EXISTENCE OF ENDO-RIGID BOOLEAN ALGEBRAS

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Abstract. How many endomorphisms does a Boolean algebra have? Can we find Boolean algebras with as few endomorphisms as possible? Of course from any ultrafilter of the Boolean algebra we can define an endomorphism, and we can combine finitely many such endomorphisms in some reasonable ways. We prove that in any cardinality $\lambda = \lambda^{\aleph_0}$ there is a Boolean algebra with no other endomorphisms. For this we use the so called "black boxes", but in a self contained way. We comment on how necessary the restriction on the cardinal is.

0. Introduction

In this paper we prove the existence of a Boolean algebra of any cardinality $\lambda = \lambda^{\aleph_0}$ which has as few endomorphisms as possible, in some natural sense. Note that every ultrafilter D of a Boolean algebra $\mathbf{B}$ induces an endomorphism h_D of $\mathbf{B}$: $h_D(x)$ is $1_{\mathbf{B}}$ for $x \in D$ and $0_{\mathbf{B}}$ otherwise. Also we can combine endomorphisms: if h_ℓ is a homomorphism from $\mathbf{B} \restriction a_\ell$ into $\mathbf{B} \restriction b_\ell$ for $\ell = 1, 2$ and $a_1 \cup a_2 = 1_{\mathbf{B}} = b_1 \cup b_2$, $a_1 \cap a_2 = 0_{\mathbf{B}} = b_1 \cap b_2$, *then* there is a unique endomorphism h of $\mathbf{B}$ extending both h_1 and h_2, and for any endomorphism h of $\mathbf{B}$ and $a \in \mathbf{B}$, $h \restriction (\mathbf{B} \restriction a)$ is a homomorphism from $\mathbf{B} \restriction a$ into the Boolean algebra $\mathbf{B} \restriction h(a)$.

Also if $\dot{\mathcal{I}}_1, \dot{\mathcal{I}}_2$ are ideals of $\mathbf{B}$ satisfying $\dot{\mathcal{I}}_1 \cap \dot{\mathcal{I}}_2 = \{0_{\mathbf{B}}\}$ and $\{a_1 \cup a_2 : a_1 \in \dot{\mathcal{I}}_1, a_2 \in \dot{\mathcal{I}}_2\}$ is a maximal ideal of $\mathbf{B}$, then there is an endomorphism h of $\mathbf{B}$ such that $h \restriction \dot{\mathcal{I}}_1 = \mathrm{id}_{\mathcal{I}_1}$ and $h \restriction \dot{\mathcal{I}}_2$ is constantly zero; but possibly there are no such non–zero ideals $\dot{\mathcal{I}}_1, \dot{\mathcal{I}}_2$, (then we call $\mathbf{B}$ indecomposable).

In §2 we define the family of such endomorphisms (those defined by a schema and those defined by a simple schema) and investigate this a little. Our main result (in §3) is that for any $\lambda > \aleph_0$ there is a Boolean algebra of cardinality $\lambda^{\aleph_0}$ (and even density character λ) with only endomorphisms as above, of course there are $2^{\lambda^{\aleph_0}}$ such Boolean algebras with no non trivial homomorphism from one to a distinct other (see 3.1, 3.15, 3.16); we also show that "cardinality $\lambda^{\aleph_0}$" is a reasonable restriction (see 3.17, 3.18, 3.19).

For simplicity, we concentrate on the case of $\mathrm{cf}(\lambda) > \aleph_0$; note that this affect only the density character as $\mathrm{cf}(\lambda) = \aleph_0 \Rightarrow \lambda^{\aleph_0} = (\lambda^+)^{\aleph_0}$. How do we construct such $\mathbf{B}$? The algebra $\mathbf{B}$ extends the Boolean algebra $\mathbf{B}_0$ which is freely generated by $\{x_\eta : \eta \in {}^{\omega>}\lambda\}$ and is a subalgebra of its completion $\mathbf{B}_0^c$. In fact, $\mathbf{B} = \langle \mathbf{B}_0 \cup \{a_\alpha : \alpha < \alpha^*\} \rangle_{\mathbf{B}_0^c}$, with a_α chosen by induction on α, has the form $\bigcup_n (d_n^\alpha \cap s_n^\alpha)$, where $\langle d_n^\alpha : n < \omega \rangle$ is a maximal anti-chain of $\mathbf{B}$, for each d_n^α we have already decided that

Publication E58; last revised 2011.4.29. This is a revised version of [Sh:229], exist since early nineties. It was supposed to be Chapter I of the book "Non-structure" and probably will be if it materializes.

it will belong to $\mathbf{B}$ and is based on (= in the completion of the subalgebra generated by) $\{x_\eta : \eta \in {}^{\omega>}\xi\}$ for some $\xi < \dot\zeta(\alpha) < \lambda$, and for some increasing η_α with the limit $\dot\zeta(\alpha)$, $s^\alpha_n \in \langle x_\nu : \eta_\alpha \restriction n \triangleleft \nu\rangle_{\mathbf{B}^c_0}$. Why these restrictions? We would like to "kill" undesirable endomorphisms and we shall omit appropriate countable (quantifier free) types which the image of a_α, if exists, has to realize, so such restrictions give us tight control and so helps us to "diagonalize" over all possible endomorphisms. To diagonalize we use a black box — it is presented in §1, but its existence is not proved here (it is proved in [Sh:309]).

* * *

In [Sh:89], answering a question of Monk, we have explicated the notion of "a Boolean algebra with no endomorphisms except the ones induced by ultrafilters on it" (see §2 here) and proved the existence of one with density character $\aleph_0$, assuming first $\diamondsuit_{\aleph_1}$ and then only CH. The idea was that if h is an endomorphism of $\mathbf{B}$, not among the "trivial" ones, then there are pairwise disjoint $d_n \in \mathbf{B}$ with $h(d_n) \not\subseteq d_n$. Then we can add, for some $S \subset \omega$, an element x such that $d_n \leq x$ for $n \in S, x \cap d_n = 0$ for $n \notin S$ while forbidding a solution for

$$\{y \cap h(d_n) = h(d_n) : n \in S\} \cup \{y \cap h(d_n) = 0 : n \notin S\}.$$

Later, further analysis had showed that the point is that we are omitting positive quantifier free types. Continuing this, Monk succeeded to prove in ZFC, the existence of such Boolean algebras of cardinality $2^{\aleph_0}$. In his proof he replaced some uses of the countable density character by the $\aleph_1$–chain condition. Also, generally it is hard to omit $< 2^{\aleph_0}$ many types but because of the special character of the types (as said above, positive quantifier free) and models involved, using $2^{\aleph_0}$ almost disjoint subsets of ω, he succeeded in doing this. Lastly, for another step in the proof (ensuring idecomposability - see Definition 2.1) he (and independently Nyikos) found it is in fact easier to do this when for every countable set $Y \subseteq \mathbf{B}$ there is $x \in \mathbf{B}$ free over it.

The question of the existence of such Boolean algebras in other cardinalities remained open (See [vDMR80] and a preliminary list of problems for the handbook of Boolean algebras by Monk).

In [Sh:229] it is proved (in ZFC) that there exist such $\mathbf{B}$ of density character λ and cardinality $\lambda^{\aleph_0}$ whenever $\lambda > \aleph_0$; from this follows answers to some other questions from Monk's list, (combining 3.1 with 2.7).

Almost all the present work is a revised version of [Sh:229] but 3.17 - 3.19 were added; here as in [Sh:229] §2 repeats [Sh:89].

1. A black box

{1.2} **Explanation 1.1.** We shall let $\mathbf{B}_0$ be the Boolean algebra freely generated by $\{\eta : \eta \in {}^{\omega>}\lambda\}$, $\mathbf{B}^c_0$ its completion and we can interpret $\mathbf{B}^c_0$ as a subset of $\mathcal{M} = \mathcal{H}_{<\aleph_1}(\lambda)$ (each $a \in \mathbf{B}^c_0$ has the form $\bigcup_{n<\omega} s_n$ where s_n is a Boolean combination of members of ${}^{\omega>}\lambda$). As the $\eta \in {}^{\omega>}\lambda$ may be over-used, we replace them for this purpose by x_η (for example below let $F \in \tau_0$ be a unary function symbol, $x_\eta = F(\eta)$).

Our desired Boolean algebra $\mathbf{B}$ will be a subalgebra of the competition $\mathbf{B}^c_0$ of $\mathbf{B}_0$ hence it extend $\mathbf{B}_0$. For our diagonalization, i.e. the omitting type, we need the following case (we shall use $\kappa = \aleph_0$). That is we need a family of subalgebras with endomorphism, for each we add an element and promise to omit the type of the supposed image. The family is sparse enough so that we can do it (i.e. with the different promises not hindering one another too much), but dense enough so that every endomorphism of the Boolean Algebra we construct is approximated. See more accurate explanation in 1.4.

{1.2d}

Convention 1.2. *We fix $\kappa \geq \aleph_0$ for this section.*

{1.3}

Definition 1.3. **(1):** *Let τ_n, for $n < \omega$, be fixed vocabularies (= signatures), $|\tau_n| \leq \kappa$, $\tau_n \subseteq \tau_{n+1}$, (with each predicate and function symbol finitary for simplicity). Let $P_n \in \tau_{n+1} \setminus \tau_n$ be unary predicates. Let $\mathcal{M} = (\mathcal{H}_{<\kappa^+}(\lambda), \in)$.*

(2): *For $n < \omega$ let $\mathcal{F}_n$ be the family of sets (or sequences) of the form $\{(f_\ell, N_\ell) : \ell \leq n\}$ satisfying:*

- **(a):** *$f_\ell : {}^{\ell\geq}\kappa \longrightarrow {}^{\ell\geq}\lambda$ is a* tree embedding, *i.e.,*
  - **(i):** *f_ℓ is length preserving, that is, η, $f_\ell(\eta)$ have the same length,*
  - **(ii):** *f_ℓ is order preserving (of $\triangleleft$), moreover, for $\eta, \nu \in {}^{\ell\geq}\kappa$ we have $\eta \triangleleft \nu$ iff $f_\ell(\eta) \triangleleft f_\ell(\nu)$,*
- **(b):** *$f_{\ell+1}$ extends f_ℓ (when $\ell + 1 \leq n$),*
- **(c):** *N_ℓ is an τ'_ℓ–model of cardinality $\leq \kappa$, $|N_\ell| \subseteq |\mathcal{M}|$, where $\tau'_\ell \subseteq \tau_\ell$,*
- **(d):** *$\tau'_{\ell+1} \cap \tau_\ell = \tau'_\ell$ and $N_{\ell+1} \restriction \tau'_\ell$ extends N_ℓ,*
- **(e):** *if $P_m \in \tau'_{m+1}$, then $P_m^{N_\ell} = |N_m|$ when $m < \ell \leq n$, and*
- **(f):** *if $x, y \in N_\ell$ then $\{x, y\} \in N_\ell$ and $\emptyset \in N_\ell$.*
- **(g):** $\mathrm{Rang}(f_\ell) \subseteq N_\ell$

(3): *Let $\mathcal{F}_\omega$ be the family of pairs (f, N) such that for some sequence $\langle (f_\ell, N_\ell) : \ell < \omega \rangle$ the following hold:*

- **(i):** *$\{(f_\ell, N_\ell) : \ell \leq n\}$ belongs to $\mathcal{F}_n$ for $n < \omega$,*
- **(ii):** *$f = \bigcup\limits_{\ell<\omega} f_\ell$, $N = \bigcup\limits_{n<\omega} N_n$, (i.e., $|N| = \bigcup\limits_{n<\omega} |N_n|$, $\tau(N) = \bigcup\limits_{n} \tau(N_n)$, and $N \restriction \tau(N_n) = \bigcup\limits_{n<m<\omega} N_m \restriction \tau(N_n)$).*

(4): *For any $(f, N) \in \mathcal{F}_\omega$ let $\langle (f_n, N_n) : n < \omega \rangle$ be as above (if $P_n \in \tau'_{n+1}$ for $n < \omega$ then it is easy to show that (f_n, N_n) is uniquely determined by (f, N)- notice clauses (d), (e) in (2)), so for each (f^α, N^α) as in 1.10 below (f^α_n, N^α_n) for $n < \omega$ are defined as above.*

(5): *A branch of* $\mathrm{Rang}(f)$ *or of f (for f as in (3)) is just any $\eta \in {}^\omega\lambda$ such that for every $n < \omega$ we have $\eta \restriction n \in \mathrm{Rang}(f)$.*

{1.4}

Explanation of our Intended Plan 1.4. (of Constructing for example the Boolean algebra)

We will be given $\mathbf{W} = \{(f^\alpha, N^\alpha) : \alpha < \alpha^*\}$, so that every branch η of f^α converges to some $\dot\zeta(\alpha)$, $\dot\zeta(\alpha)$ non-decreasing (in α). We have a free object generated by ${}^{\omega>}\lambda$ (i.e., by $\langle x_\eta : \eta \in {}^{\omega>}\lambda \rangle$, this is $\mathbf{B}_0$ in our case), and by induction on α we define $\mathbf{B}_\alpha$ and a_α for $\alpha < \alpha^*$, such that $\mathbf{B}_\alpha$ is increasing and continuous, $\mathbf{B}_{\alpha+1}$ is an extension of $\mathbf{B}_\alpha$, $a_\alpha \in \mathbf{B}_{\alpha+1} \setminus \mathbf{B}_\alpha$ (usually $\mathbf{B}_{\alpha+1}$ is generated by $\mathbf{B}_\alpha$ and a_α, and is included in the completion of $\mathbf{B}_0$). Every element will depend on few (usually $\leq \kappa$) members of ${}^{\omega>}\lambda$, and a_α "depends" in a peculiar way: the set $Y_\alpha \subseteq {}^{\omega>}\lambda$ on which it "depends" is $Y^0_\alpha \cup Y^1_\alpha$, where Y^0_α is bounded below $\dot\zeta(\alpha)$ (i.e., $Y^0_\alpha \subseteq {}^{\omega>}\zeta$ for some $\zeta < \dot\zeta(\alpha)$) and Y^1_α is a branch of f^α or something similar. See more in 1.8.

{1.5}

Definition of the Game 1.5. *We define for $\mathbf{W} \subseteq \mathcal{F}_\omega$ a game $\partial(\mathbf{W})$, which lasts ω-moves.*

In the n-th move:

Player II: Chooses f_n, a tree-embedding of ${}^{n\geq}\kappa$ into ${}^{n\geq}\lambda$, extending $\bigcup_{\ell<n} f_\ell$, such that $\mathrm{Rang}(f_n) \setminus \bigcup_{\ell<n} \mathrm{Rang}(f_\ell)$ *is disjoint to* $\bigcup_{\ell<n} |N_\ell|$*;*
then
Player I: chooses N_n such that $\{(f_\ell, N_\ell) : \ell \leq n\} \in \mathcal{F}_n$.
In the end player II wins i __*when*__$(\bigcup_{n<\omega} f_n, \bigcup_{n<\omega} N_n) \in \mathbf{W}$.

{1.6}
Remark 1.6. We shall be interested in $\mathbf{W}$ such that player II wins (or at least does not lose) the game, but $\mathbf{W}$ is "thin". Sometimes we need a strengthening of the first player in two respects: he can demand (in the n-th move) that $\mathrm{Rang}(f_{n+1}) \setminus \mathrm{Rang}(f_n)$ is outside a "small" set, and in the zero move he can determine an arbitrary initial segment of the play.

{1.7}
Definition 1.7. *We define, for $\mathbf{W} \subseteq \mathcal{F}_\omega$, a game $\partial'(\mathbf{W})$ which lasts ω-moves.*
In the zero move:
Player I chooses $k < \omega$ and $\{(f_\ell, N_\ell) : \ell \leq k\} \in \mathcal{F}_k$, and $X_0 \subseteq {}^{\omega>}\lambda$, $|X_0| < \lambda$.
In the n-th move, $n > 0$:
Player II chooses f_{k+n}, a tree embedding of ${}^{(k+n)\geq}\kappa$ into ${}^{(k+n)\geq}\lambda$, with $\mathrm{Rang}(f_{k+n}) \setminus \bigcup_{\ell<k+n} \mathrm{Rang}(f_\ell)$ *disjoint to* $\bigcup_{\ell<k+n} N_\ell \cup \bigcup_{\ell<n} X_\ell$.
Player I chooses N_{k+n} such that $\{(f_\ell, N_\ell) : \ell \leq k+n\} \in \mathcal{F}_{k+n}$ and he chooses $X_n \subseteq {}^{\omega>}\lambda$ satisfying $|X_n| < \lambda$.
In the end of the play, player II wins __*when*__ $(\bigcup_{n<\omega} f_n, \bigcup_{n<\omega} N_n) \in \mathbf{W}$

{1.8}
Remark 1.8. What do we want from $\mathbf{W}$? First that by adding an element (to $\mathbf{B}_0$) for each (f, N), we can "kill" every undesirable endomorphism, for this $\mathbf{W}$ has to "encounter" every possible endomorphism, and this will be served by "$\mathbf{W}$ a barrier" defined below. For this $\mathbf{W} = \mathcal{F}_\omega$ is O.K. but we also want $\mathbf{W}$ to be thin enough so that various demands will have small interaction. For this, disjointness and more are demanded.

{1.6A}
Definition 1.9. **(1):** *We call $\mathbf{W} \subseteq \mathcal{F}_\omega$ a* strong barrier __*if*__ *Player II wins in the game $\partial(\mathbf{W})$ and even $\partial'(\mathbf{W})$ (which just means he has a winning strategy.)*
(2): *We call $\mathbf{W}$ a* barrier __*if*__ *Player I does not win in the game $\partial(\mathbf{W})$ and even does not win in $\partial'(\mathbf{W})$.*
(3): *We call $\mathbf{W}$ disjoint* __*if*__ *for any distinct $(f^\ell, N^\ell) \in \mathbf{W}$ ($\ell = 1, 2$), f^1 and f^2 have no common branch.*

{1.7A}
The Existence Theorem 1.10. **(1):** *If $\lambda^{\aleph_0} = \lambda^\kappa$, $\mathrm{cf}(\lambda) > \aleph_0$* __*then*__ *there is a strong disjoint barrier.*
(2): *Suppose $\lambda^{\aleph_0} = \lambda^\kappa$, $\mathrm{cf}(\lambda) > \aleph_0$.* __*Then*__ *there is*

$$\mathbf{W} = \{(f^\alpha, N^\alpha) : \alpha < \alpha^*\} \subseteq \mathcal{F}_\omega$$

and a non-decreasing function $\dot{\zeta} : \alpha^* \longrightarrow \lambda$ such that:
(a): $\mathbf{W}$ *is a strong disjoint barrier, moreover for every stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$, the set $\{(f^\alpha, N^\alpha) : \alpha < \alpha^*, \dot{\zeta}(\alpha) \in S\}$ is a disjoint barrier,*
(b): $\mathrm{cf}(\dot{\zeta}(\alpha)) = \aleph_0$ *for $\alpha < \alpha^*$,*

(c): *every branch of f^α is an increasing sequence converging to $\dot\zeta(\alpha)$,*

(d): *each N^α_n is transitive, i.e.: if $\mathcal{M} \models$ "$a \in b$", $b \in N^\alpha_n$, $b \notin \lambda$, then $a \in N^\alpha_n$, (we call $\{(f_\ell, N_\ell) : \ell \le n\}$, transitive if each N_ℓ is transitive and similarly $\{(f_\ell, N_\ell) : \ell < \omega\}$ and $\mathbf{W}$).*

(e): *if $\dot\zeta(\beta) = \dot\zeta(\alpha)$, $\beta + \kappa^{\aleph_0} \le \alpha < \alpha^*$ and η is a branch of f^α, then $\eta \restriction k \notin N^\beta$ for some $k < \omega$,*

(f): *when $\lambda = \lambda^\kappa$ we can demand: if η is a branch of f^α and $\eta \restriction k \in N^\beta$ for all $k < \omega$ (where $\alpha, \beta < \alpha^*$) then $N^\alpha \subseteq N^\beta$ (and even for every $n < \omega$, $N^\alpha_n \in N^\beta$).*

Proof: See [Sh:309, 3.11], [Sh:309, 3.16].

2. Preliminaries on Boolean Algebras

We present here some easy material.

{2.1}

Definition 2.1. **(1):** *For any endomorphism h of a Boolean Algebra $\mathbf{B}$, let*

$$\begin{aligned} \mathrm{ExKer}(h) &= \{x_1 \cup x_2 : h(x_1) = 0, \text{ and } h(y) = y \text{ for every } y \le x_2\}, \\ \mathrm{ExKer}^*(h) &= \{x \in \mathbf{B} : \text{ in } \mathbf{B}/\mathrm{ExKer}(h), \text{ below } x/\mathrm{ExKer}(h), \\ &\qquad \text{there are only finitely many elements}\}. \end{aligned}$$

(2): *A Boolean algebra is endo-rigid if for every endomorphism h of $\mathbf{B}$, $\mathbf{B}/\mathrm{ExKer}(h)$ is finite (equivalently: $1_\mathbf{B} \in \mathrm{ExKer}^*(h)$).*

(3): *A Boolean algebra is indecomposable if there are no two disjoint ideals $\mathcal{I}_0$, $\mathcal{I}_1$ of $\mathbf{B}$ (except $0_\mathbf{B}$ of course), each with no maximal member, which generate a maximal ideal of $\mathbf{B}$ (that is $\{a_0 \cup a_1 : a_0 \in \mathcal{I}_0, a_1 \in \mathcal{I}_1\}$).*

(4): *A Boolean algebra $\mathbf{B}$ is $\aleph_1$-compact if for every pairwise disjoint $d_n \in \mathbf{B}$ (for $n < \omega$) for some $x \in \mathbf{B}$, we have $x \cap d_{2n+1} = 0$, $x \cap d_{2n} = d_{2n}$.*

{2.2}

Lemma 2.2. **(1):** *A Boolean algebra $\mathbf{B}$ is endo-rigid if and only if every endomorphism of $\mathbf{B}$ is the endomorphism of some scheme (see Definition 2.4(1),(3) below).*

(2): *A Boolean algebra $\mathbf{B}$ is endo-rigid and indecomposable if and only if every endomorphism of $\mathbf{B}$ is the endomorphism of some simple scheme (see Def 2.4(2) below).*

(3): *For every scheme of an endomorphism of $\mathbf{B}$ there is one and only one endomorphism of the scheme.*

Proof. Easy.

{2.2A}

Remark 2.3. (1) In fact, for a Boolean algebra $\mathbf{B}$, we have $\{h : h$ is an endomorphism of $\mathbf{B}$ defined by a scheme$\}$ is a sub-semi-group of $\mathrm{End}(\mathbf{B})$, even a normal one (as $(\mathbf{B}, \mathrm{End}(\mathbf{B}))$ is interpretable in $\mathrm{End}(\mathbf{B})$).

(2) Similarly for simple schemes.

{2.3}

Definition 2.4. **(1):** *A scheme of an endomorphism of $\mathbf{B}$ consists of a partition $a_0, a_1, b_0, \dots, b_{n-1}, c_0, \dots, c_{m-1}$ in $\mathbf{B}$ of $1_\mathbf{B}$, with maximal non-principal ideals $\dot{\mathcal{I}}_\ell$ below b_ℓ for $\ell < n$ (in other words $\dot{\mathcal{I}}_\ell$ is a maximal ideal of $\mathbf{B} \restriction b_\ell$) and non-principal ideals $\dot{\mathcal{I}}^0_\ell, \dot{\mathcal{I}}^1_\ell$ below c_ℓ for $\ell < m$ such that $\dot{\mathcal{I}}^0_\ell \cup \dot{\mathcal{I}}^1_\ell$ generates a maximal non principal ideal below c_ℓ and $\dot{\mathcal{I}}^0_\ell \cap \dot{\mathcal{I}}^1_\ell = \{0_\mathbf{B}\}$, a number*

$k \le n$, and a partition $b^*_0, \dots, b^*_{n-1}, c^*_0, \dots, c^*_{m-1}$ of $a_0 \cup b_0 \cup \dots \cup b_{k-1}$. We assume also that

$$[k + m > 0 \;\Rightarrow\; a_0 = 0], \qquad [(n-k) + m > 0 \;\Rightarrow\; a_1 = 0]$$

and except possibly a_0, a_1 there are no zero elements in the partition $a_0, a_1, b_0, \dots, b_{n-1}, c_0, \dots, c_{m-1}$.

(2): *The scheme is simple if $m = 0$.*

(3): *The endomorphism of the scheme is the unique endomorphism $h : \mathbf{B} \longrightarrow \mathbf{B}$ such that:*

- **(i):** *$h(x) = 0$ when $x \le a_0$ or $x \in \dot{\mathcal{I}}_\ell$, $\ell < k$, or $x \in \dot{\mathcal{I}}^0_\ell$, $\ell < m$,*
- **(ii):** *$h(x) = x$ when $x \le a_1$ or $x \in \dot{\mathcal{I}}_\ell$, $k \le \ell < n$ or $x \in \dot{\mathcal{I}}^1_\ell$, $\ell < m$,*
- **(iii):** *$h(b_\ell) = b^*_\ell$ when $\ell < k$,*
- **(iv):** *$h(b_\ell) = b_\ell \cup b^*_\ell$ when $k \le \ell < n$,*
- **(v):** *$h(c_\ell) = c_\ell \cup c^*_\ell$ when $\ell < m$.*

So, an endomorphism of a scheme is a "trivial" endomorphism defined by ideals, essentially maximal ones, and finitely many elements.

{2.4}

Claim 2.5. (1) *If h is an endomorphism of a Boolean Algebra $\mathbf{B}$, and $\mathbf{B}/\mathrm{ExKer}(h)$ is infinite <u>then</u> there are pairwise disjoint $d_n \in \mathbf{B}$ (for $n < \omega$) such that $h(d_n) \not\le d_n$.*

(2) *We can demand that: $h(d_n) \cap d_{n+1} \ne 0$, and if $\mathbf{B}$ satisfies the c.c.c.,<u>then</u> $\{d_n : n < \omega\}$ is a maximal antichain.*

Proof. (1) As $\mathbf{B}/\mathrm{ExKer}(h)$ is infinite we can choose inductively $d_n \in \mathbf{B}$ such that $d_n \notin \mathrm{ExKer}(h)$, $[\ell < n \;\Rightarrow\; d_\ell \cap d_n = 0_{\mathbf{B}}]$ and $\{x/\mathrm{ExKer}(h) : x \in \mathbf{B}\ \&\ x \cap \bigcup_{\ell \le n} d_\ell = 0_{\mathbf{B}}\}$ is infinite. It is enough for each n to find $d^*_n \le d_n$ such that $h(d^*_n) \not\le d^*_n$. Since $d_n \notin \mathrm{ExKer}(h)$, clearly (by the definition of $\mathrm{ExKer}(h)$) we have $h(d_n) > 0_{\mathbf{B}}$ and for some $d'_n \le d_n$, $h(d'_n) \ne d'_n$.

<u>Case 1:</u> $h(d_n) \not\le d_n$, let $d^*_n = d_n$.

<u>Case 2:</u> $h(d_n) = d_n$.

Now if $h(d'_n) \not\le d'_n$ let $d^*_n = d'_n$ and otherwise $h(d'_n) \le d'_n$ so by the choice of d'_n we have $h(d'_n) < h(d_n)$, let $d^*_n = d_n - d'_n$, so $h(d^*_n) = h(d_n) - h(d'_n) = d_n - h(d'_n) > d_n - d'_n = d^*_n$ so d^*_n is as required.

<u>Case 3:</u> Neither case 1 nor case 2.

So $h(d_n) \le d_n$ but $h(d_n) \ne d_n$ hence $h(d_n) < d_n$. So $h(d_n - h(d_n)) \le h(d_n - 0_{\mathbf{B}}) = h(d_n)$ is disjoint from $d_n - h(d_n)$, so if $h(d_n - h(d_n)) > 0_{\mathbf{B}}$ we let $d^*_n = d_n - h(d_n)$. So assume not, so $d_n - h(d_n) \in \mathrm{Ker}(h) \subseteq \mathrm{ExKer}(h)$, and hence $h(h(d_n)) = h(0_{\mathbf{B}} \cup h(d_n)) = h((d_n - h(d_n)) \cup h(d_n)) = h(d_n)$ and necessarily $h(d_n) \notin \mathrm{ExKer}(h)$ (as $d_n \notin \mathrm{ExKer}(h)$, $d_n - h(d_n) \in \mathrm{ExKer}(h)$), hence case 2 apply to $h(d_n)$ and we are done.

(2) Let $c_n = h(d_n) - d_n > 0$, so $m \ne n \;\Rightarrow\; d_n \cap d_m = 0_{\mathbf{B}} \Rightarrow c_m \cap c_n = 0_{\mathbf{B}}$ and

$$d_n \cap c_n = d_n \cap (h(d_n) - d_n) \le d_n \cap (-d_n) = 0_{\mathbf{B}}$$

so $d_n \cap c_n = 0_{\mathbf{B}}$.

By Ramsey theorem,without loss of generality, for all $m < n$ the truth value of $d_m \cap c_n = 0_{\mathbf{B}}$ is the same and of $c_n \cap d_m = 0_{\mathbf{B}}$ is the same.

Now we prove

$(*)$: for some $\langle d'_n : n < \omega \rangle$ we have $d'_n \in \mathbf{B}$, $h(d'_n) \not\le d'_n$ moreover $h(d'_n) \cap d'_{n+1} > 0$ and $n < m \Rightarrow d'_n \cap d'_m = 0_{\mathbf{B}}$.

Case 1: $c_0 \cap d_1 > 0_{\mathbf{B}}$.

Let $d'_n = d_{2n+2} \cup (c_{2n} \cap d_{2n+1})$; now $\langle d'_n : n < \omega \rangle$ are pairwise disjoint as the d_n's are. Now as $h(d_m) \geq c_m$ for $m < \omega$ clearly

$$h(d'_n) \geq h(d_{2n+2}) \geq c_{2n+2} \geq c_{2n+2} \cap d_{2n+3} > 0_{\mathbf{B}}$$

$$d'_{n+1} \geq c_{2n+2} \cap d_{2n+3} > 0_{\mathbf{B}},$$

so $h(d'_n) \cap d'_{n+1} \geq c_{2n+2} \cap d_{2n+3} > 0_{\mathbf{B}}$.

Case 2: $c_1 \cap d_0 > 0_{\mathbf{B}}$.

Let $d'_n = d_{2n+3} \cup (c_{2n+1} \cap d_{2n})$. Now $\langle d'_n : n < \omega \rangle$ are pairwise disjoint (as $\langle d_{2n} \cup d_{2n+3} : n < \omega \rangle$ are), $h(d'_n) \geq h(d_{2n+3}) \geq c_{2n+3} \geq c_{2n+3} \cap d_{2n+2} > 0$ and $d'_{n+1} \geq c_{2(n+1)+1} \cap d_{2(n+1)} = c_{2n+3} \cap d_{2n+2} > 0$. So clearly $h(d'_n) \cap d'_{n+1} \geq c_{2n+3} \cap d_{2n+2} > 0$.

Case 3: Neither Case 1 nor Case 2.

As we have noted above $d_n \cap c_n = 0_{\mathbf{B}}$ by the case assumption's we have $d_n \cap c_m = 0_{\mathbf{B}}$ for every $m, n < \omega$ and of course $n \neq m \Rightarrow d_n \cap d_m = 0_{\mathbf{B}}$ & $c_n \cap c_m = 0_{\mathbf{B}}$. Lastly let $d'_n = d_{n+1} \cup c_n$, they are as required e.g. $h(d'_n) \cap d'_{n+1} = (h(d_{n+1}) \cup h(c_n)) \cap (d_{n+2} \cup c_{n+1}) \geq h(d_{n+1}) \cap c_{n+1} = c_{n+1} > 0_{\mathbf{B}}$.

So we have proved $(*)$. Now renaming d'_n as d_n, $\langle d_n : n < \omega \rangle$ satisfies (part (1) and) the first demand of part (2).

If $\mathbf{B}$ satisfies the c.c.c., we can find $\alpha \in [\omega, \omega_1)$ and d_β for $\beta \in [\omega, \alpha)$ such that $\langle d_\beta : \beta < \alpha \rangle$ is a maximal antichain of $\mathbf{B}$, without loss of generality, $\alpha \leq \omega + \omega$. Now let d'_n be $d_n \cup d_{\omega+n}$ if $\omega + n < \alpha$, and d_n otherwise. So $n \neq m \Rightarrow d'_n \cap d'_m = 0$ and $h(d'_n) \cap d'_{n+1} \geq h(d_n) \cap d_{n+1} > 0$, so $\langle d'_n : n < \omega \rangle$ is as required. $\blacksquare_{2.5}$

{2.4u}

Definition 2.6. *A Boolean Algebra* $\mathbf{B}$ *is Hopfian if every onto endomorphism of* $\mathbf{B}$ *is one-to-one. A Boolean Algebra* $\mathbf{B}$ *is dual Hopfian if every one to one endomorphism is onto.*

{2.5}

Lemma 2.7. **(1):** *Every atomless endo-rigid Boolean Algebra* $\mathbf{B}$ *is Hopfian and dual Hopfian.*

(2): *Also* $\mathbf{B} + \mathbf{B}$ *is Hopfian (and dual Hopfian), however it is not rigid.*

Proof: Easy to check using 2.2, 2.4.

3. The Construction

{3.1}

Main Theorem 3.1. *Suppose* $\mathrm{cf}(\lambda) > \aleph_0$. *Then there is a Boolean algebra* $\mathbf{B}$ *such that:*

(a): $\mathbf{B}$ *satisfies the c.c.c and is atomless,*

(b): $\mathbf{B}$ *has power* $\lambda^{\aleph_0}$ *and has algebraic density* λ *(in the Boolean cardinal invariant notation,* $\pi(\mathbf{B}) = \lambda$*), this means:*

$$\min\{|X| : X \subseteq \mathbf{B} \setminus \{0_{\mathbf{B}}\} \text{ and } (\forall y \in \mathbf{B})(\exists x \in X)(y > 0 \Rightarrow x \leq y)\},$$

(c): $\mathbf{B}$ *is endo-rigid and indecomposable.*

Proof. Let τ_n for $n < \omega$ be as in §1 for $\kappa = \aleph_0$, we use τ'_n with τ'_0 having unary predicate Q, binary predicate $\leq$, individual constants 1, 0, binary function symbols $\cup, \cap, -$ and unary function symbol H (and more) and $P_n \notin \tau'_{n+1} \setminus \tau'_n$. We shall use Theorem 1.10(2) for λ and $\kappa = \aleph_0$, and let $\mathbf{W} = \{(f^\alpha, N^\alpha) : \alpha < \alpha^*\}$, the function $\dot\zeta$, the model $\mathcal{M} = (\mathcal{H}_{<\aleph_1}(\lambda), \in)$ and $\mathcal{T} = {}^{\omega>}\lambda$ be as there. We call $\alpha < \alpha^*$ a candidate if

$$\mathbf{B}^{N^\alpha} = \mathbf{B}[N^\alpha] = (Q^{N^\alpha}, 1^{N^\alpha}, 0^{N^\alpha}, \cup^{N^\alpha}, \cap^{N^\alpha}, -^{N^\alpha}, \leq^{N^\alpha})$$

is a Boolean algebra and $h_\alpha = H^{N^\alpha} \restriction \mathbf{B}^{N^\alpha}$ is an endomorphism of $\mathbf{B}^{N^\alpha}$; of course $\cup^{N^\alpha}$ means $\cup^{N^\alpha} \restriction Q^{N^\alpha}$ etc and we are demanding that all the relevant predicates and function symbols belongs to τ_{N^α}.

We will think of the game as follows: Player I tries to produce a non trivial endomorphism h. Player II supplies (via the range of f_ℓ) elements in $\mathbf{B}_0$ (see Stage A below) and challenges Player I for defining h on them. So Player I plays models N_ℓ in the vocabulary τ'_ℓ which is mainly a subalgebra of the Boolean algebra we are constructing, with additional elements and expanded by, in particular, the distinguished function symbol $H \in \tau'_0$ which is interpreted as an endomorphism of Boolean algebras. In the end, as $\mathbf{W}$ is a barrier, for some such play we will get a model $N^\alpha \in \mathbf{W}$, in the vocabulary $\bigcup\limits_{\ell<\omega} \tau'_\ell$ which includes a function symbol H. We can think of N^α as a Boolean algebra $\subseteq \mathbf{B}_0^c$ with an endomorphism $h_\alpha = H^{N_\alpha}$.

Stage A Let $\mathbf{B}_0$ be the Boolean algebra freely generated by $\{x_\eta : \eta \in {}^{\omega>}\lambda\}$, and $\mathbf{B}_0^c$ be its completion. For $A \subseteq \mathbf{B}_0^c$ let $\langle A \rangle_{\mathbf{B}_0^c}$ be the Boolean subalgebra of $\mathbf{B}_0^c$ that A generates. As $\mathbf{B}_0$ satisfies the c.c.c. every element of $\mathbf{B}_0^c$ can be represented as a countable union of members of $\mathbf{B}_0$, and as $\mathbf{B}_0$ is free we get $\|\mathbf{B}_0^c\| = \lambda^{\aleph_0}$. We say $x \in \mathbf{B}_0^c$ is based on (or supported by) $J \subseteq {}^{\omega>}\lambda$ if it is based on (or supported by) $\{x_\nu : \nu \in J\}$ that is $\mathbf{B}_0^c \models$ "$x = \bigcup\limits_{n<\omega} y_n$", where each y_n is in the subalgebra generated by $\{x_\nu : \nu \in J\}$; we shall also say that J is a support of x. Let $\mathrm{supp}(x)$ be the minimal such J; it is easy to prove its existence. [Why? Let $x = \bigcup\limits_{n<\omega} y_n$, where $y_n = \sigma_n(\ldots, x'_{\eta_{n,\ell}} \ldots)_{\ell \leq k_n}$; as if $y_n = \bigcup\limits_{\ell<k} y_{n,\ell}$ we can replace y_n by $y_{n,0}, \ldots, y_{n,k-1}$, hence without loss of generality, for each n, for some disjoint finite $u_n, v_n \subseteq {}^{\omega>}\lambda$ we have $y_n = \bigcap\limits_{\eta\in u_n} x_\eta \cap \bigcap\limits_{\eta\in v_n} (-x_\eta)$. Also we can replace u_n by any $u \subseteq u_n$ such that $y' = \bigcap\limits_{\eta\in u} x_\eta \cap \bigcap\limits_{\eta\in v_n} (-x_\eta)$ satisfies $y' \leq x$. So without loss of generality

$$u \subseteq u_\eta \ \&\ u \neq u_\eta \Rightarrow \bigcap_{\eta\in u} x_\eta \cap \bigcap_{\eta\in v_n} (-x_\eta) - x > 0.$$

Similarly without loss of generality

$$v \subseteq v_n \ \&\ v \neq v_n \ \Rightarrow \bigcap_{\eta\in u_n} x_\eta \cap \bigcap_{\eta\in v} (-x_\eta) - x > 0.$$

Lastly let $J = \bigcup\limits_{n<\omega} u_n \cup \bigcup\limits_{n<\omega} v_n$, clearly J is a support of x. If J is not minimal then let J' be a support of x such that $J \not\subseteq J'$ as witness e.g. by $\langle y'_n : n < \omega \rangle$. So for some n, $u_n \cup v_n \not\subseteq J'$, by symmetry without loss of generality $u_n \not\subseteq J'$, but then $u' = u_n \cap J'$ contradicts the statement above.]
Without loss of generality, not only $\mathbf{B}_0^c \subseteq \mathcal{M}$ but $x \in \mathbf{B}_0^c$ implies that the transitive closure of $\{x\}$ in $\mathcal{M}$ includes $\mathrm{supp}(x)$. We shall now define by induction on $\alpha < \alpha^*$, the truth value of "$\alpha \in Y_\ell$" ($\ell = 1, 2, 3$), "$\alpha \in Y$", "$\alpha \in Y'$", the sequence η_α,

and members a_α, b^α_m, c^α_m, d^α_m, s^α_m of $\mathbf{B}^c_0$ for $m < \omega$ when $\alpha \in Y'$ such that, $Y \cup Y_1 \cup Y_2 \cup Y_3 \subseteq Y'$ and letting $\mathbf{B}_\alpha = \langle \mathbf{B}_0 \cup \{a_\gamma : \gamma < \alpha,\ \gamma \in Y'\}\rangle_{\mathbf{B}^c_0}$ we have [1]:

$\circledast_\alpha$**:** (1) η_α is a branch of $\mathrm{Rang}(f^\alpha)$, and $\eta_\alpha \neq \eta_\beta$ for $\beta < \alpha$;

(2) if $\alpha \in Y'$, then for some $\xi < \dot\zeta(\alpha)$:
$a_\alpha = \bigcup_{m<\omega} (s^\alpha_m \cap d^\alpha_m)$, where $\langle d^\alpha_m : m < \omega\rangle$ is a maximal antichain of non zero elements (of $\mathbf{B}^c_0$), $d^\alpha_n \in \mathbf{B}_\alpha$, $\bigcup_m \mathrm{supp}(d^\alpha_m) \subseteq {}^{\omega>}\xi$, $s^\alpha_m \in \langle x_\rho : \eta_\alpha \restriction m \lhd \rho,\ \rho \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0}$, and $d^\alpha_m > s^\alpha_m \cap d^\alpha_m > 0$, and $b^\alpha_n, c^\alpha_n \in \mathbf{B}_\alpha$ are based on ${}^{\omega>}\dot\zeta(\alpha)$ and $a_\alpha \notin \mathbf{B}_\alpha$;

(3) if $\alpha \in Y$, then b^α_n, $d^\alpha_n \in N^\alpha_0$, c^α_n, $s^\alpha_n \in N^\alpha$ (hence, by clause (g) of Definition 1.3(2) each is based on $\{x_\nu : \nu \in {}^{\omega>}\lambda,\ \nu \in N^\alpha\}$), and $b^\alpha_n \cap b^\alpha_m = 0$ for $n \neq m$;

(4) if $\beta < \alpha$, $\beta \in Y$, then ($\beta \in Y'$ and) $\mathbf{B}_\alpha$ omits the type

$$p_\beta = \{x \cap b^\beta_n = c^\beta_n :\ n < \omega\}.$$

Before we carry out the construction observe:

Crucial Fact 3.2. *For any $x \in \mathbf{B}_\alpha$ letting $\zeta = \dot\zeta(\alpha)$ there are a finite subset J of ${}^{\omega>}\lambda$, $k < \omega$, $\xi < \zeta$, and $\alpha_0 < \ldots < \alpha_{k-1} < \alpha$ such that* {3.2}

(a): $\dot\zeta(\alpha_0) = \dot\zeta(\alpha_1) = \dot\zeta(\alpha_2) = \cdots = \dot\zeta(\alpha_{k-1}) = \zeta$,

(b): *x is based on*

$$\{x_\nu : \nu \in J \cup {}^{\omega>}\xi \text{ or } \nu \in \mathrm{supp}(s^{\alpha_\ell}_m) \text{ for some } \ell < k,\ m < \omega\}.$$

(c): *$x = \sigma(a_{\alpha_0}, \ldots, a_{\alpha_{k-1}}, b_0, \ldots, b_{n-1})$, for some Boolean term σ, and $b_0, \ldots, b_{n-1} \in \langle \mathbf{B}_0 \cup \{a_\alpha : \dot\zeta(\alpha) < \xi\}\rangle$, and if $x \in \mathbf{B}_0$ then $k = 0$ and n is minimal.*

Continuation of the proof of 3.1 Stage B Let us carry out the construction on α. For $\xi < \lambda$, $w \subseteq \alpha^*$ let

$$I_{\xi,w} = \{\nu :\ \nu \in {}^{\omega>}\xi \text{ or } \ \nu \in \bigcup_{m<\omega,\gamma\in w} \mathrm{supp}(s^\gamma_m)\}.$$

We call α a good candidate if ($\alpha < \alpha^*$ and) $\mathbf{B}[N^\alpha]$ is a subalgebra of $\mathbf{B}_\alpha$, $x \in \mathbf{B}[N^\alpha] \Rightarrow \mathrm{supp}(x) \subseteq N^\alpha$ of course and $h_\alpha = H^{N_\alpha} \restriction \mathbf{B}^{N_\alpha}$ is an endomorphism of $\mathbf{B}[N^\alpha]$ (note that h_α maps $\mathbf{B}[N^\alpha_n]$ into $\mathbf{B}[N^\alpha_n]$ for $n < \omega$). We let $\alpha \in Y_1$ if and only if

$\otimes^1_\alpha$ (α)**:** α is a good candidate

(β)**:** there are $d^\alpha_m \in N^\alpha_0 \cap \mathbf{B}_\alpha$ for $m < \omega$, $d^\alpha_m \neq 0$, $d^\alpha_m \cap d^\alpha_\ell = 0$ for $m \neq \ell$, such that $\langle d^\alpha_m : m < \omega\rangle$ is a maximal antichain of $\mathbf{B}^c_0$ and for some $\xi < \dot\zeta(\alpha)$ each d^α_m is based on ${}^{\omega>}\xi$, and there are a branch η_α of $\mathrm{Rang}(f^\alpha)$ and $s^\alpha_m \in N^\alpha \cap \mathbf{B}_\alpha$ ($m < \omega$) as in (1), (2) above,

(γ)**:** in addition if we add $\bigcup_{n<\omega} (s^\alpha_n \cap d^\alpha_n)$ to $\mathbf{B}_\alpha$ then

(a): each p_β ($\beta \in Y \cap \alpha$) is still omitted

(b)$_1$: $p_\alpha =: \{x \cap h_\alpha(d^\alpha_m) = h_\alpha(d^\alpha_m \cap s^\alpha_m) : m < \omega\}$ is omitted.

Let $\alpha \in Y_2$ if and only if $\alpha \notin Y_1$ and $\otimes^2_\alpha$ holds where

[1]actually the $\alpha \in Y' \setminus Y = Y' \setminus Y_1$ have no real role here, but have in 3.15 later.

$\otimes^2_\alpha$ is defined like $\otimes^1_\alpha$ replacing clause $(\mathbf{b})_1$ by

$(\mathbf{b})_2$: the type p'_α is realized where $p'_\alpha = \{x \cap h_\alpha(d_{2n} \cup d_{2n+1}) = h_\alpha(d_{2n}) : n < \omega\}$ and $h_\alpha(s_{2n}) = 1$, $h_\alpha(s_{2n+1}) = 0$ and $\langle d_n : n < \omega\rangle \in N^\alpha_0$ but $\mathbf{B}[N^\alpha]$ omit p'_α.

Let $\alpha \in Y_3$ iff $\alpha \notin Y_1 \cup Y_2$ and

$\otimes^3_\alpha$ we have (α) + (β) + (γ)(a) from $\otimes^1_\alpha$.

Let $Y = Y_1$, $Y' = Y_1 \cup Y_2 \cup Y_3$ and for $\alpha \in Y_\ell$ let $\otimes_\alpha$ mean $\otimes^\ell_\alpha$.

If $\alpha \in Y'$ we choose η^α, d^α_n, s^α_m, satisfying $\otimes_\alpha$ so also $\mathbf{B}_{\alpha+1}$ is well defined and if $\ell = 1$ let $b^\alpha_m = h_\alpha(d^\alpha_m)$, $c^\alpha_m = h_\alpha(d^\alpha_m \cap s^\alpha_m)$ for $m < \omega$, if $\ell \in \{2,3\}$ we can still choose b^α_m, c^α_m for $m < \omega$ such that $\circledast_\alpha$ holds (e.g. $\{\langle n\rangle : n < \omega\} \subseteq N^\alpha_0$ by clause (f) of 1.3(2), so $b^\alpha_n = x_{\langle 2n+1\rangle} - \bigcup\limits_{m<n} x_{\langle 2n+1\rangle}, c^\alpha_n = b^*_n \cap x_{\langle 2n\rangle})$.

If $\alpha \in \alpha^* \setminus Y'$ we leave a_α, η_α and d^α_n, s^α_n (for $n < \omega$) undefined, and so $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha$. So we have carried the induction.

So "$\alpha \in Y'$" means that Player I played Boolean Algebras and endomorphisms as in the previous remark and we get in the end a Boolean Algebra with the same properties.

The desired Boolean algebra $\mathbf{B}$ is $\mathbf{B}_{\alpha^*} = \cup\{\mathbf{B}_\alpha : \alpha < \alpha^*\}$. We shall investigate it and eventually prove that it is endo-rigid (in 3.11) and indecomposable (in 3.14) thus proving $\mathbf{B}_{\alpha^*}$ is as required in clause (c) of 3.1, where (3.1(a), 3.1(b) hold trivially noting that $|\mathbf{B}_{\alpha^*}|$ is $\le |\mathbf{B}^c_0| \le |\mathbf{B}_0|^{\aleph_0} = \lambda^{\aleph_0}$ and is $\ge |Y'| \ge \lambda^{\aleph_0}$ which will be proved later (see 3.13) and $a_\alpha \notin \mathbf{B}_\alpha$ by (2) from stage A). The rest of the proof is broken to facts and claims in this framework.

Note also

{3.3}
Fact 3.3. **(1):** *For $\nu \in {}^{\omega>}\lambda$, x_ν is free over $\{x_\eta : \eta \in {}^{\omega>}\lambda, \eta \neq \nu\}$ in $\mathbf{B}_0$ hence also over the subalgebra of $\mathbf{B}^c_0$ of those elements based on $\{x_\eta : \eta \in {}^{\omega>}\lambda,\ \eta \neq \nu\}$.*

(2): *If η is a branch of f^α hence necessarily $\eta \neq \eta_\beta$ for $\beta \in Y' \cap \alpha$, $\xi < \dot\zeta(\alpha)$, and $w \subseteq \alpha \cap Y'$, is finite <u>then</u> there is $k < \omega$ such that $\{\rho : \eta \restriction k \lhd \rho \in {}^{\omega>}\lambda\}$ is disjoint to*

$$({}^{\omega>}\xi) \cup \bigcup\{N^\beta \cap {}^{\omega>}\lambda : \beta \in w,\ \beta + 2^{\aleph_0} \le \alpha\} \cup \bigcup\{\mathrm{supp}(s^\beta_n) :\ n < \omega, \beta \in w\}.$$

Proof. (1) Should be clear.

(2) Remember clauses (a),(c),(e) of Theorem 1.10(2) and clause (1) of $\circledast_\alpha$ from stage A. $\blacksquare_{2.5}$

From 3.2 we can derive:

{3.4}
Fact 3.4. *If $\xi < \dot\zeta(\beta)$, $\beta < \alpha$, and $J \subseteq {}^{\omega>}\lambda$ is finite, <u>then</u> every element of $\mathbf{B}_\alpha$ which is based on $J \cup {}^{\omega>}\xi$ belongs to $\mathbf{B}_\beta$.*

Proof. We now prove by induction on $\gamma \in [\beta, \alpha]$ that $[x \in \mathbf{B}_\gamma \backslash \mathbf{B}_\beta \Rightarrow \mathrm{supp}(\lambda) \backslash {}^{\omega>}\xi$ is infinite]. For $\gamma = \beta$ this is empty, and for γ limit it follows as $\mathbf{B}_\gamma = \cup\{\mathbf{B}_\xi : \xi < \gamma\}$. For $\gamma + 1 \le \alpha$, let x be a counterexample;without loss of generality $x \notin \mathbf{B}_\gamma$; if $\mathbf{B}_{\gamma+1} = \mathbf{B}_\gamma$ this is impossible so a_γ, $\langle (d^\gamma_n, s^\gamma_n) : n < \omega\rangle$ are well defined. Now x is necessarily of the form $y_0 \cup (y_1 \cap a_\gamma) \cap (y_2 - a_\gamma)$ where y_0, y_1, y_2 are disjoint members of $\mathbf{B}_\gamma$. Clearly $y_1 \cap a_\gamma \notin \mathbf{B}_\gamma$ or $y_2 - a_\gamma \notin \mathbf{B}_\gamma$ so without loss of generality the former (otherwise use $-x$ which also $\in \mathbf{B}_{\gamma+1} \setminus \mathbf{B}_\gamma$ and has the same support). We can (by 3.2) find n such that $J^* = \{\rho : \eta_\gamma \restriction n \lhd \rho \in {}^{\omega>}\lambda\}$ is disjoint to J and to $\mathrm{supp}(y_1)$.

As $a_\gamma \cap \bigcup\limits_{\ell \le n} d^\gamma_\ell \in \mathbf{B}_\gamma$ clearly $y_2 \cap a_\gamma - \bigcup\limits_{\ell \le n} d^\gamma_\ell \notin \mathbf{B}_\gamma$. As $\langle d^\gamma_m : m < \omega \rangle$ is a maximal antichain of $\mathbf{B}^c_0$, we can find $m \in (n, \omega)$ such that $d^\gamma_m \cap (y_2 \cap a_\gamma - \bigcup\limits_{\ell \le n} d^\gamma_\ell) > 0_{\mathbf{B}_{\gamma+1}}$ hence $y_2 \cap d^\gamma_m > 0_{\mathbf{B}_{\gamma+1}}$ and s^γ_n has support $\subseteq J^*$ whereas y_1, d^γ_m, x has support disjoint to J^*. But $y_1 \cap d^\gamma_m \cap x = (y_1 \cap a_\gamma) \cap d^\gamma_m = (y_1 \cap d^\gamma_m) \cap s^\gamma$ easy contradiction. $\blacksquare_{3.4}$

{3.5}

Notation 3.5. **(1):** Let $\mathbf{B}^\xi$ be the set of $a \in \mathbf{B}^c_0$ based on ${}^{\omega>}\xi$.

(2): For $x \in \mathbf{B}^c_0$, $\xi < \lambda$ let $\mathrm{pr}_\xi(x) = \cap\{a \in \mathbf{B}^\xi : x \le a\}$.

(3): For $\xi \le \lambda$ and $\nu \in {}^{\omega>}\xi$ let $\mathbf{B}^\xi_\nu$ be the set of $a \in \mathbf{B}^c_0$ based on $J_{\xi,\nu} =: \{\rho \in {}^{\omega>}\xi : \neg(\nu \lhd \rho)\}$. For $x \in \mathbf{B}^c_0$ let $\mathrm{pr}_{\xi,\nu}(x) = \bigcap\{a \in \mathbf{B}^\xi_\nu : x \le a\}$.

(4): For $\gamma < \alpha^*$ let $\mathbf{B}_{\langle\gamma\rangle} = \langle \{x_\eta : \eta \in {}^{\omega>}\dot\zeta(\gamma)\} \cup \{a_\beta : \beta \in \gamma \cap Y'\}\rangle_{\mathbf{B}^c_0}$.

(5): For $I \subseteq {}^{\omega>}\lambda$ and $w \subseteq \alpha^*$ let
$\mathbf{B}(I, w) = \langle \{x_\eta : \eta \in I\} \cup \{a_\beta : \beta \in w \cap Y'\}\rangle_{\mathbf{B}^c_0}$ and for $x \in \mathbf{B}^c_0, \xi \le \lambda$ we let

$$\mathrm{pr}_{\xi,w}(x) = \cap\{y \in \langle \mathbf{B}^\xi \cup \{x_\nu : \nu \in w\}\rangle_{\mathbf{B}^c_0} : x \le y\}$$

(6): For $\xi < \lambda$ let $\mathbf{B}_{[\xi]} = \langle \{x_\eta : \eta \in {}^{\omega>}\xi\} \cup \{a_\beta : \dot\zeta(\beta) \le \xi \text{ and } \beta \in Y'\}\rangle_{\mathbf{B}^c_0}$.

(7): For $J \subseteq {}^{\omega>}\lambda$ and $\xi \le \lambda$ let $\mathrm{pr}_{\xi,J}(\lambda) = \cap\{y \in \mathbf{B}^\xi_J : x \le y\}$ where $\mathbf{B}^\xi_J = \langle \mathbf{B}^\xi \cup \{x_\nu : \nu \in J\rangle_{\mathbf{B}^c_0}$, when well defined.

{3.6}

Fact 3.6. **(1):** *For $\xi < \lambda$, $\mathbf{B}^\xi$ is a complete Boolean subalgebra of $\mathbf{B}^c_0$. For $\xi < \lambda$ and $\nu \in {}^{\omega>}\xi$, $\mathbf{B}^\xi_\nu$ is a complete subalgebra of $\mathbf{B}^c_0$.*

(2): *If $\xi < \lambda$ and $x \in \mathbf{B}^c_0$ <u>then</u> $\mathrm{pr}_\xi(x)$ is well defined and belongs to $\mathbf{B}^\xi$. Similarly, if $\xi < \lambda$, $\nu \in {}^{\omega>}\xi$ and $x \in \mathbf{B}^c_0$ <u>then</u> $\mathrm{pr}_{\xi,\nu}(x)$ is well defined and belongs to $\mathbf{B}^\xi_\nu$.*

(3): *If $\xi_0 \le \xi_1 < \lambda$, $x \in \mathbf{B}^c_0$ then $\mathrm{pr}_{\xi_0}(\mathrm{pr}_{\xi_1}(x)) = \mathrm{pr}_{\xi_0}(x)$.*

(4): *If $\xi < \lambda$ and $w \subseteq \alpha^*$ is finite <u>then</u> the function $x \mapsto \mathrm{pr}_{\xi,w}(x)$ is well defined for $x \in \mathbf{B}^c_0$ and the value is in $\langle \mathbf{B}^\xi \cup \{a_\alpha : \alpha \in w\}\rangle_{\mathbf{B}^c_0}$, of course which is a complete subalgebra of $\mathbf{B}^c_0$.*

(5): *If $\xi < \lambda$ and $\nu \in {}^{\omega>}\xi$ and $x \in \mathbf{B}^c_0$ <u>then</u> $\mathrm{pr}_{\xi,\nu}(\mathrm{pr}_\xi(x)) = \mathrm{pr}_{\xi,\nu}(x)$. If in addition $\xi_0 < \xi$ and $\nu \notin {}^{\omega>}(\xi_0)$ <u>then</u> $\mathrm{pr}_{\xi_0}(x) = \mathrm{pr}_{\xi_0}(\mathrm{pr}_{\xi,\nu}(x))$.*

(6): *$\mathbf{B}_{[\xi]} \subseteq \mathbf{B}^\xi$ and if $\xi < \dot\zeta(\alpha)$ <u>then</u> $\mathbf{B}_{[\xi]} \subseteq \mathbf{B}_\alpha$ and $\mathbf{B}(I, w) \subseteq \mathbf{B}_{\alpha^*}$*

Proof. Easy.

{3.7}

Fact 3.7. **(1):** *For $x \in \mathbf{B}_{\alpha^*}$, $\xi < \lambda$, the element $\mathrm{pr}_\xi(x)$ belongs to $\mathbf{B}_{[\xi]}$.*

(2): *If $x \in \mathbf{B}_{\alpha^*}$, $\xi < \lambda$ and $J \subseteq {}^{\omega>}(\xi + 1)$ not necessarily finite, <u>then</u> the element $\mathrm{pr}_{\xi,J}(x)$ belongs to $\langle \mathbf{B}_{[\xi]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}^c_0}$.*

(3): *Like part (2) but $J \subseteq {}^{\omega>}\lambda$ (and not necessarily $J \subseteq {}^{\omega>}(\xi + 1)$) and J is finite*

Proof. (1) We prove this for $x \in \mathbf{B}_\alpha$, by induction on α (for all ξ). Note that

$$\boxdot \quad \mathrm{pr}_\xi(\bigcup_{\ell<n} x_\ell) = \bigcup_{\ell<n} \mathrm{pr}_\xi(x_\ell) \quad \text{for} \quad x_0, \dots, x_{n-1} \in \mathbf{B}^c_0.$$

<u>Case i:</u> $\alpha = 0$, or even just $(\forall \beta < \alpha)[\dot\zeta(\beta) \le \xi]$.
Easy. Clearly we can find σ, y_ℓ, ν_k ($\ell < n$, $k < m$) such that $x = \sigma(y_0, \dots, y_{n-1}, x_{\nu_0}, \dots, x_{\nu_{m-1}})$, where σ is a Boolean term, $y_\ell \in \mathbf{B}_{[\xi]}$, $\nu_\ell \in {}^\omega\lambda \setminus {}^{\omega>}\xi$; by the remark

above without loss of generality $x = \bigcap_{\ell<n+m} s_\ell$, where $s_\ell \in \{y_\ell, 1-y_\ell\}$ when $\ell < n$, and $s_\ell \in \{x_{\nu_{\ell-n}}, 1-x_{\nu_{\ell-n}}\}$ when $n \le \ell < n+m$, and the sequence $\langle x_{\nu_0}, \dots, x_{\nu_{n-1}}\rangle$ is without repetitions. Now by 3.3 clearly $\mathrm{pr}_\xi(x) = \bigcap_{\ell<n} s_\ell$ which belongs to $\mathbf{B}_{[\xi]}$;

Case ii: α is limit.
Trivial as $\mathbf{B}_\alpha = \bigcup_{\beta<\alpha} \mathbf{B}_\beta$.

Case iii: $\alpha = \beta+1$.
By the induction hypothesis without loss of generality $x \notin \mathbf{B}_\beta$ hence $\beta \in Y'$. As $x \in \mathbf{B}_\alpha$ there are disjoint $\dot y_0$, $\dot y_1$, $\dot y_2 \in \mathbf{B}_\beta$ such that $x = \dot y_0 \cup (\dot y_1 \cap a_\beta) \cup (\dot y_2 - a_\beta)$. It suffices to prove that $\mathrm{pr}_\xi(\dot y_0), \mathrm{pr}_\xi(\dot y_1 \cap a_\beta), \mathrm{pr}_\xi(\dot y_2 - a_\beta) \in \mathbf{B}_{[\xi]}$; the first holds by the induction hypothesis and without loss of generality we concentrate on the second. Remembering clause $\circledast(1)$ of stage (A), by 3.2 applied to $\mathbf{B}_\alpha$, $\dot y_1$ we have: there are $\xi_0 < \dot\zeta(\beta)$ and $k < \omega$ such that $\dot y_1$ is based on $J \stackrel{\text{def}}{=} {}^{\omega>}\lambda \setminus \{\rho : \eta_\beta \restriction k \lhd \rho \in {}^{\omega>}\lambda\}$. Now without loss of generality each d^β_n $(n<\omega)$ is based on ${}^{\omega>}\xi_0$ (recall clause $\circledast_\beta$ (2) of stage A) and ${}^{\omega>}\xi_0 \subseteq J$ (this holds if $\eta_\beta \restriction k \notin {}^{\omega>}\xi_0$, and as η_β is increasing with limit $\dot\zeta(\beta)$ this is easy to obtain). By Case i, we can assume that $\xi < \dot\zeta(\beta)$ hence (as we can increase k and ξ_0) without loss of generality $\xi < \xi_0$, and by the induction hypothesis and 3.6(3),(5), letting $\nu =: \eta_\beta \restriction k$, it suffices to prove $\mathrm{pr}_{\xi_0,\nu}(\dot y_1 \cap a_\beta) \in \mathbf{B}_\beta$. As $m < \omega \Rightarrow a_\beta \cap d^\beta_m \in \mathbf{B}_{[\dot\zeta(\beta)]}$ and $\boxdot$ above, without loss of generality $\dot y_1 \cap d^\beta_m = 0$ for $m < k$. Now clearly for proving $\mathrm{pr}_{\xi_0,\nu}(\dot y_1 \cap a_\beta) = \dot y_1$ it is enough to show, for each $m<\omega$, that $\mathrm{pr}_\xi(\dot y_1 \cap d^\beta_m \cap \dot s^\alpha_m) = \dot y_1 \cap d^\beta_m$ as $\langle d^\beta_n : n<\omega\rangle$ is a maximal antichain of $\mathbf{B}^c_0$ and as $a_\beta \cap d^\beta_m = \dot s^\beta_m$ both by $\circledast_\beta(2)$. If $m<k$ then $\dot y_1 \cap d^\beta_m = 0$ so this is trivial. If $m \ge k$ this holds because $d^\beta_m, \dot y_1$ are based on J, ${}^{\omega>}\xi_0 \subseteq J$ and $\dot s^\beta_m$ is based on ${}^{\omega>}\lambda \setminus J$ and is $\dot s^\beta_m > 0$.
(2),(3) Same proof. $\blacksquare_{3.7}$

{3.8}
Lemma 3.8. *1) Suppose that I, w satisfy:*

> $(*)_{I,w}$**:** *$I \subseteq {}^{\omega>}\lambda$, $w \subseteq \alpha^* \cap Y'$, I is closed under initial segments,*
> $$\alpha \in w \ \&\ n<\omega \quad \Rightarrow \quad \eta_\alpha \restriction n \in I,$$
> *and for every $\alpha < \alpha^*$, if $\bigwedge_{m<\omega}(\eta_\alpha \restriction m \in I)$ then s^α_m, d^α_m are based on I and belong to $\mathbf{B}(I,w)$ and $\alpha \in w$; see Definition 3.5(5).*

Then for any countable $C \subseteq \mathbf{B}_{\alpha^*}$ there is a projection h from $\langle \mathbf{B}(I,w), C\rangle_{\mathbf{B}^c_0}$ onto $\mathbf{B}(I,w)$.
2) If $(*)_{I,w}$ holds then every member of $\mathbf{B}(I,w)$ is based on I.
3) We can add

- **(a):** *if $a_\alpha \in C \setminus \mathbf{B}(I,w)$, and $\{d^\alpha_n : n<\omega\} \subseteq \mathbf{B}(I,w)$ and $\{\eta_\alpha \restriction n : n<\omega\} \subseteq I$ then $h(a_\alpha)$ has support $\subseteq {}^{\omega>}\zeta$ for some $\zeta < \dot\zeta(\alpha)$.*
- **(b):** *if $\nu \in {}^{\omega>}\lambda$ and $x_\nu \in C$ then $h(x_\nu) \in \{0, x_\nu, 1\}$*
- **(c):** *if $c = \sigma(a_{\alpha_0}, \dots, a_{\alpha_{k-1}}, b_0, \dots, b_{n-1})$ where σ is a Boolean term $\alpha_0, \dots \alpha_{k-1} \in w$ and $\dot\zeta(\alpha_0) \le \dots \le \dot\zeta(\alpha_{k-1}) \le \xi$ and $b_\ell \in \mathbf{B}(I,w)$ and $\mathrm{supp}(b_\ell) \subseteq {}^{\omega>}\zeta$ for some $\zeta < \xi$ and $a_{\alpha_\ell} \in C$ then $h(c) = \sigma(a'_0, \dots, a'_{k-1}, b_0, \dots, b_{n-1})$ (where $\ell < k \Rightarrow a'_\ell = h(a_{\alpha_\ell})$) is in $\bigcup_{\varepsilon<\xi} \mathbf{B}_{[\varepsilon]}$.*

Remark: In $(*)_{I,w}$ the last phrase can be weakened to "for some $m_\alpha < \omega$, for every $m \in [m_\alpha, \omega)$ the elements s^α_n, d^α_m are based on I (and belong to $\mathbf{B}(I,w)$ and $(\alpha \in w)$".

Proof. 1) We can easily find $I(*)$, $w(*)$ such that $C \subseteq \mathbf{B}(I(*), w(*))$, $w \subseteq w(*) \subseteq \alpha^*$, $|w(*)\backslash w| \le \aleph_0$, $I \subseteq I(*) \subseteq {}^{\omega>}\lambda$, $I(*)$ is closed under initial segments, $|I(*)\backslash I| \le \aleph_0$, and if $\alpha \in w(*) \setminus w$, then $\dot s^\alpha_m, d^\alpha_m \in \mathbf{B}(I(*), w(*))$ hence $\{\eta_\alpha{\restriction}m : m < w\} \subseteq I(*)$. Let $w(*) \setminus w = \{\alpha_\ell : \ell < \omega\}$ for notational simplicity, and we choose by induction on ℓ a natural number $k_\ell < \omega$, such that the sets

$$\{\nu \in {}^{\omega>}\lambda : \nu \text{ appears in } \dot s^{\alpha_\ell}_m \text{ for some } m \ge k_\ell\}$$

are pairwise disjoint and disjoint to I (possible by the demand $s^\alpha_m \in \langle x_\rho : \eta_\alpha \restriction m \lhd \rho, \rho \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0}$ in clause $\circledast_\alpha(2)$ of stage A in the beginning of the proof of 3.1 and $(*)_{I,w}$). First assume that

$\boxdot$ for every α, $\langle \mathrm{supp}(s^\alpha_m) : m < \omega\rangle$ is a sequence of pairwise disjoint sets.

Now we can extend the identity map on $\mathbf{B}(I,w)$ to a projection h_0 from $\mathbf{B}(I(*), w)$ onto $\mathbf{B}(I,w)$ such that

- **(a):** if $\nu \in I(*) \setminus I$ then $h_0(x_\nu) \in \{0,1\}$
- **(b):** if $\ell < \omega$, $m > k_\ell$, then $h_0(s^{\alpha_\ell}_m) = 0$.

This is possible as $\mathbf{B}(I(*), w)$ is generated by $\mathbf{B}(I,w) \cup \{x_\nu : \nu \in I(*) \setminus I\}$ freely except the equations which hold in $\mathbf{B}(I,w)$ and $\boxdot$ above as $\langle d^{\alpha_\ell}_m : m \in (k_\ell, \omega)\rangle$ is a sequence of pairwise disjoint elements. Now we can define by induction on $\alpha \in (w(*) \setminus w) \cup \{\alpha^*\}$ a projection h_α from $\mathbf{B}(I(*), w \cup (w(*) \cap \alpha))$ onto $\mathbf{B}(I,w)$ extending h_β for any $\beta < \alpha$ satisfying $\beta \in (w(*) \setminus w) \cup \{0\}$. For $\alpha = 0$ we have defined it, for $\alpha = \alpha^*$ we get the desired conclusion, and in limit stages take the union. In successive stages there is no problem by the choice of h_0, and of the k_ℓ's (and $\circledast(2)$ of stage A).

If $\boxdot$ fails, we just define $h_\xi = h{\restriction}(\mathbf{B}(I(*), w(*)) \cap \mathbf{B}^\xi)$ by induction on $\xi \le \lambda$ such that (it is the identity on $\mathbf{B}(I,w) \cap \mathrm{Dom}(h_\xi)$) and

- $(a)'$ if $\nu \in I(*) \setminus I$ and $x_\nu \in \mathrm{Dom}(h_\xi)$ then $h_\xi(x_\nu) \in \{0,1\}$
- $(b)'$ if $\ell < \omega, m > k_\ell$ and $h_\xi(d^{\alpha_\ell}_m)$ is well defined then $s^{\alpha_\ell}_m \cap d^{\alpha_\ell}_m$ does not belong to the filter on $\mathbf{B}^c_0$ generated by $\{d \in \mathrm{Dom}(h_\xi) : h_\xi(d) = 1\}$.

2) The proof of part 1) gives this.
3) Note that by clause $(a)'$, in clause $(b)'$, if $h_\xi(s^{\alpha_\ell}_m \cap d^{\alpha_\ell}_m)$ is well defined then it is $0_{\mathbf{B}_0}$ This is possible by the choice of $\langle k_\ell : \ell < \omega\rangle$ and as $\langle d^{\alpha_\ell}_m : m < \omega\rangle$ is a sequence of pairwise disjoint elements of $\mathbf{B}^c_0$. $\blacksquare_{3.8}$

Claim 3.9. *If $\mathbf{B}'$ is an uncountable subalgebra of $\mathbf{B}_{\alpha^*}$ then there is an antichain $\{d_n : n < \omega\} \subseteq \mathbf{B}'$ such that for no $x \in \mathbf{B}_{\alpha^*}$ do we have $x \cap d_{2n} = 0$, $x \cap d_{2n+1} = d_{2n+1}$ for every n, provided that* {3.9}

- **(*):** *no single countable $I \subseteq {}^{\omega>}\lambda$ is a support for every $a \in \mathbf{B}'$.*

Proof: We choose by induction on $\alpha < \omega_1$, d_α, I_α, such that:

- **(i):** $I_\alpha \subseteq {}^{\omega>}\lambda$ is countable, closed under initial segments
- **(ii):** $\bigcup_{\beta<\alpha} I_\beta \subseteq I_\alpha$ and for α limit, equality holds,
- **(iii):** $d_\alpha \in \mathbf{B}'$ is based on $I_{\alpha+1}$ but not on I_α.

There is no problem doing this as we are assuming (*).

By clause (iii), for each α there are a non zero $s^0_\alpha \in \langle x_\eta : \eta \in I_\alpha\rangle_{\mathbf{B}^c_0}$ and non-zero s^1_α, $s^2_\alpha \in \langle x_\eta : \eta \in I_{\alpha+1} \setminus I_\alpha\rangle_{\mathbf{B}^c_0}$ such that $s^1_\alpha \cap s^2_\alpha = 0$, $s^0_\alpha \cap s^1_\alpha \le d_\alpha$, $s^0_\alpha \cap s^2_\alpha \le 1 - d_\alpha$.

By Fodor's lemma, as we can replace I_α by $I_{h(\alpha)}$ if $h : \omega_1 \to \omega_1$ is increasing continuous, without loss of generality, $s^0_\alpha = s^0$ (i.e., does not depend on α). For each α there is $n(\alpha) < \omega$ such that

$$s^0 = s^0_\alpha \in \langle x_\eta : \eta \in I_\alpha \cap {}^{n(\alpha)\ge}\lambda\rangle_{\mathbf{B}^c_0},$$

$$s^1_\alpha, s^2_\alpha \in \langle x_\eta : \eta \in (I_{\alpha+1} \setminus I_\alpha) \cap {}^{n(\alpha)\ge}\lambda\rangle_{\mathbf{B}^c_0}.$$

Again, by renaming without loss of generality $n(\alpha) = n(*)$ for every α. For $n < \omega$ let $d^n = d_n - \bigcup\limits_{\ell<n} d_\ell$, $s^n = s^0 \cap \bigcap\limits_{\ell<n} s^2_\ell \cap s^1_n$, so easily $d^n \in \mathbf{B}'$, $\langle d^n : n < \omega\rangle$ is an antichain, $s^n \le d^n$ and $s^n \in \langle x_\eta : \ \eta \in {}^{n(*)\ge}\lambda\rangle_{\mathbf{B}^c_0}$ and by the choice of $\mathbf{B}_0$ easily $0 < s^n$. Suppose $x \in \mathbf{B}_\alpha$ satisfies: for each $n < \omega$, we have $x \cap d^{2n} = 0$, $x \cap d^{2n+1} = d^{2n+1}$. Then for $n < \omega$, $x \cap s^{2n} = 0$, $x \cap s^{2n+1} = s^{2n+1}$. But by 3.8(1) (for $I = {}^{n(*)\ge}\lambda$, $w = \emptyset$ and $C = \{x\}$), there is such x in $\langle x_\eta : \eta \in {}^{n(*)\ge}\lambda\rangle_{\mathbf{B}^c_0}$, an easy contradiction. $\blacksquare_{3.9}$

Hence we have proved in particular that for every $\aleph_1$- compact $\mathbf{B}' \subseteq \mathbf{B}_{\alpha^*}$, some countable $I \subseteq {}^{\omega>}\lambda$ supports every $x \in \mathbf{B}'$.

{3.10}
Claim 3.10 (Crucial Claim). *No infinite subalgebra* $\mathbf{B}'$ *of* $\mathbf{B}_{\alpha^*}$ *is* $\aleph_1$*-compact.*

Proof. Suppose that there is such subalgebra, and let ξ be minimal such that there is an infinite $\aleph_1$–compact $\mathbf{B}' \subseteq \mathbf{B}_{[\xi]}$. The proof is broken into five parts.

Part I If

- **(a):** $\mathbf{B}' \subseteq \mathbf{B}_{\alpha^*}$ is $\aleph_1$-compact and infinite (subalgebra)
- **(b):** $\mathbf{B}' \subseteq \mathbf{B}_{[\xi]}$,

then

- **(c):** for every $\zeta < \xi$, finite $J \subseteq {}^{\omega>}\lambda$ and $x \in \mathbf{B}' \setminus \{y : \{z \in \mathbf{B}' : z \le y\}$ is finite$\}$, there is $x_1 \in \mathbf{B}'$, $x_1 \le x$ such that for no $y \in \langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}^c_0}$, do we have $y \cap x = x_1$.

So toward contradiction assume $\mathbf{B}'$ satisfies (a) and (b), but it fails (c) for $\zeta < \xi$, a finite $J \subseteq {}^{\omega>}\lambda$ and $x \in \mathbf{B}'$, hence $\{y : \ y \le x, \ y \in \mathbf{B}'\}$ is infinite. So for every $z \in \mathbf{B}'$, there is $g(z) \in \langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}^c_0}$. such that $g(z) \cap x = z \cap x$ (otherwise we can use $x_1 = z \cap x$). Let $\mathbf{B}^a$ be the Boolean subalgebra of $\langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}^c_0}$ generated by $\{g(z) : z \in \mathbf{B}'\}$, so $z \in \mathbf{B}' \Rightarrow z \cap x \in \mathbf{B}^a$. Clearly

$$\{y \in \mathbf{B}' : y \le x\} = \{t \cap x : t \in \mathbf{B}^a\}.$$

Let $x^* = \mathrm{pr}_{\zeta,J}(x)$ (it is in $\mathbf{B}_{[\zeta]}$ by 3.7(1) if $J = \emptyset$, 3.7(3) otherwise), and let

$$\mathbf{B}^b = \{t \cap x^* : \ t \in \mathbf{B}^a\} \cup \{t \cup (1 - x^*) : t \in \mathbf{B}^a\}.$$

Clearly $\mathbf{B}^b$ is a subalgebra of $\langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}^c_0}$, and $1 - x^*$ is an atom of $\mathbf{B}^b$. Now $\mathbf{B}^b$ is infinite, why? there are distinct $x_n \le x$ in $\mathbf{B}'$ (for $n < \omega$), so $g(x_n) \in \mathbf{B}^a$ and hence $g(x_n) \cap x^* \in \mathbf{B}^b$. As $x \le x^*$ and

$$n \ne m \quad \Rightarrow \quad g(x_n) \cap x = x_n \cap x = x_n \ne x_m = x_m \cap x = g(x_m) \cap x,$$

clearly $[n \ne m \ \Rightarrow \ g(x_n) \cap x^* \ne g(x_n) \cap x^*]$ so $\mathbf{B}^b$ is really infinite. We shall prove that $\mathbf{B}^b$ is $\aleph_1$-compact, thus contradicting the choice of ξ. Let $d_n \in \mathbf{B}^b$ be pairwise disjoint, and we would like to find $t \in \mathbf{B}^b$ satisfying $t \cap d_{2n} = 0$, $t \cap d_{2n+1} = d_{2n+1}$ for $n < \omega$. Clearly without loss of generality $d_n \le x^*$ (as $1 - x^*$ is an atom of $\mathbf{B}^b$).

So $d_n = t_n \cap x^*$ for some $t_n \in \mathbf{B}^a$, hence easily $t_n \cap x \in \mathbf{B}'$ so for some $x_n \in \mathbf{B}'$, $x_n \le x$ and $t_n \cap x = x_n \cap x = x_n$. So $x_n = g(x_n) \cap x$.

For $n \ne m$,

$$x_n \cap x_m = (t_n \cap x) \cap (t_m \cap x) \le (t_n \cap x^*) \cap (t_m \cap x^*) = d_n \cap d_m = 0.$$

As $\mathbf{B}'$ is $\aleph_1$-compact there is $y \in \mathbf{B}'$ satisfying $y \cap x_{2n} = 0$, $y \cap x_{2n+1} = x_{2n+1}$ for $n < \omega$. Now $g(y)$, d_n, t_n belong to $\langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}_0^c}$ and (as $x_n \le x \le x^*$ and $d_n = t_n \cap x^*$, $t_n \cap x = x_n$):

(i): $g(y) \cap d_{2n} \cap x = g(y) \cap t_{2n} \cap x = g(y) \cap x_{2n} \cap x = y \cap x_{2n} \cap x = 0$,

(ii): $g(y) \cap d_{2n+1} \cap x = g(y) \cap t_{2n+1} \cap x = g(y) \cap x_{2n+1} \cap x = y \cap x_{2n+1} \cap x = x_{2n+1} \cap x = t_{2n+1} \cap x = d_{2n+1} \cap x$.

Now, by the definition of $x^* = \mathrm{pr}_{\zeta,J}(x)$,

$$s \in \langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}_0^c} \& s \cap x = 0 \quad \Rightarrow \quad s \cap x^* = 0$$

(as $1 - s \in \langle\langle \mathbf{B}'_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}_0^c}$ and by the left side $x \le 1 - s$), hence by clause (i) (for $s = g(y) \cap d_{2n}$):

(iii): $g(y) \cap d_{2n} \cap x^* = 0$.

Also, by the definition of $x^* = \mathrm{pr}_{\zeta,J}(x)$,

$$s_1, s_2 \in \langle \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}_{\mathbf{B}_0^c} \ \& \ s_1 \cap x = s_2 \cap x \quad \Rightarrow \quad s_1 \cap x^* = s_2 \cap x^*$$

(as $s_1 - s_2 \in \mathbf{B}_{[\zeta]} \cup \{x_\nu : \nu \in J\}\rangle_{\mathbf{B}_0^c}$ and by the left side $x \le 1 - (s_1 - s_2)$ hence as above $x^* \le 1 - (s_1 - s_2)$ and similarly $x^* \le 1 - (s_2 - s_1)$). Hence by clause (ii)

(iv): $g(y) \cap d_{2n+1} \cap x^* = d_{2n+1} \cap x^*$.

But $d_n \le x^*$, so from (iii) and (iv), $(g(y) \cap x^*) \cap d_{2n} = 0$, $(g(y) \cap x^*) \cap d_{2n+1} = d_{2n+1}$, and $g(y) \in \mathbf{B}^a$, hence $g(y) \cap x^* \in \mathbf{B}^b$. So $\mathbf{B}^b$ is $\aleph_1$–compact and this contradicts the minimality of ξ, hence we finish proving Part I.

Part II: If $\mathbf{B}^1 \subseteq \mathbf{B}_{\alpha^*}$ is $\aleph_1$– compact, $\mathbf{B}^1 \subseteq \mathbf{B}^2$, $\mathbf{B}^2 = \langle \mathbf{B}^1 \cup \{z\}\rangle_{\mathbf{B}^2}$ then $\mathbf{B}^2$ is $\aleph_1$–compact.

The proof is straightforward. [If $d_n \in \mathbf{B}^2$ are pairwise disjoint, let $d_n = d_n^0 \cup (d_n^1 \cap z) \cup (d_n^2 - z)$ for some disjoint $d_n^0, d_n^1, d_n^2 \in \mathbf{B}^1$. As $\mathbf{B}_{\alpha^*}$ satisfies the c.c.c. also $\mathbf{B}^1$ satisfies the c.c.c., hence being $\aleph_1$-compact, is complete. Now the each of the sets each $\mathbf{J}_\ell (\ell < 3)$ is an ideal of $\mathbf{B}^1$ and their union $\mathbf{J}_0 \cup \mathbf{J}_1 \cup \mathbf{J}_2$ is a dense subset of $\mathbf{B}^1$ where $\mathbf{J}_\ell = \{x \in \mathbf{B}^1 : x > 0$ satisfies $\ell = 0 \Rightarrow \mathbf{B}^2 \models x \cap z = 0$ and $\ell = 1 \Rightarrow \mathbf{B}^2 \models x \le z$ and $\ell = 2 \Rightarrow \mathbf{B}^2 \models (\forall y)(0 < y \le x \ \& \ y \in \mathbf{B}^1 \Rightarrow y \cap z \ne 0 \ne y - z)\}$. As $\mathbf{B}^1$ is complete without loss of generality $d_m^\ell \in \mathbf{J}_0 \cup \mathbf{J}_1 \cup \mathbf{J}_2$ for $m < \omega, \ell < 3$. Also there is a maximal antichain $\langle x_n : n < \gamma \le \omega\rangle_{\mathbf{B}^1}$ of $\mathbf{B}^1$ consisting of elements of this family. Similarly without loss of generality for each n we have $x_n \le d_m^\ell$ for some $m < \omega_1$ and $\ell < 3$; or $x \cap d_m^\ell = 0$ for every m. Without loss of generality $d_n^1 \ne 0 \Rightarrow d_n^1 \in \mathbf{J}_2$ and $d_n^2 \ne 0 \Rightarrow d_n^2 \in \mathbf{J}_2$ and necessarily $d_n^0 \cap (d_m^1 \cup d_m^2) = 0$ for $n, m < \omega$. Now, necessarily $d_n^0 \cap d_m^0 = 0$ for $n \ne m$ and without loss of generality, $d_n^1 \cap d_m^1 = 0$ for $n \ne m$ — otherwise replace them by $d_n^1 - \bigcup_{\ell < n} d_\ell^1$; similarly $d_n^2 \cap d_m^2 = 0$, for $n \ne m$.

So, for $\ell = 0, 1, 2$, there is $y^\ell \in \mathbf{B}^1$ such that for every $n < \omega$ we have:

$$y^\ell \cap d_{2n}^\ell = 0, \quad y^\ell \cap d_{2n+1}^\ell = d_{2n+1}^\ell.$$

Hence $y^0 \cup (y^1 \cap z - y') \cup (y^2 \cap z - y')$ is a solution.]

Part III: ξ cannot be a successor ordinal.

Proof: Let $\mathbf{B}'$ satisfy clauses (a), (b) (hence (c)) of Part I.

Suppose toward contradiction that $\xi = \zeta + 1$, and by 3.9 there is a countable $I \subseteq {}^{\omega>}\xi$ which supports every $a \in \mathbf{B}'$; without loss of generality, I is closed under initial segments and, under those demands, $|I \setminus {}^{\omega>}\zeta| \leq \aleph_0$ is minimal. Now, by applying Part I we get

- **$\boxdot$:** for every finite $J \subseteq {}^{\omega>}\lambda$, and $x \in \mathbf{B}'$ for which $\{y \in \mathbf{B}' : y \leq x\}$ is infinite, there is $x_1 \in \mathbf{B}'$, $x_1 \leq x$ such that for no $y \in \langle \mathbf{B}_{[\zeta]} \cup \{x_\eta : \eta \in J\}\rangle_{\mathbf{B}_0^c}$ do we have $y \cap x = x_1$.

Now, $I \setminus {}^{\omega>}\zeta$ is infinite. [Why? Otherwise let $\mathbf{B}'' = \langle \mathbf{B}' \cup \{x_\eta : \eta \in I \setminus {}^{\omega>}\zeta\}\rangle_{\mathbf{B}_0^c}$; it is infinite and $\aleph_1$- compact by Part II, and we shall we apply Part I to it. Let $k = |I \setminus {}^{\omega>}\zeta|$ and let $I \setminus {}^{\omega>}\zeta = \{\eta_0, \ldots, \eta_{k-1}\}$ and for $u \subseteq \{0, \ldots, k-1\}$, let

$$x_u \stackrel{\text{def}}{=} \bigcap\{x_{\eta_\ell} : \ell \in u\} \cap \bigcap\{1 - x_{\eta_\ell} : \ell < k, \quad \text{and} \quad \ell \notin u\}.$$

So $x_u \in \mathbf{B}''$, $1 = \bigcup\{x_u : u \subseteq \{0, \ldots, k-1\}\}$, hence for some u, $\{y \in \mathbf{B}'' : y \leq x_u\}$ is infinite; now ζ, x_u contradict the conclusion of Part I.]

As $\mathbf{B}'$ is $\aleph_1$–compact, for any $x \in \mathbf{B}'$ such that $\{y \in \mathbf{B}' : y \leq x\}$ is infinite, x can be splitted in $\mathbf{B}'$ to two elements satisfying the same, i.e., $x = x^1 \cup x^2$, $x^1 \cap x^2 = 0$, $\{y \in \mathbf{B}' : y \leq x^\ell\}$ is infinite for $\ell = 1, 2$. Let $I \setminus {}^{\omega>}\zeta = \{\eta_\ell : \ell < \omega\}$, so we can find pairwise disjoint $\dot{y}_n \in \mathbf{B}'$ such that $\{y \in \mathbf{B}' : y \leq \dot{y}_n\}$ is infinite. Now, by $\boxdot$ above, for each n we can find d_{2n}, d_{2n+1} satisfying $\dot{y}_n = d_{2n} \cup d_{2n+1}$, $d_{2n} \cap d_{2n+1} = 0$ and such that for no $y \in \langle \mathbf{B}_{[\zeta]} \cup \{x_{\eta_\ell} : \ell < n\}\rangle_{\mathbf{B}_0^c}$ do we have $y \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$.

Since $\mathbf{B}'$ is $\aleph_1$–compact there is $y \in \mathbf{B}'$ such that $y \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$ for every $n < \omega$. As $y \in \mathbf{B}'$ clearly $y \in \mathbf{B}_{[\xi]} = \mathbf{B}_{[\zeta+1]}$, and y is based on $\{x_\nu : \nu \in {}^{\omega>}\zeta\} \cup \{x_{\eta_\ell} : \ell < \omega\}$, so by 3.7(2) we have $y' = \mathrm{pr}_{\zeta,\{\eta_\ell : \ell<\omega\}}(x)$ belong to $\langle \mathbf{B}_{[\zeta]} \cup \{x_{\eta_\ell} : \ell < \omega\}\rangle_{\mathbf{B}_0^c}$. Hence $y' \in \langle \mathbf{B}_{[\zeta]} \cup \{x_{\eta_\ell} : \ell < n\}\rangle_{\mathbf{B}_0^c}$ for some $n < \omega$. This is a contradiction to $y' \cap (d_{2n} \cup d_{2n+1}) = d_{2n+1}$ which holds as by the choice of d_{2n}, d_{2n+1}, so $y \cap d_{2n} = 0$, $y \cap d_{2n+1} = d_{2n+1}$ and $d_{2n}, d_{2n+1} \in \langle \mathbf{B}_{[\zeta]} \cup \{x_{\eta_\ell} : \ell < \omega\}\rangle_{\mathbf{B}_0^c}$ so $y' \cap d_{2n} = 0, y' \cap d_{2n+1} = d_{2n+1}$.

Part IV: Let $\mathbf{B}'$ satisfy clauses (a), (b) of Part I (and hence clause (c) too). By 3.9, for some countable $I \subseteq {}^{\omega>}\xi$, every $b \in \mathbf{B}'$ is based on I. By Part III, ξ is not a successor ordinal and trivially it is not zero hence ξ is a limit ordinal. Now by 3.5(6) (i.e. the definition of $\mathbf{B}_{[\zeta]}$ for $\zeta \leq \lambda$) for no $\zeta < \xi$ is $I \subseteq {}^{\omega>}\zeta$, hence necessarily $\mathrm{cf}(\xi) = \aleph_0$. Let

$$Fi(\mathbf{B}') = \{x \in \mathbf{B}' : \{y \in \mathbf{B}' : y \leq x\} \text{ is finite}\}.$$

Next we shall show:

- **$(**)$:** for some finite $w^* \subseteq \{\alpha < \alpha^* : \dot{\zeta}(\alpha) = \xi\}$ and $x^* \in \mathbf{B}' \setminus Fi(\mathbf{B}')$, for every $y \leq x^*$ from $\mathbf{B}'$, for some $z \in \langle \cup_{\zeta<\xi} \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w^*\}\rangle_{\mathbf{B}_0^c}$ we have $z \cap x^* = y$.

Suppose $(**)$ fails and we choose by induction on $n < \omega$, x_n, y_n, w_n such that:

- **(i):** $x_n \in \mathbf{B}'$, and $m < n \ \Rightarrow \ x_m \cap x_n = 0$,
- **(ii):** $1 - \bigcup_{i<n} x_i \notin Fi(\mathbf{B}')$,
- **(iii):** $w_n \subseteq \{\alpha : \dot{\zeta}(\alpha) = \xi\}$ is finite,
- **(iv):** $w_n \subseteq w_{n+1}$,
- **(v):** $y_n \leq x_n$ and $y_n \in \mathbf{B}'$,
- **(vi):** for no $z \in \langle \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\rangle_{\mathbf{B}_0^c}$ do we have $z \cap x_n = y_n$.

For $n = 0$ we have $1 \notin Fi(\mathbf{B}')$, hence (ii) is satisfied.
For each n let w_n be a finite subset of $\{\alpha : \dot{\zeta}(\alpha) = \xi\}$ extending $\bigcup_{\ell<n} w_\ell$ such that for every $\ell < n$, x_ℓ, $y_\ell \in \langle \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\rangle_{\mathbf{B}_0^c}$, it exists by 3.5(6). Then, as $1 - \bigcup_{\ell<n} x_\ell \notin Fi(\mathbf{B}')$, and as $\mathbf{B}'$ is $\aleph_1$–compact, there is $x_n \leq 1 - \bigcup_{i<n} x_i$ satisfying $x_n \in \mathbf{B}'$ such that $1 - \bigcup_{\ell\leq n} x_\ell \notin Fi(\mathbf{B}')$ and $x_n \notin Fi(\mathbf{B}')$. Now, as (**) fails, necessarily w_n, x_n do not satisfy the requirements on w^*, x^* in $(**)$, so there is $y_n \in \mathbf{B}'$, $y_n \leq x_n$ such that for no $z \in \langle \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w_n\}\rangle_{\mathbf{B}_0^c}$ do we have $z \cap x_n = y_n$. So we can carry the definition. As $\mathbf{B}'$ is $\aleph_1$– compact, for some $z^* \in \mathbf{B}'$ we have $z^* \cap x_n = y_n$ for every n.

As $z^* \in \mathbf{B}'$ and $\mathbf{B}' \subseteq \mathbf{B}_{[\xi]}$, for some finite $w^* \subseteq \{\alpha < \alpha^* : \dot{\zeta}(\alpha) = \xi\}$ we have

$$z^* = \sigma(\ldots, a_\alpha, \ldots, \ldots, b_\ell, \ldots)_{\alpha\in w^*,\ell<n} \in \langle \bigcup_{\varepsilon<\xi} \mathbf{B}_{[\varepsilon]} \cup \{a_\alpha : \alpha \in w^*\}\rangle_{\mathbf{B}_0^c}$$

where σ is a Boolean term, and $\ell < n \Rightarrow b_\ell \in \bigcup_{\epsilon<\xi} \mathbf{B}_{[\epsilon]}$. As w^* is finite, for some $n(*) < \omega$ we have $w^* \cap (\bigcup_{n<\omega} w_n) \subseteq w_{n(*)}$.

Let $k^* < \omega$ be such that there are no repetitions in $\langle \eta_\alpha \restriction k^* : \alpha \in w_{n(*)+1}\rangle$ and $k^* > n(*)$. Let $\zeta < \xi$ be such that: $\mathrm{supp}(d_n^\alpha) \subseteq {}^{\omega>}\zeta$ for $\alpha \in w_{n(*)+1} \cup w^*$, $n < \omega$ and $\mathrm{supp}(\dot{s}_k^\alpha) \subseteq {}^{\omega>}\zeta$ for $\alpha \in w_{n(*)+1} \cup w^*$, $k < k^*$, and

$$x_n, y_n \in \langle \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w_{n(*)+1}\}\rangle_{\mathbf{B}_0^c}$$

for $n < n(*) + 1$ and $z^* \in \langle \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w^*\}\rangle_{\mathbf{B}_0^c}$

We shall now apply 3.8 with I, w, C there standing for

$$I' = \{\eta : \eta \in {}^{\omega>}\zeta \text{ or } \eta \triangleleft \nu \text{ where } \nu \in \mathrm{supp}(\dot{s}_n^\alpha) \text{ for some } \alpha \in w_{n(*)+1}, n < \omega\},$$

$w' =: \{\alpha < \alpha^* : (\forall n < \omega)(\eta_\alpha \restriction n \in I)\}$ and $C' =: \{z^*\}$ here; clearly the demands there hold, recalling $\mathrm{supp}(\dot{s}_n^\alpha)$ is a finite subset of $\{\rho \in {}^{\omega>}\dot{\zeta}(\alpha) : \eta_\alpha{\restriction}n \triangleleft \rho\}$ by $\circledast_\alpha(2)$. So there is a projection f from $\langle \mathbf{B}(I', w') \cup \{z^*\}\rangle_{\mathbf{B}_0^c}$ onto $\mathbf{B}(I', w')$, and so by 3.8(2) clearly $f(z^*)$ is based on I'. As clearly $w' \subseteq \{\alpha : \dot{\zeta}(\alpha) < \xi\} \cup w_{n(*)+1}$, we get

$$f(z^*) \in \mathbf{B}(I', w') \subseteq \langle \bigcup_{\varepsilon<\xi} \mathbf{B}_{[\varepsilon]} \cup \{a_\alpha : \alpha \in w_{n(*)+1}\}\rangle_{\mathbf{B}_0^c},$$

So $f(z^*)$ belongs to $\mathbf{B}(I', w')$, which is $\subseteq \langle \bigcup_{\varepsilon<\xi} \mathbf{B}_{[\varepsilon]} \cup \{a_\alpha : \alpha \in w_{n(*)}\}\rangle_{\mathbf{B}_0^c}$. Also $f(x_{n(*)}) = x_{n(*)}$ and $f(y_{n(*)}) = y_{n(*)}$ as $x_{n(*)}, y_{n(*)} \in \mathbf{B}(I', w')$, so as $z^* \cap x_{n(*)} = y_{n(*)}$ by the choice of z^*, necessarily $f(z^*) \cap x_{n(*)} = y_{n(*)}$, so by the previous sentence we get a contradiction to clause (vi) for $n(*)$. So $(**)$ holds.

<u>Part V.</u> We continue the first paragraph of Part IV, and let $(**)$ of Part IV hold for w^* and x^*.

Let $d_0, \ldots, d_m \in \mathbf{B}_{[\xi]}$ be such that

$\boxtimes$**:** (a) $\bigcup_{\ell=0}^{m} d_\ell = 1$ and
(b) $(\forall \ell \leq m)(\forall \alpha \in w^*)(d_\ell \leq a_\alpha \vee d_\ell \cap a_\alpha = 0)$.

There is an $\ell \leq m$ such that $\{y \cap d_\ell : y \leq x^* \text{ and } y \in \mathbf{B}'\}$ is infinite. It is clear (by Part II) that $\mathbf{B}'' = \langle \mathbf{B}', d_\ell\rangle_{\mathbf{B}_0^c}$ is $\aleph_1$– compact; also $x^* \cap d_\ell \in \mathbf{B}'' \setminus Fi(\mathbf{B}'')$.

Now, assume that $y \in \mathbf{B}''$, $y \le x^* \cap d_\ell$. Clearly for some $y' \in \mathbf{B}'$ we have $y = y' \cap d_\ell$ and without loss of generality $y' \le x^*$. By $(**)$, that is the choice of w^*, x^* for some $z \in \langle \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]} \cup \{a_\alpha : \alpha \in w^*\}\rangle_{\mathbf{B}_0^c}$ we have $z \cap x^* = y'$. Hence $z \cap (x^* \cap d_\ell) = y$, and by the choice of d_ℓ that is $\boxtimes(b)$ and the choice of z, for some $z' \in \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]}$, the equation $z' \cap (x^* \cap d_\ell) = z \cap (x^* \cap d_\ell) = y$ holds.

So by the previous paragraph, in $\mathbf{B}''$ the element $x^{**} \stackrel{\text{def}}{=} x^* \cap d_\ell$ satisfies the requirements in $(**)$ for $w^{**} =: \emptyset$. Now we use (c) of part I. As $\mathrm{cf}(\xi) = \aleph_0$, let $\xi = \bigcup_{n<\omega} \zeta_n$ with $\zeta_n < \zeta_{n+1} < \omega$, and by induction on $n < \omega$ we choose x_n, y_n such that:

(i): $x_n \in \mathbf{B}''$, $x_n \le x^{**}$, and $m < n \;\Rightarrow\; x_m \cap x_n = 0$,
(ii): $x^{**} - \bigcup_{\ell<n} x_i \notin Fi(\mathbf{B}'')$,
(iii): $y_n \in \mathbf{B}''$, $y_n \le x_n$,
(iv): for no $z \in \mathbf{B}_{[\zeta_n]}$ do we have $z \cap x_n = y_n$.

As $\mathbf{B}''$ is $\aleph_1$–compact, for some $z^* \in \mathbf{B}''$ we have $z^* \cap x_n = y_n$ for every n.

Now, as $\mathbf{B}''$, x^{**}, $w^{**} = \emptyset$ satisfy $(**)$, for some $z^{**} \in \bigcup_{\zeta<\xi} \mathbf{B}_{[\zeta]}$ we have $z^* \cap x^{**} = z^{**} \cap x^{**}$. So for some n, $z^{**} \in \mathbf{B}_{[\zeta_n]}$, contradicting clause (iv) above. Thus we have finished the proof of 3.10. $\blacksquare_{3.10}$

{3.11} **Claim 3.11.** $\mathbf{B}_{\alpha^*}$ *is endo-rigid.*

Before proving 3.11 we prove the subclaim 3.12 (For endomorphism h of $\mathbf{B}_{\alpha^*}$ we shall try to find $\alpha \in Y'$ such that $h(a_\alpha)$ has to realize p_α to get contradiction, but before choosing α we try to choose appropriate $\langle d_n^\alpha : n < \omega\rangle$, this is what 3.12 does for us):

{3.11A} **Subclaim 3.12.** *Assume that h is an endomorphism of $\mathbf{B}_{\alpha^*}$ and $\mathbf{B}_{\alpha^*}/\mathrm{ExKer}(h)$ is an infinite Boolean algebra. Then we can find ρ^* and $\bar d$ such that*

(A): $\bar d = \langle d_n : n < \omega\rangle$ *and* $\rho^* \in {}^{\omega>}\lambda$,
(B): $\{d_n : n < \omega\}$ *is a maximal antichain of* $\mathbf{B}_{\alpha^*}$ *and* $d_n > 0$ *of course,*
(C): *at least one of* $\boxtimes_1$, $\boxtimes_2$, $\boxtimes_3$ *hold, where*
- $\boxtimes_1$: **(a):** *if* $\rho^* \lhd \rho^{**} \in {}^{\omega>}\lambda$ *and* $n \in (0,\omega)$ *then for some* $s \in \langle x_\nu : \rho^{**} \lhd \nu \in {}^{\omega>}\lambda\rangle_{\mathbf{B}_0^c} \setminus \{0,1\}$ *we have* $h(s) \cap d_n = 0$,
  (b): *for no* $x \in \mathbf{B}_0^c$ *do we have* $n < \omega \;\Rightarrow\; x \cap h(d_{2n} \cup d_{2n+1}) = h(d_{2n+1})$.
- $\boxtimes_2$: *for no* $x \in \mathbf{B}_0^c$ *do we have: for every* $n \in (2,\omega)$

$$n \text{ is odd} \;\Rightarrow\; x \cap h(d_n) \cap d_0 = h(d_n) \cap d_0, \quad \text{and}$$
$$n \text{ is even} \;\Rightarrow\; x \cap h(d_n) \cap d_0 = 0.$$

Proof. As in the proof of 2.5(2), we can ignore the maximality requirement in clause (B) (call it $(\mathrm{B})^-$).

Recall

$$\mathrm{ExKer}^*(h) = \{a \in \mathbf{B}_{\alpha^*} : \{x/\mathrm{ExKer}(h) : x \le a\} \text{ is finite}\}.$$

Let $\dot{\mathcal{I}}_h = \{a \in \mathbf{B}_{\alpha^*} : \text{the set } \{h(d) : d \le a \text{ in } \mathbf{B}_{\alpha^*}\} \text{ is finite}\}$, clearly it is an ideal of $\mathbf{B}_{\alpha^*}$ included in $\mathrm{ExKer}^*(h)$ hence $1_{\mathbf{B}_{\alpha^*}} \notin \dot{\mathcal{I}}_h$

Case α: For some $\rho^* \in {}^{\omega>}\lambda$ and $a^* \in \mathbf{B}_{\alpha^*} \setminus \dot{\mathcal{I}}^*(h)$ we have: for every ρ satisfying $\rho^* \lhd \rho \in {}^{\omega>}\lambda$ there is $s \in \langle\{x_\eta : \rho \lhd \eta \in {}^{\omega>}\lambda\}\rangle_{\mathbf{B}_0^c} \setminus \{0_{\mathbf{B}_0^c}\}$ such that $h(s \cap a^*) = 0$.

Without loss of generality $\mathrm{supp}(a^*) \subseteq \{\rho \in {}^{\omega>}\lambda : \neg(\rho^* \triangleleft \rho)\}$, hence above $s \cap a^* \neq 0_{\mathbf{B}^c_0}$. Let $\mathbf{B}^a = \{h(d) : d \leq a^*\} \cup \{1 - h(d) : d \leq a^*\}$. This is a Boolean subalgebra of $\mathbf{B}^c_0$ and $1 - h(a^*)$ is an atom in it (or zero). As $a^* \notin \dot{\mathcal{I}}_h$, clearly $\mathbf{B}^a$ is infinite, hence by 3.10 there is an antichain $\langle y_n : n < \omega\rangle$ of $\mathbf{B}^a$ such that for no $x \in \mathbf{B}^c_0$ do we have $x \cap (y_{2n} \cup y_{2n+1}) = y_{2n}$. Without loss of generality$y_n \leq h(a^*)$ as at most one y_n fails this. Let d_n be such that $h(d_n) = y_n$ and without loss of generality $d_n \leq a^*$; of course $y_n > 0$, hence $d_n > 0$. Without loss of generality $\{d_n : n < \omega\}$ is an antichain (as we can use $d_n - \bigcup\limits_{m<n} d_m$).

Let $d'_0 = 1 - a^*$, $d'_1 = d_0 \cup d_1$ and $d'_{2+n} = d_{2+n}$. So clearly clauses (A), (B)$^-$, (C)$\boxtimes_1$ hold for $\langle d'_n : n < \omega\rangle$, and our ρ^*.

<u>Case β</u>: For some $a^* \in \mathbf{B}_\alpha$, $\{h(x) - a^* : x \in \mathbf{B}_{\alpha^*}, x \leq a^*\}$ is infinite.
<u>Clearly</u>

$$\mathbf{B}^a = \{h(x) - a^* : x \in \mathbf{B}_{\alpha^*} \ \&\ x \leq a^*\} \cup \{1 - (h(x) - a^*) : x \in \mathbf{B}_{\alpha^*} \ \&\ x \leq a^*\}$$

is a subalgebra of $\mathbf{B}_{\alpha^*}$ (and a^* is an atom in it). By the assumption (of this case) $\mathbf{B}^a$ is infinite. So by 3.10 there are pairwise disjoint $y_n \in \mathbf{B}_a \setminus \{0\}$ such that $\neg(\exists x \in \mathbf{B}_a) \bigwedge\limits_{n<\omega} (x \geq y_{2n+1} \ \&\ x \cap y_{2n} = 0)$. As a^* is an atom of $\mathbf{B}^a$, without loss of generality $y_n \leq 1 - a^*$, hence there are $d_n \in \mathbf{B}_{\alpha^*}$ such that $d_n \leq a^*$ and $h(d_n) - a^* = y_n$. Clearly

$$h(d_n - \bigcup_{\ell<n} d_\ell) = y_n - \bigcup_{\ell<n} y_\ell = y_n$$

hence without loss of generality the d_n are pairwise disjoint. Let $d'_0 = 1 - a^*, d'_1 = d_0 \cup d_1$ and $d'_{2+n} = d_{2+n}$, so $\langle d'_n : n < \omega\rangle$ is an antichain and $h(d'_n) \cap d_0 = h(d_n) - a^* = \dot{y}_n$ for $n = 2, 3, \ldots$, hence for no $x \in \mathbf{B}_{\alpha^*}$ do we have $n < \omega \Rightarrow x \cap h(d'_{2+2n} \cup d'_{2+2n+1}) = h(d'_{2+2n+1})$. So $\langle d'_n : n < \omega\rangle$ are as requested in $\boxtimes_2$.

Why the two sub-cases exhaust all possibilities?

Suppose none of Cases (α), (β) occurs. As case (α) fail for $a^* = 1_{\mathbf{B}^{\alpha^*}}$ necessarily for some $\rho_0 \in {}^{\omega>}\lambda$ we have

(a) $h(s) > 0$ for every $s \in \langle\{x_\eta : \rho_0 \trianglelefteq \eta \in {}^{\omega>}\lambda\}\rangle_{\mathbf{B}^c_0} \setminus \{0, 1\}$.

Clearly $a \in \langle\{x_\eta : \rho_0 \triangleleft \eta \in {}^{\omega>}\lambda\{\rangle_{\mathbf{B}^c_0} \setminus \{0\}$ implies that $a \notin \dot{\mathcal{I}}_h$. As clause (β) fail clearly for every $a \in \mathbf{B}_{\alpha^*}$ the set $\{h(d) - a : d \leq a, d \in \mathbf{B}_{\alpha^*}\}$ is finite. Next we note that:

$\boxplus$ if $\rho_0 \triangleleft \rho \in {}^{\omega>}\lambda$ then for some $s \in \langle\{x_\eta : \rho \triangleleft \eta \in {}^{\omega>}\lambda\}\rangle_{\mathbf{B}_0} \setminus \{0, 1\}$ we have $h(s) = s$.

[Why? for each $\alpha < \omega_1$ let $n_\alpha = (\{h(d) - x_{\rho^\frown\langle\alpha\rangle} : d \leq x_{\rho^\frown\langle\alpha\rangle}\}) + |\{h(d) - (-x_{\rho^\frown\langle\alpha\rangle}) : d \leq (-x_{\rho^\frown\langle\alpha\rangle})\}|$, so we know that n_α is enough, so for some $n(*)$ the set $Z = \{\alpha < \omega_1 : n_\alpha = n(*)\}$ is infinite. By Ramsey theorem for some a we have: if $\alpha < \beta$ are from Z and $\mathbf{t}_1, \mathbf{t}_2 \in \{0, 1\}$ are truth values then $h(x^{\mathbf{t}_1}_{\rho^\frown\langle\alpha\rangle} \cap x^{\mathbf{t}_2}_{\rho^\frown\langle\beta\rangle}) - x^{\mathbf{t}_1}_{\rho^\frown\langle\alpha\rangle} = a_{\mathbf{t}_1,\mathbf{t}_2} \in \mathbf{B}_\alpha$, $h(x^{\mathbf{t}_1}_{\rho^\frown\langle\alpha\rangle} \cap x^{\mathbf{t}_2}_{\rho^\frown\langle\beta\rangle}) \cap x^{\mathbf{t}_2}_{\rho^\frown\langle\beta\rangle} = b_{\mathbf{t}_1,\mathbf{t}_2} \in \mathbf{B}_\alpha$ where $x^{\mathbf{t}}$ is x if $\mathbf{t}$ is 1 and is $-x$ if $\mathbf{t}$ is 0. Let $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ be from Z and let $s = x_{\rho^\frown\langle\alpha_0\rangle} \cap (-x_{\rho^\frown\langle\alpha_1\rangle}) \cap x_{\rho^\frown\langle\alpha_2\rangle} \cap (-x_{\rho^\frown\langle\alpha_3\rangle})$. Now $h(s) \leq x_{\rho^\frown\langle\alpha_0\rangle}$ as $h(x_{\rho^\frown\langle\alpha_2\rangle}) - x_{\rho^\frown\langle\alpha_0\rangle} = h(x_{\rho^\frown\langle\alpha_3\rangle}) - x_{\rho^\frown\langle\alpha_0\rangle}$ and the equation above, similarly $h(s) \leq x_{\rho^\frown\langle\alpha_1\rangle}$ and also $h(s) \leq x_{\rho^\frown\langle\alpha_2\rangle}$ (using $x_{\rho^\frown\langle\alpha_0\rangle}, x_{\rho^\frown\langle\alpha_1\rangle}$) and $h(s) \leq (-x_{\rho^\frown\langle\alpha_3\rangle})$ together $h(s) \leq s$. Now $s_1 = s - h(0) > 0$. Easily $s > 0$ and s is disjoint to $b^* = \cup\{a_{\mathbf{t}_1,\mathbf{t}_2} \cup b_{\mathbf{t},\mathbf{t}_2} : \mathbf{t}_1, \mathbf{t}_2$

truth values}. If $(\mathbf{t}_0, \mathbf{t}_1) \neq (1, 0)$ and $s_1 \cap h(x^{\mathbf{t}_0}_{\rho\frown\langle\alpha_0\rangle} \cap x^{\mathbf{t}_1}_{\rho\frown\langle\alpha_1\rangle}) > 0$, as $s \cap b^* = 0_{\mathbf{B}_\alpha}$ we get easy contradiction. Similarly for $(x_{\rho\frown\langle\alpha_2\rangle}, x_{\rho\frown\langle\alpha_3\rangle})$ hence $h(s) = s$.

So $\boxplus$ holds. Let $a^* \in \mathbf{B}_{\alpha^*}$ be such that $h(a^*) \nleq a^*$ (exists by 2.5), and let $a^{**} = h(a^*) - a^* > 0$. By "not Case (α)" and $\boxplus$, for some $\rho_0 \in {}^{\omega>}\lambda$,

(b) $h(s) \cap a^{**} \neq 0$ for every $s \in \langle x_\eta : \rho_0 \triangleleft \eta \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0} \setminus \{0, 1\}$.

Possibly increasing ρ_0 the set $\{\eta : \rho_0 \triangleleft \eta \in {}^{\omega>}\lambda\}$ is disjoint to $\mathrm{supp}(a^*) \cup \mathrm{supp}(h(a^*))$. Let $s_n = x_{\rho^0\frown\langle n\rangle} - \bigcup\limits_{m<n} x_{\rho^0\frown\langle m\rangle}$ for $n < \omega$, so the s_n's are pairwise disjoint non-zero members of $\mathbf{B}_{\alpha^*}$ and by (a) we have $h(s_n) \cap a^{**} \neq 0$. But $h(s_n) \cap h(a^*) - a^* = h(s_n) \cap (h(a^*) - a^*) = h(s_n) \cap a^{**} > 0$. So clearly the assumption of case (β) holds (for a^*).

$\blacksquare_{3.12}$

<u>Proof of 3.11.</u> Suppose h is a counterexample, i.e., h is an endomorphism of $\mathbf{B}_{\alpha^*}$ but $\mathbf{B}_{\alpha^*}/\mathrm{ExKer}(h)$ is infinite, and we shall get a contradiction.

Clearly if for some good candidate α, $h_\alpha \subseteq h$ and $\alpha \in Y_1$ (see Stage B) then $h(a_\alpha)$ realizes the type $p_\alpha = \{x \cap b^\alpha_n = c^\alpha_n : n < \omega\}$, a contradiction (as by clause $\circledast(4)$ of stage A, $\mathbf{B}_{\alpha^*}$ omits p_α). So we shall try to find such α which satisfies the requirements $\circledast^1_\alpha$ of stage B (hence implicitly $\circledast_\alpha$ of stage A) for belonging to Y_1. Let ρ^*, $\langle d_n : n < \omega\rangle$ be as in 3.12 (ρ^* is needed only if $\boxtimes_1$ of (C) of 3.12 holds otherwise we can let $\rho^* = \langle\rangle$) and let $\xi < \lambda$ be such that

$$\bigcup\{\mathrm{supp}(d_n) \cup \mathrm{supp}(h(d_n)) : n < \omega\} \subseteq {}^{\omega>}\xi$$

Let $n(*) = \lg(\rho^*)$. Let $Z \subseteq {}^{\omega>}\lambda$, it will be used only in 3.13. We can find a good candidate $\alpha < \alpha^*$ such that

(a): $h_\alpha \subseteq h$, and $\dot{\zeta}(\alpha) > \xi$
(b): $d_n \in \mathbf{B}[N^\alpha_0]$ for $n < \omega$ and $\rho^* \in \mathrm{Rang}(f^\alpha)$, and $\langle d_n : n < \omega\rangle \in N^\alpha_0$
(c): N^α_0 is an elementary submodel of the expansion $(\mathcal{H}_{<\aleph_1}(\lambda), \in, \mathbf{B}_{\alpha^*}, h, \{(\eta, x_\eta) : \eta \in {}^{\omega>}\lambda\}, Z)$ of $\mathcal{M}$

so in particular N^α_n is closed under the functions implicit in the choice of $\mathbf{B}^\alpha_0$ and ρ^*, $\langle d_n : n < \omega\rangle$, i.e.,

(d): $a \in \mathbf{B}[N^\alpha_0] \ \Rightarrow\ \mathrm{supp}(a) \subseteq N^\alpha_0$,
(e): $\eta \in N^\alpha_m \cap {}^{\omega>}\lambda \ \Leftrightarrow\ x_\eta \in \mathbf{B}[N^\alpha_m]$,
(f): $({}^{\omega>}\lambda) \cap N^\alpha_m$ is closed under initial segments, and each node has infinitely many immediate successors,
(g): if $\boxtimes_1$ of clause C of 3.12 holds and if $\rho^* \triangleleft \rho^{**} \in N^\alpha_n$, and n is large enough then there is $s \in N^\alpha_n$ as required in 3.12(C)$\boxtimes_1$ so $h(s) - d_0 = 0$.

As $\mathbf{W}$ is a barrier this is possible (using the game $\eth'(\mathbf{W})$ and not $\eth(\mathbf{W})$ because of the requirement $\rho^* \in \mathrm{Rang}(f^\alpha)$ recalling Definition 1.7, that is we choose a strategy for player I, choosing the N_n-s and in the zeroth move also f_ℓ for $\ell = 1, \ldots, \ell g(\rho^*) + 1$. So for some play of the game, player II wins while player I uses the strategy described above so the play is $\langle(N^\alpha_n, f^\alpha_n) : n < \omega\rangle$ for some $\alpha < \alpha^*$, so we are done). Note that the proof of 3.13 below use the rest of the present proof, only ignoring case III below. We then will choose η_α, an ω–branch of $\mathrm{Rang}(f^\alpha)$ above ρ^*; but $\mathbf{W}$ is a disjoint barrier (see Definition 1.9(3)) hence η_α hence distinct from η_β for $\beta < \alpha$ and we will choose $s_n \in N^\alpha$ in $\langle x_\nu : \eta_\alpha \restriction n \triangleleft \nu \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0} \setminus \{0, 1\}$ and let $b_n = h(d_n)$, $c_n = h(d_n \cap s_n)$ for $n < \omega$, $p_\alpha = \{x \cap b_n = c_n : n < \omega\}$, and

$a_\alpha = \bigcup_{n<\omega} (d_n \cap s_n) \in \mathbf{B}_0^c$. All should have superscript $\overline{s}$ (where $\overline{s} = \langle s_n : n < \omega\rangle$), but we usually omit it or write $a_\alpha[\overline{s}], p_\alpha[\overline{s}]$ etc. It is enough to prove that for at least one such $\bar{s}$ we have $a_\alpha[\bar{s}]$, $\bar{s}[d_n : n < \omega]$ exemplify that $\alpha \in Y_1$.

The choice of $\overline{s}$ (and η_α which is determined by $\overline{s}$) is done by listing the demands on them (see Stage B) and showing that a solution exists. The only problematic one is (4) (omitting p_β for $\beta \leq \alpha$, $\beta \in Y_1$) and we partition it to three cases:

(I): $\dot\zeta(\beta) < \dot\zeta(\alpha)$ or $\dot\zeta(\beta) = \dot\zeta(\alpha)$, $\beta + 2^{\aleph_0} \leq \alpha$,
(II): $\dot\zeta(\beta) = \dot\zeta(\alpha)$, $\beta < \alpha < \beta + 2^{\aleph_0}$,
(III): $\beta = \alpha$.

We shall prove first that every $\overline{s}$ is O.K. for (I), second that for any family $\{(\eta^i, \overline{s}^i) : i < 2^{\aleph_0}\}$ (η^i is a branch of Rang(f^α) above ρ^*, etc.) with pairwise distinct η^i's, all except $< 2^{\aleph_0}$ many are O.K. for any instance of (II), and third that for every η (a branch of Rang(f^α) above ρ^*) there is $\overline{s}$ such that $(\overline{\eta}, \overline{s})$ satisfies (III). This clearly suffices (as for each branch η of Rang(f^α) choose $\overline{s}_f$ such that $(\eta, \overline{s}_\eta)$ satisfies III, and then chose η such that $(\eta, \overline{s}_\eta)$ satisfies II) .

Case I: $\dot\zeta(\beta) < \dot\zeta(\alpha)$ or $\dot\zeta(\beta) = \zeta(\alpha)$, $\beta + 2^{\aleph_0} \leq \alpha$
Let $\overline{s}$ be as above. Suppose some $x \in \langle \mathbf{B}_\alpha, a_\alpha[\overline{s}]\rangle_{\mathbf{B}_0^c}$ realize p_β. Clearly there is a partition $\langle y_\ell : \ell < 4\rangle$ of 1 (in $\mathbf{B}_\alpha$) such that $x = y_0 \cup (y_1 \cap a_\alpha[\overline{s}]) \cup (y_2 - a_\alpha[\overline{s}])$. Choose $\xi < \dot\zeta(\alpha)$ large enough and finite $k < \omega$ so that

⊡ $[\dot\zeta(\beta) < \dot\zeta(\alpha) \Rightarrow \dot\zeta(\beta) < \xi]$, and $d_n, h_\alpha(d_n), b_n^\beta$, are based on $\{x_\nu : \nu \in {}^{\omega>}\xi\}$ (for $n < \omega$) and c_n^β (for $n < \omega$), y_0, y_1, y_2, y_3 are based on $J = \{x_\nu : \nu \in {}^{\omega>}\lambda,\ \eta_\alpha \restriction k \ntrianglelefteq \nu\}$, where $k < \omega$ also satisfies that $\eta_\alpha(k) > \xi$, $\eta_\alpha \restriction k \notin N^\beta$ (where $\eta_\alpha \in {}^\omega\lambda$ is the one determined by $\overline{s}$).

These are possible because of 3.2 and 1.10(2)(e).

We claim:

(*): there is $m < \omega$ such that $b^* = (b_m^\beta \cap (y_1 \cup y_2)) - \bigcup_{n\leq k} d_n \neq 0$.

For suppose (*) fails, then as $a_\alpha[\bar{s}] \cap (\bigcup_{n\leq k} d_n) \in \mathbf{B}_\alpha$; without loss of generality

$$(y_1 \cup y_2) \cap \bigcup_{n\leq k} d_n = 0$$

[Why? otherwise let

$$y_0' = y_0 \cup (y_1 \cap a_\alpha[\overline{s}] \cap \bigcup_{n\leq k} d_n) \cup (y_2 \cap (\bigcup_{n\leq k} d_n - a_\alpha[\overline{s}]))$$
$$y_1' = y_1 - \bigcup_{n\leq k} d_n,$$
$$y_2' = y_2 - \bigcup_{n\leq k} d_n \quad .$$

So for every $m < \omega$, $b_m^\beta \cap (y_1' \cup y_2') = 0$. We should now check that the demands on k in $\boxdot_{\xi,k}$ are still satisfied (for ξ there is no change)].

Thus, if x realizes p_β then so does y_0, but $y_0 \in \mathbf{B}_\alpha$ contradicting the induction hypothesis. So (*) holds.

Now as $\langle d_n : n < \omega\rangle$ is a maximal antichain in $\mathbf{B}_\alpha$, for some $\ell < \omega$,

$$d_\ell \cap b^* = d_\ell \cap (b_m^\beta \cap ((y_1 \cup y_2) - \bigcup_{n\leq k} d_n)) \neq 0.$$

Necessarily $\ell > k$. So for some $i \in \{1,2\}$ we have $d_\ell \cap b^* \cap y_i \neq 0$. As x realizes p_β, necessarily $x \cap (d_\ell \cap b^\beta_m \cap y_i) = d_\ell \cap c^\beta_n \cap y_i$, which is based on J. But we know that $x \cap (d_\ell \cap b^\beta_m \cap y_i)$ is

$$\begin{array}{ll} d_\ell \cap b^\beta_m \cap y_1 \cap a_\alpha[\overline{s}] = d_\ell \cap b^\beta_m \cap y_1 \cap s_\ell & (\text{if } i=1) \\ \text{or } d_\ell \cap b^\beta_m \cap y_2 \cap (1 - a_\alpha([\overline{s}]) = d_\ell \cap b^\beta_m \cap y_2 \cap (1 - s_\ell) & (\text{if } i=2). \end{array}$$

As $d_\ell \cap b^\beta_m \cap y_i \neq 0$ is based on J, $\ell > k$, $\eta_\alpha(k) > \xi$, clearly s_ℓ is free over $\{x_\nu : \nu \in J\}$ (see Fact 3.3(1)). As $d_\ell \cap b^\beta_m \cap y_i \geq d_\ell \cap b^* \cap y_i > 0$ and $s_n \notin 0, 1$ necessarily $x \cap (d_\ell \cap b^\beta_m \cap y_i)$ is not based on J, contradiction.

Case II: $\beta < \alpha < \beta + 2^{\aleph_0}$.
We shall prove that if η^i, $\bar{s}^i$ are appropriate (for $i = 1, 2$) and $\eta^1 \neq \eta^2$ then p_β cannot be realized in both $\langle \mathbf{B}_\alpha, a_\alpha[\bar{s}^i]\rangle_{\mathbf{B}^c_0}$. (So as $\beta < \alpha < \beta + 2^{\aleph_0}$, there are less than $2^{\aleph_0}$ non-appropriate pairs $(\eta^i, \bar{s}^i)$).

So toward contradiction, for $i = 1, 2$, let $x^i \in \langle \mathbf{B}_\alpha, a_\alpha[\bar{s}^i]\rangle_{\mathbf{B}^c_0}$ realize p_β. Clearly there is a partition $\langle y^i_\ell : \ell < 4\rangle$ of 1 (in $\mathbf{B}_\alpha$) such that

$$x^i = y^i_0 \cup (y^i_1 \cap a_\alpha[\overline{s}^i]) \cup (y^i_2 - a_\alpha[\overline{s}^i]).$$

Choose $\xi < \dot{\zeta}(\alpha)$ large enough and finite $k < \omega$ such that

(i): $d_n, h_\alpha(d_n), b^\beta_n$ (for $n < \omega$) are based on $\{x_\eta : \eta \in {}^{\omega >}\xi\}$,

(ii): y^i_ℓ (for $i = 1, 2$ and $\ell < 4$) and c^β_n (for $n < \omega$) are based on

$$J = \{x_\nu : \nu \in {}^{\omega >}\lambda \;\&\; \eta^1{\restriction}k \ntrianglelefteq \nu \;\&\; \eta^2{\restriction}k \ntrianglelefteq \nu\},$$

(iii): $\eta^1(k) > \xi$, $\eta^2(k) > \xi$ and $\eta^1 \restriction k \neq \eta^2 \restriction k$.

We claim that

$(*)$**:** there is $m < \omega$ such that $0 < b^* =: b^\beta_m - (y^1_0 \cup y^1_3 \cup y^2_0 \cup y^2_3) - \bigcup\limits_{n \leq k} d_n$.

[Why? Otherwise $a^i =: a_\alpha[\bar{s}^i] \cap (y^i_0 \cup y^i_3 \cup \bigcup\limits_{n\leq k} d_n)$ belongs to $\mathbf{B}_\alpha$ for $i = 1, 2$ and $a^1 \cup a^2$ realizes p_β, a contradiction.]

Clearly b^* is based on J.

As $\langle d_n : n < \omega\rangle$ is a maximal antichain in $\mathbf{B}^c_0$ (and hence in $\mathbf{B}_\alpha$), for some $\ell < \omega$ we have $0 < d_\ell \cap b^*$; clearly $\ell > k$. So for some $j(1), j(2) \in \{1, 2\}$ we have $0 < b^{**} =: d_\ell \cap b^* \cap y^1_{j(1)} \cap y^2_{j(2)}$ (just recall $y^i_1 \cup y^i_2 = 1 - (y^i_0 \cup y^i_3)$). Clearly also b^{**} is based on J and $b^{**} \leq d_\ell \cap b^* \leq d_\ell \cap b^\beta_m$ by the choice of b^{**}, b^* respectively. So by the last two sentences, as x^i realizes p_β, clearly $x^i \cap (d_\ell \cap b^\beta_m) = d_\ell \cap c^\beta_m$, but the latter does not depend on i. Hence $x^1 \cap b^{**} = x^2 \cap b^{**}$. But as $\overline{a}_\alpha[\overline{s}^i] = \bigcup\limits_n (d_n \cap s^i_n)$ we know that $x^i \cap d_\ell$ is $d_\ell \cap s^i_2$ if $j(i) = 1$ and is $d_\ell - s^i_\ell$ if $j(i) = 2$. We can conclude that

$$\text{either} \quad b^{**} \cap s^1_\ell = b^{**} \cap s^2_\ell \quad \text{or} \quad b^{**} \cap s^1_\ell = b^{**} \cap (-s^2_\ell)$$

(the other 2 possibilities are reduced to those). But b^{**} is based on J whereas $\text{supp}(s^1_\ell)$, $\text{supp}(s^2_\ell)$ are disjoint subsets of $\{x_\eta : \eta \in {}^{\omega >}\lambda\} \setminus J$ and $0 < s^i_\ell < 1$, a contradiction.

Case III: $\beta = \alpha$.
This case is splitted into two sub-cases. Let η_α be any ω-branch of f^α such that $\rho^* \triangleleft \eta_\alpha$, so necessarily $\eta_\alpha \neq \eta_\beta$ whenever $\beta < \alpha$. The proof splits to cases according

which $\ell \in \{1,2\}$ is such that $\boxtimes_\ell$ of 3.12(2) is satisfied by $\rho^*,\bar{d}$. Choose $\rho^*_n \in N^\alpha \cap ({}^{\omega>}\lambda)$ such that

$$\ell g(\rho^*_n) = \ell g(\rho^*) + n + 1, \quad \rho^*_n \restriction (\ell g(\rho^*_n) - 1) \lhd \eta_\alpha \quad \text{and} \quad \rho^*_n \ntriangleleft \eta_\alpha.$$

Sub-case 1: $\boxtimes_1$ holds.
We choose $s_n \in \mathbf{B}[N^\alpha]$, satisfying $s_n \in \langle x_\nu : \rho^*_n \lhd \nu \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0}$, $s_n \neq 0, 1$ and

$$\boxplus \quad n = 2m \Rightarrow h(s_n) = 0, \qquad n = 2m+1 \Rightarrow h(s_n) = 1$$

this means $x \cap h(d_{2n} \cup d_{2n+1}) = h(d_{2n})$ (this is possible by $\boxtimes_1$ applied to ρ^*_n and using s_n or $1 - s_n$).

Assume toward contradiction that $x \in \langle \mathbf{B}_\alpha \cup a_\alpha[\bar{s}]\rangle_{\mathbf{B}^c_0}$ satisfies $x \cap h(d_n) = h(d_n \cap s_n)$ hence $x \cap h(d_{2n} \cup d_{2n+1}) = h(d_{2n})$ for $n < \omega$. Let $\langle y_\ell : \ell < 4\rangle$ be a partition of 1 in $\mathbf{B}_\alpha$ such that $x = y_0 \cup (y_1 \cap a_\alpha[\bar{s}]) \cup (y_2 - a_\alpha[\bar{s}])$. As the type $q = \{x' \cap h(d_{2n} \cup d_{2n+1}) = h(d_{2n}) : n < \omega\}$ is not realized in $\mathbf{B}_\alpha$, and $\langle y_0, y_1, y_2, y_3\rangle$ is a partition of 1 in $\mathbf{B}_\alpha$ clearly for some $i < 4$ the type

$$q_i = \{x' \cap h(d_{2n} \cup d_{2n+1}) \cap y_i = h(d_{2n+1}) \cap y_i : n < \omega\}$$

Let $k(*) < \omega$ be such that $\{\eta : \eta_\alpha \restriction k(*) \trianglelefteq \eta \in {}^{\omega>}\lambda\}$ is disjoint to $\text{supp}(y_i)$ for $i < 4$ and $\xi < \eta_\alpha(k(*) - 1)$ is not realized in $\mathbf{B}_\alpha$. By the choice of the y_ℓ's and the choice of x, necessarily $i \in \{1,2\}$ and for notational simplicity let $i = 1$. So $\mathcal{U} =: \{n : h(d_{2n} \cup h_{2n+1}) \cap y_1 \cdot > 0\}$ is infinite, and as $\langle d_k : k < \omega\rangle$ is a maximal antichain of $\mathbf{B}^c_0$ hence of $\mathbf{B}_\alpha$, clearly for each $n \in \mathcal{U}$, the set

$$\mathcal{U}_n = \{k : h(d_{2n} \cup d_{2n+1}) \cap d_k \cap y_1 > 0\}$$

is nonempty. Clearly $k \in U_n \Rightarrow x \cap h(d_{2n} \cup d_{2n+1}) \cap d_k \cap y_1 = s_k \cap h(d_{2n} \cup d_{2n+1}) \cap d_k \cap y_i$ hence $k(*) \leq k \in U_n \Rightarrow \text{supp}(s_k) \subseteq \text{supp}((x \cap h(d_{2n}) \cup d_{2n+1}) \cap y_i)$. Hence if $n \in \mathcal{U}$ and $\mathcal{U}_n$ is infinite then $x \cap h(d_{2n} \cup d_{2n+1}) \cap y_1$ is not in $\mathbf{B}_\alpha$, easy contradiction as $h(d_{2n}) \in \mathbf{B}_\alpha$; so $n \in \mathcal{U} \Rightarrow 1 \leq |\mathcal{U}_n| < \aleph_0$. If $\cup\{\mathcal{U}_m : n < \omega\}$ is finite then $d^* = \cup\{d_\ell : \ell \in \mathcal{U}_n$ for some $n < \omega$ (so $n \in \mathcal{U}$)$\}$ belong to $\mathbf{B}_\alpha$ (as a finite union of members) and $n < \omega \Rightarrow h(d_{2n} \cup d_{2n+1}) \leq d^*$ and $x \cap d^* \in \mathbf{B}_\alpha$ so q_i is realized and we get easy contradiction. Let $f : \mathcal{U} \to \omega, f(n) = \max(\mathcal{U}_n)$. Recall that $k(*) < \omega$ and $\xi < \dot{\zeta}(\alpha)$ are large enough. For $n \in \mathcal{U}$ with $f(n) \geq k(*)$, clearly.

$$x \cap (h(d_{2n} \cup d_{2n+1}) \cap y_i \cap d_{f(n)} \in \{h(d_{2n} \cup d_{2n+1}) \cap y_i \cap s_{f(n)},$$
$$h(d_{2n} \cup d_{2n+1}) \cap y_i \cap (-s_{f(n)})\}$$

by the choice of the $\bar{a}_\alpha[\bar{s}]$'s and

$$x \cap h(d_{2n} \cup d_{2n+1}) \cap y_i = h(d_{2n}) \cap y_i$$

by the choice of x. But the latter, $h(d_{2n}) \cap y_i$ is supported by $\{x_\nu : \rho^*_n \ntriangleleft \nu \in {}^{\omega>}\lambda\}$ (as the ρ^*_m's are pairwise $\lhd$–incomparable), whereas the former is not by the choice of ξ, and $k(*)$.

Sub-case 2: $\boxtimes_2$ holds.
We choose $s_n \in \langle x_\nu : \rho^*_n \lhd \nu \in {}^{\omega>}\lambda\rangle_{\mathbf{B}^c_0}$, $s_n \in \mathbf{B}[N^\alpha] \setminus \{0_{\mathbf{B}_\alpha}, 1_{\mathbf{B}_\alpha}\}$. Now for $i \in \{1,2,3,4\}$ we let $\bar{s}^i = \langle s^i_n : n < \omega\rangle$ be defined as follows: s^i_n is s_n if $n + i$ is even and $-s^i_n$ if $n + i$ is odd. If for some i the Boolean algebra $\langle \mathbf{B}_\alpha \cup \bar{a}_\alpha[\bar{s}^i]\rangle_{\mathbf{B}^c_0}$ omit the type $p^i_\alpha = \{x \cap h(d_n) \cap d_0 = h(s^i_n) \cap h(d_n) \cap d_0 : n < \omega\}$ then we are done, so assume that $x^i \in \langle \mathbf{B}_i \cup \{a_\alpha[\bar{s}^i]\}\rangle_{\mathbf{B}^c_0}$ realizes p^i_α, hence $y_i =: x^i \cap d_0$ realizes p^i_α and belong

to $\mathbf{B}_\alpha$. But then y_1 realizes $\{x \cap h(d_{2n} \cup d_{2n+1}) \cap d_0 = h(d_{2n+1}) \cap d_0 : n < \omega\}$ but this contradict the choice of $\langle d_n : n < \omega\rangle$, see $\boxtimes_2$ of 3.12, so we are done.

So we finish the proof of 3.11; so $\mathbf{B}_{\alpha^*}$ is endo-rigid. $\blacksquare_{3.11}$

{3.12AT} **Claim 3.13.** *There are $\lambda^{\aleph_0}$ ordinal $\alpha < \alpha^*$ which belongs to Y' (even to Y_2).*

Proof: Let h be the identity on $\mathbf{B}_{\alpha^*}$. In the proof of 3.11, guessing the good candidate α we have $\lambda^{\aleph_0}$ possible choices as $Z \subseteq {}^{\omega>}\lambda$ was arbitrary and we could use $Z = \{\eta \restriction n : n < \omega\}$ for any $\eta \in {}^\omega\lambda$. We can find a maximal antichain $\langle d_n : n < \omega\rangle$ of $\mathbf{B}_0$ included in $\langle\{x_{\langle\gamma\rangle} : \gamma < \lambda$ moreover $<\gamma> \in N_0^\alpha\}\rangle_{\mathbf{B}_0}$. For any $\eta \in \lim(f^\alpha)$ and $\bar{s} = \langle s_n : n < \omega\rangle$ where $s_n \in \langle\{x_\rho : (\eta\restriction n)^\frown\langle\eta(n)+1\rangle \triangleleft \rho\}\rangle_{\mathbf{B}_0^c}$ we can define $\bar{a}_\alpha[\bar{s}] = \cup\{d_n \cap s_n : n < \omega\} \in \mathbf{B}_0^c$. As in the proof of 3.11 for some such $(\eta, \bar{s})$ the fitting $\eta_\alpha = \eta$, $a_\alpha = \bar{a}_\alpha[\bar{s}]$, all the demands for $\alpha \in Y_2$ (see $\otimes$ in Stage B) are satisfied.

{3.12} **Lemma 3.14.** *$\mathbf{B}_{\alpha^*}$ is indecomposable.*

Proof: Suppose $\mathbf{J}_0, \mathbf{J}_1$ are disjoint ideals of $\mathbf{B}_{\alpha^*}$, each with no maximal member, which generate a maximal ideal of $\mathbf{B}_{\alpha^*}$. For $\ell = 0, 1$ let $\{d_n^\ell : \ell < \omega\}$ be a maximal antichain $\subseteq \mathbf{J}_\ell \setminus \{0_{\mathbf{B}_{\alpha^*}}\}$ (maximal as subset of $\mathscr{J}_\ell \setminus \{0_{\mathbf{B}_{\alpha^*}}\}$), they are countable as $\mathbf{B}_{\alpha^*}$ satisfies the c.c.c., and may be chosen infinite as $\ell < 2 \Rightarrow \mathbf{J}_\ell \neq \{0\}$ (and $\mathbf{B}_{\alpha^*}$ is atomless). Let $\mathscr{J}$ be the ideal generated by $\mathbf{J}_0 \cup \mathbf{J}_1$.

Now, for example for some $\xi < \lambda$, $\{d_n^\ell : \ell < 2,\ n < \omega\} \subseteq \mathbf{B}_{[\xi]}$, so easily for some $\alpha \in Y_2$, $\dot\zeta(\alpha) > \xi$. Clearly $a_\alpha \in \mathbf{J}$ or $1 - a_\alpha \in \mathbf{J}$. For notational simplicity assume $a_\alpha \in \mathbf{J}$. So $a_\alpha = b^0 \cup b^1$, $b^\ell \in \mathbf{J}_\ell$. Now, $\mathrm{pr}_\xi(b^\ell) \in \mathbf{B}_{[\xi]}$ and is disjoint to each $d_n^{1-\ell}$ so by the maximality of $\{d_n^{1-\ell} :\ n < \omega\}$, $\mathrm{pr}_\xi(b^\ell)$ is disjoint to every member of $\mathbf{J}_{1-\ell}$. As $\mathbf{J}_0 \cup \mathbf{J}_1$ generates a maximal ideal of $\mathbf{B}_{\alpha^*}$, clearly $\mathrm{pr}_\xi(b^\ell) \in \mathbf{J}_\ell$ [otherwise $\mathrm{pr}_\xi(b^\ell) = 1 - c^0 \cup c^1$, for some $c^0 \in \mathbf{J}_1$, $c^1 \in \mathbf{J}_1$, and then $c^{1-\ell}$ is necessarily a maximal member of $\mathbf{J}_{1-\ell}$, so $\mathbf{J}_{1-\ell}$ is principal, a contradiction]. So $\mathrm{pr}_\xi(b^0) \cup \mathrm{pr}_\xi(b^1) < 1$, but $1 = \mathrm{pr}_\xi(a_\alpha) = \bigcup_{\ell=0}^{2} \mathrm{pr}_\xi(b^\ell)$, a contradiction. $\blacksquare_{3.14}$

$\blacksquare_{3.1}$

Of course,

{3.13} **Claim 3.15.** *If $\mathrm{cf}(\lambda) > \aleph_0$ __then__ there are $\mathbf{B}_i$ for $i < 2^{\lambda^{\aleph_0}}$ such that*

(a) *$\mathbf{B}_i$ is Boolean algebra of cardinality $\lambda^{\aleph_0}$, density character λ, and this holds even for $\mathbf{B}_i \restriction a$ for $a \in \mathbf{B}_i \setminus \{0_{\mathbf{B}_i}\}$,*
(b) *$\mathbf{B}_i$ is endo-rigid indecomposable,*
(c) *any homomorphism from any $\mathbf{B}_i$ to $\mathbf{B}_j$ $(i \neq j)$ has a finite range.*

Proof: We can repeat the proof of 3.1. Now we build $\mathbf{B}_\alpha\langle Z\rangle$ for every $Z \subseteq {}^\omega\lambda$, such that for each α we try to guess not $\mathbf{B}_{\alpha^*}$ and an endomorphism of it but we try to guess $\mathbf{B}^1[N^\alpha] = (\mathbf{B}_\alpha\langle Z_1\rangle)\restriction N^\alpha$, $\mathbf{B}^2[N^\alpha] = (\mathbf{B}_\alpha\langle Z_2\rangle)\restriction N^\alpha$ and $h = H^{N^\alpha}$ an homomorphism from $\mathbf{B}^1[N^\alpha]$ into $\mathbf{B}^2[N^\alpha]$, and we "kill" i.e., guarantee h cannot be extended to a homomorphism from $\mathbf{B}\langle Z^1\rangle$, to $\mathbf{B}\langle Z^2\rangle$ when $\mathbf{B}\langle Z^\ell\rangle\restriction N^\alpha = \mathbf{B}^\ell[N^\alpha]$

$\blacksquare_{3.15}$

{3.14} **Claim 3.16.** *In 3.1, 3.15 we can replace the assumption $\mathrm{cf}(\lambda) > \aleph_0$ by $\lambda > \aleph_0$.*

Proof: We replace ${}^{\omega>}\lambda, S$ by ${}^{\omega>}(\lambda \times \omega_1), \{\lambda \times \delta : \delta < \omega_1$ a limits ordinal$\}$ so we use [Sh:309, 3.17] instead of [Sh:309, 3.11, 3.16]. $\blacksquare_{3.16}$

Also note:

{3.15}

Observation 3.17. *Assume* $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$, *and* $\mathbf{B}$ *is c.c.c. Boolean algebra of cardinality* λ, *and there is* $\mu, \mu < \lambda < \mu^{\aleph_0}$, *hence without loss of generality* $\lambda > \mu = \min\{\mu : \mu^{\aleph_0} \geq \lambda\}$.

1. *There is a free Boolean algebra* $\mathbf{B}_x$ *of cardinality* μ *such that* $\mathbf{B}_0 \subseteq \mathbf{B}$.
2. *There is* $\bar{\mathbf{B}}$ *such that*
   - (a) $\bar{\mathbf{B}} = \langle \mathbf{B}_n : n < \omega \rangle$,
   - (b) $\mathbf{B}_n$ *is a Boolean subalgebra of* $\mathbf{B}$,
   - (c) $\mathbf{B}_n \subseteq \mathbf{B}_{n+1}$ *and* $\mathbf{B} = \bigcup_{n<\omega} \mathbf{B}_n$,
   - (d) *there is* $A_n \subseteq \mathbf{B}_{n+1}$ *independent over* $\mathbf{B}_n$[2] *of cardinality* μ.
3. $\mathbf{B}$ *is not endo-rigid.*
4. *There are projections*[3] *of* $\mathbf{B}$ *whose range are atomless countable Boolean algebra.*
5. *there are* $\lambda^{\aleph_0}$ *atomless Boolean subalgebras* $\mathbf{B}'$ *of* $\mathbf{B}$ *such that there is a projection from* $\mathbf{B}$ *onto* $\mathbf{B}'$

Proof: By cardinal arithmetic $(\forall \kappa < \mu)(\kappa^{\aleph_0} < \mu)$ and $\mathrm{cf}(\mu) = \aleph_0$. Let $\mu = \sum_{n<\omega} \mu_n$, $\mu_n < \mu_{n+1}$, $\mu_n^{\aleph_0} = \mu_n$.

(1), (2) By [Sh:92, Lemma 4.9, p.88], we can find $\langle b_\alpha : \alpha < \mu \rangle$ independent in $\mathbf{B}$. Let $\mathbf{B}_*$ be the subalgebra of $\mathbf{B}$ generated by $\{b_\alpha : \alpha < \mu\}$, and let $\mathbf{B}_*^c$ be the completion of $\mathbf{B}_*$. Let h^* be a homomorphism from $\mathbf{B}$ into $\mathbf{B}_*^c$ extending $\mathrm{id}_{\mathbf{B}_*}$, (it is well known that such homomorphism exists) and let $\mathbf{B}' = \mathrm{Rang}(h^*)$, so $\mathbf{B}_* \subseteq \mathbf{B}' \subseteq \mathbf{B}_*^c$, $|\mathbf{B}_*| \leq |\mathbf{B}'| \leq \lambda$. For each $a \in \mathbf{B}'$ there is a countable $u_a \subseteq \mu$ such that a is based on (i.e., belongs to the completion inside $\mathbf{B}_*^c$ of) the set $\{b_\alpha : \alpha \in u_a\}$.

We can find pairwise distinct $\eta_\alpha \in \prod_{n<\omega} \mu_n^+$ for $\alpha < \lambda^{\aleph_0}$ such that $\eta_\alpha \restriction (n+1) \neq \eta_\beta \restriction (n+1) \Rightarrow \eta_\alpha(n) + \mu_n \neq \eta_\beta(n) + \mu_n$. Now for each $a \in \mathbf{B}'$ the set

$$w_a = \{\alpha : (\exists^\infty n)(u_a \cap [\mu_n \times \eta_\alpha(n), \mu_n \times \eta_\alpha(n) + \mu_n) \neq \emptyset)\}$$

has cardinality $\leq 2^{\aleph_0}$. But $|\mathbf{B}'| + 2^{\aleph_0} \leq \lambda < \lambda^{\aleph_0}$, hence for some $\alpha < \lambda^{\aleph_0}$ we have

$$\eta_\alpha \notin \bigcup \{w_a : a \in \mathbf{B}'\}$$

Let

$$\mathbf{B}'_m =: \quad \{x \in \mathbf{B}' : h^*(x) \text{ is based on } A'_n\}$$

where

$$A'_m =: \{b_\beta : \text{ if } n \geq m \text{ then } \beta \notin [\mu_n \times \eta_\alpha(n), \mu_n \times \eta_\alpha(n) + \mu_n)\}\}.$$

The sequence $\langle \mathbf{B}'_n : n < \omega \rangle$ is as required except that in clause (d) if we naturally let $A''_n = \{b_\beta : \beta \in [\mu_n \times \eta_\alpha(n), \mu_n \times \eta_\alpha(n) + \mu_n)\}$ we get $|A''_n| \geq \mu_n$ instead

[2]i.e., for every $a \in \mathbf{B}_n \setminus \{0_{\mathbf{B}_n}\}$ and a non-trivial Boolean combination b of members of A_n we have $a \cap b > 0$

[3]i.e homomorphism h from $\mathbf{B}$ into $\mathbf{B}$ such that $x \in \mathbf{B} \Rightarrow h(h(x)) = h(x)$

$|A''_n| \geq \mu$. So let ω be the disjoint union of the infinite sets v_n for $n < \omega$, and let $\mathbf{B}_m = \{x \in \mathbf{B} : h^*(x)$ is based on $A_m\}$, where

$$A_m = \{b_\beta : \quad \text{if } n < \omega \text{ and } n \notin \bigcup_{k \leq m} v_k \text{ then } \beta \notin [\mu_n \times \eta_\alpha(n), \mu_n \times \eta_\alpha(n) + \mu_n)\}.$$

Then the sequence $\langle \mathbf{B}_m : m < \omega \rangle$ is as required.

(3) Follows by (4).

(4) Choose $a_n \in A_n$ for $n < \omega$. Now we define by induction on n, a projection h_n from the Boolean algebra $\mathbf{B}_n$ onto the subalgebra $\mathbf{B}^*_n$ of $\mathbf{B}_n$ generated by $\{a_\ell : \ell < n\}$ freely and extending h_m for $m < n$. For $n = 0$, let D_0 be any ultrafilter of $\mathbf{B}_0$ and let $h_0(x)$ be $1_{\mathbf{B}_0} = 1_{\mathbf{B}}$ if $x \in D_0$ and $0_{\mathbf{B}_0} = 0_{\mathbf{B}}$ if $x \in \mathbf{B}_0 \setminus D_0$. For $n = m+1$ let $\langle a^m_k : k < 2^m \rangle$ list the atoms of $\mathbf{B}^*_m$, which is a finite Boolean algebra, and for $k < 2^m$ let $D^m_k = \{x \in \mathbf{B}_m : a^m_k \subseteq h(x) \in \mathbf{B}^*_m\}$, this is an ultrafilter of $\mathbf{B}_m$. For each k we can find two ultrafilters $D^m_{k,0}, D^m_{k,1}$ of $\mathbf{B}_n = \mathbf{B}_{m+1}$ extending D^m_k such that $a_m \in D^m_{k,1}$ and $a_m \notin D^m_{k,0}$. Lastly define $h_n = h_{m+1} : \mathbf{B}_n \to \mathbf{B}^*_n$ by $h_n(x) = \bigcup\{a^m_k \cap a_m : x \in D^m_{k,1}\} \cup \bigcup\{a^m_k - a_m : x \in D^m_{k,0}\}$, it is easy to check that h_n is a homomorphism from $\mathbf{B}_n$ onto $\mathbf{B}^*_n$ and is the identity on $\mathbf{B}^*_n$ and extend h_m.

Clearly $h = \cup\{h_n : n < \omega\}$ is a projection of $\mathbf{B} = \cup\{\mathbf{B}_n : n < \omega\}$ onto $\mathbf{B}^* = \cup\{\mathbf{B}^*_n : n < \omega\}$, so $h, \mathbf{B}^*$ are as required.

(5) By the proof of part (4), that is the arbitrary choice of $\langle a_n : n < \omega \rangle \in \prod\limits_{n<\omega} A_n$.

$\blacksquare_{3.17}$

{3.16AT}

Discussion 3.18. (1) *In 3.17 the only use of the c.c.c. is to find a free subalgebra of* $\mathbf{B}$ *of cardinality* μ.

(2) *What about* $|\mathbf{B}| < 2^{\aleph_0}$*? S.Koppelberg and the author noted (independently) that under* MA *(or just* $\mathfrak{p} = 2^{\aleph_0}$*) such Boolean algebras are not endo-rigid. Why? let* $a_n \in \mathbf{B} \setminus \{0_{\mathbf{B}}\}$ *be pairwise disjoint, let* D_n *be an ultrafilter of* $\mathbf{B}$ *to which* a_n *belong, and for* $x \in \mathbf{B}$ *let* $\mathcal{U}_x = \{n : x \in D_n\}$*. By* MA *there is an infinite* $\mathcal{U} \subseteq \omega$ *such that for every* $x \in \mathcal{U}$ *the set* $\mathcal{U} \cap \mathcal{U}_x$ *is finite or the set* $\mathcal{U} \setminus \mathcal{U}_x$ *is finite. Let* $h \in \text{Ext}(\mathbf{B})$ *be* $h(x) = \cup\{a_n : n \in \mathcal{U}_x\}$ *if* $\mathcal{U} \cap \mathcal{U}_x$ *is finite and* $h(x) = 1_{\mathbf{B}} - \cup\{a_n : n \in \mathcal{U} \setminus \mathcal{U}_x\}$ *if* $\mathcal{U} \setminus \mathcal{U}_x$ *is finite.*

(3) *Assume* $\mu = \sum\{\mu_n : n < \omega\}$, $\mu_n^\kappa = \mu_n < \mu_{n+1}$*. If* $\mathbf{B}$ *is a Boolean algebra satisfying the* κ^+*- c.c. such that* $\mu < |\mathbf{B}| < \mu^{\aleph_0}$ *then the construction of 3.17 holds. The proof is similar.*

{4.20}

Discussion 3.19. *We may wonder whether Claims 3.9, 3.10 can be improved to: if* $d_n \in \mathbf{B}_{\alpha^*}$ *(for* $n < \omega$*) are pairwise disjoint non–zero, then for some* $w \subseteq \omega$ *there is no* $x \in \mathbf{B}_{\alpha^*}$ *satisfying*

$$[n \in w \;\Rightarrow\; x \cap d_n = d_n] \quad \textit{and} \quad [n \in \omega \setminus w \;\Rightarrow\; x \cap d_n = 0].$$

The problem is that $\{\eta_\alpha : \alpha < \alpha^*\} \subseteq {}^\omega\lambda$ *may contain a perfect set and if we are not careful about the* s^α_n*-s mentioned above we may fail. If* $\lambda = \mu^+$, $\mu^{\aleph_0} = \mu$*, then we may try to demand, for each* $\zeta^* < \lambda$*, that*

$$\langle \bigcup_{n<\omega} \text{supp}(s^\alpha_n) : \alpha < \alpha^*, \zeta(\alpha) = \zeta^* \rangle$$

is a sequence of pairwise disjoint sets. Alternatively we may look for a thinner black box (of course, preferably more then just no perfect set of η_α*'s), see* [Sh:309, §3].

References

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