

**INNER PRODUCT SPACE WITH NO ORTHO-NORMAL BASIS  
WITHOUT CHOICE  
E68**

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ABSTRACT. We prove in ZF that there is an inner product space, in fact, nicely definable with no orthonormal basis.

§ 1

The theorem below is known in ZFC, but probably not in ZF; really we use the simple black box (see [Sh:309]).

**Theorem 1.1.** (ZF) *There is an inner-product space  $V$  over  $\mathbb{R}$  with no orthonormal basis.*

{a1}

*Remark 1.2.* In fact, nicely definable one, here - Borel.

*Proof. Stage A:*

Let  $V_1$  be the Hilbert space over  $\mathbb{R}$  with orthonormal basis  $\{x_\eta : \eta \in {}^\omega \omega\}$ , so an element  $x$  has a unique representation as  $x = \sum \{a_{x,\eta} x_\eta : \eta \in {}^\omega \omega\}$  with  $a_{x,\eta} \in \mathbb{R}$  and norm  $< \infty$  so  $\text{supp}_1(x) := \{\eta : a_{x,\eta} \neq 0\}$  is countable and  $\text{supp}_k^1(x) := \{\eta : |a_\eta| \geq \frac{1}{k+1}\}$  finite for every  $k < \omega$  where the norm is  $\sum \{a_{x,\eta}^2 : \eta \in {}^\omega \omega\}$ . The inner product is  $((\sum a_\eta x_\eta), (\sum a'_\eta x_\eta)) = \sum \{a_\eta a'_\eta : \eta \in {}^\omega \omega\} \in \mathbb{R}$ .

For  $\eta \in {}^\omega \omega$  let  $y_\eta = x_\eta + \sum_{n < \omega} \frac{1}{2^n} x_{\eta \upharpoonright n}$ .

Let  $V$  be the subspace of  $V_1$  generated by  $\{x_\eta : \eta \in {}^\omega \omega\} \cup \{y_\eta : \eta \in {}^\omega \omega\}$  so as a vector space it is  $\bigoplus_{\eta \in {}^\omega > \omega} \mathbb{R} x_\eta \oplus \bigoplus_{\eta \in {}^\omega \omega} \mathbb{R} y_\eta$  and it “inherits” the inner product from  $V_1$ .

Toward contradiction assume that  $\{z_s : s \in S\}$  is an ortho-normal basis of  $V$ . So every  $x \in V$  has the unique representation  $\sum_{s \in S} b_{x,s} z_s$ , where  $b_{s,n} \in \mathbb{R}$  and for  $k \in [1, \omega)$  and  $x \in V$  let  $\text{supp}_k^2(x) := \{s \in S : |b_{x,s}| \geq \frac{1}{2^{k+1}}\}$ , so finite and  $\text{supp}_2(x) := \{s \in S : b_{x,s} \neq 0\}$  so countable.

Stage B:

We choose  $\eta_n$  by induction on  $n$  such that:

- ⊕<sub>1</sub> (a)  $\eta_n \in {}^n \omega$
- (b)  $\eta_m = \eta_n \upharpoonright m$  if  $m < n$

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- (c) if  $n = m + 1$  then  $\eta_n = \eta_m \hat{\ } \langle i \rangle$  with  $i < \omega$  minimal such that:  
 if  $l \leq m, s \in \text{supp}_m^2(x_{\eta_l})$  and  $\nu \in \text{supp}_m^1(z_s) \subseteq {}^\omega \omega$  then  
 $\neg(\eta_m \hat{\ } \langle i \rangle \trianglelefteq \nu)$ .

This is well defined as in clause (c),  $\text{supp}_m^2(x_{\eta_\ell})$  is a finite subset of  $S$  and for each  $s \in \text{supp}_m^2(x_{\eta_\ell})$ , the set  $\text{supp}_m^1(z_s)$  is a finite subset of  ${}^\omega \omega$ .

Lastly, let

- $\boxplus_2$  (a)  $\eta_\omega := \cup\{\eta_n : n < \omega\} \in {}^\omega \omega$   
 (b)  $S_1 = \cup\{\text{supp}_2(x_\rho) : \rho \triangleleft \eta_\omega\}$   
 (c)  $S_2 = S \setminus S_1$   
 (d)  $X_\ell$  is the closure inside  $V$  of  $\oplus\{\mathbb{R}z_s : s \in S_\ell\}$  for  $\ell = 1, 2$   
 (e)  $S_{1,n} := \cup\{\text{sup}_m^2(x_{\eta_\ell}) : m, \ell \leq n\}$ .

Note

- $\boxplus_3$   $V = X_1 \oplus X_2$ , i.e.  $X_1, X_2$  are orthogonal but  $X_1 + X_2$  is  $V$   
 $\boxplus_4$   $S_1 = \cup\{\text{supp}_m^2(x_{\eta_n}) : n < \omega, m < n\} = \cup\{S_{1,n} : n < \omega\}$   
 $\boxplus_5$   $\eta_n \in S_1$  for  $n < \omega$ .

Stage C: As  $y_{\eta_\omega} \in V$  see Stage A and  $\boxplus_2(a)$  of Stage B, recalling  $\boxplus_3$

- $\otimes_1$  there are  $y^1 \in X_1, y^2 \in X_2$  such that  $y_{\eta_\omega} = y^1 + y^2$ .

Also

- $\otimes_2$   $\{\rho : \eta_{n+1} \trianglelefteq \rho \in {}^\omega \omega\}$  is disjoint to  $\cup\{\text{supp}_m^1(z_s) : s \in S_{1,n}\}$  for every  $n < \omega$ .

[Why? By the choice of  $\eta_{n+1}$  in  $\boxplus_1(c)$ .]

- $\otimes_3$   $\eta_\omega \notin \text{supp}_1(z_s) = \cup\{\text{supp}_m^1(z_s) : m < \text{omega}\}$  for every  $s \in S_1$ .

[Why? The  $\notin$  by  $\otimes_2$ .]

Hence by  $\boxplus_3$

- $\otimes_4$  if  $s \in S_1$  then  $y_{\eta_\omega}, z_s$  are orthogonal (in  $V_1$ ).

But

- $\otimes_5$   $(y_{\eta_\omega}, x_{\eta_n}) = \frac{1}{2^n}$ .

[Why? By the choice of  $y_\eta$  is stage N.]

By  $\boxplus_5 + \otimes_4 + \otimes_5$  we get contradiction.  $\square_{1.1}$

## REFERENCES

[Sh:309] Saharon Shelah, *Black Boxes*, , 0812.0656. 0812.0656. 0812.0656.

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