

**A NOTE**  
**SHE8**

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## MAIN RESULTS

See around classification theorem: Springer 1986, October 3, 1985, A Note on  $\kappa$ -freeness.

**Theorem.** *If  $\lambda > \aleph_1$  is regular,  $|G| = \lambda$  and  $G$  is a  $\lambda$ -free abelian group then there is a free group  $G' \subseteq G$ ,  $|G'| = \lambda$  provided that*

(\*)  $\lambda$  is not the successor of a singular cardinal of cofinality  $\aleph_0$ .

*Proof.* By [Sh:52,3.9],  $G$  is strongly  $\mu$ -free for  $\mu < \lambda$ .

Let  $G = \bigcup_{i < \lambda} G_i$ ,  $G_i$  strictly increasing continuous,  $|G_i| = |i| + \aleph_1$ . Additionally without loss of generality  $G_{i+1} = \bigcup_{\alpha < |i + \omega_1|} G_{i,\alpha}$ ,  $G_{i,\alpha}$  increasing continuous,

$G/G_{i,\alpha+1}$  is  $\lambda$ -free.

[Why? Choose  $G_i$  by induction on  $i$  and for  $i = j + 1$  choose  $G_{j,\alpha}$  by induction on  $\alpha$ ; using the previous sentence and bookkeeping we are done.]

Without loss of generality  $\forall i < j, \forall \alpha < |i + \omega_1|, \exists \beta < |j + \omega_1| [G_{i,\alpha+1} \subseteq G_{j,\beta+1}]$ . Without loss of generality

⊠  $\alpha < \lambda \Rightarrow G/G$  is  $\aleph_1$ -free (forgetting the “additionality”).

[Why? If  $\lambda$  is a limit cardinal we know  $G$  is strongly  $|G_i|^+$ -free so we can demand that  $G/G_{i+1}$  is  $\aleph_1$ -free. If  $\lambda = \mu^+$ , then  $\text{cf}(\mu) > \aleph_0$  by the hypothesis (\*), so if  $G/G_{i+1}$  is not  $\aleph_1$ -free then for  $\alpha < |i + \omega_1|$  large enough,  $G/G_{\alpha+1}$  is not  $\aleph_1$ -free, contradiction.]

Let  $S = \{i < \lambda : \text{cf}(i) = \omega_1\}$ . Easily  $G/G_\delta$  is  $\aleph_1$ -free for  $\delta \in S$  (by ⊠). Let  $y_\delta \in G_{\delta+1} \setminus G_\delta$ , so there is  $H_\delta \subseteq G_\delta$ ,  $|H_\delta| \leq \aleph_0$ ,  $H_\delta$  free,  $G_\delta/H_\delta$  free and the triple  $(H_\delta, \langle H_\delta, y_\delta \rangle, G_\delta)$  is free. (See [Sh 161] so for some stationary  $S_1 \subseteq S$  and  $\alpha(*) < \delta, (\forall \delta \in S_1) H_\delta \subseteq G_{\alpha(*)}$ . We are allowed to increase  $H_\delta$ , so by (\*) without loss of generality  $H_\delta = H$ . Now  $\langle H, y_\delta : \delta \in S_1 \rangle$  is free.

*Remark.* The proof works for general classes.

## REFERENCES.

- [Sh 161] Saharon Shelah. Incompactness in regular cardinals. *Notre Dame Journal of Formal Logic*, **26**:195–228, 1985.