp}_{\epsilon,\phi}(\bar a_{t_{\beta(\epsilon)}}, A_\alpha, M) : \epsilon < {\rm cf}(\delta)\rangle$ is essentially constant,
(ϵ) if $B \subseteq M$, $|B| < {\rm cf}(\delta) + h(\delta)$ then $p^*_{M,\langle \bar a_{t_{\beta(i)}} : i < {\rm cf}(\delta)\rangle} \restriction B$ is realized in M, see Definition 3.26(1),$(*)_3$, so in Definition 3.26(1),$(*)_2$'s notation, $[M]^{<({\rm cf}(\delta)+h(\delta))} \subseteq \mathscr{P}_0$, {3.15}
(ζ) if $B \subseteq M$, $|B| < h(\delta)$ then $p^*_{M,\langle \bar a_{t_{\beta(i)}} : i < {\rm cf}(\delta)\rangle} \restriction (B \cup A_\delta)$ is realized in M, so in Definition 3.26(1)$(*)_2$'s notation, $[M]^{<h(\delta)} \subseteq \mathscr{P}_1$a {3.15}

(η) there are $B^- \subseteq A_\delta$ of cardinality $\mathrm{cf}(\delta)$ and $B^+ \subseteq M$ of cardinality $h(\delta)$ such that $p^*_{M,\langle \bar a_{t_{\beta(i)}}: i<\mathrm{cf}(\delta)\rangle} \restriction (B^- \cup B^+)$ is omitted by M, actually $\{\varphi(\bar a_{t_{\beta(i)}}, \bar x) : i < \mathrm{cf}(\delta)\} \cup \{\varphi(\bar x, a_t) : t \in J\}$ is omitted for some $J \in [I]^{\mathrm{cf}(\delta)}$.

Proof. Continuation of the proof of Theorem 3.28.

Case D: $\lambda = \kappa^+ > \chi^{<\theta}$.

Similar to Case C, but we have to allow $h(\delta)$ to be $\kappa^+ = \lambda$ in addition to κ. So I_h, defined similarly using $J^{[\lambda]}$ (not $J^{[\kappa]}$), is no longer λ-like, $\bar b_\alpha \in {}^{\partial}(M_{I_h})$, if the rest is not obvious look at the proof of Case E.

Case E: $0 < \gamma^*$, $\chi^{<\kappa} + |\alpha| < \mu_i < \lambda$, μ_i $(i < \alpha^*)$ strictly increasing, each μ_i regular, $\mu_{i+1} > \mu_i^{+++}$, $\mu_i > \chi + \partial^+ + \theta$, $(\forall \mu < \mu_i)\mu^{<\kappa} < \mu_i$, $\prod\limits_i 2^{\mu_i} = 2^\lambda$ (without the last assumption we just get a smaller number of models; note that if $(\forall \alpha < \lambda)(\chi + |\alpha|^{<\kappa} < \lambda)$, then there is such $\langle \mu_i : i < \alpha\rangle$).

{3c.17} Let $J^i \cong J^{[\mu_i^{++}]}$ for $i < \alpha^*$ be from Fact 3.30 below, and for each $i < \gamma^*$ define $J_h \in K^{\mathrm{or}}_{\mu_i^{+3}}$ for $h : \{\delta < \mu_i^{+++} : \mathrm{cf}(\delta) = \mu_i^{++}\} \longrightarrow \{\mu_i^+, \mu_i^{++}\}$ to be $\sum\limits_{\zeta<(\mu_i^{+3}+\kappa)} (J^i_\zeta)^*$, where: $\mu_i^{+3} + \kappa$ is ordinal addition, the J^i_ζ are pairwise disjoint, J^i_ζ is isomorphic to J^i except when $h(\zeta)$ is well defined and equal to μ_i^+, then J^i_ζ is isomorphic to $J^i \times (\mu_i^+)^*$.

Lastly, for every

$$\bar h \in \prod_i \big\{h : \mathrm{Dom}(h) = S_i =: \{\delta < \mu_i^{++} : \mathrm{cf}(\delta) = \mu_i^+\}, \quad h \text{ as above } \big\},$$

we let $I_{\bar h} =: \sum\limits_i J_{h_i} + \lambda \times J^{[\kappa]}$.

For each $i < \alpha$ we have to prove that $h_i/\mathscr{D}_{\mu_\gamma^{+++}}$ is an invariant of the isomorphic type of $M_{I_{\bar h}}$. For this it is enough to prove, for each $\gamma_* < \gamma^*$, that

($*$) $M_{I_{\bar h}}$ exactly semi κ–obeys $(0, h_{\gamma_*}\varphi)$.

It is enough to prove "semi κ–obeys $(0, h_\gamma, \varphi)$", as then the exactness follows by Fact α above. Let $\bar b_\alpha \in {}^{\partial}(M_{I_{\bar h}})$ for $\alpha < \mu_\gamma^{+++}$, so $\bar b_\alpha = \bar\sigma^\alpha(\bar t^\alpha)$, $\bar t^\alpha \in {}^{\kappa >}(I_{\bar h})$. We can find a stationary set $Y \subseteq \{\delta < \mu_{\gamma_*}^{+++} : \mathrm{cf}(\delta) = \kappa\}$ such that

$$\alpha \in Y \Rightarrow \bar\delta^\alpha = \sigma^* \wedge \ell g(\bar t^\alpha) = \epsilon^*,$$

as $\{(\epsilon, \zeta) : t^\alpha_\epsilon < \zeta^\alpha_\zeta\} = v, u_{i,\gamma} = \{\epsilon < \epsilon^* : t^\alpha_\epsilon \in J_{h_i}\} = u_\gamma$. By clauses (i)+(h), without loss of generality $\langle \bar t^\alpha : \alpha \in Y\rangle$ is order indiscernible, as in the proof of Case C.

So for each $\epsilon < \epsilon^*$, $\langle t^\alpha_\epsilon : \alpha \in Y\rangle$ is constant, or strictly increasing, or strictly decreasing, and for some $\gamma < \gamma^*$ they are all in on one I_{h_γ}, moreover if $\langle t^\alpha_\epsilon : \alpha \in Y_\epsilon\rangle$ is not constant necessarily $\gamma \geq \gamma_*$. So if $\langle t^\alpha_\epsilon : \alpha \in Y\rangle$ is strictly increasing, $\delta < \mu_{\gamma_*}^{+++}$, $\mathrm{cf}(\delta) = \mu_i^+$, then

$$\mathrm{cf}(I^*_{\bar h} \restriction \{t : t < t^*_\kappa \text{ for every } \alpha \in Y\})$$

is μ^+_γ or μ^{++}_γ when $\epsilon \in u_\gamma$, so is $\geq \mu^{++}_{\gamma_*}$ except when $\epsilon \in u_{\gamma_*}$ and $h(\delta) = \mu^+_i$. The situation is similar when $\langle t^\alpha_\epsilon : \alpha \in Y \rangle$ is strictly decreasing, except that now $\epsilon \in u_{\gamma_*}$ is impossible.

Case F: λ is regular $> \chi^{<\theta} + \chi^\partial + \kappa^+$, without loss of generality $\lambda > (2^\partial)^+$.

(Why the without loss of generality? Otherwise Case C applies.)

First proof:

Let

$$S = \{\delta < \lambda : \mathrm{cf}(\delta) = (2^\partial)^+\},$$

and for $h : S \longrightarrow \mathrm{Reg} \cap [\kappa, \lambda)$ we define I_h as in Case C. It suffices to prove

(∗) M_{I_h} exactly semi κ-obeys $(0, h, \varphi)$.

It suffices to prove M_{I_h} semi κ-obeys $(0, h, \varphi)$ as the exactly follows by Fact α. Let $\bar{b}_\alpha \in {}^\partial(M_{I_h})$ for $\alpha < \lambda$ be such that $\langle \bar{b}_\alpha : \alpha < \lambda \rangle$ is (κ, φ)-skeleton like and let $\bar{b}_\alpha = \bar{\sigma}^\alpha(\bar{t}^\alpha)$, and we choose a stationary set $Y_0 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ such that $\alpha \in Y \Rightarrow \sigma^\alpha = \sigma^*$ and $\{(\epsilon, \zeta) : t^\alpha_\epsilon < t^\alpha_\zeta\} = v, \ell g(t^\alpha) = \epsilon^* < \kappa$ (but no Δ-system!).

Let $\langle A_i : i < \lambda \rangle$, $\langle I_i = (\gamma_i \times \delta \times J^*) \cap I : i < \lambda \rangle$, $\mathscr{C}$ be as there.

For $\delta \in S \cap \mathrm{acc}(C)$ let $Y_1 \subseteq Y \cap \delta \cap \mathscr{C}$ be unbounded of order type $\mathrm{cf}(\delta)$, and $Y_2 \subseteq Y_1$ be unbounded and $\langle t^\alpha : \alpha \in Y_2 \rangle$ be indiscernible (for $<_I$) (exists as $\mathrm{otp}(Y_1) = (2^\partial)^+$).

Let

$u_0 = \{\epsilon < \epsilon^* : \langle t^\alpha_\epsilon : \alpha \in Y_2 \rangle$ is constant$\}$,
$u_1 = \{\epsilon < \epsilon^* : \langle t^\alpha_\epsilon : \alpha \in Y_2 \rangle$ is increasing and $(\forall \beta < \delta)(\exists \alpha \in Y_2)(t^\alpha_\epsilon \notin I_\beta)\}$,
$u_2 = \epsilon^* \setminus u_0 \setminus u_1$.

Choose $\beta_0 < \beta_1 < \beta_2$ in Y_2 such that $\{t^\alpha_\epsilon : \alpha \in Y_1,\ \epsilon \in u_2 \cup u_0\} \subseteq I_{\beta^*_0}$.

For each $\beta \in Y_2 \setminus \beta_2$ define $\bar{s}^\beta \in {}^{\epsilon^*}I$, $\bar{s}^\beta \restriction u_0 = \bar{t}^\alpha \restriction u_0$ for $\alpha \in Y_2$, $\bar{s}^\beta \restriction u_1 = \bar{t}^\beta \restriction u_1$, $\bar{s}^\beta \restriction u_2 = \bar{t}^{\beta_2} \restriction u_2$. Now we can continue as in Case C when we note

(⊗) if $\beta_3 < \beta_4$ are from $Y_2 \setminus \beta_2$ then $\bar{\sigma}^*(\bar{t}^{\beta_4})$, $\bar{\sigma}^*(\bar{s}^{\beta_4})$ realize the same $\{\varphi, \psi\}$–type over A_{β_3}.

[Why? Let $\bar{d} \in {}^\partial(A_{\beta_3})$ so $\bar{d} = \bar{\sigma}'(\bar{t}')$, $\bar{t} \in {}^{\kappa>}(I_{\beta_3})$. If, e.g.,

$$M_{I_h} \models \vartheta[\bar{\sigma}^*(\bar{t}^{\beta_4}), \bar{d}) \equiv \neg\vartheta[\bar{\sigma}^*(\bar{s}^{\beta_4}), \bar{d}]$$

then

$$M \models \vartheta[\bar{\sigma}^*(\bar{t}^{\beta_4}), \bar{\sigma}'(\bar{t}')] \equiv \neg\vartheta[\bar{\sigma}^*(\bar{t}^{\beta_4}), \bar{\sigma}'(\bar{t}')].$$

So (⊗) holds.

Now we can find $\bar{t}'' \in {}^{\kappa>}(I_{\beta_1})$ such that $\bar{t}'', \bar{t}'$ realizes the same quantifier free type (in I!) over I_{β_0}, hence over $(\bar{t}^{\beta_4} \restriction (u_0 \cup u_2))\hat{\ }\bar{t}^{\beta_2} \restriction (u_0 \cup u_2)$. Hence

$$M_{I_h} \models \vartheta[\bar{\sigma}^*(\bar{t}^{\beta_4}), \bar{\sigma}'(\bar{t}'')] \equiv \neg\vartheta[\bar{\sigma}^*(\bar{s}^{\beta_4}), \bar{\beta}'(\bar{t}'')].$$

Similarly $\bar{s}^{\beta_4}$, $\bar{t}^{\beta_2}$ realize the same quantifier free type (in I) over I_{β_1}, hence

$$M_{I_h} \models \vartheta[\bar{\sigma}^*(\bar{s}^{\beta_4}), \bar{\sigma}'(\bar{t}'')] \equiv \vartheta[\bar{\sigma}^*(\bar{t}^{\beta_2}), \bar{\sigma}'(\bar{t}''),$$

so together

$$M_{I_h} \models \vartheta[\bar\sigma^*(\bar t^{\beta_4}), \bar\sigma'(\bar t'')] \equiv \neg\vartheta[\bar\sigma^*(\bar t^{\beta_2}), \bar\sigma'(\bar t'')].$$

But this contradicts the choice of $\mathscr{C}$ (as $Y \subseteq \mathscr{C}$).

Second proof:
Similar to case C using [Sh:E62, 3.7=Lc2].

Case G:λ is regular $> \chi^{<\theta} + \chi^\partial$.
If cases (C) + (F) do not occur then $\lambda = \kappa^+$, so case D applies.

Case H: λ is singular $> \chi^{<\theta} + \chi^\partial$ (hence $> (2^\partial)^+$).
Combine the proof of cases E and F. $\square_{3.28}$

{3c.18} **Fact 3.31.** Assume $\chi \le \mu = \mu^{<\theta} < \lambda$ and the linear order $J^{[\lambda]}$ are from [Sh:E62, 2.21=Lc73] with $(\mu, \mu^+, \mu^+, \aleph_0)$ here standing for $(\lambda, \mu_1, \mu_2, \theta)$ there and for $I \in K^{\rm or}_\mu$ we define M_I naturally, as $M_{I+J^{[\lambda]}} \restriction \{\sigma(\bar t) : \sigma \text{ a } \tau_{\chi,\theta}\text{-term}, \bar t \in {}^{\theta>}(I + J^{[\mu]})\}$ (using the fullness of the representations).
Then

- $\boxtimes_1$ if I_1, $I_2 \in K^{\rm or}_\mu$, and $M_{(I_1+J^{[\mu]})} \ncong M_{(I_2+J^{[\mu]})}$, then $M_{(I_1+J^{[\lambda]})} \ncong M_{(I_2+J^{[\lambda]})}$, so $M'_I \ncong M'_J$
- $\boxtimes_2$ $|\{M_I/\cong: I \in K^{\rm or}_\lambda\}| \ge |\{M_{I+J^{[\mu]}}/\cong: I \in K^{\rm or}_\mu\}| = |\{M'_I \ncong: I \in K^{\rm or}_M\}|$.

Proof. The first clause by clause (j) of [Sh:E62, 2.21=Lc73(4)] below, the second clause follows. $\square$

* * *

{3.1 17} *Remark* 3.32. Note that if we use strongly κ-homogeneous $J^{[\kappa]}$ and M_I is weakly fully represented in $\mathscr{M}_{\chi,\theta}(I)$ then this form of I helps to "eliminate quantifiers" is $\mathscr{M}_{\chi,\theta}(I)$, i.e. $\mathrm{tp}(\bar\sigma, \bar t), \emptyset, M_I)$ is determined by $\bar\sigma$ and the order of $\bar t$ if $\bar t \in {}^{\kappa>}I$. The order $I^{[\kappa]}$ is not really so homogeneous but it close too, see [Sh:E62, §2].

{3.27new} **Claim 3.33.** *In the theorems above in the assumption we can restrict ourselves to linear order I satisfying*

- $(*)_I$ (a) *for every infinite $J \subseteq I$, the number of Dedekind cuts of J realized by elements of I is at most $|J|$ (i.e., stable in θ for every θ),*
- (b) *for every infinite $J_0 \subseteq I$ there is an J_1, satisfying $J_0 \subseteq J_1 \subseteq I$ such that $|J_0| = |J_1|$ and: if $s, t \in I \setminus J_1$ realize the same Dedekind cuts of J_1 then there is an automorphism h of I over J_1 (i.e. $h{\restriction}J_1 = \mathrm{id}_{J_1}$) mapping s to t (i.e., almost homogeneous for every θ). See Definition* [Sh:E62, 2.15=Lb56] *and* [Sh:E62, 2.16=Lb60].

{3.29new} *Proof.* By 3.35. $\square_{??}$

{3.1} {3.28new} We may weaken a little the definition of weakly κ-skeleton like (Definition 3.1(1)).

Definition 3.34. 1) We say $\langle \bar a_s : s \in I\rangle$ is pseudo κ-skeleton like for Λ when: for every $\varphi(\bar x, \bar a) \in \Lambda$ and a Dedekind cut (I_0, I_1) of I such that $I_1 \ne \emptyset \Rightarrow \mathrm{cf}(I_1) \ge \kappa$ and $I_2 \ne \emptyset \Rightarrow \mathrm{cf}(I_2^*) \ge \kappa$ there are J_0, J_1 such that

- $(*)_1$ J_0 is an end segment of I_0 non empty if $I_0 \ne \emptyset$,

$(*)_2$ J_1 is an initial segment of I_1, non empty if $I_1 \neq \emptyset$,

$(*)_3$ if $s, t \in J_0 \cup J_1$ then $M \models \varphi[\bar{a}_s, \bar{a}] \equiv \varphi[\bar{a}_t, \bar{a}]$; clearly this is a weaker demand than the "weakly" version.

2) Similarly we adopt Definition 3.1(2),(4). {3.1}

What is the difference? E.g., for $\kappa = \aleph_0$, $J_{\bar{a}}$ instead of being countable it may be a Suslin order or Specker order.

{3.29new}

Claim 3.35. *We can through all this section ask (a) or (a)+(b) or (a)+(b)′, where*

(a) *replace weakly in "weakly ... skeleton likeq" by pseudo (including the definitions) and all claims remain true;*

(b) *restricting ourselves to $\lambda \geq 2^{<\kappa}$, we can replace linear orders by strongly κ-dense linear order (see below);*

(b)′ *we can demand that all our linear orders are θ-stable and almost θ-homogeneous, see Definition* [Sh:E62, 2.21=Lc73].

{3.30new}

Definition 3.36. 1) A linear order I is κ-homogeneous if $\mathrm{cf}(I) \geq \kappa$, $\mathrm{cf}(I^*) \geq \kappa$ for any subsets J_0, J_1 of I of cardinality $< \kappa$ (possibly empty) satisfying $(\forall s_0 \in J_0)(\forall s_1 \in J_1)(s_0 <_I s_1)$ there is $t \in I$ such that $(\forall s_0 \in J_0)(s_0 <_I t)$ and $(\forall s_1 \in J_1)(t <_I s_1)$.
2) A linear order I is strongly κ-dense if it is κ-dense and every partial one-to-one function from I to I of cardinality $< \kappa$ can be extended to an automorphism.
3) A linear order I is θ-stable if for every $J \subseteq I$ of cardinality $\leq \theta$, the number of Dedekind cuts of J induced by elements of I is at most $\bar{\theta}$.

Proof. Straightforward, we rely on [Sh:E62, 2.21=Lc73(5)]. $\square_{3.36}$

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