BUILDING COMPLICATED INDEX MODELS AND BOOLEAN ALGEBRAS

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Abstract. We build models using an indiscernible model sub-structures of ${}^{\kappa \geq}\lambda$ and related more complicated structures. We use this to build various Boolean algebras.

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Printed version of this exists since the early nineties and somewhat circulated. This was supposed to be Ch.VII of the book "Non-structure" and probably will be if the book will materialize. In the reference like [Sh:E62, 1.16=L1.15] we have 1.16 is the number of the claim and 1.15 its label so intended just to help the author correct it if the numbers will be changed. The author thanks Alice Leonhardt for the beautiful typing. Pub.511.

§ 0. Introduction

We begin with an example that motivates our need to pass beyond the framework of trees with $\omega+1$ levels. Suppose that we are asked to construct a rigid Boolean algebra of cardinality λ. We can take a sequence $\langle I_\alpha : \alpha < \lambda\rangle$ exemplifying that $K^\omega_{\rm tr}$ has the full $(\lambda,\lambda,\aleph_0,\aleph_0)$–bigness property and build a Boolean algebra $BA(I_\alpha)$ for each α. We then construct a rigid Boolean algebra $\mathbf{B}_\lambda$ by choosing an increasing, continuous sequence $\langle \mathbf{B}_\alpha : \alpha \le \lambda\rangle$, where $\mathbf{B}_0$ is trivial and $\mathbf{B}_{\alpha+1}$ is obtained from $\mathbf{B}_\alpha$ by "planting" a copy of $BA(I_\alpha)$ below $a_\alpha \in \mathbf{B}_\alpha$, and our bookkeeping will ensure that $\mathbf{B}_\lambda \setminus \{0\} = \{a_\alpha : \alpha < \lambda\}$. This seems to be a reasonable strategy, and it works, see a little more below. Now, suppose that we are moreover asked to construct a complete Boolean algebra $\mathbf{B}$ of cardinality λ with no non-trivial one-to-one endomorphism. We should assume that $\lambda^{\aleph_0} = \lambda$ (as the cardinality of any complete Boolean algebra satisfies this equality) and it is natural to demand in addition that $\mathbf{B}$ satisfies the ccc. It is not hard to modify the construction above so that $\mathbf{B}_\lambda$ has the ccc, so let $\mathbf{B}$ be its completion. Assume toward a contradiction that $f : \mathbf{B} \longrightarrow \mathbf{B}$ is a non-trivial, one-to-one endomorphism. We can find $a \in \mathbf{B}\setminus\{0\}$ with $a \cap f(a) = 0$ and $\alpha < \lambda$ such that $a = a_\alpha$. Then I_α is embedded in some sense in $\mathbf{B} \restriction a_\alpha$, say by $\eta \mapsto a^\alpha_\eta$. Hence $\eta \mapsto f(a^\alpha_\eta)$ is a similar embedding into $\mathbf{B} \restriction f(a)$ that is constructed from $\langle I_\beta : \beta \ne \alpha\rangle$ alone. It seems reasonable that the "I_α strongly unembeddable into $\sum\{I_\beta : \beta \ne \alpha\}$" in the sense of Definition [Sh:E59, 2.5=L2.3] to deduce a contradiction; this works in the case above. However in the present case $f(a^\alpha_\eta)$ is not in general a member of $\mathbf{B}_\lambda$, but rather is a countable union $\bigcup\limits_{n<\omega} b^\alpha_{\eta,n}$ of members of $\mathbf{B}_\lambda$. We would like to find an appropriate unembeddability condition of I_α into $\sum\limits_{\beta\ne\alpha} I_\beta$ to handle this complication. At some price, our original notion can be modified to handle this complication when η has finite length, but not when η has length ω. Instead, in this latter case, we replace it by an "approximation" $b^\alpha_{\eta,n(\alpha,\eta)} > 0$; this was part of the motivation of having the "strongly finitary on P^I_ω" Definition [Sh:E59, 2.5=L2.3]. Previously, we could use demands like "$a^\alpha_{\eta\restriction\ell} \ge a^\alpha_\eta$" but now we have to use demands like $a^\alpha_\nu \cap a^\alpha_\eta = 0$, $\ell g(\eta) = \omega$, $\ell g(\nu) < \omega$, but such demands tend to contradict the ccc.

Our solution is to replace subtrees of ${}^{\omega\ge}\lambda$ by index sets I of the form

$$I = I' \cup \{(\eta\restriction n)\hat{\ }\langle\alpha_\ell\rangle : n < \omega,\ \eta \in I',\ \eta(n) = (\alpha_0,\alpha_1) \text{ and } \ell \in \{0,1\}\},$$

where $I' \subseteq {}^\omega\{(\alpha_0,\alpha_1) : \alpha_0 < \alpha_1 < \lambda\}$, and choose $BA(I)$ to be generated by $\{a^I_\eta : \eta \in I\}$ freely except that

$$\eta \in I' \ \wedge\ \eta(n) = (\alpha_0,\alpha_1) \quad \Rightarrow \quad a^I_{\eta\restriction n\hat{\ }\langle\alpha_0\rangle} - a^I_{\eta\restriction n\hat{\ }\langle\alpha_1\rangle} \ge a^I_\eta.$$

(Actually, to ensure the ccc it is better to use a more complicated variant.) But now, the bigness properties have to be proved in this context. For other aims, we use subtrees of ${}^{\omega\ge}2$ of cardinality $\kappa \in [\aleph_1, 2^{\aleph_0})$, originally to deal with number of non-isomorphic models.

In this work we deal with more complicated index sets as motivated above.

In §1 we introduce classes like $K^\omega_{{\rm tr}(n)}$, which are close to being trees with $\omega+1$ levels, together with bigness properties (related to $\psi_{{\rm tr}(n)}$) for them. We prove some existence theorems of the form "for many λ there is a sequence $\langle I_\alpha : \alpha < \lambda\rangle$, where each $I_\alpha \in K^\omega_{\mathrm{tr}(n)}$ has cardinality λ and is strongly $\psi_{\mathrm{tr}(n)}$–unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$." We also define "super" versions of these bigness properties related to the ones in [Sh:331, 1.1=L7.1,1.5=L7.3].

In §2 we construct Boolean algebras with few appropriate morphisms for several versions.

In §3 we construct a ccc Boolean algebra of cardinality $2^{\aleph_0}$ of pregiven length (see Definition 3.3) such that any infinite homomorphic image has cardinality $2^{\aleph_0}$. We {5.1}
use a Boolean algebra constructed from a single $I \in K^\omega_{\mathrm{tr}(n)}$ as in §2. As it happens, the complicated $I \in K^\omega_{\mathrm{tr}(h)}$ are not needed, just non trivial ones. Our point is that $K^\omega_{\mathrm{tr}(h)}$ is not good just for the constructions in §2, it is a quite versatile way to build structures with preassumed properties (not to speak of varying the index model).

The main result is (3.6): {5.4}

- (∗) for $\mu \in [\aleph_0, 2^{\aleph_0})$, there is a ccc Boolean algebra **B** with length μ (see Definition 3.2 below) and every infinite homomorphic image of **B** being of {5.0A}
cardinality $2^{\aleph_0}$.

If $\mathrm{cf}(\mu) > \aleph_0$ we can demand the length is not obtained (see Definition 3.2), if {5.0A}
$\mathrm{cf}(\mu) = \aleph_0$ this is impossible.
Also we can replace $\aleph_0$ here by any strong limit cardinal κ of cofinality $\aleph_0$ (see {5.8}
3.14).

In §4 we deal with tree of the form $S \cup {}^{\omega>}2$, where $S \subseteq {}^\omega 2$ of cardinality λ.

Note that §1, §2 are revised versions of parts of [Sh:136] and parallel to [Sh:331] and §4 is a revised version of parts of [Sh:262]. The results in §2 answer problems of Monk (presented in Oberwolfach 1980).

In §3, We solve a problem of Boolean algebras of Monk on which the author earlier give a consistency result, using ideas from §2.

§4 supersedes [Sh:a, VIII 1.8] and repeats [Sh:262, 1.2, 1.3]. Baldwin [Bal89] has continued [Sh:262, 1.2-1.3]. We can apply this to models of $\varphi \in \mathbb{L}_{\aleph_1,\aleph_0}$, probably using [Sh:522].

Recall that in [Sh:a, Ch.VIII,1.8+1.7(2)], we proved that for pairs of first order complete theories (T, T_1) satisfying the hypothesis of Theorem 4.1 below {3.1}

$$\dot{\mathbb{I}}(\lambda, T_1, T) \geq \min\{2^\lambda, \beth_2\}.$$

We shall improve the result replacing $\dot{\mathbb{I}}(\lambda, T_1, T)$ by $\dot{I}\dot{E}(\lambda, T_1, T)$. We improve the proof from [Sh:a, VIII 1.8], in particular we use the trees U_η defined in Fact 4.9. {3.3B}
They are subtrees of ${}^{\omega>}2$ as disjoint as we can.

We can use trees similar to $({}^{\omega\geq}2, \triangleleft)$ with finite or countable levels and heavier structure (i.e., like pure conditions in forcing notions as in [Sh:326, §2]). As in 1.4(3), we use here a weak form of representation: the amount of similarity depends on the terms and formulas.

We can use such trees as in §2 to build "complicated", rigid like structures. In [Sh:98, claim 1.2, claim 1.1(3)] (more [Sh:105, 1.4, 1.1]) this was done for abelian groups, one step: getting $\mathbb{Z} \subseteq G$ such that G is $\aleph_1$–free of cardinality $\aleph_1$, $\mathbb{Z}$ not a direct summand of G). This was continued in Göbel and Shelah [GbSh:519] and Göbel-Shelah-Ströngmann [GShS:785].

We hope in subsequent work to get 1.11(1) for every $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ (hence {2.7}{1.14}
2.17(2) too).

{x2}

Definition 0.1. 1) We say a structure M is atomically $(<\mu)$-stable <u>when</u>: if $A \subseteq M, |A| < \mu$ then $\{\mathrm{tp}_{qf}(\bar{a}, A, M)| : \bar{a} \in {}^{\omega>}M\}$ has cardinality $< \mu$.
2) We may write μ instead of $< \mu^+$.

§ 1. Trees with structure

{par1}

We deal here with "relatives" of $K^\omega_{\rm tr}$ which are more complicated, strengthening our ability to construct and still the existence proofs work at least partially.

In this section (and the next) we define and see what we can do for $K^\omega_{\rm ptr}$, $\varphi_{\rm ptr}$, $K^\omega_{{\rm tr}(n)}$, $\varphi_{{\rm tr}(n)}$, $K^\omega_{{\rm tr}(*)}$, $\varphi_{{\rm tr}(*)}$ (which were introduced in [Sh:136]) getting the parallel of [Sh:331, 2.15=L7.11]. The reason for their introduction was for constructing some Boolean algebras; we shall deal with these constructions later.

{1.1}

Definition 1.1. 1) $K^\kappa_{\rm ptr}$ is the class of I such that:

(a) the set of elements of I is, for some linear order J, a subset of

$$\{\eta : \ \eta \text{ is a sequence of length } \le \kappa, \text{ such that: } \text{if } i+1 < \lg(\eta),\ \eta(i) \text{ has the form } \langle s,t\rangle,\ s <_J t, \text{ and } \text{if } i+1 = \lg(\eta),\ \eta(i) \in J\}.$$

Also if $\eta \in I$, $i+1 < \lg(\eta)$, $\eta(i) = \langle s,t\rangle$ then $(\eta \restriction i){}^\frown\langle s\rangle \in I$ and $(\eta \restriction i){}^\frown\langle t\rangle \in I$; and also the empty sequence belongs to I and if $\delta < \lg(\eta)$, δ is a limit ordinal then $\eta \restriction \delta \in I$.

(b) The relations of I are:

(α) $\eta \trianglelefteq \nu$ meaning η is an initial segment of ν, i.e. $\eta = \nu{\restriction}\lg\eta$

(β) $P_i = \{\eta : \lg(\eta) = i\}$,

(γ) $<_1 = \{\langle \eta,\nu\rangle : \lg(\eta) = \lg(\nu) = i+1, \eta(i) <_J \nu(i), \eta \restriction i = \nu \restriction i\}$,

(δ) $Eq_i = \{\langle\eta,\nu\rangle : \eta \restriction i = \nu \restriction i\}$ and

(ε) ${\rm Suc}_L = \{\langle\eta,\nu\rangle : \eta \restriction i = \nu \restriction i, i+1 = \lg(\eta) < \lg(\nu),\ \nu(i) = \langle s,t\rangle \text{ and } \eta(i) = s \text{ for some } i<\kappa \text{ and } s <_J t\}$,

(ζ) ${\rm Suc}_R =] \{\langle\eta,\nu\rangle : \eta \restriction i = \nu \restriction i,\ i+1 = \lg(\eta) < \lg(\nu),\ \nu(i) = \langle s,t\rangle \text{ and } \eta(i) = t \text{ for some } i<\kappa \text{ and } s <_J t\}$,

(η) an individual constant $\langle\rangle$,

(θ) functions ${\rm Res}^L_\alpha(\eta) = (\eta\restriction\alpha){}^\frown\langle s\rangle, {\rm Res}^R_\alpha(\eta) = (\eta\restriction\alpha){}^\frown\langle t\rangle$, when $\eta(\alpha) = \langle s,t\rangle$, $\alpha+1 < \lg(\eta)$ and ${\rm Res}^L_\alpha(\eta) = {\rm Res}^R_\alpha(\eta) = \eta$ otherwise.

2) Let

$$\psi_{\rm ptr}(x_0,x_1;y_0,y_1) = \bigvee_{i+1<\kappa}\big[P_{i+1}(x_1)\ \&\ P_{i+1}(y_1)\ \&\ P_\kappa(x_0)\ \&\ x_0 = y_0\ \&\ {\rm Suc}_L(x_1,x_0)\ \&\ {\rm Suc}_R(y_1,y_0)\ \&\ x_1 <_0 y_0\big].$$

This depends on κ but we usually suppress this parameter.

3) $I \in K^\kappa_{\rm ptr}$ is standard <u>iff</u> in (a) of (1), J is a set of ordinals with the natural order, or at least a well ordering (usually we shall use those).

{1.2}

Definition 1.2. 1) For $h : \kappa \longrightarrow \omega \setminus \{0\}$, the class $K^\kappa_{{\rm tr}(h)}$ is defined like $K^\kappa_{\rm ptr}$, but replacing pairs by increasing $h(i)$–tuples at level i, that is:

(a) the set of elements of I is, for some linear order J, a subset of

$$\big\{\eta : \quad \eta \text{ is a sequence of length } \le \kappa, \text{ for } i+1 < \lg(\eta),\ \eta(i) \text{ has the form } \langle s_0, \ldots, s_{h(i)-1}\rangle \text{ such that } s_0 <_J s_1 <_J \ldots <_J s_{h(i)-1} \text{ and for } i+1 = \lg(\eta),\ \eta(i) \in J\big\}.$$

Also if $\eta \in I$, $i+1 < \lg(\eta)$, $m < h(i)$ and $\eta(i) = \langle s_0, \ldots, s_{h(i)-1}\rangle$ then $(\eta \restriction i)\char`\^\langle s_m\rangle \in I$; and also if $\delta < \lg(\eta)$, δ a limit ordinal then $\eta \restriction \delta \in I$, and lastly the empty sequence belongs to I.

(b) The relations of I are:

(α) $\eta \trianglelefteq \nu$ which holds iff $\eta = \nu \restriction \lg(\eta)$,

(β) $P_i := \{\eta : \lg(\eta) = i\}$,

(γ) $<_1 := \{\langle \eta, \nu\rangle : \lg(\eta) = \lg(\nu) = i+1, \eta(i) <_J \nu(i), \eta \restriction i = \nu \restriction i\}$,

(δ) $Eq_i = \{\langle \eta, \nu\rangle : \eta \restriction i = \nu \restriction i\}$,

(ε) for $m < h(i)$, $i < \kappa$:

$$\mathrm{Suc}_{i,m} = \{\langle \eta, \nu\rangle : \quad \eta \restriction i = \nu \restriction i,\ i+1 = \lg(\eta), \nu(i) = \langle s_0, \ldots, s_{h(i)-1}\rangle, \eta(i) = s_m\}$$

(ζ) an individual constant $\langle\rangle$,

(θ) functions $\mathrm{Res}^m_\alpha(\eta) = (\eta \restriction \alpha)\char`\^\langle s_m\rangle$, when

$$\eta(\alpha) = \langle s_0, \ldots, s_{h(\alpha)-1}\rangle,\ \alpha < \lg(\eta) \text{ and } m < h(\alpha).$$

If $n \ge h(\lg(\eta(\alpha)))$ or $\lg(\eta) = \alpha + 1 \& \eta(\alpha) = s_0$ we stipulate

$$\mathrm{Res}^n_\alpha(\eta) = (\eta \restriction \alpha)\char`\^\langle s_0\rangle.$$

2) $\psi_{\mathrm{tr}(h)}(\bar{x}; \bar{y})$ where $\bar{x} = (x_0, x_1), \bar{y} = (y_0, y_1)$ is[1]

$$x_0 = y_0 \ \ \& \ \ P_\kappa(y_0) \ \ \& \bigvee_{i<\kappa} (P_{i+1}(x_1) \& P_{i+1}(y_1) \& x_1 <_1 y_1 \& \mathrm{Suc}_{i,0}(x_0, x_1) \& \mathrm{Suc}_{i,h(i),1}(y_1, y_0)$$

3) We define $\psi'_{\mathrm{tr}(h)}$ as follows: $\bigvee\limits_{i<\kappa} \big(x_0 = y_0 \& \ P_\kappa(y_0) \& \bigwedge\limits_{\ell=1}^{h(i)-1} [P_{i+1}(x_\ell) \& \ P_{i+1}(y_\ell) \& \ \mathrm{Res}^{\ell-1}_i(x_0) = x_\ell \& \ \mathrm{Res}^\ell_i(y_0) = y_\ell]\big)$ so if $\alpha = \sup(\mathrm{Rang}(h)$ then $\bar{x} = \langle x_\ell : \ell < 1+\alpha\rangle, \bar{y} = \langle y_\ell : \ell < 1+\alpha\rangle)$.

4) If $\bigwedge\limits_{i<\kappa} h(i) = n$ we may write $K^\kappa_{\mathrm{tr}(n)}$, so for $n = 2$ we get K^κ_{ptr} up to some renaming.

If $\bigwedge\limits_{i<\kappa} h(i) = i \mod \omega$ we may write $K^\kappa_{\mathrm{tr}(*)}$. We say "$I \in K^\kappa_{\mathrm{tr}(h)}$ is standard" in case the underlying set J is well ordered (usually a set of ordinals). Writing $\eta(\alpha)(\ell)$ if $\lg(\eta) = \alpha + 1$, this means $\eta(\alpha)$ (of course it $\alpha + 1 < \lg(\eta)$ this means $\mathrm{Res}^\ell_\alpha(\eta)$).

[1] below the intention is: $y_0 \restriction i = x_\ell \restriction i \ \& \ y_0(i) = \langle x_0(i), \ldots, x_{h(i)-1}(n)\rangle$

{1.3}
Remark 1.3. Dealing with $K^\omega_{\rm ptr}$ (or $K^\omega_{{\rm tr}(n)}$, $K^\omega_{{\rm tr}(*)}$, $K^\omega_{{\rm tr}(h)}$, those are parallel cases) here, we below introduce "super*" version, parallel to Definitions [Sh:331, 1.1=L7.1], [Sh:331, 1.4=L7.2] (and see more versions in [Sh:331, 1.5=L7.3,1.6=L7.3A], so the easy [Sh:331, 1.6=L7.5] has to be redone, hence claim [Sh:331, 1.8(2)=L7.5(2)] is no longer of any help and we should prove a parallel. The role of $\bar{\mathbf e}$ here correspond in the role $\psi_{\rm tr}$ in [Sh:E59, §2], [Sh:331, §1].

Definition 1.4. Let $h : \omega \longrightarrow \omega \setminus \{0\}$, and $\bar{\mathbf e}$ be a function with domain ω, $\bar{\mathbf e}(n)$ an equivalence relation on $\mathscr P(h(n))$ satisfying: {1.4}

$$u_1 \bar{\mathbf e}(n) u_2 \quad \Rightarrow \quad |u_1| = |u_2|.$$

For this definition we identify a set (of natural numbers or ordinals) with an increasing sequence enumerating it. Defining $\bar{\mathbf e}$ we may ignore classes which are singleton; see clause (5) on defualt values.
1) For $I \in K^\omega_{{\rm tr}(h)}$, $J \in K^\omega_{{\rm rtr}(h')}$ and cardinals μ, κ we say I is (μ,κ)-super–$\bar{\mathbf e}$–unembeddable into J (for $K^\omega_{\rm tr}(h)$) <u>when</u>:

$(*)_{I,J,\mu,\kappa,\bar{\mathbf e}}$ for every large enough regular cardinal χ and $x \in \mathscr H(\chi)$ and a fixed well ordering $<^*_\chi$ of the set $\mathscr H(\chi)$ and $f_1 : I \to {}^{\kappa>}J$, there are $\langle M_n, N_n : n < \omega\rangle$ such that:

- (i) $M_n \prec N_n \prec M_{n+1} \prec (\mathscr H(\chi), \in, <^*_\chi)$,
- (ii) $M_n \cap \mu = N_n \cap \mu$ and $\kappa \subseteq M_0$,
- (iii) I, J, μ, κ, h, x belong to M_0,
- (iv) there is $\eta \in P^I_\omega$ such that for every n, $\eta \restriction n \in M_n$ and for n large enough for $\ell < h(n)$ we have ${\rm Res}^\ell_n(\eta) \in N_n \setminus M_n$ and they realize the same Dedekind cut by $<^I_1$ on

$$\{\nu : \nu \in I \cap M_n \text{ and } \nu, {\rm Res}^0_n(\eta) \text{ are } <^I_1 \text{–comparable}\},$$

this is equivalent to: ${\rm Res}^0_n(\eta)$, ${\rm Res}^1_n(\eta), \ldots, {\rm Res}^{h(n)-1}_n(\eta)$ realize the same Dedekind cut on

$$\{(\eta \restriction n)\hat{\ }\langle s\rangle \in I : s \in M_n\}$$

(recall that $<^I_1$ linearly orders $\{(\eta{\restriction}n)\hat{\ }\langle s\rangle : s\} \cap I$) and moreover: if $h(n) > 1$ and $u_1 \bar{\mathbf e}(n) u_2$ then

(α) if $\ell_1 \in u_1$ & $\ell_2 \in u_2$ & $|u_1 \cap \ell_1| = |u_2 \cap \ell_2|$ then $f_1({\rm Res}^{\ell_1}_n(\eta)), f_1({\rm Res}^{\ell_2}_n(\eta))$ has the same length

(β) the sequences $\nu_{\eta,n,u_1}, \nu_{\eta,n,u_2} \in {}^{\kappa>}J$ realizes the same atomic type over $J \cap M_n$ in J where for $u \subseteq h(n)$ we let $\nu_{\eta,n,u}$ be the concatanation of the sequences $f_1({\rm Res}^\ell_n(\eta))$ for $\ell \in u$.

- (v) for every $\nu \in P^J_\omega$

$$\Big(\bigcup_{n<\omega} M_n\Big) \cap \bigcup\{{\rm Res}^\ell_n(\nu) : \ell < h(n), n < \omega\}$$

is included in some M_m

2) For $I, J \in K^\omega_{\mathrm{tr}(h)}$ and cardinals μ, κ we say[2] that I is (μ, κ) -super- $\bar{\mathbf{e}}$ - unembeddable$^-$ into J (for $K^\omega_{\mathrm{tr}(h)}$) <u>when</u>:

$(*)'_{I,J,\mu,\aleph_0}$ for every large enough χ and $x \in \mathscr{H}(\chi)$ and a fix well ordering $<^*_\chi$ if $\mathscr{H}(\chi)$ there is M such that

- (i) $M \prec (\mathscr{H}(\chi), \in, <^*_\chi)$
- (ii) $x \in M$
- (iii) M is countable
- (iv) there is $\eta \in P^I_\omega$ such that $m < \omega \& \ell < h(n) \Rightarrow \mathrm{Res}^\ell_n(\eta) \in M$ and for every function $f \in M$ from ${}^{\omega >}I$ to μ for infinitely many n we have:
  $\circledast$ if $\ell'_0 < \ldots < \ell'_{k-1} < h(n)$, and $\ell''_0 < \ldots < \ell''_{k-1} < h(n)$ and $\{\ell'_0, \ldots, \ell'_{k-1}\} \bar{\mathbf{e}}(n) \{\ell''_0, \ldots, \ell''_{k-1}\}$ <u>then</u> $f(\langle \mathrm{Res}^{\ell'_i}_n(\eta) : i < k\rangle) = f(\langle \mathrm{Res}^{\ell''_i}_n(\eta) : i < k\rangle)$
- (v) if $\nu \in P^J_\omega$ then $\nu \in M$ or for some $k < \omega, \nu \restriction k \in M, \nu \restriction (k+1) \notin M$.

3) Let $\bar{\mathbf{e}}_0$ be defined by $\bar{\mathbf{e}}_0(n) = \{(\{\ell\}, \{k\}) : \ell, k < h(n)\}$, let $\bar{\mathbf{e}}_1$ be defined by $\{(\{0, \ldots, h(n)-2\}, \{1, \ldots, h(n)-1\})$. Let $\bar{\mathbf{e}}_2$ be defined by $\bar{\mathbf{e}}_2(n) = \{(\{0, \ldots \lfloor h(n)/2 \rfloor - 1\}, \{\lfloor h(n)/2 \rfloor, \ldots, 2\lfloor h(n)/2 \rfloor - 1\})\}$. If $\bar{\mathbf{e}} = \bar{\mathbf{e}}_0$ we may omit it.
4) For $\langle I_\xi : \xi \in w\rangle$, w a set of ordinals, $I_\xi \in K^\omega_{\mathrm{tr}(h)}$, standard for simplicity, letting $\zeta(*) = \sup(w \cup \{\eta(0)(\ell) : \eta \in \cup\{I_\xi : \xi \in w\}) + 1$ we define $\sum\limits_{\xi \in w} I_\xi \in K^\omega_{\mathrm{rtr}(h)}$ as $\{\langle\rangle\} \cup \{\langle \xi \rangle \underset{\zeta(*)}{\otimes} d\eta : \xi \in w \text{ and } \eta \in I_\xi\}$ on $\otimes$ see below.

{1.4D}

Remark 1.5. 1) We can define this also for trees with more than ω levels (as in Def
{1.2} 1.1, 1.2) but feel we have enough parameters anyhow.
2) Recall $\xi \underset{\zeta(*)}{\otimes} \eta$ is $\langle\rangle$ if $\eta = \langle\rangle$, and is $\langle \zeta(*) \times \xi + \eta(0), \eta(1), \eta(2), \ldots\rangle$ otherwise.

{1.5}

Definition 1.6. 1) $K^\omega_{\mathrm{tr}(h)}$ has the $(\chi, \lambda, \mu, \kappa)$–super $\bar{\mathbf{e}}$-bigness property <u>when</u> there are standard $I_\zeta \in K^\omega_{\mathrm{tr}(h)}$ for $\zeta < \chi, |I_\zeta| = \lambda$ such that I_ζ is (μ, κ)–super–$\bar{\mathbf{e}}$–unembeddable into I_ϵ for each $\zeta \neq \epsilon < \chi$.
2) $K^\omega_{\mathrm{tr}(h)}$ has the full $(\chi, \lambda, \mu, \kappa)$-super $\bar{\mathbf{e}}$–bigness property <u>when</u> there are standard $I_\zeta \in K^\omega_{\mathrm{tr}(h)}$ for $\zeta < \chi$, $|I_\zeta| = \lambda$ such that I_ζ is (μ, κ)–super–$\bar{\mathbf{e}}$–unembeddable into $J_\zeta = \sum\limits_{\epsilon < \chi, \epsilon \neq \zeta} I_\epsilon$ for each $\zeta < \chi$.
3) We write above super^x <u>when</u> $x = \mathrm{nr}$ and use the above definition, or $x = \mathrm{vr}$
{1.4} and we replace unembeddable by unembeddable$'$, i.e. (in Definition 1.4) we replace $(*)_{I_\zeta, J_\zeta, \mu, \kappa}$ by $(*)'_{I_\zeta, J_\zeta, \mu, \kappa, \theta}$. We replace λ by $\bar{\lambda} = (\lambda_0, \lambda_1)$ if $|I_\xi| = \lambda$ is replaced by $|\{\eta \in I_\xi : \lg(\eta) < \omega\}| = \lambda_1$, $|I_\xi| = \lambda_0$.

Remark 1.7. Also K^ω_{tr} can be brought into the framework above as a specific case, i.e., h is constantly 1.

{1.6A}

Claim 1.8 (Monotonicity)**.** *For every given h we have:*
1) If $K^\omega_{\mathrm{tr}(h)}$ has the [full] $(\chi_1, \lambda_1, \mu, \kappa)$–super $\bar{\mathbf{e}}$–bigness properties, $\chi_2 \leq \chi_1$ and $\lambda_2 \geq \lambda_1$, <u>then</u> $K^\omega_{\mathrm{tr}(h)}$ has the [full] $(\chi_2, \lambda_2, \mu, \kappa, \theta)$–super $\bar{\mathbf{e}}$–bigness property; similarly for super.

[2]this is helpful in constructing Boolean algebras as in §2 in more cardinals without using Definition 1.4, even $\psi'_{\mathrm{rtr}(h)}$; but this is the minor version and the reader can ignore it. {1.4}

2) If $K^\omega_{\mathrm{tr}(h)}$ has the full $(\chi,\lambda,\mu,\kappa)$–super $\bar{\mathbf{e}}$–bigness property and $\chi_1 = \min\{\chi,\lambda\}$ <u>then</u> $K^\omega_{\mathrm{tr}(h)}$ has the $(2^{\chi_1},\lambda,\mu,\kappa)$-super $\bar{\mathbf{e}}$-bigness. Similarly for super.

Proof. 1) Straightforward.
2) Similar to [Sh:331, 1.8(2)=L7.5].

If $\langle I_\alpha : \alpha < \chi\rangle$ exemplifies "$K^\omega_{\mathrm{tr}(h)}$ has the full $(\chi,\lambda,\mu,\kappa,\theta)$–super^x $\bar{\mathbf{e}}$–bigness property", $\chi_1 = \min\{\chi,\lambda\}$ and $h(0) = n(*)$, then we let for $A \subseteq \chi_1$, $J_A = \sum\{I_\alpha : \alpha \in A\}$, see Definition 1.4(4). {1.4}

Let $\langle A_\alpha : \alpha < 2^{\chi_1}\rangle$ be such that $A_\alpha \subseteq \lambda$, and $\alpha \neq \beta \Rightarrow A_\alpha \not\subseteq A_\beta$. Now $\langle J_{A_\alpha} : \alpha < 2^{\chi_1}\rangle$ exemplifies "$K^\omega_{\mathrm{tr}(h)}$ has the $(2^{\chi_1},\lambda,\mu,\kappa,\theta)$-super^x $\bar{\mathbf{e}}$-bigness property". $\square_{1.8}$

On the [full] strong $(\chi,\lambda,\mu,\kappa)$–bigness property (and strongly finitary) see [Sh:E59, 2.5=L2.3]; by 1.9 below for $\psi_{tr(h)}$ from Definition 1.2(2) it is a consequence of the super version and as in [Sh:E59], [Sh:331] it is useful. {1.2} {1.6B}

{1.6B}

Claim 1.9. *If $K^\omega_{\mathrm{tr}(h)}$ has the [full] $(\chi,\lambda,\mu^{<\kappa},2^{<\kappa})$–super–bigness property and $\chi \le \lambda$ <u>then</u> $K^\omega_{\mathrm{tr}(h)}$ has the [full] strong $(\chi,\lambda,\mu,\kappa)$–bigness property for $\psi_{\mathrm{rtr}(h)}$ for functions f which are strongly finitary on P_ω.*

Proof. The result follows by the definitions and 1.10 below. $\square_{1.9}$ {1.6C} {1.6C}

Claim 1.10. *If $(*)_{I,J,\mu_1,\kappa_1}$ (where $\mu_1 = \mu^{<\kappa}$, $\kappa_1 = 2^{>\kappa}$, $\{I,\ J\} \subseteq K^\omega_{\mathrm{tr}(h)}$ are standard, $h \in {}^\omega\omega$; see Definition 1.4(1),(3), <u>then</u> I is strongly $(\mu,\kappa,\psi_{\mathrm{tr}(h)})$–unembeddable into J for embeddings strongly finitary on P^I_ω.* {1.4}

Proof. Recalling 1.4(3) we have $\bar{\mathbf{e}} = \bar{\mathbf{e}}_0$. Without loss of generality I, J are subsets of ${}^{\omega\ge}(\bigcup\limits_{n<\omega}({}^n\theta))$ for some cardinal θ, and let $<^*$ be a well ordering of $\mathscr{M}_{\mu,\kappa}[J]$ (respecting being a subterm). Suppose f is a function from I into $\mathscr{M}_{\mu,\kappa}(J)$, so for $\eta \in I$, {1.4}

$$f(\eta) = \sigma_\eta(\nu_{\eta,0},\dots,\nu_{\eta,i},\dots)_{i<\alpha_\eta}$$

for some term σ_η, ordinal $\alpha_\eta < \kappa$, $\nu_{\eta,i} \in J$ and f is strongly finitary on P_ω, i.e.,

$$\eta \in P^I_\omega \quad \Rightarrow \quad \alpha_\eta < \omega \ \&\ [\ \sigma_\eta \text{ has finitely many subterms }].$$

Let χ be regular large enough, and define for $\eta \in P^I_\omega$,

$$g(\eta) = \{\alpha : \text{ the } \alpha\text{-th element by } <^* \text{ is a subterm of } f(\eta)\}$$

(so we use "the strongly finitary" only so that $g(\eta)$ is finite).

Let $\langle M_n, N_n : n < \omega\rangle$ be as in the conclusion of Definition 1.4(1); and let $\eta \in P^I_\omega$ be as in clause (iv) of Definition 1.4(1). Let $m = \alpha_\eta$ and $\nu_\ell =: \nu_{\eta,\ell} \in J$. Apply clause (v) of Definition 1.4(1) to each ν_ℓ. For $\ell < m$ define $k_\ell = \min\{k \le \omega :$ if $k < \omega \ \vee\ \lg(\nu_\ell) > k$ then $\nu_\ell \restriction (k+1) \notin \bigcup\limits_{n<\omega} M_n\}$. If $k_\ell = \omega$ by clause (v) for some $n(\ell) < \omega$ we have $\{\nu_\ell \restriction k : k < \omega\} \subseteq M_{n(\ell)}$. If $k_\ell < \omega$ clearly for some $n(\ell) < \omega$ we have: {1.4} {1.4} {1.4}

(∗) $\nu_\ell \restriction k_\ell \in M_{n(\ell)}$ and $\lg(\nu_\ell) > k_\ell \Rightarrow \nu_\ell{\restriction}(k_\ell+1) \notin \bigcup\limits_{n<\omega} M_n$ and if $\nu_\ell(k_\ell) = \langle \alpha_{\ell,k_\ell,i} : i < h(k_\ell)\rangle$, and $i < h(k_\ell), \alpha_{\ell,k_\ell,i} \notin M_{n(\ell)}$ then:

(*i*) $\alpha_{\ell,k_\ell,i} \notin \bigcup_{n<\omega} M_n$ hence $n<\omega \Rightarrow \alpha_{\ell,k_\ell,i} \notin N_n$ and

(*ii*) $z_\ell =: \min_{<^I_1}\{y \in \bigcup_{n<\omega} M_n : (\nu \restriction k_\ell)\hat{}\langle y\rangle \in J$ and $(\nu{\restriction}k_\ell)\hat{}\langle \alpha_{\ell,k_\ell,i}\rangle <^I_1 y\}$ belongs to $M_{n(\ell)}$ (we can arrange that there are such y's or allow ∞ as a value).

Let $n_* = \omega$ be such that $n_* \geq \max\{n(0),\dots,n(m-1)\} < \omega$, and

$$\lg(\nu_\ell) < \omega \quad \Rightarrow \quad \bigcup\{\text{Rang}(\nu_\ell(k)) : k < \lg(\nu_\ell)\} \cap \bigcup_{k<\omega} M_k \subseteq M_{n(*)}.$$

Let $y_\ell = \eta$ (for $\ell < \omega$) and $x_\ell = (\eta{\restriction}n_*)\hat{}\langle\alpha_\ell\rangle$ for $\ell < h(n_*)$ where $\eta(n_*) = \langle \alpha_\ell : \ell < h(n)\rangle$ (and $x_{h(n_*)+\ell} = x_0$) and the rest should be clear. $\square_{1.10}$

{1.7} **Lemma 1.11.** *1) $K^\omega_{\text{ptr}}, K^\omega_{\text{tr}(h)}$ when $h \in {}^\omega(\omega\backslash\{0,1\}$ have the full $(\lambda,\lambda,\mu,\kappa)$–super–bigness property* *when:*

$\oplus_0$ *λ regular, $\lambda > \mu \geq \kappa, \lambda > \mu^\kappa$ and $(\forall\theta)[\theta < \lambda \Rightarrow \theta^{\aleph_0} < \lambda]$.*

{x2} *2) We can get the above bigness to be exemplified by atomically $\aleph_0$–stable standard I_ζ's. (See Definition 1.1(1) on standard, and Definition 0.1 on atomically stable which is characterized below).*
3) $K^\omega_{\text{tr}(h)}$ has the full $(\lambda,\lambda,\mu,\kappa)$–super–bigness property when:

($\oplus_1$) *λ regular, $\lambda > \mu \geq \kappa$, $\lambda^{\aleph_0} = \lambda$.*

4) Above we can deduce that $K^\omega_{\text{tr}(h)}$ has the full $(\lambda,\lambda,\mu,\kappa)-\psi_{\text{tr}(h)}-$ bigness property.

Proof. Similar to [Sh:331, §1]. $\square_{1.11}$

{1.8} **Claim 1.12.** *1) Let $I \in K^\omega_{\text{tr}(h)}$ then I is atomically μ–stable iff*

(*a*) *for $n<\omega, \eta \in P^I_{n+1}$ the linear order $(\{\nu \in P^I_{n+1} : \nu \restriction n = \eta \restriction n\}, <^I_1)$ is atomically μ–stable (i.e., for every subset of cardinality $\leq \mu$ only $\leq \mu$ many Dedekind cuts are realized),*

(*b*) *for any $I' \subseteq I$, $|I'| \leq \mu$ the set $\{\eta \in P^I_\omega : n < \omega \;\&\; \ell < h(n) \Rightarrow \text{Res}^\ell_n(\eta) \in I'\}$ has cardinality $\leq \mu$.*

2) For $\mu = \text{cf}(\mu) > \aleph_0$, "[standard] atomically $(<\mu)$–stable" is characterized similarly (for $\mu = \chi^+$, this means "atomically χ–stable").
3) If $I \in K^\omega_{\text{tr}(h)}$ is standard, $\mu = \text{cf}(\mu)$, $[\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu]$ then I is atomically $(<\mu)$–stable.
4) The family of "atomically $(<\mu)$–stable $I \in K^\omega_{\text{tr}(h)}$" is closed under well ordered sums.

Proof. 1) Let $J \subseteq I$ be of cardinality $\leq \mu$. Without loss of generality

$\boxtimes_1$ $\eta \in J \;\&\; n < \lg(\eta) \;\&\; \ell < h(n) \Rightarrow \text{Res}^\ell_n(\eta) = (\eta \restriction n)\hat{}\langle(\eta(n))(\ell)\rangle \in J$,

$\boxtimes_2$ if $\eta \in P^I_\omega$ and $(\forall \ell, n)[\ell < h(n),\ n < \omega \Rightarrow \text{Res}^\ell_n(\eta) \in J]$, then $\eta \in J$ (see clause (b) of the assumption).

Let $J' = \{\eta \restriction \ell : \eta \in J, \ell < \lg(\eta)\}$, and for $\nu \in J'$ let $J^*_\nu = \{\eta : \eta \in I,\ \eta \notin J,\ \lg(\eta) \geq \lg(\nu)+1 \text{ and } \nu \lhd \eta\}$.

So

(∗) $\langle J^*_\nu : \nu \in J'\rangle$ is a partition of $I \setminus J$.

For $\eta \in I \setminus J$ let $k(\eta) = \max\{k : \eta \restriction k \in J\}$, it is well defined (and $< \omega$) by $\boxtimes_2$ above and clearly $\eta \in J^*_{\eta \restriction k(\eta)}$.

We now observe:

⊗ if $n < \omega$, $\bar\eta' = \langle \eta'_\ell : \ell < n\rangle$, $\bar\eta'' = \langle \eta''_\ell : \ell < n\rangle$, and $\eta'_\ell, \eta''_\ell \in I$, then a sufficient condition for $\mathrm{tp}_{\mathrm{qf}}(\bar\eta', J, I) = \mathrm{tp}_{\mathrm{qf}}(\bar\eta'', J, I)$ is:

(∗) (a) *quad* if $\eta'_\ell \in J$ or $\eta''_\ell \in J$ <u>then</u> $\eta'_\ell = \eta''_\ell$,

(b) $\lg(\eta'_\ell) = \lg(\eta''_\ell)$,

(c) if $\eta'_\ell \notin J$ (equivalently $\eta''_\ell \notin J$) <u>then</u> $k(\eta'_\ell) = k(\eta''_\ell)$ call it k_ℓ and $\eta'_\ell \restriction k_\ell = \eta''_\ell \restriction k_\ell$,

(d) for $\ell_1, \ell_2 < n < \omega$ and $k < \omega$ we have

(α) $\eta'_{\ell_1} \restriction k = \eta'_{\ell_2} \restriction k \Leftrightarrow \eta''_{\ell_1} \restriction k = \eta''_{\ell_2} \restriction k$,

(β) if in (α) both hold and $k < \lg(\eta'_\ell) \& k < \lg(\eta'_{\ell_2})$ and $m_1, m_2 < h(k)$ and for $i = 1,2$ we have

$$k+1 < \lg(\eta'_{\ell_i}) \&\ t'_i = (\eta'_{\ell_i}(k))(m_i) \&\ t''_i = (\eta''_{\ell_i}(k))(m_i)$$

or

$$k+1 = \lg(\eta'_{\ell_i}) \&\ t'_i = \eta'_{\ell_i}(k) \&\ t''_i = \eta''_{\ell_i}(k)$$

<u>then</u>

$$(\eta'_{\ell_1} \restriction k)\char`^\langle t'_1\rangle <^I_1 (\eta'_{\ell_1} \restriction k)\char`^\langle t'_2\rangle \Leftrightarrow (\eta'_{\ell_1} \restriction k)\char`^\langle t''_1\rangle <^I_1 (\eta''_{\ell_1} \restriction k)\char`^\langle t''_2\rangle$$

(e) if $\eta'_\ell \in I^*_\nu, \eta'_\ell \restriction k \in J', \eta'_\ell \restriction (k+1) \notin J'$ (hence similarly for η''_ℓ) and $\nu \lhd \rho \in J$ and $m_1, m_2 < h(\lg \nu)$ and

(α) we have $\bullet_1$ or $\bullet_2$ where

$\bullet_1$ $k+1 < \lg(\eta'_\ell)\ \&\ t' = (\eta'_\ell(k_\ell))(m_1)\ \&\ t'' = (\eta''_\ell(k_\ell))(m_1)$,

$\bullet_2$ $k+1 = \lg(\eta'_\ell)\ \&\ t' = \eta'_\ell(k)\ \&\ t'' = \eta''_\ell(k)$

and

(β) $k+1 < \lg(\rho)\ \&\ s = (\rho(k))(m_2)$ or $k+1 = \lg(\rho)\ \&\ s = \rho(k)$

<u>then</u>

(α) $\nu\char`^\langle s\rangle <^I_1 \nu\char`^\langle t'\rangle \Leftrightarrow \nu\char`^\langle s\rangle <^I_1 \nu\char`^\langle t''\rangle$, and

(β) $s = t' \Leftrightarrow s = t''$.

It is easy to check that this is true. Also (⊗) defines the equivalence relation (equality of q.f.–type in I over J) as various pieces of information being the same, now in all cases we have $\leq \mu$ choices (for clause (e), (f) in (⊗) recall clause (a) in the assumption) we are done.

2) Similarly.

3) Follows as well orders are atomically μ-stable.

4) Straight. $\square_{1.12}$

Claim 1.13. *If $I \in K^{\omega}_{\mathrm{tr}(h)}$ is standard and λ satisfy $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$ then I is $(<\lambda)$-atomically stable.* {1.10}

{1.8} *Proof.* Obvious by 1.12(1) by λ successor and by 1.12(2) for λ a limit cardinal.

§ 2. Applications to Boolean Algebras

{par2}

We here construct some Boolean algebras with "no non-trivial morphism".

We shall use mainly $\mathrm{BA}_{\mathrm{tr}}(I), I \in K^{\omega}_{\mathrm{tr}}$ for constructing mono-rigid ccc Boolean algebra; $\mathrm{BA}_{\mathrm{tr}(h)}(I), I \in K^{\omega}_{\mathrm{rtr}(h)}, h \in {}^{\omega}(\omega \setminus \{0,1,2\})$ for constructing complete mono-rigid ccc Boolean algebras and $\mathrm{BA}_{\mathrm{rtr}r}(I), I \in K^{\omega}_{\mathrm{rtr}}(I)$ for constructing Bonnet Rigid Boolean algebras. In each case for every I from a relevant family (which exemplify full bigness in the relevant case), we derive a Boolean algebra $\mathrm{BA}_x(I)$, chosen to fit the proof of the case of rigidity we are interested in, this is Definition 2.1. We {2.1} then build a Boolean algebra $\mathbf{B}$ of cardinality λ planting a copy of $\mathrm{BA}_x(I_a)$ below enough elements $a \in \mathbf{B}$ such that $a \neq b \Rightarrow I_a \neq I_b$. (see 2.4). We mainly show that {2.4} $\mathrm{BA}_{\mathrm{rtr}(h)}(I)$ satisfies a strong version of the ccc hence the ccc is preserved (see 2.6), {2.6} hence the outcome of the construction 2.4 is as required with respect to the ccc, {2.4} completeness and cardinality. We then observe the relevant weak representability results (see 2.12). Note that if we consider the completion of a ccc Boolean algebra $\mathbf{B}$ {2.10} and $\mathbf{B}$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ then its completion is weakly represented in $\mathscr{M}_{\aleph_1,\aleph_1}(J)$. Next (in 2.14) we deal with deducing unembedability of $\mathrm{BA}_x(I)$ into {2.11} a Boolean algebra $\mathbf{B}$ which is weakly represented in $\mathscr{M}_{\mu,\kappa}(J)$, the main case is part (2). We deduce as conclusions that there are mono-rigid [complete] Boolean algebra (2.16, 2.17). We then deal with Bonnet rigid Boolean algebra (2.18 till the end). {2.18} {2.1}

Definition 2.1. 1) For $I \in K^{\omega}_{\mathrm{tr}}$ let $BA_{\mathrm{tr}}(I)$ be the Boolean algebra generated by x_η(for $\eta \in I$) freely except that

$(*)_1$ $\eta \lhd \nu \in P^I_\omega \Rightarrow\ x_\eta \geq x_\nu$.

2) For $I \in K^{\omega}_{\mathrm{ptr}}$ let $BA_{\mathrm{ptr}}(I)$ be the Boolean algebra generated by x_η (for $\eta \in I$) freely except that for $\eta \in I$, $\lg(\eta) = \omega$ and $n < \omega$, letting $\eta = \langle <\alpha_0, \beta_0>, \ldots, <\alpha_n, \beta_n> \ldots, \rangle$, the following holds:

$(*)_2$ $x_\eta \leq x_{\eta \restriction n \hat{}\langle \alpha_n \rangle}$ and $x_\eta \cap x_{\eta \restriction n \hat{}\langle \beta_n \rangle} = 0$.

3) For $h \in {}^{\omega}(\omega \setminus \{0\})$ and $I \in K^{\omega}_{\mathrm{tr}(h)}$ let $BA_{\mathrm{tr}(h)}(I)$ be the Boolean Algebra generated by x_η (for $\eta \in I$) freely except that for $\eta \in P^I_\omega$ and $n < \omega$, letting $\eta(n) = \langle s_0, \ldots, s_{h(n)-1} \rangle\rangle$ we have:

$(*)_3$ $x_\eta \leq x_{\eta \restriction n \hat{}\langle s_0 \rangle}$ and $x_\eta \cap \bigcap\limits_{\ell=1}^{h(n)-1} x_{\eta \restriction n \hat{}\langle s_\ell \rangle} = 0$.

The second equality is trivial if $h(n) = 1$, so usually $h \in {}^{\omega}(\omega \setminus \{0,1\})$; if $\forall n (h(n) = 1)$ this is like the case of $I \in K^{\omega}_{\mathrm{tr}}$ and if $(\forall n)(h(n) = 2)$ this is like the case of $I \in K^{\omega}_{ptr}$.
4) For $I \in K^{\omega}_{\mathrm{tr}}$ or just I is a set of sequences of ordinals closed under initial segments let $\mathrm{BA}_{\mathrm{trr}}(I)$ be the Boolean algebra generated by x_η (for $\eta \in I$) freely except that:

- (a) $x_{\eta \hat{}\langle \alpha \rangle} \cap x_{\eta \hat{}\langle \beta \rangle} = 0$ for[3] $\alpha \neq \beta$,
- (b) $x_\eta \leq x_\nu$ for $\nu \lhd \eta$,
- (c) if η has finitely many immediate successors, $\{\eta \hat{}\langle \alpha_\ell \rangle : \ell < k_\eta\}$ and $k_\eta \geq 2$ <u>then</u> $x_\eta = \cup \{x_{\eta \hat{}\langle \alpha_\ell \rangle} : \ell < k_\eta\}$

[3] we are, of course, assuming $\eta \hat{}\langle \alpha \rangle, \eta \hat{}\langle \beta \rangle \in I$, similarly in other cases.

(*d*) if $\eta \lhd \nu$, and every ρ satisfying $\eta \unlhd \rho \lhd \nu$ has a unique successor, then $x_\eta = x_\nu$.

5) We define $BA_{\mathrm{tr}(h,g)}(I)$ for $g \in {}^\omega\omega$, $h \in {}^\omega(\omega \setminus \{0,1\})$, $I \in K^\omega_{\mathrm{tr}(h)}$ satisfying $g \le h$, i.e., $(\forall n)(g(n) \le h(n))$, as the Boolean algebra generated by x_η $(\eta \in I)$ freely except that:

$(*)_5$ if $\eta \in I$, $\lg(\eta) = \omega$, $\ell < \omega$, $\eta(\ell) = \langle \alpha_0, \dots, \alpha_{k-1} \rangle$ where $k = h(\ell)$, <u>then</u>

(α) $x_\eta \le \bigcup\limits_{m=0}^{g(\ell)} x_{(\eta \restriction \ell)\hat{}\langle \alpha_m \rangle}$,

(β) if $g(\ell) < h(\ell) - 1$ then $x_\eta \cap \bigcap\limits_{m=g(\ell)+1}^{h(\ell)-1} x_{(\eta \restriction \ell)\hat{}\langle \alpha_m \rangle} = 0$; (if $g(\ell) = g(\ell)$ <u>then</u> this is trivial so usually we assume $g < h$).

6) Assume that $h \in {}^\omega(\omega \setminus \{0,1\}), \bar{\mathbf{e}}$ an ω-sequence with $\bar{\mathbf{e}}(n) = \{\{u_{1,n}, u_{2,n}\}\}$ where $u_{1,n}, u_{2,n}$ are non-empty distinct subsets of $h(n)$ with the same number of elements. For $I \in K^\omega_{\mathrm{tr}(h)}$ we define $\mathrm{BA}_{\mathrm{tr}(h),\bar{\mathbf{e}}}(I)$ as the Boolean algebra generated by $\{x_\eta : \eta \in I\}$ freely except that for $\eta \in P^I_\omega$ and $n < \omega$ letting $\eta(n) = \langle s_0, \dots, s_{h(n)-1} \rangle$ we have

$(*)'_3$ $x_\eta \le \bigcup\limits_{\ell \in u_{1,n}} x_{(\eta \restriction n)\hat{}\langle s_\ell \rangle}$ and $x_\eta \cap \bigcup\limits_{\ell \in u_{2,n}} x_{(\eta \restriction n)\hat{}\langle s_\ell \rangle} = 0$

(We have much freedom in this case).

{2.2} *Notation* 2.2. 1) Let $K^\omega_{\mathrm{tr}(h,g)} = K^\omega_{\mathrm{tr}(h)}$, note that for $I \in K^\omega_{\mathrm{tr}(h)}$, if $g = h$ then $\mathrm{BA}_{\mathrm{tr}(h,g)}(I)$ essentially is $BA_{\mathrm{tr}(h)}(I)$; also if $h = 1$, then $K^\omega_{\mathrm{tr}(h)} = K^\omega_{\mathrm{tr}}$, $BA_{\mathrm{tr}(h)}(I) = BA_{\mathrm{tr}}(I)$.
2) We let x stand for: tr <u>or</u>ptr <u>or</u>trr <u>or</u>tr(h), <u>or</u>tr(h,g) (where h, g are as above).
3) Note that when we say "a Boolean algebra is generated by $X = \{x_i : i \in U\}$ freely except the set equations. . ." we have 0 and 1 in the Boolean algebra.
4) For a Boolean algebra $\mathbf{B}$ and $a \in \mathbf{B}, \mathbf{B}\restriction a$ is naturally defined Boolean algebras but $1_{\mathbf{B}\restriction a} = a$.

{2.3} **Definition 2.3.** For Boolean algebras $\mathbf{B}$, $\mathbf{B}_1$ and $a^* \in \mathbf{B}_1 \setminus \{0_{\mathbf{B}_1}\}$ we define the "$\mathbf{B}$–surgery of $\mathbf{B}_1$ at a^*" or "surgery of $\mathbf{B}_1$ at a^* by $\mathbf{B}$", called $\mathbf{B}_2$, as a Boolean algebra extending $\mathbf{B}_1$, $\mathbf{B}_2 = [\mathbf{B}_1 \restriction (-a^*)] \times [(\mathbf{B}_1 \restriction a^*) * \mathbf{B}]$ where $\times$ is a direct product and $*$ free product. Alternatively $\mathbf{B}_2$ is generated as follows: first make $\mathbf{B}$ disjoint to $\mathbf{B}_1$ (by taking an isomorphic copy) and then $\mathbf{B}_2$ is generated freely by $\mathbf{B}_1 \cup \mathbf{B}$ except the relations

$$\begin{array}{ll} 0_{\mathbf{B}_1} = 0_{\mathbf{B}} = 0, & \\ a \cap b = c & (\text{for } a,b,c \in \mathbf{B}_1 \text{ such that } a \cap b = c \text{ in } \mathbf{B}_1), \\ a \cup b = c & (\text{for } a,b,c \in \mathbf{B}_1 \text{ such that } a \cap b = c \text{ in } \mathbf{B}_1) \end{array}$$

$$\begin{array}{ll} 1_{\mathbf{B}_1} - b = c & (\text{for } b,c \in \mathbf{B}_1 \text{ such that } 1_{\mathbf{B}_1} - b = c \text{ in } \mathbf{B}_1), \\ a \cap b = c & (\text{for } a,b,c \in \mathbf{B} \text{ such that } a \cap b = c \text{ in } \mathbf{B}), \\ a \cup b = c & (\text{for } a,b,c \in \mathbf{B} \text{ such that } a \cup b = c \text{ in } \mathbf{B}), \\ 1_{\mathbf{B}} - b = c & (\text{for } b,c \in \mathbf{B} \text{ such that } 1_{\mathbf{B}} - b = c) \end{array}$$

and

$$1_{\mathbf{B}} = a^*$$

{2.4}

Construction 2.4. Let x be one of $\{\mathrm{tr}, \mathrm{ptr}, \mathrm{tr}(h), \mathrm{trr}, \mathrm{tr}(h,g)\}$, λ a cardinal, $\alpha < \lambda^+$ (usually $\alpha = \lambda$, always $\alpha > 0$). The idea is to construct a Boolean algebra by defining an increasing continuous sequence $\mathbf{B}_i$ $(i \le \alpha)$, $\mathbf{B}_0$ trivial, and we get $\mathbf{B}_{i+1}$ by a surgery of $\mathbf{B}_i$ at $a_i^* \in \mathbf{B}_i$ by $\mathbf{B}_i^* = \mathrm{BA}_x(I_i)$ (see Definition 2.1 and 2.2(2)), {2.2} where $|I_i| = \lambda$, $I_i \in K_x^\omega$ and I_i is strongly ψ_x–unembeddable into $\sum\limits_{j<\alpha, j\ne i} I_j$ (or e.g. $\mathrm{super}^y - \bar{\mathbf{e}}$-unembeddable into it, $y \in \{\mathrm{nr}, \mathrm{vr}\}$).

We denote $\mathbf{B} = \mathbf{B}_\alpha$ by $\mathrm{Sur}_x\langle I_i, a_i^* : i < \alpha\rangle$. Usually we would like to have $\mathbf{B}_\alpha \setminus \{0\} = \{a_i^* : i < \alpha\}$. If there are $\langle I_i : i < \alpha\rangle$ as above and α is divisible by λ then this is clearly possible.

{2.5}

Definition 2.5. 1) A Boolean algebra satisfies the λ–chain condition (or the λ-c.c.) <u>iff</u> there are no λ elements which form an antichain (i.e., they are $\ne 0$, the intersection of any two is zero).
2) A Boolean algebra satisfies the strong λ–chain condition or the λ-Knaster condition <u>iff</u> among any λ elements there are λ which are pairwise not disjoint.

{2.6}

Claim 2.6. *Let $x \in \{\mathrm{tr}, \mathrm{ptr}, \mathrm{tr}(n), \mathrm{tr}(h), \mathrm{tr}(*)\}$, $I \in K_x^\omega$, λ uncountable regular.*
1) If $x = \mathrm{tr}$, <u>then</u> $\mathrm{BA}_x(I)$ satisfies the strong λ–chain condition.
2) If $x = \mathrm{ptr}$, <u>then</u> $\mathrm{BA}_x(I)$ satisfies the strong $\left(2^{\aleph_0}\right)^+$–chain condition.
3) If $x = \mathrm{tr}(k)$, $k \ge 3$, $I \in K_{\mathrm{tr}(k)}^\omega$ is standard, <u>then</u> $\mathbf{B} = \mathrm{BA}_{\mathrm{tr}(k)}(I)$ satisfies the strong λ–chain condition; similarly for $K_{\mathrm{tr}(*)}^\omega$ and for $K_{\mathrm{tr}(h)}^\omega$ for $h \in {}^\omega(\omega \setminus 3)$ and $K_{\mathrm{tr}(h,g)}^\omega$ (for $h \in {}^\omega(\omega \setminus 3)$, $g \in {}^\omega\omega$ such that $g \le h$ & $(\forall^\infty n)(g(n) + 1 < h(n))$).

Instead $h \in {}^\omega(\omega \setminus 3)$ we can demand: $h \in {}^\omega(\omega \setminus 1)$ and $h(n) \ge 3$ for every large enough n.
4) If $x = \mathrm{ptr}$, $BA_x(I)$ satisfies the strong λ–chain condition provided that I is atomically $(< \lambda)$-stable; for example if $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$.
5) If $h, \bar{\mathbf{e}}$ are as in 2.1(6), and for every n large enough, $(*)_{\bar{\mathbf{e}}}^n$ below holds and λ is {2.1} *regular uncountable and $I \in K_{\mathrm{rtr}(h)}^\omega$ <u>then</u> $\mathrm{BA}_{\mathrm{rtr}(h),\bar{\mathbf{e}}}(I)$ satisfies the strong λ-chain condition where*

> $(*)_{\bar{\mathbf{e}}}^n$ $\bar{\mathbf{e}}(n) = \{(u_1^n, u_2^n)\}$, *where $u_1^n, u_2^n \subseteq \{0, \ldots h(n) - 1\}$ are non-empty disjoint of the same cardinality and not both singletons.*

Remark 2.7. Clearly we can similarly phrase sufficient condition for "any family of λ non-zero elements there is an uncountable subfamily such that any k members of the subfamily have non-zero intersection".

Before we prove 2.6 recall the well known ($\mathbf{B}_0 = \{0, 1\}$ is the two elements Boolean {2.6} algebra):

{2.6A}

Fact 2.8. 1) If $\mathbf{B}$ is the Boolean algebra generated by $\{x_t : t \in I\}$ freely except the set Λ of equations in $\{x_t : t \in I\}$, (so each member of Λ has the form $\sigma(x_{t_0}, \ldots, x_{t_{n-1}}) = 0$, where $\sigma(y_0, \ldots, y_{n-1})$ is a Boolean term, $t_0, \ldots, t_{n-1} \in I$) <u>then</u>:

> (α) $\mathbf{B} \models \sigma^*(x_{s_0}, \ldots, x_{s_{n-1}}) > 0$
> <u>iff</u>

(β) for some function $f : I \longrightarrow \{0,1\}$ we have:

(a) f respects Λ, i.e., $\sigma(x_{t_0},\dots,x_{t_{m-1}}) \in \Lambda \Rightarrow \mathbf{B}_0 \models 0 = \sigma(f(t_0),\dots,f(t_{m-1}))$,

(b) $\mathbf{B}_0 \models \sigma^*(f(s_0),\dots,f(s_{n-1})) = 1$.

2) In fact if $f : I \to \{0,1\}$ satisfies clause (a) then there is one and only one homomorphism $\hat{f}$ from $\mathbf{B}$ into $\mathbf{B}_0$ such that $s \in I \Rightarrow \hat{f}(x_s) = f(s)$

Proof. 1) So we take $x = \text{tr}$ and check the strong λ–chain conditions. Note that by
{2.6A} 2.8 and the definition of $\text{BA}_{\text{tr}}(I)$ we have

$(*)_1$ $x_{\eta_1} \cap \dots \cap x_{\eta_k} \cap (-x_{\nu_1}) \cap \dots \cap (-x_{\nu_m}) = 0$ iff $(\exists i,j)(\nu_i \triangleleft \eta_j \in P^I_\omega \vee \nu_i = \eta_j \in P^I_\omega)$.

{2.1} [Why? The if implication is trivial recalling Definition 2.1(1). For proving the "only if" implication, assume that the second statement holds, define $f : I \to \{0,1\}$ by $f(\eta) = 1$ iff $(\exists \ell)(\eta = \eta_\ell \vee \eta \triangleleft \eta_\ell \in P^I_\omega)$; clearly it respect the equations in the definition of $\text{BA}_{\text{rtr}}(I)$ and $\hat{f}$ maps $x_{\eta_1} \cap \dots \cap x_{\eta_k} \cap (-x_{\nu_1}) \cap \dots \cap (-x_{\nu_n})$ to 1 so we are done.]

Now for $u \in [I]^{<\omega}$ let $x_u = \bigcap_{\eta \in u} x_\eta$ and $x_{-u} = \bigcap_{\eta \in u} (-x_\eta)$. Clearly if $a \in BA_x(I) \setminus \{0\}$ then for some $u,\ v \in [I]^{<\aleph_0}$, we have $0 < x_u \cap x_{-v} \leq a$ (hence u and v are disjoint), in fact a is a finite union of such elements. To check the strong λ–chain condition it suffices to take $\{(u_i,v_i) : i < \lambda\} \subseteq [I]^{<\aleph_0} \times [I]^{<\aleph_0}$ such that $(\forall i < \lambda)[x_{u_i} \cap x_{-v_i} \neq 0]$ and to find $A \in [\lambda]^\lambda$ such that

$$(\forall i,j \in A)[x_{u_i} \cap x_{-v_i} \cap x_{u_j} \cap x_{-v_j} \neq 0].$$

We may assume that $\langle u_i : i \in A\rangle$ and $\langle v_i : i \in A\rangle$ are Δ–systems, say with hearts u^*, v^* respectively, so as $u_i \cap v_i = \emptyset$ necessarily $u_i \cap v^* = \emptyset$ and $u^* \cap v_i = \emptyset$ and $u^* \cap v^* = \emptyset$. We may assume $i \neq j \in A$ implies $u_i \cap v_j = \emptyset$ and $u_i \neq u_j$ and $v_i \neq v_j$. We may assume that for some non-zero $m,\ n < \omega$ for every $i \in A$ we have $|u_i| = m$ & $|v_i| = n$. Say $u_i = \{\eta_{i,\ell} : \ell < m\}$, $v_i = \{\nu_{i,\ell} : \ell < n\}$ (without repetitions) and for each $\ell < m$ the sequence $\langle \eta_{i,\ell} : i \in A\rangle$ is constant or is without repetitions, and similarly $\langle \nu_{i,\ell} : i \in A\rangle$. We may assume

$(*)_2$ $\langle \lg(\eta_{i,\ell}) : \ell < m\rangle$, $\langle \lg(\nu_{i,\ell}) : \ell < n\rangle$ is the same for all $i \in A$.

Clearly then, using the Δ–system assumption,

$(*)_3$ for $i \in A$, $\ell < m$, $k < n$ there is at most one $j \in A$ such that $\nu_{j,k} \triangleleft \eta_{i,\ell} \in P^I_\omega$.

[Why? If we have $\nu_{j,k} \trianglelefteq \eta_{i,\ell} \in P^I_\omega$, note that $\neg(\nu_{i,k} \trianglelefteq \eta_{i,\ell})$ by $(*)_1$, hence $\nu_{j,k} \neq \nu_{i,k}$ so $i \neq j$ and hence $\nu_{j,k} \notin v^*$, and $\nu_{j,k} = \eta_{i,\ell} \restriction \lg(\nu_{j,k})$. Thus $j \neq j_1 \in A \Rightarrow \nu_{j_1,k} \neq \nu_{j,k}$ and hence $j \neq j_1 \in A \Rightarrow \nu_{j_1,k} \neq \eta_{i,\ell} \restriction \lg(\nu_{j,k}) = \eta_{i,\ell} \restriction \lg(\nu_{j_1,k})$. Hence $j \neq j_1 \in A \Rightarrow \neg(\nu_{j_1,k} \trianglelefteq \eta_{i,\ell})$ and we have finished.]

So for $i \in A$, the set

$$w_i := \{j : \text{for some } \ell < m,\ k < n \text{ we have } \nu_{j,k} \triangleleft \eta_{i,\ell} \in P^I_\omega\}$$

has at most $mn < \aleph_0$ members. So by $(*)_1$ it suffices to find $A' \in [A]^\lambda$ such that $[i \neq j \in A' \;\Rightarrow\; j \notin w_i]$. By Hajnal free subset theorem [Haj62] or see [Sh:E62, 3.14=L4.Ha] there[4] is such A'.
2) The case $x = \text{ptr}$ is similar, but more complicated. First note

$(*)_4$ assume $I \in K^\omega_{\text{ptr}}$ and $\mathbf{B} = \text{BA}_{\text{ptr}}(I)$; $m, n < \omega$, and $\nu_k, \eta_\ell \in I$ for $\ell < m, k < n$ <u>then</u> $\mathbf{B} \models x_{\eta_0} \cap \ldots \cap x_{\eta_{m-1}} \cap (-x_{\nu_0}) \cap \ldots \cap (-x_{\nu_{n-1}}) = 0$ iff one of the following conditions holds:

- (a) $(\exists \ell, k < m)[\text{lg}(\eta_\ell) = \omega \;\&\; \text{Suc}_R(\eta_k, \eta_\ell)]$,
- (b) $(\exists \ell < m)(\exists k < n)[\text{lg}(\eta_\ell) = \omega \;\&\; \text{Suc}_L(\nu_k, \eta_\ell)]$,
- (c) $(\exists \ell, k < m)\big(\text{lg}(\eta_\ell) = \text{lg}(\eta_k) = \omega \;\&\; (\exists j < \omega)(\eta_\ell \restriction j = \eta_k \restriction j \;\&\; (\exists \alpha, \beta, \gamma)[\eta_\ell(j) = \langle \alpha, \beta \rangle \;\&\; \eta_k(j) = \langle \beta, \gamma \rangle])\big)$
- (d) $(\exists \ell < m)(\exists k < n)(\eta_\ell = \nu_k)$.

[Why? If (a) or (b) or (c) or (d) holds, the intersection is zero by the equations we have imposed defining $BA_{\text{ptr}}(I)$ in Definition 2.1(2); so the "if" implication holds. {2.1}
Next we prove the other implication, so we assume (a), (b), (c) and (d) fail, and we shall use 2.8, so we have to define $f(\rho)$ for $\rho \in I$; we do it by cases. {2.6A}

Case 1: $\text{lg}(\rho) = \omega$, $\rho \in \{\eta_0, \ldots, \eta_m\}$.
Let $f(\rho) = 1$.

Case 2: $\text{lg}(\rho) = \omega$, not case 1.
Let $f(\rho) = 0$.

Case 3: $\text{lg}(\rho) = k < \omega$ and for some $\ell < m$, $\text{lg}(\eta_\ell) = \omega \& \text{Suc}_L(\rho, \eta_\ell)$. Let $f(\rho) = 1$.

Case 4: $\text{lg}(\rho) = k < \omega$ and for some $\ell < m$, $\text{lg}(\eta_\ell) = \omega \& \text{Suc}_R(\rho, \eta_\ell)$.
Let $f(\rho) = 0$.

Case 5: $\text{lg}(\rho) < \omega$, $\rho \in \{\eta_\ell : \ell < m\}$.
Let $f(\rho) = 1$.

Case 6: no previous case.
Let $f(\rho) = 0$.

First f is well defined, i.e., there are no contradiction between cases 3, 4, cases 3, 5, cases 4, 5 as clauses (c), (b), (a) respectively fail, (actually cases 3,5 cannot contradict). Second we show that f respect the equations from Definition 2.1(2), {2.1}
that is from $(*)_2$ there. If $x_\eta \leq x_{\eta \restriction n \hat{}\langle \alpha_n \rangle}$ is an instance of $(*)_2$ of 2.1(2), and f {2.1}
fails it, that is $f(\eta) = 1$, $f(\eta \restriction n \hat{}\langle \alpha_n \rangle) = 0$, then necessarily by $\text{lg}(\eta) = \omega$ case 1 occurs for η, hence case 3 occurs for $(\eta \restriction n)\hat{}\langle \alpha_n \rangle$. So $f((\eta \restriction n)\hat{}\langle \alpha_n \rangle) = 1$, hence f has to satisfy the equation. Similarly for the other equation in $(*)_2$ of 2.1(2) using {2.1}
case 4 instead case 3. Third: $f(x_{\eta_\ell}) = 1$ for $\ell < m$ by cases 1, 5, and $f(\nu_k) = 0$ for $k < n$ as by failure of clause (d), case 2 occur if $\text{lg}(\nu_k) = \omega$, and case 6 occurs if $\text{lg}(\nu_k) < \omega$. So by 2.8 we are done proving $(*)_4$] {2.6A}

Let $a_\alpha \in BA_x(I) \setminus \{0\}$ for $\alpha < \lambda = (2^{\aleph_0})^+$, so as before without loss of generality $a_\alpha = x_{\eta_{\alpha,0}} \cap \ldots \cap x_{\eta_{\alpha,n_\alpha-1}} \cap (-x_{\eta_{\alpha,n_\alpha}}) \cap \ldots \cap (-x_{\eta_{\alpha,m_\alpha-1}}) \neq 0$, without loss of generality $n_\alpha = n^*$, $m_\alpha = m^*$ and $P^I_\omega \cap \{\eta_{\alpha,\ell} : \ell < m^*\} \neq \emptyset$ (for notational simplicity below). We can define $\eta_{\alpha,\ell}$ $(m^* \leq \ell < \omega)$ such that

[4]Note that $(-x_{\nu_{j_1,\ell_1}}) \cap (-x_{\nu_{j_2,\ell_2}}) > 0$ always holds.

$$\mathrm{Suc}_L(\rho,\eta_{\alpha,\ell}) \vee \mathrm{Suc}_R(\rho,\eta_{\alpha,\ell}) \Rightarrow \rho \in \{\eta_{\alpha,k} : k < \omega\}$$

Without loss of generality the atomic type of $\langle \eta_{\alpha,\ell} : \ell < \omega \rangle$ in I does not depend on α, and they form a Δ–system, i.e.,

$(*)$ $\eta_{\alpha,\ell_1} = \eta_{\beta,\ell_2}$ & $\alpha \neq \beta \Rightarrow$ $(\forall \alpha_1, \beta_1 < \lambda)(\eta_{\alpha_1,\ell_1} = \eta_{\alpha_1,\ell_2} = \eta_{\beta_1,\ell_1} = \eta_{\beta_1,\ell_2})$.

Now we apply $(*)_4$; check that each case fails.
3) Wilog we deal with $K^\omega_{\mathrm{tr}(h,g)}$. Let $a_\alpha \neq 0$ $(\alpha < \lambda)$ be non-zero pairwise disjoint elements, let $a_\alpha = \sigma_\alpha(\bar{x}_{\bar{\eta}_\alpha})$, σ_α a Boolean term, $\bar{\eta}_\alpha$ a finite sequence from I, (i.e, we write $\bar{x}_{\langle \eta_{\alpha,0},\dots,\eta_{\alpha,k_\alpha-1}\rangle}$ instead of $\langle x_{\eta_{\alpha,0}},\dots,x_{\eta_{\alpha,k_\alpha-1}}\rangle$). Without loss of generality $\sigma_\alpha = \sigma$ and $\bar{\eta}_\alpha = \langle \eta_{\alpha,0},\dots,\eta_{\alpha,k-1}\rangle$ is without repetition, and

$$a_\alpha = \bigcap_{\ell<k(0)} x_{\eta_{\alpha,\ell}} \cap \bigcap_{k(0)\leq \ell<k} (-x_{\eta_{\alpha,\ell}})\,.$$

So there is $n(\alpha) < \omega$ such that $\lg(\eta_{\alpha,\ell}) < \omega \;\Rightarrow\; \lg(\eta_{\alpha,\ell}) \leq n(\alpha)$, and $\lg(\eta_{\alpha,\ell(1)}) = \lg(\eta_{\alpha,\ell(2)}) = \omega$, $\ell(1) \neq \ell(2)$ implies

$$\eta_{\alpha,\ell(1)} \restriction n(\alpha) \neq \eta_{\alpha,\ell(1)} \restriction n(\alpha)$$

and $(\forall n)[n \geq n(\alpha) \;\Rightarrow\; h(n) \geq 3\&\ h(n) > g(n)+1]$.

Without loss of generality if $m < n(\alpha)$, $\lg(\eta_{\alpha,i}) > m+1$, $\eta_{\alpha,i}(m) = \langle \gamma_0,\gamma_1,\dots\rangle$ then $(\eta_{\alpha,i} \restriction m)\hat{\ }\langle \gamma_j\rangle$ belongs to $\{\eta_{\alpha,0},\eta_{\alpha,1},\dots\}$ (for we can change $\bar{\eta}_\alpha$ and σ_α, and then uniformize σ_α, k again).

Now without loss of generality $n(\alpha) = n^*$ for every α, and $\lg(\eta_{\alpha,i}) = \ell_i \leq \omega$ and the truth value of $(\eta_{\alpha,i_1} \restriction m)\hat{\ }\langle \eta_{\alpha,i_1}(m)(m')\rangle = \eta_{\alpha,i_2}$ does not depend on α. Also (by the theorem on Δ–systems) for every $m < k$, $\langle \eta_{\alpha,m} : \alpha < \lambda\rangle$ is constant or is without repetition. Also there is $j_m \leq n^*$ such that $\eta_{\alpha,m}{\restriction} j_m$ is constant, but <u>either</u> $\langle \eta_{\alpha,m}(j_m) : \alpha < \lambda\rangle$ is an indiscernible sequence of pairwise distinct tuples of length $h(j_m)$ when $j_m+1 < \ell_m$ singletons when $j_m+1 = \ell_m$ <u>or</u> $\ell_m \geq n^*$, (recall that $<^I_1$ is a well ordering, that is, we use "I is standard"). It follows that: $i_1, i_2 < k$, $\alpha,\beta,\gamma < \lambda$, $\ell \leq n^*$ and $\eta_{\alpha,i_1} \restriction \ell = \eta_{\beta,i_2} \restriction \ell$ implies $\eta_{\alpha,i_1} \restriction \ell = \eta_{\alpha,i_2} \restriction \ell = \eta_{\gamma,i_1} \restriction \ell = \eta_{\gamma,i_2} \restriction \ell$.

Let $\alpha < \beta < \lambda$, and we shall prove $a_\alpha \cap a_\beta \neq 0$. For notational simplicity let $\alpha = 0, \beta = 1$. Now we shall define a function f from I to the trivial Boolean algebra $\mathbf{B}_0 = \{0,1\}$.

We define $f(\eta) = 1$ iff one of the following cases occurs:

(a) $\eta = \eta_{j,\ell}$ where $j < 2$, $\ell < k(0)$;

(b) $\eta = (\eta_{j,\ell} \restriction i)\hat{\ }\langle \alpha_0\rangle$ where $j < 2$, $\ell < k(0)$, $i \in [n^*,\omega)$, $\omega = \lg(\eta_{j,\ell})$, $\eta_{j,\ell}(i) = \langle \alpha_0,\dots\rangle$ and $h(i) - g(i) > 2 \vee \eta_{0,\ell}{\restriction} i \neq \eta_{1,\ell}{\restriction} i$;

(c) $\eta = (\eta_{j,\ell} \restriction i)\hat{\ }\langle \alpha\rangle$, where $h(i) - g(i) = 2\&\eta_{0,\ell}{\restriction} i = \eta_{1,\ell}{\restriction} i$ (hence $g(i) \geq 1$) and $j < 2$, $\ell < k(0)$, $i \in [n^*,\omega)$, $\omega = \lg(\eta_{j,\ell})$, α appears in the sequence $\eta_{0,\ell}(i)$ or in the sequence $\eta_{1,\ell}(i)$ but is not the last element in any of them.

Clearly f is well defined. Also

$(*)$ if $\ell \in [k(0),k)$ and $j \in \{0,1\}$ then $f(\eta_{j,\ell}) = 0$.

[Why? Let $\eta = \eta_{j,\ell}$; assume toward contradiction that (*) fail, there are two possible reasons for $f(\eta_{j,\ell}) = 1$. The first is $\eta = \eta_{j(1),\ell(1)}$ where $j(1) \in \{0,1\}$, $\ell(1) < k(0)$ but for $j \neq j(1)$ this is impossible by the "cleaning" above, and if $j = j(1)$ this is impossible as $a_j \neq 0$. The second is $\eta = (\eta_{j(1),\ell(1)} \restriction i)\hat{}\langle \alpha_m^{j(1),\ell(1)} \rangle$ where $i \in [n^*, \omega)$, $j(1) < 2$, $\ell(1) < k(0)$, $\omega = \lg(\eta_{j(1),\ell(1)})$, $m < h(i)$ is as in clause (b) or (c) above (so $m = 0$ for (b)), and $\eta_{j(1),\ell(1)}(i) = \langle \alpha_0^{j(1),\ell(1)}, \dots, \alpha_{h(i)-1}^{j(1),\ell(1)} \rangle$. In any case $\lg(\eta) = i + 1 < \omega$. But we have assumed $\lg(\eta_{j,\ell}) < \omega \ \Rightarrow \ \lg(\eta_{j(1),\ell(1)}) \leq n^*$, while $(\eta_{j(1),\ell(1)} \restriction i)\hat{}\langle \alpha_m^{j(1),\ell(1)} \rangle$ appears in the sequence $\langle \eta_{j(\ell),\ell} : \ell < k \rangle$, so easy contradiction].

It is enough to prove that there is a homomorphism f from $BA_{\mathrm{rtr}(h,g)}[I]$ to $\{0,1\}$ such that $\hat{f}(x_\eta) = f(\eta)$ as then we are done because clearly by $(*)$ (and f's definition) $\hat{f}(a_0) = \hat{f}(a_1) = 1$. To prove this we have to show that the identities appearing in the definition of $BA_x[I]$ are respected by f. Such an identity looks like

$$\oplus \ \ x_\rho \leq \bigcup_{m=0}^{g(i)} x_{(\rho\restriction i)\hat{}\langle \alpha_m \rangle}, \quad x_\rho \cap \bigcap_{m=g(i)+1}^{h(i)-1} x_{(\rho\restriction i)\hat{}\langle \alpha_m \rangle} = 0, \text{ where } \rho \in P_\omega^I,\ \rho(i) = \langle \alpha_0, \dots, \alpha_{h(i)-1} \rangle.$$

If $f(\rho) = 0$ they hold trivially, so we should consider only the case $f(\rho) = 1$. As $\lg(\rho) = \omega$, necessarily $\rho = \eta_{j(*),\ell(*)}$ for some $j(*) < 2$, $\ell(*) < k(0)$ (in the other cases in the definition of f where $f(x_\rho) = 1$ the sequence ρ is finite). If $i < n^*$ then $(\rho \restriction i)\hat{}\langle \alpha_m \rangle \in \{\eta_{j(*),\ell} : \ell < k\}$, for every $m < h(i)$ so as $a_j > 0$, by clause (a) of the definition of f and by $(*)$ we can finish. So assume $i \geq n^*$. Now if $\eta_{1-j(*),\ell(*)} \restriction i \neq \rho \restriction i$ then $f((\rho \restriction i)\hat{}\langle \alpha_m \rangle)$ is 1 if $m = 0$, and is 0 if $m \neq 0$, so clearly the two equations in $(\oplus)$ hold. We are left with case $\eta_{1-j(*),\ell(*)} \restriction i = \rho \restriction i$ $(= \eta_{j(*),\ell(*)} \restriction i)$. First assume $h(i) - g(i) > 2$. Now $f((\rho \restriction i)\hat{}\langle \alpha_m \rangle)$ is 1 iff α_m is the first element in $\eta_{j(*),\ell(*)}(i)$ or is the first element in $\eta_{1-j(*),\ell(*)}(i)$. So the first equation in $(\oplus)$ holds trivially; in the second equation at most one of the $\alpha \in \{\alpha_{g(i)+1}, \dots, \alpha_{h(i)-1}\}$ satisfies $f((\rho \restriction i)\hat{}\langle \alpha \rangle) = 1$ (as α_0 is not among them) and there are at least 2 (as $h(i) - g(i) > 2$), so the second equation holds.

Now the second possibility, $h(i) - g(i) \leq 2$ (so necessarily $h(i) = g(i) = 2$), implies $f((\rho \restriction i)\hat{}\langle \alpha_{h(i)-1} \rangle) = 0$ hence to second equation is $(\oplus)$ holds. Now $X = \{x_{(\rho\restriction i)\hat{}\langle \alpha_0 \rangle}, \dots, x_{(\rho\restriction i)\hat{}\langle \alpha_{g(i)} \rangle}\}$ has at least two members and $x_{(\rho\restriction i)\hat{}\langle \alpha_{h(i-1)} \rangle}$ is not among them. So, by clause (c) in the definition of f, for at least one of the members of X is mapped by f to 1, so the first equation holds.
4) Like part 2).
5) Like part (3). $\square_{2.6}$

{2.7}

Claim 2.9. *1) If $\mathbf{B}_1, \mathbf{B}$ satisfy the strong λ–chain condition, $a^* \in \mathbf{B}_1 \setminus \{0_{\mathbf{B}_1}\}$, $\mathbf{B}_2$ is the result of a $\mathbf{B}$–surgery of $\mathbf{B}_1$ at a^*, then $\mathbf{B}_2$ satisfies the strong λ–chain condition. If one of $\mathbf{B}_1$, $\mathbf{B}$ satisfies the strong λ–chain condition, and the other only the λ–chain condition, then $\mathbf{B}_2$ satisfies the λ–chain condition.*
2) If $\mathbf{B}_2$ is the result of a $\mathbf{B}$–surgery of $\mathbf{B}_1$ at a^*, then $\mathbf{B}_1 \lessdot \mathbf{B}_2$ (i.e., $\mathbf{B}_1$ is a subalgebra of $\mathbf{B}_2$, and every maximal antichain of $\mathbf{B}_1$ is a maximal antichain of $\mathbf{B}_2$. This is also called "$\mathbf{B}_2$ is a regular extension of $\mathbf{B}_1$").

Proof. Well known (and easy). $\square_{2.9}$

{2.8}

Claim 2.10. *The relation $\lessdot$ between Boolean algebras is a partial order and if a sequence $\langle \mathbf{B}_i : i < \alpha \rangle$ is $\lessdot$–increasing continuous, <u>then</u> $\mathbf{B}_0 \lessdot \bigcup\limits_{i<\alpha} \mathbf{B}_i$, and if each $\mathbf{B}_i$ satisfies the strong χ–chain condition (for a regular χ), <u>then</u> so does $\bigcup\limits_{i<\alpha} \mathbf{B}_i$.*

Proof. Well known, Solovay-Tenenbaum [ST71] for the χ–chain condition, and Kunen-Tall [KT79, p.179] for the strong χ–chain condition. $\square_{2.9}$

{2.9}
{2.4} **Claim 2.11.** *1) In the construction 2.4, if $|I_i| = \lambda$ hence $|BA_x(I_i)| = \lambda$ for $i < \alpha$, <u>then</u> $\|\mathbf{B}_i\| = \lambda$ for $i > 0$, $i \le \alpha$.*
{2.4} *2) In 2.4, if each $BA_x(I_i)$ satisfies the strong χ–chain condition, χ is regular, <u>then</u> $\mathbf{B} = \mathrm{Sur}_x\langle I_i, a^*_i : i < \alpha \rangle$ satisfies the (strong) χ–chain condition.*
{2.4} *3) Assume that in 2.4 we use non-trivial $\mathbf{B}_0$, $|I_i| = \lambda$. <u>then</u> $\|\mathbf{B}\| = \lambda + ||\mathbf{B}_0||$; if in addition $\mathbf{B}_0$ satisfies the λ–cc, and each $BA_x(I_i)$ satisfies the strong λ–chain condition, <u>then</u> $\mathbf{B}$ satisfies the λ-cc; if in addition $\mathbf{B}_0$ satisfies the strong λ–cc, <u>then</u> so does $\mathbf{B}$.*

Proof. 1) Trivial.
{2.5}{2.6} 2) By 2.5, 2.6, 2.9, 2.10.
3) Similar. $\square_{2.11}$

{2.10}
{2.4} **Lemma 2.12.** *1) For the construction in 2.4, $\mathbf{B}_\alpha$ is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}(\sum\limits_{i<\alpha} I_i)$ (see Definition* [Sh:E59, 2.4=L2.2(c),(d)]*).*
2) Moreover, $\mathbf{B}_\alpha \restriction (1 - a^*_i)$ is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}(\sum\limits_{j<\alpha, j\neq i} I_j)$.
3) If $\mathbf{B}_\alpha$ satisfies the θ–chain condition, <u>then</u> $\mathbf{B}^c_\alpha$ (the completion of $\mathbf{B}_\alpha$) can be weakly represented in $\mathscr{M}^*_{\theta,\theta}(\sum\limits_{j<\alpha} I_j)$. This representation can extend the one from
{2.10} *2.12(1).*
{2.10} *4) Similarly for 2.12(2).*
{2.4} *5) If in 2.4 we use $\mathbf{B}_0$ non-trivial we have to adapt, for example assume $\mathbf{B}_0$ is weakly representable in a relevant way (e.g., for (1) assume $\mathbf{B}_0$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J + \sum\limits_{i<\alpha} I_i)$).*

Proof. 1) Define $f(0) = 0$, $f(1) = 1$. For $b \in \mathbf{B}_\alpha$ and $b \neq 0, 1$, say b first appears in $\mathbf{B}_{i+1}$.

Say

$$b = \big(b', \bigcup_{j<m} (c_j \cap d_j)\big)$$

with $b' \in \mathbf{B}_i \restriction (-a^*_i)$, $c_j \in \mathbf{B}_i \restriction a^*_i, d_j \in BA_x(I_i)$. Say (by induction hypothesis) $f(b') = x'$, $f(c_j) = x_j$, $f(a^*_i) = x, d_j = \sigma_j(x_{\eta_0}, \dots, x_{\eta_{m-1}})$ where σ is a Boolean term and $\eta_0, \dots, \eta_{m-1} \in I_i$.

Then we set

$$f(b) = F_k(x, x', x_0, \dots, x_{m-1}, \eta_0, \dots, \eta_{m-1}),$$
$$k \ \text{ codes } \ \langle m, n, \sigma_0, \dots, \sigma_{m-1} \rangle,$$

where F_k is a suitable function symbol. Thus, $f(b)$ codes all the relevant information about b.

2) We may assume that $a^*_i \neq 0, 1$. We go exactly as in (1), up to $\mathbf{B}_i$, for $\alpha > i$, we use $(-a^*_i)$ in place of 1, and working always with $\mathbf{B}_\alpha \restriction (-a^*_i)$. Note that no terms involving I_i appear then.
3) For each $a \in \mathbf{B}^c_\alpha$ we can fix $\kappa < \theta$ and a sequence $\langle b_\gamma : \gamma < \kappa\rangle$ of elements of $\mathbf{B}_\alpha$ such that $a = \bigcup\limits_{\gamma<\kappa} b_\gamma$. Then let $f_\alpha = F(\sigma_\gamma : \gamma < \kappa)$, where $f(b_\gamma) = \sigma_\gamma$ for all $\gamma < \kappa$.
4),5) Similarly. $\square_{2.12}$

{2.10Anew} {2.13}

Remark 2.13. 1) In 2.16-2.15 below we can omit the weak from representation and the strong from unembeddability.
2) Why weakly represented? As the order of the construction and the choice of the a^*_i play a role in the definition, we can overcome this in various ways but there is no real reason for doing this

{2.11}

Lemma 2.14. *1) Suppose $I \in K^\omega_{\rm tr}$ is strongly $(\aleph_0, \aleph_0, \psi_{\rm tr})$–unembeddable into $J \in K^\omega_{\rm tr}$, and $\mathbf{B}$ is a Boolean algebra weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$. Then $BA_{\rm tr}(I)$ is not embeddable into $\mathbf{B}$.*
2) Suppose $I \in K^\omega_{\rm tr}$ is strongly $(\mu, \kappa, \psi_{\rm tr})$–unembeddable into J for embedding strongly finitary on P^I_ω and $\mathbf{B}$ a Boolean algebra weakly represented in $\mathscr{M}_{\mu,\kappa}(J)$. Then $BA_{\rm tr}(I)$ is not embeddable into $\mathbf{B}$.

Proof. 1) Let $g : B \longrightarrow \mathscr{M}_{\aleph_0,\aleph_0}(J)$ be a weak representation of $\mathbf{B}$ into $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ (with the well ordering $<^*$), and h be an embedding of $BA_{\rm tr}(I)$ into $\mathbf{B}$. For $\eta \in I$ define $f(\eta) = g(h(x_\eta))$. As I is strongly $(\aleph_0, \aleph_0, \psi_{\rm tr})$–unembeddable into J, there are ν_1, ν_2, η, n such that $\eta \in P^I_\omega, \nu_1 = \eta \restriction (n + 1)$, $\nu_1 \restriction n = \nu_2 \restriction n$, $\nu_2(n) <^J_1 \nu_1(n), \lg(\nu_1) = \lg(\nu_2) = n + 1$ and

$$\langle f(\nu_1), f(\eta)\rangle \approx \langle f(\nu_2), f(\eta)\rangle \qquad \mod\ (\mathscr{M}^*_{\aleph_0,\aleph_0}(J), <^*).$$

Hence (because g is a weak representation)

$$h(x_\eta) < h(x_{\nu_1}) \iff h(x_\eta) < h(x_{\nu_2}) \qquad \text{(in } \mathbf{B}).$$

But h is an embedding and hence $x_\eta < x_{\nu_1} \iff x_\eta < x_{\nu_2}$ in $BA_{\rm tr}(I)$, contradicting the definition of $BA_{\rm tr}(I)$.
2) Similar. $\square_{2.14}$

{2.12}

Lemma 2.15. *1) Suppose I, $J \in K^\omega_{\rm ptr}$ and I is standard, strongly $(\mu, \kappa, \psi_{\rm ptr})$–unembeddable into J by f strongly finitary on P^I_ω. If $\mathbf{B}$ is a Boolean algebra weakly representable in $cM_{\aleph_0,\aleph_0}(J)$ say by g, $\mathbf{B} \subseteq \mathbf{B}_1$, $\mathbf{B}$ dense in $\mathbf{B}_1$ (e.g. $\mathbf{B}_1$ the completion of $\mathbf{B}$, the case that interest use), g_1 extends g and is a weak representation of $\mathbf{B}_1$ in $\mathscr{M}_{\mu,\kappa}(J)$, then $\mathrm{BA}_{\rm ptr}(I)$ is not embeddable into $\mathbf{B}_1$.*
2) Parallely for $K^\omega_{{\rm tr}(h)}$, $\psi_{{\rm tr}(h)}$, $BA_{{\rm tr}(h)}(-)$ (for $h \in {}^\omega(\omega \setminus 2)$) and $K^\omega_{{\rm tr}(h)}$, $\psi_{{\rm tr}(h,g)}$, $BA_{{\rm tr}(h,g)}(-)$.
3) If $I \in K^\omega_{{\rm tr}(h)}$ is standard, $(\aleph_0, \aleph_0)$-super$^{\rm vr}$ unembeddable into $J \in K^\omega_{{\rm tr}(h)}$, $\mathbf{B}$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ and satisfies the ccc (for example $\mathrm{Rang}(h) \subseteq [3, \omega)$) then $\mathrm{BA}_{{\rm tr}(h)}(I)$ is not embeddable into the completion of $\mathbf{B}$.

Proof. 1) Suppose $\mathbf{f}$ is an embedding of $\mathrm{BA}_{\rm ptr}(I)$ into $\mathbf{B}_1$. For $\eta \in I$ define $f(\eta)$ as follows: if $\lg(\eta) < \omega$, $f(\eta) = g_1(\mathbf{f}(x_\eta))$; if $\lg(\eta) = \omega$, choose $a_\eta \in \mathbf{B}, 0 < a_\eta \leq \mathbf{f}(x_\eta)$

(possible as $\mathbf{B}$ is dense in $\mathbf{B}_1$) and let $f(\eta) = g(a_\eta)$. As I is strongly $(\mu, \kappa, \psi_{\rm ptr})$–unembeddable into J by a function f which is strongly finitary on P^I_ω, there are ν_1, ν_2, η, n such that $\eta \in P^I_\omega, \nu_1 = \eta \restriction n\hat{\ }\langle \alpha \rangle$, $\nu_2 = \eta \restriction n\hat{\ }\langle \beta \rangle$, $\eta(n) = \langle \alpha, \beta \rangle$, $\alpha < \beta$ and

$$\langle f(\nu_1), f(\eta) \rangle \approx \langle f(\nu_2), f(\eta) \rangle \mod (M_{\mu,\kappa}(J), <^*).$$

Hence, as g_1 is a weak representation

$$(*) \qquad \begin{array}{rcl} \mathbf{B}_1 \models \mathbf{f}(a_\eta) < \mathbf{f}(x_{\nu_1}) & \Leftrightarrow & \mathbf{B}_1 \models \mathbf{f}(a_\eta) < \mathbf{f}(x_{\nu_2}), \\ \mathbf{B}_1 \models \mathbf{f}(a_\eta) \cap \mathbf{f}(x_{\nu_1}) = 0 & \Leftrightarrow & \mathbf{B}_1 \models \mathbf{f}(a_\eta) \cap \mathbf{f}(x_{\nu_2}) = 0. \end{array}$$

But in $\mathrm{BA}_{\rm ptr}(I)$, $x_{\nu_1} \ge x_\eta$, $x_{\nu_2} \cap x_\eta = 0$. Hence, as $\mathbf{f}$ is an embedding,

$$\mathbf{B}_1 \models \text{“}\mathbf{f}(x_{\nu_1}) \ge \mathbf{f}(x_\eta) \,\&\, \mathbf{f}(x_{\nu_2}) \cap \mathbf{f}(x_\eta) = 0\text{”}.$$

But $0 < a_\eta \le \mathbf{f}(x_\eta)$ so $\mathbf{f}(x_{\nu_1}) \ge a_\eta$, $\mathbf{f}(x_{\nu_2}) \cap a_\eta = 0$, a contradiction to $(*)$ above.

We have proved that $BA_{\rm ptr}(I)$ is not embeddable into $\mathbf{B}_1$.

2) Similar proof (the extra details appear in the proof of part (3)).

3) Note that this is not used. Assume toward contradiction that $\mathbf{f}$ is an embedding of $\mathrm{BA}_{\mathrm{tr}(h)}(I)$ into $\mathbf{B}_1$, the completion of $\mathbf{B}$. Let $g : \mathbf{B} \to \mathscr{M}_{\aleph_0,\aleph_0}(J)$ be a weak representation say for the well ordering $<^*$ of $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ which respect subterms. So
{2.4} by 2.4(3) there is $g_1 : \mathbf{B}_1 \to \mathscr{M}_{\aleph_1,\aleph_1}(J)$ which extend g and is a weak representation of $\mathbf{B}_1$ in $(\mathscr{M}_{\aleph_1,\aleph_1}(J), <^*)$. Choose a function $f : I \to \mathscr{M}_{\aleph_1,\aleph_1}(J)$ as in the proof of part (1). Let $x = \langle h, g, g_1, f, I, J, \mathbf{B}, \mathbf{B}_1 \rangle$, and χ large enough.

As it is assumed in part (3) that “I is $(\aleph_0, \aleph_0)$-super$^{\rm vr}$ unembeddable into J”
{1.4} there are M, η as in $(*)'$ of Definition 1.4(2). Let $f(\eta) = \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}})$ where $\nu_{\eta,k} \in J$ are pairwise distinct for $k < k(\eta)$. For each k let $n_k \le \omega$ be
{1.4} maximal such that $\nu_{\eta,k} \restriction n_k \in M$, it exists by clause (v) in $(*)'$ of Definition 1.4(2). If $n_k < \lg(\nu_{k,\ell})$, for each $m < h(n_k)$ let $\nu^*_{k,m} = (\nu_{\eta,k} \restriction n_k)\hat{\ }\langle s_{k,m} \rangle \in M$ be $<^J_1$-minimal such that $\mathrm{Res}^m_{n_k}(\nu_{\eta,k}) <^J_1 \nu^*_{k,m}$, clearly it exists except when $\mathrm{Res}^m_{n_k}(\nu_{\eta,k})$ is $<^J_1$-above all $\{(\nu_{\eta,k} \restriction n_k)\hat{\ }\langle s \rangle : s \in M\}$, then we let $s_{k,m} = \infty$ with the obvious conventions. We define $Y^* = \{\langle \nu_k : k < k(\eta) \rangle : \langle \nu_k : k < k(\eta) \rangle$ is similar in J to $\langle \nu_{\eta,0}, \ldots, \nu_{\eta,k(\eta)-1} \rangle$ over $\{\nu_{\eta,k} : \nu_{\eta,k} \in M\} \cup \{\nu^*_{k,m} : k < k(\eta), m < h(k(\eta))\}$ which is a finite subset of M. We define a filter D on $Y^* : Y \in D$ iff there are $\nu'_{k,m} <^J_1 \nu^*_{k,m}$ for all relevant k, m such that if $\langle \nu''_k : k < k(\eta) \rangle \in Y$ satisfies $\nu'_{k,m} \le^J_1 \nu''_{k,m}$ for all relevant k, m then $\langle \nu''_k : k < k(\eta) \rangle \in Y$. Clearly $(Y^*, D) \in M$, and by weak representably the following function f_1 belongs to M:

$\mathrm{Dom}(f_1) = \{\varrho \in I : \lg(\varrho) < \omega\}, \mathrm{Rang}(f_1) \le \{0, 1\}$

$f_1(\varrho) = 1$ <u>iff</u>: $\{\langle \nu_k : k < k(\eta) \rangle \in Y^* : \mathrm{BA}_{\mathrm{tr}(h)}(J) \models \mathbf{f}(x_\varrho) \ge \sigma_\eta(x_{\nu_0}, \ldots, x_{\nu_{k(\eta)-1}})\} \in D$ <u>iff</u> that set is $\neq \emptyset \mod D$.

Recall that σ_η is a $\tau_{\aleph_0,\aleph_0}$- term hence $\in M$. So by the choice of M and η
{1.4} for infinitely many n as $\tau = \tau_0$ (see Definition 1.4), we have: the truth values of $\mathrm{BA}_{\mathrm{tr}(h)}(J) \models \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta)}) \ge \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}})$ is the same for all $\ell < h(n)$. As $\mathbf{f}$ is an embedding and $\mathbf{B}_1 \models \mathbf{f}(x_\eta) \ge \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}}) > 0$ and $\mathrm{BA}_{\mathrm{tr}(h)}(I) \models x_{\mathrm{Res}^0_n(\eta)} \ge x_\eta$ we have $\mathbf{B}_1 \models$ “$\mathbf{f}(x_{\mathrm{Res}(\eta^0_n(\eta))}) \ge \mathbf{f}(x_\eta) \ge f(\eta) = \sigma_\eta(x_{\nu_{\eta,0}}, \ldots x_{\nu_\eta,k(\eta)-1} > 0.$” So $f_1(\mathrm{Res}^0_n(\eta)) = 1$ hence by the choice of n we have $\ell < h(n) \Rightarrow f_1(\mathrm{Res}^\ell_n(\eta) = 1$. So $\mathbf{B}_1 \models$ “$\bigcap\limits_{\ell < h(n)} \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta)}) \cap f(x_\eta) > 0$”

but $\mathbf{f}$ is an embedding and $\mathrm{BA}_{\mathrm{rtr}(h)}(J) \models'' 0 < f(\eta) \le \mathbf{f}(x_\eta)$ hence $\mathrm{BA}_{\mathrm{tr}(h)}(I) \models \bigcap\limits_{\ell<n} x_{\mathrm{Res}^\ell_n(\eta)} \cap x_\eta > 0$ contradicting the definition of $\mathrm{BA}_{\mathrm{tr}(h)}(I)$. $\square_{2.15}$

Conclusion 2.16. *Suppose $\lambda > \aleph_0$. Then :* {2.13}

1. *There is a rigid Boolean algebra $\mathbf{B}$ satisfying the $\aleph_1$–chain condition λ.*
2. *Moreover, if $a, b \in \mathbf{B}$ are $\neq 0$, $a - b \neq 0$, then $\mathbf{B} \restriction a$ cannot be embedded into $\mathbf{B} \restriction b$ (hence $\mathbf{B}$ has no one-to one endomorphism $\neq \mathrm{id}$).*
3. *Moreover, we can find such $\mathbf{B}_i$ (for $i < 2^\lambda$), $|\mathbf{B}_i| = \lambda$; and if $a \in \mathbf{B}_i$, $b \in \mathbf{B}_j$, $i \neq j$ or $a - b \neq 0$ then $\mathbf{B}_i \restriction a$ cannot be embedded into $\mathbf{B}_j \restriction b$.*

Proof. We leave it to the reader as the next proof is similar (but here we should use $(\lambda, \lambda, \aleph_0, \aleph_0)$–$\psi_{\mathrm{tr}}$–bigness, Theorem [Sh:331, 2.20=L7.11], $x = \mathrm{tr}$ instead $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$–$\psi_{\mathrm{tr}(h)}$-bigness, [Sh:331, 1.11=L7.6] $x = \mathrm{tr}(h)$ there respectively. (Also we have dealt with it in [Sh:E59, 2.16=L2.7]). $\square_{2.16}$

Conclusion 2.17. *1) There is a complete Boolean algebra $\mathbf{B}$ satisfying the* ccc*, having density λ (in fact* {2.14}

$$a \in \mathbf{B} \setminus \{0\} \quad \Rightarrow \quad \mathbf{B} \restriction a \text{ has density } \lambda,$$

so $|\mathbf{B}| = \lambda^{\aleph_0}$) and monorigid *(i.e., every one-to-one endomorphism is the identity) provided that.*

- $(*)_1$ *K^ω_{ptr} has the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$– ψ_{ptr} – bigness property for f strong finitary on P_ω, by standard atomically $(< \aleph_1)$–stable $I \in K^\omega_{\mathrm{ptr}}$.*

2) We can replace $(*)_1$ by, for some $h \in {}^\omega(\omega \setminus 3)$:

- $(*)_2$ *λ is as in 1.11(1) or* {1.7}
- $(*)_3$ *$K^\omega_{\mathrm{tr}(h)}$ has the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$ – $\psi_{\mathrm{tr}(h)}$ – bigness property.*
  or
- $(*)_4$ *$K^\omega_{\mathrm{tr}(h)}$ has the full super$^{\mathrm{vr}}(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$–bigness property.*

3) Moreover, we can find such $\mathbf{B}_i$ (for $i < 2^\lambda$) satisfying: if $a \in \mathbf{B}_i \setminus \{0\}$, $b \in \mathbf{B}_j \setminus \{0\}$, $[i \neq j \vee (i = j \wedge a - b \neq 0_{\mathbf{B}_i})]$, then $\mathbf{B}_i \restriction a$ cannot be embedded into $\mathbf{B}_j \restriction b$.

Proof. We first prove parts (1), (2); for part (1) let $h \in {}^\omega\omega$ be constantly 2. First note that if f is a one-to-one endomorphism $\neq \mathrm{id}$ of any Boolean algebra $\mathbf{B}$, then there is an element $a \neq 0$ with $a \cap f(a) = 0$. For, choose x with $x \neq f(x)$. If $x \cap -f(x) \neq 0$ we can take $a = x \cap -f(x)$; if $-x \cap f(x) \neq 0$ we can take $a = -x \cap f(x)$. Hence for **(1)** and **(2)** we only need to find $\mathbf{B}$ of power λ such that if $a, b \in \mathbf{B}$ are non-zero and $a - b \neq 0$ (and even $a \cap b = 0$), then $\mathbf{B} \restriction a$ cannot be embedded in $\mathbf{B} \restriction b$.

Now let $\langle I_\alpha : \alpha < \lambda \rangle$ exemplify the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$–$\psi_{\mathrm{tr}(h)}$–bigness property for f strongly finitary on P_ω; such a sequence exist by $(*)_1$ or $(*)_2$ or $(*)_3$ or $(*)_4$ by 1.11(1), 1.9 for any $h \in {}^\omega(\omega \setminus 3)$. Let $\mathbf{B} = \mathrm{Sur}_x\langle I_\alpha, a^*_\alpha : \alpha < \lambda \rangle$ be as {1.7}{1.6B} in the construction 2.4 for $x = \mathrm{tr}(h)$, such that $\mathbf{B} \setminus \{0\} = \{a^*_\alpha : \alpha < \lambda\}$. Then {2.4} by 2.11(1), $|\mathbf{B}| = \lambda$. By 2.6(3), 2.6(4), each $\mathrm{BA}_{\mathrm{tr}(h)}(I_\alpha)$ satisfies the strong $\aleph_1$-cc {2.6}{2.8} hence by 2.11 the Boolean algebra $\mathbf{B}$ satisfies the $\aleph_1$–chain condition and let $\mathbf{B}^*$ {2.9} be its completion. Now let $a, b \in \mathbf{B}^*$ be non-zero, with $c = a - b \neq 0$. Toward contradiction suppose f is an embedding of $\mathbf{B}^* \restriction a$ into $\mathbf{B}^* \restriction b$. Then $f(c) \cap c = 0$,

and $f \restriction (\mathbf{B} \restriction c)$ is an embedding of $\mathbf{B}^* \restriction c$ into $\mathbf{B}^* \restriction f(c)$. But $\mathbf{B}$ is dense in $\mathbf{B}^*$ hence $a^*_\alpha \le c$ for some α, hence $BA_{\mathrm{tr}(h)}(I_\alpha)$ is embeddable into $\mathbf{B}^* \restriction c$, hence into $\mathbf{B}^* \restriction f(c)$, hence into $\mathbf{B}^* \restriction (-c) = \mathbf{B}^* \restriction (-a^*_\alpha)$. But by 2.12(3), $\mathbf{B} \restriction (-a^*_\alpha)$ {2.10}
is weakly representable in $\mathscr{M}^*_{2^{\aleph_0},\aleph_1}\left(\sum\limits_{\beta \ne \alpha, \beta < \lambda} I_\beta\right)$. This contradicts 2.15 when we {2.12}
assume $(*)_4$.

For part (3) let $\langle I_{\alpha,\beta} : \alpha, \beta < \lambda \rangle$ rename $\langle I_\alpha : \alpha < \lambda \rangle$ and we shall choose for $\xi < 2^\lambda$, functions f_ξ, g_ξ from λ to λ and $A_\xi \in [\lambda]^\lambda$ such that g_ξ is one-to-one and $\mathrm{Rang}(f_\xi) = A_\xi$ and $(\forall \alpha \in A_\xi)(\exists^\lambda \beta < \lambda)(f_\xi(\beta) = 1)$ and $\xi_1 \ne \xi_2 \Rightarrow A_{\xi_1} \not\subseteq A_{\xi_2}$. Consider for $\xi < 2^\lambda$, $\mathbf{B}^\xi$ constructed as $\mathrm{Sur}_x \langle I_{f_\xi(\alpha), g_\xi(\alpha)}, a^\xi_\alpha : \alpha < \lambda \rangle$, for simplicity assume that for every $a \in \mathbf{B}^\xi \setminus \{0\}$ and $\zeta \in A_\xi$ for some ξ, $a^\xi_\alpha = a$ and $f_\xi(\alpha) = \zeta$. Let $\mathbf{B}^{\xi,*}$ be the completion of $\mathbf{B}^\xi$. As g_ξ is one-to-one clearly $\mathbf{B}^\xi$ satisfies the demand in (2), and as $\xi \ne \zeta < 2^\lambda \Rightarrow A_\xi \not\subseteq A_\zeta$ also the demands in (3) hold. $\square_{2.17}$

{2.15} **Conclusion 2.18.** *1) For $\lambda > \aleph_0$, there is a Boolean algebra $\mathbf{B}$ of cardinality λ with no non-trivial endomorphism onto itself, moreover it is Bonnet rigid (defined below).*
2) Moreover, we can find such $\mathbf{B}_i$ (for $i < 2^\lambda$) such that for $i, j < 2^\lambda$, $a \in \mathbf{B}_i \setminus \{0\}$, $b \in \mathbf{B}_j \setminus \{0\}$ there is no embedding of $\mathbf{B}_i \restriction a$ into a homomorphic image of $\mathbf{B}_j \restriction b$ except when $i = j$ & $a \le b$.

We prove it later.

{2.15Anew} *Remark* 2.19. We shall use Boolean algebras built from cases of $BA_{\mathrm{trr}}(I)$, see
{2.1} Definition 2.1(4), hence has no long chains. We can go in the inverse direction using Boolean algebras built from orders, using, for example, $LO(I)$ is the linear order with elements $\{x_\eta, y_\eta : \eta \in I\}$ such that:

(1) $\lg(\eta) < \omega$ implies $x_\eta < y_\eta$, and $y_{\eta\hat{}\langle\alpha\rangle} < x_{\eta\hat{}\langle\beta\rangle}$ for $\alpha < \beta$ and $x_{\eta \restriction n} < x_\eta < y_\eta < y_{\eta \restriction n}$ for $n < \lg(\eta)$.
(2) $\lg(\eta) = \omega$ implies $x_{\eta\restriction n} < x_\eta = y_\eta < y_{\eta\restriction n}$ for $n < \omega$.

{2.19} In such cases we need a parallel to Lemma 2.23, which is true.

{2.15} {2.16} We make some preparations to the proof of 2.18.

Definition 2.20. A Boolean algebra $\mathbf{B}$ is called Bonnet–rigid iff there are no Boolean algebra $\mathbf{B}'$ and homomorphisms $\mathbf{f}_\ell : \mathbf{B} \to \mathbf{B}'$ (for $\ell = 0, 1$) such that $\mathbf{f}_0$ is one-to-one and $\mathbf{f}_1$ is onto $\mathbf{B}'$, except when $\mathbf{f}_0 = \mathbf{f}_1$.

{2.17} **Observation 2.21.** *1) If $\mathbf{B}$ is Bonnet–rigid then it has no onto endomorphism $\ne \mathrm{id}_{\mathbf{B}}$.*
2) A Boolean algebra $\mathbf{B}$ is Bonnet–rigid iff:

$(*)$ *For no disjoint non-zero $a, b \in \mathbf{B}$ is there an embedding of $\mathbf{B} \restriction a$ into a homeomorphic image of $\mathbf{B} \restriction b$.*

Proof. 1) Otherwise choose $\mathbf{B}' = \mathbf{B}$, h_0 the identity and h_1 the given endomorphism.
2) Suppose $\mathbf{f}_\ell : \mathbf{B} \longrightarrow \mathbf{B}'$ (for $\ell = 0, 1$) contradict Bonnet–rigidity. Suppose first $\mathbf{f}_1$ is not one-to-one, so for some $a \in \mathbf{B}$, $a \ne 0$, $\mathbf{f}_1(a) = 0$.

For any $b \in \mathbf{B}$, $\mathbf{f}_1(b - a) = \mathbf{f}_1(b) - \mathbf{f}_1(a) = \mathbf{f}_1(b)$. So $\mathbf{B}'$ is a homomorphic image of $\mathbf{B} \restriction (1 - a)$ and $\mathbf{B} \restriction a$ can be embedded into it, so we are finished.

Second, assume $\mathbf{f}_1$ is one-to-one, then $\mathbf{f}_1$ is an isomorphism from $\mathbf{B}$ onto $\mathbf{B}'$ hence $\mathbf{f}_1^{-1}\mathbf{f}_0 : \mathbf{B} \longrightarrow \mathbf{B}$ is an embedding (well defined as f_1 is one to one and onto). It is not the identity (otherwise $\mathbf{f}_0 = \mathbf{f}_1$) so for some $a \in \mathbf{B}$, the elements $a, \mathbf{f}_1^{-1}\mathbf{f}_0(a)$ are disjoint non-zero; choose $b = \mathbf{f}_1^{-1}\mathbf{f}_0(a)$. $\square_{2.21}$

We shall use $\mathrm{BA}_{\mathrm{trr}}(I)$ (see Definition 2.1(4)). Note: {2.1} {2.18}

Claim 2.22. *1) The only atoms of* $\mathrm{BA}_{\mathrm{trr}}(I)$ *are* x_η *where* η $(\in I)$ *has no immediate successor, or at least*

$$\eta \lhd \nu_1 \ \&\ \nu \lhd \nu_2 \quad \Rightarrow \quad \nu_1,\ \nu_2 \text{ are } \lhd\text{–comparable.}$$

2) The set $\{x_\eta : \eta \in I\}$ *is a dense subset of* $\mathrm{BA}_{\mathrm{trr}}(I)$.

Proof. Check. $\square_{2.22}$

{2.19}

Lemma 2.23. *If* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{B}_0 = \mathrm{BA}_{\mathrm{trr}}(I)$, *<u>then</u>* $\mathbf{B}$ *is isomorphic to some* $BA_{\mathrm{trr}}(J)$, J *weakly representable in* $\mathscr{M}_{\aleph_0,\aleph_0}(I)$ *hence* $\mathbf{B}$ *is weakly representable in* $\mathscr{M}_{\aleph_0,\aleph_0}(I)$.

Proof. So let $\mathbf{J}$ be an ideal of $\mathbf{B}_0$ such that $\mathbf{B}$ is isomorphic to $\mathbf{B}_0/\mathbf{J}$.
Let

$$I_1 = \{\eta \in I : x_\eta \notin \mathbf{J}\};$$

I_1 is an approximation to J. (Clearly I_1 is closed under initial segments by 2.1(4)(b).) Let {2.1}

$$\begin{aligned} A_0 = \ & \{\eta \in I_1 : \ \eta \text{ has } < \aleph_0 \text{ immediate successors in } I_1, \text{ say} \\ & \qquad \eta\hat{\ }\langle \alpha_\ell \rangle \ \text{ for } \ell < m, \ \text{ and } (x_\eta - \bigcup_\ell x_{\eta\hat{\ }\langle \alpha_\ell \rangle}) \in \mathbf{J}\}, \\ A_1 = \ & \{\eta \in I_1 : \ \eta \text{ has } < \aleph_0 \text{ immediate successors in } I_1, \text{ say} \\ & \qquad \eta\hat{\ }\langle \alpha_\ell \rangle \ \text{ for } \ell < m, \ \text{ and } (x_\eta - \bigcup_\ell x_{\eta\hat{\ }\langle \alpha_\ell \rangle}) \notin \mathbf{J}\}, \\ A_3 = \ & \{(\eta,\nu) : \ \eta \in A_0,\ \eta \lhd \nu \in I_1,\ \lg(\nu) \text{ is limit},\ x_\eta - x_{\nu\restriction i} \in \mathbf{J}, \\ & \qquad \text{when } \lg(\eta) \le i < \lg(\nu) \text{ and for no } \eta' \lhd \eta \text{ does } (\eta',\nu) \\ & \qquad \text{have those properties } \}, \end{aligned}$$

and let

$$A_4 = \{(\eta,\nu) \in A_3 : \nu \in I_1,\ x_\eta - x_\nu \notin \mathbf{J}\}.$$

Now for $\eta \in I$ let $\alpha_\eta = \mathrm{Min}\{\alpha : \eta\hat{\ }\langle \alpha \rangle \notin I\}$.
Put

$$\begin{aligned} J = \ \ I_1 \ & \cup \{\eta\hat{\ }\langle \alpha_\eta \rangle : \eta \in A_1\} \cup \\ & \cup \{\eta\hat{\ }\langle \alpha_\eta + 1 \rangle : (\eta,\nu) \in A_4\}. \end{aligned}$$

Now $\mathrm{BA}_{\mathrm{trr}}(J)$ is isomorphic to $\mathbf{B}$, and the lemma should be clear. $\square_{2.23}$

Now we can turn to

Proof. Proof of 2.18:
1) Let $\langle I_\alpha : \alpha < \lambda \rangle$ exemplify that $K^\omega_{\rm tr}$ has the full strong $(\lambda, \lambda, \aleph_0, \aleph_0)$–bigness property, I_α standard.

Without loss of generality

$(*)_1$ $\alpha \neq \beta \ \Rightarrow \ I_\alpha \cap I_\beta = \{\langle\rangle\}$

$(*)_2$ if $\nu \in I_\alpha$, then for some η we have $\nu \trianglelefteq \eta \in I_\alpha$ and $\lg(\eta) = \omega$.

{2.4} We construct as in 2.4, using $\mathrm{BA}_{\rm trr}(I_\alpha)$ (i.e., $x = $ trr there) but making the surgeries on atoms only getting $\mathbf{B} = \mathrm{Sur}\langle I_\alpha, a^*_\alpha : \alpha < \lambda\rangle$. Looking at the construction, it is clear that $\mathbf{B} = \mathrm{BA}_{\rm trr}(I^*)$, where $I^* = \{\eta_1 \hat{\ } \eta_2 \hat{\ } \ldots \hat{\ } \eta_n : n < \omega,\ \eta_\ell \in I_{\alpha_\ell}$ for some $\alpha_\ell < \lambda$, and for $\ell < n$ we have $\lg(\eta_\ell) = \omega$ and $a^*_{\alpha_\ell + 1}$ is $x_{\eta_\ell}\}$.

{2.17} By 2.21(2), it suffices to prove:

$(**)$ if a, b are disjoint non-zero, $\mathbf{B}'$ is a homomorphic image of $\mathbf{B} \restriction b$, then $\mathbf{B} \restriction a$ cannot be embedded into $\mathbf{B}'$.

{2.18} Suppose $(**)$ fails and a, b, $\mathbf{B}'$ exemplify this. By Claim 2.22 and $(*)$, there is $\eta \in I^*, x_\eta \leq a$, $\lg(\eta)$ limit, and let α be such that $a^*_\alpha = x_\eta$. Clearly $\mathbf{B}'$ is also
{2.19} a homomorphic image of $\mathbf{B} \restriction (1 - x_\eta)$, hence by 2.23 it is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}(\sum\limits_{j<\lambda, j\neq\alpha} I_j)$ and $\mathbf{B}' \cong BA_{\rm trr}(I^+)$ for some I^+ weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(\sum\limits_{j<\lambda, j\neq\alpha} I_j)$.

We can conclude:

$(***)$ $BA_{\rm trr}(I_\alpha)$ is weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(\sum\limits_{j<\lambda, j\neq\alpha} I_j)$.

But from this the contradiction is trivial (we could avoid the "weakly").
2) No new point. $\square_{2.18}$

§ 3. Arbitrary length of a Boolean Algebra with no small infinite homomorphic image

We recall the definition of the length (and length$^+$) of a Boolean algebra (Definition 3.2). Our aim is to construct a Boolean algebra of cardinality continuum with {5.0A} no infinite homomorphic image of smaller cardinality. Toward this, for a Boolean algebra $\mathbf{B}$, an ω-sequence $\langle a_n : n < \omega \rangle$ of pairwise disjoint members of $\mathbf{B} \setminus \{0_\mathbf{B}\}$ and $I \in K^\omega_{\mathrm{tr}(h)}$, we define in Definition 3.3 an extension $\mathbf{B}' = \mathrm{ba}[\mathbf{B}, \bar{a}, I]$ of $\mathbf{B}$, we {5.1} shall use it for h with $\langle h(n) : n < \omega \rangle$ going to infinity. The properties we need are: $\mathbf{B} \lessdot \mathbf{B}'$, $\|\mathbf{B}'\| \le 2^{\aleph_0}$, $\mathbf{B}'$ satisfies the ccc (see 3.4(1),(3), 3.5, and inside the proof {5.3} of 3.6), moreover a stronger version of $\mathbf{B} \lessdot \mathbf{B}'$ holds (see 3.4(5)). Also if $\mathbf{f}$ is a {5.2} homomorphism from $\mathbf{B}'$ into any Boolean algebra $\mathbf{B}''$ satisfying $n < \omega \Rightarrow \mathbf{f}(a_n) > 0$ (in $\mathbf{B}''$) then $\mathbf{B}'$ has at least $2^{\aleph_0}$ elements (see inside the proof of 3.6). Theorem {5.4} 3.6 is the main result: if $\mu \in [\aleph_1, 2^{\aleph_0}]$ then some ccc Boolean algebra $\mathbf{B}$ of cardinality $2^{\aleph_0}$ and length μ, has no infinite homomorphic image of cardinality $< 2^{\aleph_0}$, for this we take care of every antichain $\langle a_n : n < \omega \rangle$ by an extension $\mathrm{ba}[-, \bar{a}, I]$. We start with a ccc Boolean algebra of length and cardinality μ. In this framework we need to show that the length has not increased by the construction. For this we prove by induction on the length of the construction that for any family of μ^+ finite sequences from the Boolean algebra and $m < \omega$, there is a subfamily of μ^+ finite sequences which is an indiscernible set. We may like to consider a limit $\mu \in [\aleph_1, 2^{\aleph_0})$ and ask above that its length is μ but the suprimum is not obtained; by similar construction (of length $2^{\aleph_0} \times \mu$) we get such Boolean algebra provided that $\mathrm{cf}(\mu)$ is uncountable (see **??**); if $\mathrm{cf}(\mu) = \aleph_0$ this is impossible, (3.12). We then {5.7bis} generalize the results, replacing $\aleph_0$ by any strong limit κ of cofinality $\aleph_0$.

Convention 3.1. *h will be from ${}^\omega(\omega \setminus \{0\})$ and for simplicity ${}^\omega(\omega \setminus \{0, 1\})$. Actually* {5.0} *$h = 2$ (i.e. $(\forall n) h(n) = 2$, so using K^ω_{ptr}) suffices but if we like to have the ccc, better use $h \ge 3$.*

Definition 3.2. For a Boolean algebra $\mathbf{B}$ let {5.0A}

$$\begin{aligned} \mathrm{length}(\mathbf{B}) &= \sup\{|A| : A \subseteq \mathbf{B},\ A \text{ is linearly ordered by } <_\mathbf{B}\}, \\ \mathrm{length}^+(\mathbf{B}) &= \sup\{|A|^+ : A \subseteq \mathbf{B},\ A \text{ is linearly ordered by } <_\mathbf{B}\}. \end{aligned}$$

Definition 3.3. For a Boolean Algebra $\mathbf{B}^*$, $\bar{a} = \langle a_n : n < \omega \rangle$ with $a_n \in \mathbf{B}^* \setminus \{0_{\mathbf{B}^*}\}$ {5.1} for $n < \omega$, such that $\bigwedge\limits_{n<m} a_n \cap a_m = 0$, and $I \in K^\omega_{\mathrm{tr}(h)}$, we define a Boolean Algebra $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ as follows: it is generated by $\mathbf{B}^* \cup \{x_\eta : \eta \in I\}$ freely except the following equalities:

(*a*) all the equalities which $\mathbf{B}^*$ satisfies and $x_\eta \le 1_{\mathbf{B}^*}$,

(*b*) if $n < \omega$ is even, $k = h(n) - 1$, $\eta \in P^I_\omega$, $\nu = \eta \restriction n$, $\eta(n) = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \rangle$ then

$$a_n - \bigcup_{\ell < k/2} \left((x_{\nu\hat{\ }\langle \alpha_{2\ell} \rangle} - x_{\nu\hat{\ }\langle \alpha_{2\ell+1} \rangle} \right) \le x_\eta,$$

(*c*) if $n < \omega$ is odd, $k = h(n) - 1$, $\eta \in P^I_\omega$, $\nu = \eta \restriction n$, and $\eta(n) = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \rangle$ then

$$\Big(a_n \cap \bigcap_{\ell < k/2} \left(1 - (x_{\nu\hat{\ }\langle \alpha_{2\ell} \rangle} - x_{\nu\hat{\ }\langle \alpha_{2\ell+1} \rangle}) \right) \Big) \cap x_\eta = 0,$$

(*d*) $x_{\langle\rangle_I} = 0$, ($\langle\rangle_I$ is the root of I).

{5.2}
{5.1} **Claim 3.4.** *1) For* $\mathbf{B}^*, \bar{a}, I$ *as in Definition 3.3,* $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ *is an extension of* $\mathbf{B}^*$ *(so the equalities do not cause members of* $\mathbf{B}^*$ *to become identified and of course* $1_{\mathrm{ba}[\mathbf{B}^*,\bar{a},I]} = 1_{\mathbf{B}^*}$*).*

{5.1} *2) For* $I_1, I_2 \in K^{\omega}_{\mathrm{tr}(h)}$*,* $\mathbf{B}^*, \bar{a}$ *as in Definition 3.3, if* $I_1 \subseteq I_2$*, then* $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ *is a subalgebra of* $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$*.*

3) In (1), $\mathbf{B}^* \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$*.*

4) In (2), if also $I_1 \subseteq^* I_2$*, which means that* $I_1 \subseteq I_2$ *and*

$$\eta \in P^{I_2}_{\omega} \setminus I_1 \quad \Rightarrow \quad \bigvee_{n,\ell} \mathrm{Res}^{\ell}_n(\eta) \notin I_1,$$

then $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$*.*

5) In (4) for every non-zero $c \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ *there is* d^* *such that:*

(*i*) $c \le d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$,

(*ii*) *if* $0 < b \le d^*, b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ *then* $c \cap b \neq 0$*.*

Proof. 1) It is a particular case of (2) for $I_1 = \{\langle\rangle\}, I_2 = I$.

2) Let $d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \setminus \{0\}$. We would like to prove that $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2] \models d^* \neq 0$. Clearly without loss of generality for some $\alpha(*) \le \omega$ we have:

$$\alpha(*) < \omega \;\&\; d^* \le a_{\alpha(*)} \text{ or } \alpha(*) = \omega \;\&\; (\forall n)(d^* \cap a_n = 0).$$

Now we shall define a function $\mathbf{f} : \mathbf{B}^* \cup \{x_\eta : \eta \in I_2\} \longrightarrow \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \restriction d^*$, which will map all the equations appearing in the definition of $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ to ones satisfied in $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]\restriction d^*$ and maps d^* to itself; this suffices.

Now we define $\mathbf{f} = \mathbf{f}^{d^*}$ as follows:

(*A*) for $b \in \mathbf{B}^*, \mathbf{f}(b) = b \cap d^*$, or more exactly the interpretation of $b \cap d^*$ in $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$,

(*B*) for $\eta \in I_1$, $\mathbf{f}(x_\eta) = x_\eta \cap d^*$,

(*C*) if $\eta \in P^{I_2}_{\omega}$, $\eta \notin I_1$, let:

$$\begin{array}{ll} \mathbf{f}(x_\eta) = d^* & \text{if } \alpha(*) \text{ is even (including } \alpha(*) = \omega), \\ \mathbf{f}(x_\eta) = 0 & \text{if } \alpha(*) \text{ is odd (and } < \omega), \end{array}$$

(*D*) for $\eta \in I_2 \setminus I_1$ such that (C) does not apply let $\mathbf{f}(x_\eta) = 0$.

Now check, the main point being that the equations in clauses (b)+(c) of Definition
{5.1} 3.3 hold trivially by the present choice in clause (C).

3) Again, it suffices to prove this for the context of (2), i.e., to prove (4).

4) The proof of part (2) above will give it provided that we are given $c \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2] \setminus \{0\}$ and we then find $d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$, $d^* \neq 0$, such that we can construct a function $\mathbf{f}$ as there satisfying that the homomorphism $\hat{\mathbf{f}}$ which $\mathbf{f}$ induces from $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ into $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \restriction d^*$ (which is the identity on the latter by its definition) will satisfy $\hat{\mathbf{f}}(c) \ge d^*$. Now as we can decrease c, without loss of generality $c \notin \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ and c has the form

$(*)$ $c = d \cap \bigcap_{\ell < m_0} x_{\eta_\ell} \cap \bigcap_{\ell \in [m_0, m)} (1 - x_{\eta_\ell})$,

with $d \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \setminus \{0\}$, $\eta_\ell \in I_2 \setminus I_1$ for $\ell < m$. We shall show that more then necessary here holds (used in part (5)):

⊡ if $0 < d' \leq d, d' \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ then some d^* satisfying $0 < d^* \leq d', d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ is as required. i.e. there is $\mathbf{f}$ as in the previous proof.

Choose d^* and $\alpha(*) \leq \omega$ satisfying:

(i) $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \models 0 < d^* \leq d'$

(ii) $d^* \leq a_{\alpha(*)}$ & $\alpha(*) < \omega$ or $\bigwedge_{n<\omega} d^* \cap a_n = 0$ & $\alpha(*) = \omega$.

For $k < m$ let $I^*_{2,k} = I_1 \cup \{\eta_k\} \cup \{\mathrm{Res}^\ell_i(\eta_k) : \mathrm{Res}^\ell_i(\eta_k)$ is well defined$\}$ and for $k \leq m$ let $I_{2,k} = \cup\{I^*_{2,\ell} : \ell < k\} \cup I_1$. Easily $I_1 = I_{2,0} \subseteq^* I_{2,1} \subseteq^* \ldots \subseteq^* I_{2,m} \subseteq I_2$. Clearly $c \in \mathrm{ba}[\mathbf{B}, \bar{a}, I_{2,m}]$ and it suffice to prove that $\mathbf{B}^* \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_{2,m}]$, so without loss of generality $I_2 = I_{2,m}$. If $\lg(\eta_k) < \omega$, we can add the $\mathrm{Res}^\ell_i(\eta_k)$ to $I_{2,k}$ one by one.

As $\lessdot$ is transitive, and by part (2) without loss of generality $m = 1$, and one of the following occurs:

(a) $I_2 \setminus I_1 = \{\eta_0\}$ and $\lg(\eta_0) < \omega$,

(b) $I_2 \setminus I_1 \subseteq \{\eta_0,\ \mathrm{Res}^\ell_n(\eta_0) :\ n_0 \leq n < \omega,\ \ell < h(n)\}$ and $\lg(\eta_0) = \omega$.

In case (a), let $\mathbf{f}(x_{\eta_0})$ be d^* if $m_0 = 1$, and let $\mathbf{f}(x_{\eta_0})$ be 0 if $m_0 = 0$ (and $\mathbf{f}(b) = b \cap d^*$ if $b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}.I_1]$). In case (b), if $\alpha(*) = \omega$ act similarly, i.e., define $\mathbf{f}(x_\nu) = d^*$ for $\nu \in I_2 \backslash I_1$ if $m_0 = 1$, and 0 if $m_0 = 0$ for $\eta \in I_2 \setminus I_1$. In case (b), if $\alpha(*) < \omega$, by repeated use of case (a) without loss of generality (but we do not have to use it)

$$(\forall n \leq \alpha(*))(\forall \ell < h(n))(\mathrm{Res}^\ell_n(\eta_0) \in I_1).$$

Let

$$\begin{array}{ll} \mathbf{f}(b) = b \cap d^* & \text{for } b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1], \\ \mathbf{f}(x_{\eta_0}) = d^* & \text{if } \alpha(*) \text{ is even}, \\ \mathbf{f}(x_{\eta_0}) = 0 & \text{if } \alpha(*) \text{ is odd}, \\ \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta_0)}) = 0 & \text{whenever } n < \omega,\ \ell < h(n) \text{ and } \mathrm{Res}^\ell_n(\eta_0) \notin I_1. \end{array}$$

Now check.

5) Again without loss of generality I_1, I_2 satisfy **(a)** or **(b)** from the proof of (4) (use 3.4(2) and the transitivity of the conclusion) and even c is in subalgebra of {5.2}
$\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ generated by $\{x_{\eta_0}\} \cup \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$. Note also that if $c = c_1 \cup c_2$ it suffices to prove the conclusion for c_1 and for c_2.

So without loss of generality $(*)$ in the proof of part (4) holds, so $c \leq d$, and by the proof of part (4), d is as required. $\square_{3.4}$

{5.3}
Claim 3.5. *Assume $h \geq 3$ or just for n large enough, $h(n) \geq 3$. If $\mathbf{B}^*, \bar{a}, I$ are as in Definition 3.3, I standard, $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and $\mathbf{B}^*$ satisfies the [strong] λ–cc, {5.1}
then $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ satisfies the [strong] λ–cc.*

Proof. Let $c_i \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ for $i < \lambda$, $c_i \neq 0$. Without loss of generality c_i has the form

$$c_i = d_i \cap \bigcap_{\ell < m_{i,0}} x_{\eta_{i,\ell}} \cap \bigcap_{\ell \in [m_{i,0}, m_{i,1})} (1 - x_{\eta_{i,\ell}}),$$

where $\eta_{i,\ell} \in I$, $d_i \in \mathbf{B}^* \setminus \{0\}$. Without loss of generality $d_i \leq a_{n_i}$ for some $n_i < \omega$ or $n_i = \omega \& \bigwedge\limits_{n<\omega} d_i \cap a_n = 0$. Without loss of generality $m_{i,0} = m_0$, $m_{i,1} = m_1$, $\lg(\eta_{i,\ell}) = n_\ell$, $n_i = n^*$; $\langle \eta_{i,\ell} : \ell < m_1 \rangle$ is with no repetition for every i.

Also letting $k_i < \omega$ be the minimal k such that

(∗) (a) $\ell g(\eta_{i,\ell}) < \omega \Rightarrow \lg(\eta_{i,\ell}) \leq k$,
(b) $n^* < \omega \Rightarrow n^* < k$,
(c) $\ell_1 < \ell_2 < m_1 \Rightarrow \eta_{i,\ell_1} \restriction k \neq \eta_{i,\ell_2} \restriction k$,
(d) $(\forall n)(k \leq n \to h(n) \geq 3)$

and without loss of generality $k_i = k^*$ and if $\lg(\eta_{i,\ell}) = \omega$, $k < k^*, \ell < h(k)$ then $\mathrm{Res}^\ell_k(\eta_{i,\ell}) \in \{\eta_{i,m} : m < m_{i,1}\}$.

By the Δ–system argument, without loss of generality

(∗) if $i \neq j < \lambda$, $k \leq k^* + 1$, and $m', m'' < m_1$, $\ell', \ell'' < h(k)$ and $\mathrm{Res}^{\ell'}_k(\eta_{i,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{j,m'',j})$, <u>then</u> for every $\alpha, \beta < \lambda$ we have:

$$\mathrm{Res}^{\ell'}_k(\eta_{\alpha,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{\alpha,m''}) = \mathrm{Res}^{\ell'}_k(\eta_{\beta,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{\beta,m''}).$$

{2.6} {5.4} We can now check, (similarly to 2.6). $\square_{3.5}$

Theorem 3.6. *Let $\aleph_1 \leq \mu < 2^{\aleph_0}$. There is a Boolean Algebra* $\mathbf{B}$ *such that:*

(A) $\mathbf{B}$ *has cardinality $2^{\aleph_0}$ and satisfies the* ccc*(and even the strong λ-c.c. if $\lambda = c\ell(\lambda) > \aleph_0$)*

(B) $\mathbf{B}$ *has length μ (i.e., there is in* $\mathbf{B}$ *a chain of length μ but no chain of length μ^+),*

moreover

$(B)^+$ *if $n, m < \omega$, $\bar{c}^\zeta \in {}^m\mathbf{B}$ for $\zeta < \mu^+$* <u>*then*</u> *for some $Y \in [\mu^+]^{\mu^+}$ (i.e., $Y \subseteq \mu^+$ of cardinality μ^+), the* sequence *$\langle \bar{c}^\zeta : \zeta \in Y \rangle$ is a* (qf, n)*-indiscernible set*
{5.4A} *(in the Boolean algebra* $\mathbf{B}$*; set not sequence; see 3.7(2) below),*

(C) *every infinite homomorphic image of* $\mathbf{B}$ *has cardinality $2^{\aleph_0}$.*

{5.4A}

Remark 3.7. 1) Note that $(B)^+ \Rightarrow (B)$, for it $m = 1$ suffices, for this constant h is O.K., but the proof here is simpler.
2) $\langle \bar{c}^\zeta : \zeta \in Y \rangle$, a sequence of m–tuples from a model M (for example a Boolean algebra) is an (Δ, n)-indiscernible set <u>iff</u> for any $\zeta_0, \ldots, \zeta_{n-1}$ from Y with no repetitions and $\xi_0, \ldots, \xi_{n-1}$ from Y with no repetitions, the Δ-type of $\bar{c}^{\zeta_0} {}^\frown \ldots {}^\frown \bar{c}^{\zeta_{n-1}}$ in M is equal to the Δ-type of $\bar{c}^{\xi_0} {}^\frown \ldots {}^\frown \bar{c}^{\xi_{n-1}}$ in M. For Δ the set of quantifier free formulas we write qf.

Proof. Let $h : \omega \longrightarrow \omega$ be for example $h(n) = 2n + 2$.

Let $I_\beta \in K^\omega_{\mathrm{tr}(h)}$, for $\beta < 2^{\aleph_0}$, be standard, have cardinality continuum and be such that:

$(*)_{I_\beta}$ for every $f : I_\beta \to \theta$, $\theta < 2^{\aleph_0}$, for some $\eta \in P^{I_\beta}_\omega$ for every $n < \omega$,

$$(\forall \ell < h(n))(f(\mathrm{Res}^0_n(\eta)) = f(\mathrm{Res}^\ell_n(\eta)))$$

$$(\text{i.e., } \eta(m) = \langle \alpha_\ell : \ell < h(n)\rangle \Rightarrow |\{f(\eta \restriction n\hat{\ }\langle \alpha_\ell\rangle) : \ell < h(n)\}| = 1.)$$

[Why such I's exist? The full tree will serve, that is we let

$$I_\beta = \{\langle \bar\alpha^\ell : \ell < \gamma\rangle : \begin{array}{l} \gamma \le \omega,\ \bar\alpha^\ell \text{ an increasing sequence of length } h(\ell) \\ \text{from } 2^{\aleph_0},\ \text{except in the case } 0 < \gamma < \omega \ \&\ \ell = \gamma - 1 \\ \text{then we demand } \bar\alpha^\ell \text{ is just an ordinal } < 2^{\aleph_0}\}. \end{array}$$

This is as required as for any $f : I_\beta \longrightarrow \theta$ we can choose by induction on $\ell < \omega$, a sequence $\bar\alpha_\ell = \langle \beta_{\ell,0}, \ldots, \beta_{\ell,h(\ell)-1}\rangle$, where $\beta_{\ell,0} < \ldots < \beta_{\ell,h(\ell)-1} < 2^{\aleph_0}$ and $f(\langle \bar\alpha^0, \ldots, \bar\alpha^{\ell-1}, \beta_{\ell,i}\rangle)$ does not depend on i, possible as $2^{\aleph_0} > |\mathrm{rang}(f)|$. So I_β's as required in $(*)_{I_\beta}$ exists indeed.]

We shall now construct $\mathbf{B}_\alpha$ (for $\alpha \le 2^{\aleph_0}$), $\bar a^\alpha = \langle a^\alpha_n : n < \omega\rangle$ such that:

(I) (a) $\mathbf{B}_0$ is a subalgebra of $\mathscr{P}(\omega)$ of cardinality μ with a chain of cardinality μ satisfying the ccc (even the strong $\aleph_1$-cc), [e.g. let A be a set of μ reals, let h be a one to one function from ω onto the rationals and $\mathbf{B}$ is the Boolean algebra of subset of ω generated by $\{\{n : h(n) < a\} : a \in A\}$. Clearly $\mathfrak{B}$ has a linearly ordered subset of cardinality μ, e.g. it sets of generators. Of course, its length is not $> \mu$ as its cardinality is μ.]

(b) $\mathbf{B}_\alpha$ is increasing continuous, of cardinality $2^{\aleph_0}$ if $\alpha > 0$,

(c) $\bar a^\alpha$ is an ω-sequence of pairwise disjoint non-zero elements of $\mathbf{B}_\alpha$,

(d) if $\alpha < 2^{\aleph_0}$, $a_n \in \mathbf{B}_\alpha \setminus \{0_{\mathbf{B}_\alpha}\}$, $\bigwedge\limits_{n \ne m} a_n \cap a_m = 0$ then for $2^{\aleph_0}$ many ordinals α we have: $\bigwedge\limits_{n<\omega} a_n = a^\alpha_n$

[you can demand $\{a^\alpha_n : n < \omega\}$ is maximal antichain; does not matter],

(e) $\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar a^\alpha, I_\alpha]$, we denote the x_η by x^α_η for $\eta \in I_\alpha$.

There is no problem to do the bookkeeping, and $\mathbf{B}_\alpha \subseteq \mathbf{B}_{\alpha+1}$ by 3.4(1). We shall show that $\mathbf{B} := \mathbf{B}_{2^{\aleph_0}}$ is as required. Obviously $\mathbf{B}$ has cardinality $2^{\aleph_0}$. {5.2}

By 3.4(3) clearly $\mathbf{B}_\alpha \lessdot \mathbf{B}_{\alpha+1}$, so we can prove by induction on α that $\beta < \alpha \Rightarrow$ {5.2}
$\mathbf{B}_\beta \lessdot \mathbf{B}_\alpha$, by 2.9, 2.10. We can also prove by induction on α that $\mathbf{B}_\alpha$ satisfies the {2.7}{2.8}
$\aleph_1$–cc (even the strong λ-cc when $\lambda = \mathrm{cf}(\lambda) > \aleph_0$): the successor stage by 3.5, the {5.3}
limits steps by 2.10. So demand (A) from 3.6 holds. If $\mathbf{f}$ is a homomorphism from {2.8}{5.4}
$\mathbf{B}$ onto some $\mathbf{B}'$, $\aleph_0 \le \|\mathbf{B}'\| < 2^{\aleph_0}$ then there are $b_n \in \mathbf{B}' \setminus \{0\}$ pairwise disjoint, now for some $a_n \in \mathbf{B}$, $\mathbf{f}(a_n) = b_n$ and without loss of generality $\bigwedge\limits_{n \ne m} a_n \cap a_m = 0$ (otherwise use $a'_n = a_n \setminus \bigcup\limits_{m<n} a_m$). Hence for every infinite co-infinite $Y \subseteq \omega$ for some $\alpha = \alpha_Y$:

$$\{a^\alpha_{2n} : n < \omega\} = \{a_n : n \in Y\} \quad \text{and} \quad \{a^\alpha_{2n+1} : n < \omega\} = \{a_n : n \in \omega \setminus Y\}.$$

Now define $g : I_\alpha \to \mathbf{B}'$ by $g(\eta) = \mathbf{f}(x^\alpha_\eta)$, so (by the choice of the I_α's, i.e., by $(*)_{I_\beta}$) for some $\eta = \eta_Y \in P^{I_\alpha}_\omega$, for every n, letting $\eta(n) = \langle \alpha_0, \alpha_i, \ldots, \alpha_{h(n)-1}\rangle$ we have

$$\bigwedge_{\ell<h(n)} f(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_\ell\rangle}) = f(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_0\rangle}).$$

Hence $\mathbf{f}\big(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_\ell\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{\ell+1}\rangle}\big) = 0_{\mathbf{B}'}$ for $\ell < h(n)-1$ and hence

$$\mathbf{f}\big(a^\alpha_n \cap \bigcap_{\ell<\frac{h(n)-1}{2}} \big(1-(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell}\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell+1}\rangle})\big)\big) = \mathbf{f}(a^\alpha_n)$$

and

$$\mathbf{f}\big(a^\alpha_n - \bigcup_{\ell<\frac{h(n)-1}{2}} \big(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell}\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell+1}\rangle}\big)\big) = \mathbf{f}(a^\alpha_n).$$

{5.1} Hence (see Definition 3.3)

$$\begin{array}{rcl} n \text{ is even} & \Rightarrow & \mathbf{B}' \models \mathbf{f}(a^\alpha_n) \le \mathbf{f}(x^\alpha_\eta), \\ n \text{ is odd} & \Rightarrow & \mathbf{B}' \models \mathbf{f}(a^\alpha_n) \cap \mathbf{f}(x^\alpha_\eta) = 0. \end{array}$$

Therefore,

$$\begin{array}{rcccl} m \in Y & \Rightarrow & \text{for some even } n,\ a^\alpha_n = a_m & \Rightarrow & \mathbf{B}' \models b_m \le \mathbf{f}(x^\alpha_\eta), \\ m \in \omega\setminus Y & \Rightarrow & \text{for some odd } n,\ a^\alpha_n = a_m & \Rightarrow & \mathbf{B}' \models b_m \cap \mathbf{f}(x^\alpha_\eta) = 0. \end{array}$$

As this occurs for every infinite co-infinite $Y \subseteq \omega$, for some $\alpha = \alpha_Y$, and $\eta = \eta_Y$, clearly we get $2^{\aleph_0}$ many distinct members of $\mathbf{B}'$, simply the $\mathbf{f}(x_{\eta_Y})$ a contradiction.
{5.4} So demand **(C)** of 3.6 holds.

What about the length, i.e, clauses **(B)** and $\mathbf{(B)}^+$? For **(B)**, first note that $\mathbf{B}_0$ has a chain of cardinality μ and hence so does $\mathbf{B}$. If $J \subseteq \mathbf{B}$ is a chain, $|J| = \mu^+$, then $(\mathbf{B})^+$ give contradiction and even the weakly "indiscernible sequence" version does because as $\mathbf{B} \models$ ccc, it has no subset of order type μ^+ or $(\mu^+)^*$; but the variant of $\mathbf{(B)}^+$ implies just this ($m = 1$ suffices).

So it suffices to prove that clause $(\mathbf{B})^+$ holds for $\mathbf{B}_\alpha$ by induction on α.

Case 1: $\alpha = 0$

Trivial (can get $\bar{c}^\zeta$ constant on $Y \in [\mu^+]^{\mu^+}$).

Case 2: α is limit, $\mathrm{cf}(\alpha) \neq \mu^+$

For some $\beta < \alpha$,

$$Y_1 = \{\zeta < \mu^+ : \bar{c}^\zeta \subseteq \mathbf{B}_\beta\} \in [\mu^+]^{\mu^+},$$

(note that if $\mathrm{cf}(\alpha) < \mu^+$, then we can get $Y_1 = \mu^+$) and use the induction hypothesis.

Case 3: $\mathrm{cf}(\alpha) = \mu^+$

Let $\langle \beta_\epsilon : \epsilon < \mu^+\rangle$ be an increasing continuous sequence with limit α. Let $n, m, \langle \bar{c}^\zeta : \zeta < \mu^+\rangle$ be given. Without loss of generality $\bar{c}^\zeta = \langle c^\zeta_\ell : \ell < m\rangle$ is a partition of $1_{\mathbf{B}_\alpha}$ (i.e., $\ell_1 \neq \ell_2 \Rightarrow c^\zeta_{\ell_1} \cap c^\zeta_{\ell_2} = 0$ and $1_{\mathbf{B}_\alpha} = \bigcup_{\ell<m} c^\zeta_\ell$). For each $\zeta < \mu^+$, we can find $a^\zeta_\ell, b^\zeta_\ell \in \mathbf{B}_{\beta_\zeta}$ such that:

(a) $a^\zeta_\ell \le c^\zeta_\ell \le b^\zeta_\ell$,

(b) $0 < x \le b^\zeta_\ell - a^\zeta_\ell$ & $x \in \mathbf{B}_{\beta_\zeta} \Rightarrow x \cap c^\zeta_\ell - a^\zeta_\ell \ne 0$.

[Why? By use of 3.4(5)). If ζ is limit, for some $f(\zeta) < \zeta$ we have $\{a^\zeta_\ell, b^\zeta_\ell : \ell < m\} \subseteq \mathbf{B}_{f(\beta_\zeta)}$. By Fodor lemma for some $\epsilon(*) < \mu^+$ and a stationary set $S \subseteq \mu^+$, we have $\bigwedge_{\zeta \in S} f(\zeta) = \epsilon(*)$. {5.2}

So

(c) $\epsilon \in S \Rightarrow \{a^\varepsilon_\ell, b^\varepsilon_\ell : \ell < m\} \subseteq \mathbf{B}_{\beta_{\dot\epsilon(*)}}$.

Also without loss of generality

(d) if $\varepsilon < \zeta \in S$ then $\{c^\varepsilon_\ell : \ell\} \subseteq \mathbf{B}_{\beta_\zeta}$.

Now apply the induction hypothesis on $\mathbf{B}_{\beta_{\dot\epsilon(*)}}, \langle \bar a^\zeta{}^\frown \bar b^\zeta : \zeta < \mu^+\rangle$ where $\bar a^\zeta = \langle a^\zeta_\ell : \ell < m\rangle, \bar b^\zeta := \langle b^\zeta_\ell : \ell < m\rangle$.

So there is $Y \in [S]^{\mu^+}$ such that $\langle \bar b^\zeta : \zeta \in Y\rangle$ is an (n, qf)-indiscernible set. So let $\zeta_0 < \ldots < \zeta_{n-1}$ be from S and for $k \le n$ let $\mathbf{B}'_k$ be the subalgebra of $\mathbf{B}$ generated by $X_k = \{\bar b^{\zeta_i}_\ell, b^{\zeta_i}_\ell : i < n, \ell < m\} \cup \{\bar c^{\zeta_i}_\ell : i < k, \ell < m\}$. We understand $\mathbf{B}'_0$ by the choice of Y and we can understand $\mathbf{B}'_n$ by clauses (c) + (d) above.

Case 4: $\alpha = \beta + 1$

Let $n, m < \omega$ and $\bar c^\zeta \in {}^m(\mathbf{B}_{\beta+1})$ for $\zeta < \mu^+$ be given, $\bar c^\zeta = \langle c^\zeta_\ell : \ell < m\rangle$. So there are $k_{\zeta,0} = k(\zeta,0) < \omega, k_{\zeta,1} = k(\zeta,1) < \omega$ and $b^\zeta_0, \ldots, b^\zeta_{k_{\zeta,0}-1} \in \mathbf{B}_\beta$, $\eta^\zeta_0, \ldots, \eta^\zeta_{k_{\zeta,1}-1} \in I_\beta$ and Boolean terms σ^ζ_ℓ (for $\ell < m$) such that

$$c^\zeta_\ell = \sigma^\zeta_\ell(b^\zeta_0, \ldots, b^\zeta_{k_{\zeta,0}-1}, x_{\eta^\zeta_0, \varrho^\zeta_0}, \ldots, x_{\eta^\zeta_{k(\zeta,1)}, \varrho^\zeta_{k(\zeta,1)-1}}).$$

Without loss of generality $\langle \eta^\zeta_\ell : \ell < k_{\zeta,1}\rangle$ is a $\triangle$-system.

Without loss of generality $k_{\zeta,0} = k_0$, $k_{\zeta,1} = k_1$, $\sigma^\zeta_\ell = \sigma_\ell$ and $\lg(\eta^\zeta_\ell) = m_\ell \le \omega$ for every $\zeta < \mu^+$.

Also there is $k_{\zeta,2} < \omega$ such that:

II (α) $\lg(\eta^\zeta_\ell) < \omega \Rightarrow \lg(\eta^\zeta_\ell) < k_{\zeta,2}$,

(β) $\eta^\zeta_{\ell_1} \ne \eta^\zeta_{\ell_2} \Rightarrow \eta^\zeta_{\ell_1} \restriction k_{\zeta,2} \ne \eta^\zeta_{\ell_2} \restriction k_{\zeta,2}$,

(γ) $\varrho^\zeta_{\ell_1} \ne \varrho^\zeta_{\ell_2} \Rightarrow \varrho^\zeta_{\ell_1} \restriction k_{\zeta,2} \ne \varrho^\zeta_{\ell_2} \restriction k_{\zeta,2}$

(δ) $2n + 2 < k_{\zeta,2}$.

Without loss of generality $\bigwedge_\zeta k_{\zeta,2} = k_2$.

Without loss of generality the statement $(*)$ (with k_2 here for k^* there and is $> n$) from the proof of 3.5 holds (essentially being Δ-system), i.e., {5.3}

$(*)$ if $i \ne j < \lambda$, $k \le k_2 + 1$, and $m', m'' < m_1$, $\ell', \ell'' < h(k)$ and $\mathrm{Res}^{\ell'}_k(\eta_{m',i}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',j})$, then for every $\alpha, \beta < \lambda$ we have:

$$\mathrm{Res}^{\ell'}_k(\eta_{m',\alpha}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',\alpha}) = \mathrm{Res}^{\ell'}_k(\eta_{m',\beta}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',\beta}).$$

Let $\bar{b}^\zeta = \langle b^\zeta_\ell : \ell < k_0\rangle$. By the induction hypothesis without loss of generality $\langle \bar{b}^\zeta{}^\frown \langle a^\beta_\ell : \ell \le k_2\rangle : \zeta < \mu^+\rangle$ is (qf, n)-indiscernible and without loss of generality the sequence $\langle\langle \eta^\zeta_\ell \restriction (k_2+1) : \ell < k_1\rangle : \zeta < \mu^+\rangle$ is indiscernible (sequence of finite sequences of ordinals).

{5.4F} To finish the proof of 3.6 it suffices to observe 3.8 below. $\square_{1.13}$

Observation 3.8. *If* $\mathbf{B}^* = \mathrm{ba}_2[\mathbf{B},\bar{a}], n^* < \omega, I^0 = \{\eta \in I : \lg(\eta) \le n^*\}$, $Z \subseteq P^I_\omega$, *and for every* $\nu \in Z$ *and* $n \ge n^*$ *the set* $\{\nu' \in Z : \nu' \restriction n = \nu \restriction n\}$ *has* $< \lfloor h(n)/2\rfloor$ *elements, then* $\{x_\eta : \eta \in Z\}$ *is independent in* $\mathbf{B}^*$ *over* $\mathbf{B}^0 =: \mathrm{ba}_2[\mathbf{B},\bar{a}, I^0]$, *except the equations* $c^+_\eta \le x_\eta$ & $c^-_\eta \cap x_\eta = 0$ *for* $\eta \in Z$ *where*

$$\begin{aligned} c^+_\eta &= \bigcup\Big\{a_{2n} - \bigcup\{x_{\mathrm{Res}^{2\ell}_{2n}(\eta)} - x_{\mathrm{Res}^{2\ell+1}_{2n}(\eta)} : \ell < h(n)/2\} : 2n < n^*\Big\},\\ c^-_\eta &= \bigcup\Big\{a_{2n+1} - \bigcup\{x_{\mathrm{Res}^{2\ell}_{2n+1}(\eta)} - x_{\mathrm{Res}^{2\ell+1}_{2n+1}(\eta)} : 2\ell+1 < h(2n+1)\} :\\ &\qquad\qquad 2n+1 < n^*\Big\}.\end{aligned}$$

(Note: $\eta_1 \restriction n^* = \eta_2 \restriction n^* \quad \Rightarrow \quad (c^+_{\eta_1}, c^-_{\eta_1}) = (c^+_{\eta_2}, c^-_{\eta_2})$.*)*

Proof. Let $\mathbf{f}_0$ be any function with domain $X = \{x_\eta : \eta \in Z\}$ and such that $\mathbf{f}_0(x_\eta) \in \{c^+_\eta, 1_{\mathbf{B}} - c^-_\eta\}$ and

$$J^1 = I^0 \cup X \cup \{\mathrm{Res}^\ell_n(\nu) : \nu \in Z,\ \ell < h(n),\ n < \omega\}.$$

{5.2} It clearly by 3.4(2) suffices to find a homomorphism from $\mathbf{B}_1 =: \mathrm{ba}[\mathbf{B},\bar{a}, J^1]$ into $\mathbf{B}^0$ extending $\mathrm{id}_{\mathbf{B}^0} \cup \mathbf{f}_0$. For this it suffices to find a mapping $\mathbf{f}$ from $\mathbf{B} \cup \{x_\eta : \eta \in J^1\}$ into $\mathbf{B}^0$ extending $\mathrm{id}_{\mathbf{B}^0}, \mathbf{f}_0$ and $\mathrm{id}_{\{x_\eta:\eta\in I^0\}}$ and preserving the equations defining $\mathrm{ba}[\mathbf{B},\bar{a}, J^1]$. As $\mathbf{f} \restriction \mathbf{B}$, $\mathbf{f} \restriction \{x_\eta \in I : \lg(\eta) \le n^*\}$, and $\mathbf{f} \restriction \{x_{(\eta,\varrho)} : \eta \in Z\}$ are defined and

$$J^1 = X \bigcup_{n\in[n^*,\omega)} \{x_\eta : \eta \in Z_n\} \cup X,$$

where $Z_n = \{\eta \in J^1 : \lg(\eta) = n+1\}$, to finish the definition of $\mathbf{f}$ we shall choose $\mathbf{f} \restriction \{x_\eta : \eta \in Z_n\}$ for each $n \in [n^*,\omega)$. Let $\mathbf{E}_n$ be the two place relation on Z_n defined by

$$\nu_1\ \mathbf{E}_n \nu_2 \Leftrightarrow \nu_1 \restriction (n-1) = \nu_2 \restriction n.$$

$$\nu_1\ \mathbf{E}_n \nu_2 \Leftrightarrow \nu_1 \restriction (n-1) = \nu_2 \restriction (n-1).$$

Clearly $\mathbf{E}_n$ is an equivalence relation and let $\langle X_{n,\ell} : \ell < \ell_n\rangle$ list the $\mathbf{E}_n$-equivalence classes. For each $\ell < \ell_n$ there is $\nu_{n,\ell}$, a sequence of length n such that $\rho \in Y_{n,\ell} \Rightarrow \nu_{n,\ell} \triangleleft \rho$. Let $\mathscr{P}_{n,\ell} = \{\{\mathrm{Res}^{\mathrm{tr}}_n(\eta) : \eta \in Z$ satisfies $\eta\restriction n = \nu_{n,\ell}\}$, so $\mathscr{P}_{n,\ell}$ is a family of $< \lfloor h(n)/2\rfloor$ subsets of $Y_{n,\ell}$ each with exactly $h(n)$ elements$\}$.

So easily there is a function $f_{n,\ell}$ such that:

- $(*)$ $f_{n,\ell}$ is a function with domain $Y_{n,\ell}$ into $\mathbf{B}^0$ such that: if $\nu_{n,\ell} \triangleleft \eta \in Z$ then for some $k < \lfloor h(n)/2\rfloor$ we have $f_{n,\ell}(\mathrm{Res}^{2k+1}_n(\eta)) = 0, f_{n,\ell}(\mathrm{Res}^{2k}_n(\eta)) = 1$.

By finite cardinality consideration this is possible. Now define $\mathbf{f}\restriction Z_n$ by:

(∗) if $\eta \in Y_{n,\ell}$ then $\mathbf{f}(x_\eta)$ is $1_{\mathbf{B}^0}$ if $f_{n,\ell}(\eta) = 1$ and is $0_{\mathbf{B}^0}$ if $f_{n,\ell}(\eta) = 0$.

Now check, (i.e. the relevant new cases of clause (d)(i)+(ii) of Definition **??** becomes trivial). $\square_{3.8}$ {5.11}

{5.16}

Discussion 3.9. 1) In the proof of clause $(B)^+$, the successor case we use the fact that $h(n)$ converges to ∞, as when the level increases we need more $\eta \in P^{I_\beta}_\omega$ to see non-freeness.
2) The proof there for limit α uses just "$\langle \mathbf{B}_i : i \le 2^{\aleph_0}\rangle$ is $\lessdot$–increasing continuous moreover with projections (i.e. 3.4(5))", and the induction hypothesis. {5.2}
3) We can vary the construction in some ways. We can demand that each $\bar{a}^\alpha$ is a maximal antichain — no difference so far. We may like to use $\langle I_\beta : \beta < 2^{\aleph_0}\rangle$ such that I_β is not super unembeddable[0] into $\sum\limits_{\gamma \ne \beta} I_\gamma$. We can construct our Boolean Algebra to be monorigid (i.e., with no one-to-one endomorphism), and even get $2^{2^{\aleph_0}}$ such Boolean Algebras no one embeddable to another, even restricting to non-zero elements, even not embeddable into the completion of another. To carry out this we need for $\lambda = 2^{\aleph_0}$: there is $\bar{I} = \langle I_\alpha : \alpha < \lambda\rangle$ exemplifying that $K^\omega_{\mathrm{tr}(h)}$ has the full $(\lambda, \lambda, \aleph_1, \aleph_1)$–supper bigness property, such that at least for one β, I_β satisfies $(*)_{I_\beta}$ from the beginning of the proof of 3.6. Now such $\bar{I}$ exists (with $(*)_{I_\beta}$ for every β), may be done elsewhere. {5.4}
4) Of course the proof works for $\mu = 2^{\aleph_0}$.
5) We can separate some parts of the proof to independent claims. We can ask for "$\mathbf{B}$ has length μ, but no chain of cardinality μ (i.e., the supremum is not obtained)" for μ limit. It is natural to demand $\mathrm{cf}(\mu) > \aleph_0$, next we address this.

{5.17}

Claim 3.10. *Assume* $2^{\aleph_0} \ge \mu$, $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$. *Then there is a Boolean algebra* $\mathbf{B}$, $|\mathbf{B}| = 2^{\aleph_0}$, $\mathbf{B}$ *has no homomorphic image of cardinality* $\in [\aleph_0, 2^{\aleph_0})$ *and* $\mathrm{length}(\mathbf{B}) = \mu$, *but the supremum is not obtained, i.e.,* $\mathrm{length}^+(\mathbf{B}) = \mu$ *and every infinite homo image* $\mathbf{B}'$ *of* $\mathbf{B}$ *has length* $\ge \mu$.

Proof. Like 3.6. {5.4}

Let $\mu = \sum\limits_{i<\kappa} \mu_i$ with $\langle \mu_i : i < \kappa\rangle$ is increasing continuous and $\kappa < \mu_i < \mu$. For $\epsilon < \kappa$, let $\mathbf{B}^\epsilon$ be a subalgebra of $\mathscr{P}(\omega)$ of cardinality μ_ϵ and length μ_ε. Let $\langle I_\alpha : \alpha < 2^{\aleph_0} \times \kappa\rangle$ be as in the proof of 3.6. We define $\mathbf{B}_\alpha$ (for $\alpha \le 2^{\aleph_0} \times \kappa$) similarly {5.4}
to the proof of 3.6, specifically: {5.4}

$$\begin{aligned}
&\mathbf{B}_0 = \mathbf{B}^0, \text{ the trivial boolean algebra}\\
&\mathbf{B}_{2^{\aleph_0} \times \epsilon + 1} \text{ is the free product } \mathbf{B}_{2^{\aleph_0} \times \epsilon} * \mathbf{B}^\epsilon,\\
&\mathbf{B}_\alpha \text{ is increasing continuous in } \alpha,\\
&\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar{a}^\alpha, I_\alpha] \text{ for } \alpha < 2^{\aleph_0} \times \kappa,\ \alpha \notin \{2^{\aleph_0} \times \epsilon : \epsilon < \kappa\},
\end{aligned}$$

where $\bar{a}_\alpha$ is $\langle a_{\alpha,n} : n < \omega\rangle, a_{\alpha,n} \in \mathbf{B}_\alpha, a_{\alpha,n} > 0, n_1 \ne n_2 \Rightarrow a_{\alpha,n_1} \cap a_{\alpha,n_2} = 0$. The choice of the $\bar{a}_\alpha$'s, i.e. the bookkeeping is as above.

So, by the proof of 4.14: {5.14}

(∗) if $0 < \alpha < 2^{\aleph_0} \times \kappa$ <u>then</u>
  (α) $\mathbf{B}_\alpha$ satisfies the strong λ-cc if $\lambda = \mathrm{cf}(\lambda) > \aleph_0$
  (β) $\mathbf{B}_{1+\alpha}$ has length $\mu_0 + \sum\{\mu_\epsilon : 2^{\aleph_0} \times \epsilon < \alpha\} < \mu$

(γ) if $\alpha = 2^{\aleph_0} \times \epsilon, \epsilon$ a successor ordinal then $\mathbf{B}_\alpha$ has no homomorphic image of cardinality $\in [\aleph_0, 2^{\aleph_0})$

(δ) if $\alpha < \beta \leq 2^{\aleph_0} \times \kappa$ and $b \in \mathbf{B}_\beta \setminus \{0_{\mathbf{B}_\beta}\}$ then for some $a \in \mathbf{B}_\alpha$ we have $\mathbf{B}_\beta \models b \leq a$ and if $\mathbf{B}_\alpha \models 0 < a' \leq a$ then $a' \cap b > 0_{\mathbf{B}_\beta}$.

{5.4} [Note: for clause (β) we use the proof of $(B)^+$ of 3.6 for $\alpha = 2^{\aleph_0} \times \epsilon + 1$ for clause (δ) we have a new clause, but easy one].

It follows that

(∗∗) $\mathbf{B} = \mathbf{B}_{2^{\aleph_0} \times \kappa}$ has length μ.

Now we just need to show

(∗∗∗) let $J \subseteq \mathbf{B}$, $|J| = \mu$ be a chain; get a contradiction.

Let $\mathbf{B}^*_\epsilon = \mathbf{B}_{2^{\aleph_0} \times \epsilon}$. Let $c_\alpha \in J$ (for $\alpha < \mu$) be pairwise distinct.

By clause (δ) of (∗) for each $\epsilon < \kappa$ and $\alpha < \mu^+_\epsilon$ we can find $b^\epsilon_\alpha \in \mathbf{B}^*_\epsilon$ such that:

(a) $c_\alpha \leq b^\epsilon_\alpha$,

(b) $0 < x \leq b^\epsilon_\alpha$ & $x \in \mathbf{B}^*_\epsilon \quad \Rightarrow \quad x \cap c_\alpha \neq 0$.

Note:

(c) b^ϵ_α is unique, and

(d) $c_\alpha \leq c_\beta \quad \Rightarrow \quad b^\epsilon_\alpha \leq b^\epsilon_\beta$.

As $\mathbf{B}^*_\epsilon$ has length $\leq \mu_\epsilon$ and J is a chain, necessarily for some $Y_\epsilon \subseteq \mu^+_\epsilon$, $|Y_\epsilon| = \mu^+_\epsilon$, and

(e) $b^\epsilon_\alpha = b^\epsilon$ for $\alpha \in Y_\epsilon$.

We can apply clause (δ) of (∗) to $-c_\alpha$ (for $\alpha \in Y_\epsilon$ and $\mathbf{B}^*_\epsilon$ and possibly shrinking Y_ϵ) to get $a^\epsilon_\alpha \in \mathbf{B}^*_\epsilon$ such that:

(f) $(-c_\alpha) \leq a^\epsilon_\alpha$ and $0 < x \leq a^\epsilon_\alpha \& x \in \mathbf{B}^*_\epsilon \Rightarrow x \cap (-c_\alpha) \neq 0$.

As above without loss of generality shrinking Y_ϵ further we get

(g) $a^\epsilon_\alpha = a^\epsilon$ for $\alpha \in Y_\epsilon$.

As the length of $\mathbf{B}^*_\epsilon$ is $\leq \mu_\epsilon < |Y_\epsilon|$, for some $\alpha \in Y_\epsilon$, $c_\alpha \notin \mathbf{B}^*_\epsilon$; as

$$a^\epsilon_\alpha \geq -c_\alpha, \qquad b^\epsilon_\alpha \geq c_\alpha, \qquad b^\epsilon_\alpha \in \mathbf{B}^*_\epsilon \qquad \text{and} \qquad a^\epsilon_\alpha \in \mathbf{B}^*_\epsilon$$

necessarily $a^\epsilon_\alpha \cap b^\epsilon_\alpha \neq 0$.

Hence:

(h) $b^\epsilon \cap a^\epsilon \neq 0$.

Let $g(\epsilon) = \text{Min}\{\zeta < \epsilon : a^\epsilon, b^\epsilon \in \mathbf{B}^*_\zeta\}$, so for limit $\varepsilon, g(\varepsilon) < \varepsilon$ hence on some stationary $S \subseteq \kappa$ the function $g \restriction S$ is constantly $\zeta(*)$, and without loss of generality

$$\zeta < \epsilon \in S \Rightarrow |\{\alpha \in Y_\zeta : c_\alpha \in \mathbf{B}^*_\epsilon\}| = \mu^+_\zeta.$$

As $\mathbf{B}$ satisfies the ccc we can find $\epsilon_1 < \epsilon_2$ in S such that

$$b^{\epsilon_1} \cap a^{\epsilon_1} \cap b^{\epsilon_2} \cap a^{\epsilon_2} \neq 0.$$

Choose $\alpha \in Y_{\epsilon_1}$ such that $c_\alpha \in \mathbf{B}_{\epsilon_1}$ and $\beta \in Y_{\epsilon_2}$. Now $\{c_\alpha, c_\beta\}$ is independent; a contradiction. $\square_{3.10}$

{5.6p}

Remark 3.11. We may further ask: is the restriction "$\mathrm{cf}(\mu) > \aleph_0$" in (3.10) necessary? {5.17}

{5.7bis}

Observation 3.12. *Assume that the infinite Boolean algebra* $\mathbf{B}$ *has the length* μ, $\mathrm{cf}(\mu) = \aleph_0$. *Then the length is obtained.*

Proof. Let $\dot{\mathscr{I}} = \{b \in \mathbf{B} : \mathrm{length}(\mathbf{B} \restriction b) < \mu\}$.

Easily

$$b_1 \le b_2 \ \&\ b_2 \in \dot{\mathscr{I}} \Rightarrow b_1 \in \dot{\mathscr{I}}.$$

Also clearly $\dot{\mathscr{I}}$ is closed under unions. [Why? If $b_1, b_2 \in \dot{\mathscr{I}}, b = b_1 \cup b_2 \notin \dot{\mathscr{I}}$ then there is a chain $\langle c_t : t \in J\rangle$, J a linear order of cardinality μ, $(s <_J t \ \Rightarrow\ c_s <_{\mathbf{B}} c_t)$ and $c_t \le b$.

Let

$$\mathbf{E}_l = \{(t, s) : t \in J,\ s \in J,\ c_t \cap b_l = c_s \cap b_l\}.$$

Then $\mathbf{E}_l$ is a convex equivalence relation on J; if $|J/\mathbf{E}_l| = \mu$ then $\{c_t \cap b_l : t \in J\}$ exemplifies $b_l \notin \dot{\mathscr{I}}$, a contradiction. So $|J/\mathbf{E}_l| < \mu$. Hence $\mathbf{E} = \mathbf{E}_1 \cap \mathbf{E}_2$ is a convex equivalence relation with $\le |J/\mathbf{E}_1| \times |J/\mathbf{E}_2| < \mu$ classes, but as $b = b_1 \cup b_2$ it is the equality.]

If $\mathbf{B}/\dot{\mathscr{I}}$ is infinite then we can find $\langle a_n/\dot{\mathscr{I}} : n < \omega\rangle$ pairwise disjoint non-zero. Now $b_n := a_n - \bigcup\limits_{\ell<n} a_\ell$ are pairwise disjoint members of $\mathbf{B}$ not in $\dot{\mathscr{I}}$. Let $\mu = \sum\limits_{n<\omega} \mu_n$, $\mu_n < \mu$. Let $\langle c^n_t : t \in J_n\rangle$ be an increasing chain in $\mathbf{B} \restriction b_n$, $|J_n| \ge \mu_n$ (note that we can invert J_n). Let $J = \sum\limits_{n<\omega} J_n$ (without loss of generality $n < m \ \Rightarrow\ J_n \cap J_m = \emptyset$) and for $t \in J_n$ let $c^*_t = b_0 \cup \ldots \cup b_{n-1} \cup c^n_t$. Now $\langle c^*_t : t \in J\rangle$ exemplifies that the length is obtained. So $\mathbf{B}/\dot{\mathscr{I}}$ is finite, so without loss of generality $\dot{\mathscr{I}}$ is a maximal ideal. Try to choose $a_n \in \dot{\mathscr{I}}$ satisfying $\bigwedge\limits_{\ell<n} a_n \cap a_\ell = 0$ such that $\mathrm{length}(\mathbf{B} \restriction a_n) > \mu_n$; if we succeed to repeat the proof for the case $\mathbf{B}/\dot{\mathscr{I}}$ is infinite; hence we necessarily fail. Hence for some n (replacing $\mathbf{B}$ by $\mathbf{B} \restriction (-(a_0 \cup \ldots \cup a_{n-1}))$) we have

$$b \in \dot{\mathscr{I}} \quad \Rightarrow \quad \mathrm{length}(\mathbf{B} \restriction b) \le \mu_n.$$

Let $J \subseteq \mathbf{B}$ be linearly ordered, $|J| > \mu_n^+$. Possibly shrinking J, without loss of generality $J \subseteq \dot{\mathscr{I}} \vee J \subseteq \mathbf{B} \setminus \dot{\mathscr{I}}$. As we can replace J by $\{1_{\mathbf{B}} - b :\ b \in J\}$ without loss of generality $J \subseteq \dot{\mathscr{I}}$, so for some $b \in J$ we have $|\{c \in J : c \le b\}| \ge \mu_n^+$, and hence $\mathrm{length}(\mathbf{B}{\restriction}b) \ge \mu_n^+$, a contradiction. $\square_{3.12}$

Remark 3.13. We may wonder if we can replace $\aleph_0$ in 3.10 by another cardinals. Most natural are κ strong limit of cofinality ω. {5.17}

{5.8}

Claim 3.14. *Assume* $\kappa \le \mu < 2^\kappa$, κ *strong limit and* $\mathrm{cf}(\kappa) = \aleph_0$. *Then there is a Boolean algebra* $\mathbf{B}$ *such that:*

(α) $|\mathbf{B}| = 2^\kappa$

(β) $\mathbf{B} \models \mathrm{ccc}$

(γ) $\mathbf{B}$ *has length* μ *(and satisfies clause (B)*$^+$ *of 3.6)* {5.4}

(δ) $\mathbf{B}$ *has no homomorphic image* $\mathbf{B}'$, $\kappa \leq |\mathbf{B}'| < 2^\kappa$.

Proof. Let $h \in {}^\omega\omega$ be $h(n) = 2(n+1)$. Let $\mathbf{B}_0 \subseteq \mathscr{P}(\kappa)$ have cardinality μ and length μ,

$$I^0_\alpha = \{\eta : \quad \eta \text{ is an } \omega\text{–sequence, } \eta(n) \text{ is an increasing sequence of ordinals} < 2^\kappa \text{ of length } h(n)\},$$

and

$$I_\alpha = I^0_\alpha \cup \{\mathrm{Suc}^\ell_n(\eta) : n < \omega,\ \ell < h(n),\ \eta \in I^0_\eta\}$$

so $|I_\alpha| = 2^\kappa$. Let $\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar{a}_\alpha, I_\alpha]$, $\mathbf{B}_\alpha$ (for $\alpha \leq 2^\kappa$) increasing continuous (again $\bar{a}_\alpha$ an ω–sequence of pairwise disjoint non-zero elements of $\mathbf{B}_\alpha$ such that each sequence appears 2^κ times).

{5.2} Again, for $\alpha < \beta$, $\mathbf{B}_\alpha \lessdot \mathbf{B}_\beta$ (and even the conclusion of 3.4(5) holds). The proof
{5.4} that $\mathbf{B} =: \mathbf{B}_{(2^\kappa)}$ satisfies (α), (β), (γ) is as in the proof of 3.6. $\square_{3.14}$

For (δ) we need

{5.19} **Observation 3.15.** *Assume that* κ *is a strong limit cardinal of countable cofinality.*
1) If $\mathbf{B}'$ *is a Boolean Algebra of cardinality* $\geq \kappa$ *but* $< 2^\kappa$ *then:*

(a) *there are pairwise disjoint non-zero* b_n *(for* $n < \omega$*) in* $\mathbf{B}'$ *such that*

($*$) *for no* $c \in \mathbf{B}'$, $\bigwedge\limits_{n<\omega} b_{2n} \leq c$ & $\bigwedge\limits_{n<\omega} b_{2n+1} \cap c = 0$.

2) For a Boolean algebra $\mathbf{B}'$ *a sufficient condition for* $\mathbf{B}'$ *to satisfy* **(a)** *(i.e., the existence of a sequence* $\langle b_n : n < \omega\rangle$ *of pairwise disjoint elements of* $\mathbf{B}'$ *satisfying* ($*$) *above) is:*

(b) $\mathbf{B}'$ *has cardinality* $< 2^\kappa$ *and there are* $b_n \in \mathbf{B}'$ *such that* $\bigwedge\limits_{n<m} b_n \cap b_m = 0$ *and* $\kappa = \liminf_n |\mathbf{B} \restriction b_n|$.

{5.19} We first prove that 3.15 suffices. Toward contradiction assume that $\mathbf{B}'$ is a Boolean algebra of cardinality $< 2^\kappa$ but $\geq \kappa$, and $\mathbf{B}'$ is homomorphic image of $\mathbf{B}$. If clause
{5.14} (a) is satisfied by $\mathbf{B}'$, then the proof is very similar to the earlier proof of 4.14: for a homomorphism $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ from $\mathbf{B}$ onto $\mathbf{B}'$ we can find pairwise disjoint $a_n \in \mathbf{B}$ (for $n < \omega$) such that $f(a_n) = b_n$. So, for some α, $\bar{a}_\alpha = \langle a_n : n < \omega\rangle$ and repeat the
{5.4} relevant part of 3.6. Using clauses (b),(c) of Definition 3.3 we get a contradiction.
{5.19} We are left with proving 3.15, first the second part.

Proof of Observation 3.15(2) We can find $\bar{c}^\zeta = \langle c^\zeta_n : n < \omega\rangle$, $c^\zeta_n \in \mathbf{B}'$, $c^\zeta_n \leq b_n$ for $\zeta < 2^\kappa$ such that the sequences $\langle c^\zeta_n : n < \omega\rangle$ pairwise distinct for $\zeta < 2^\kappa$. For each ζ let $b^\zeta_{2n} = c^\zeta_n, b^\zeta_{2n+1} = b_n - c^\zeta_n$, so if clause **(a)** fails then for every $\zeta < 2^\kappa$ there is $y_\zeta \in \mathbf{B}'$ such that for every $n < \omega$ we have

$$b^\zeta_{2n} \leq y_\zeta, \qquad b^\zeta_{2n+1} \cap y_\zeta = 0.$$

So $\bigwedge\limits_n y_\zeta \cap b_n = c^\zeta_n$ and hence $\zeta < \xi < 2^\kappa \Rightarrow\ y_\zeta \neq y_\xi$, which contradicts $|\mathbf{B}'| < 2^\kappa$.

Proof. Proof of Observation 3.15(1): Assume that the conclusion fails. For μ let

$$\dot{\mathscr{I}}_\mu = \dot{\mathscr{I}}_\mu[\mathbf{B}'] =: \{b \in \mathbf{B}' : \mathbf{B}' \restriction b \text{ has the cardinality } < \mu\}.$$

Clearly it is an ideal of $\mathbf{B}'$ increasing with μ and $1_{\mathbf{B}'} \in \dot{\mathscr{I}}_\mu \iff \mu > |\mathbf{B}|$. If $\mathbf{B}'/\dot{\mathscr{I}}_\kappa[\mathbf{B}']$ is infinite then we can easily get condition **(b)** of part (2), and we are done. If not, but for every $\mu < \kappa$, $\dot{\mathscr{I}}_\mu[\mathbf{B}] \neq \dot{\mathscr{I}}_\kappa[\mathbf{B}']$, let $\kappa = \sum\limits_{n<\omega} \mu_n$, $\mu_n < \mu_{n+1}$, choose $b_n \in \dot{\mathscr{I}}_\kappa[\mathbf{B}'] \setminus \dot{\mathscr{I}}_{\mu_n}[\mathbf{B}']$; but $\dot{\mathscr{I}}_\kappa[\mathbf{B}'] = \bigcup\limits_{\mu<\kappa} \dot{\mathscr{I}}_\mu[\mathbf{B}']$, so $\bigwedge\limits_n \bigvee\limits_m b_n \in \dot{\mathscr{I}}_{\mu_m}[\mathbf{B}']$. So without loss of generality $b_n \in \dot{\mathscr{I}}_{\mu_{n+1}}[\mathbf{B}'] \setminus \dot{\mathscr{I}}_{\mu_n}[\mathbf{B}']$ and hence $\langle b_n - \bigcup\limits_{l<n} b_l : n < \omega\rangle$ are as required. We are left with the case that for some $\mu(*) < \kappa$,

$$\dot{\mathscr{I}} := \dot{\mathscr{I}}_{\mu(*)}[\mathbf{B}'] = \dot{\mathscr{I}}_\kappa[\mathbf{B}']$$

and without loss of generality $\dot{\mathscr{I}} = \dot{\mathscr{I}}_{\mu(*)}[\mathbf{B}']$ is a maximal ideal.

Without loss of generality $2^{\mu(*)} < \mu_n < \mu_{n+1}$ for $n < \omega$. Let $b_i \in \dot{\mathscr{I}}$ for $i < \kappa$ be distinct (exist as $|\mathbf{B}'| \geq \kappa, \dot{\mathscr{I}}$ is a maximal ideal of $\mathbf{B}'$). By the proof of Erdös–Tarski theorem, without loss of generality $\langle b_i : i < \kappa\rangle$ are non-zero pairwise disjoint.

[Why? For example apply the Δ-system lemma to

$$\{\{x : x \leq b_i\} : i < (2^{\mu_n})^+\},$$

and get $Y_n \subseteq (2^{\mu_n})^+$ of cardinality $(2^{\mu_n})^+$ and a set A_n of cardinality $\leq 2^{\mu(*)}$ such that

$$i, j \in Y_n \ \&\ i \neq j \quad \Rightarrow \quad \{x : x \leq b_i\} \cap \{x : x \leq b_j\} = A_n.$$

So $|A_n| \leq \mu(*)$. Pick $Y'_n \subseteq Y_n$ of cardinality $(2^{\mu_n})^+$ such that

$$i, j \in Y'_n \ \&\ i \neq j \quad \Rightarrow \quad \{x : x \leq b_i\} \cap \bigcup_{m<n} A_m = \{x : x \leq b_j\} \cap \bigcup_{m<n} A'_m,$$

where $A'_n = \{x : (\exists i \in Y'_n)(x \leq b_i)\}$. Let $i(n) = \min(Y'_n)$. Then

$$X_n = \{x_i - x_{i(n)} : i \in Y_n, i > i(n)\} \subseteq \mathbf{B} \setminus \{0\}$$

is an antichain, and $\bigcup\limits_n X_n$ is as required.].

Let

$$\mathscr{P}_0 = \{Y \in [\kappa]^{\aleph_0} : \text{there is } b \in \dot{\mathscr{I}} \text{ such that } (\forall i \in Y)(b_i \leq b)\}.$$

This is a subset of $[\kappa]^{\aleph_0}$ of cardinality $\leq |\dot{\mathscr{I}}| \cdot \mu(*)^{\aleph_0} \leq |\mathbf{B}'| + \kappa = |\mathbf{B}'|$, but $[\kappa]^{\aleph_0} = 2^\kappa > |\mathbf{B}'|$, so there is $Y_0 \in [\kappa]^{\aleph_0} \setminus \mathscr{P}_0$.

Let

$$\mathscr{P}_1 = \{Y \in [\kappa]^{\aleph_0} : Y \subseteq \kappa \setminus Y_0 \text{ and } (\exists b \in \dot{\mathscr{I}})(\forall i \in Y)(b_i \leq b)\}.$$

By cardinality considerations as above there is $Y_1 \in [\kappa]^{\aleph_0} \setminus \mathscr{P}_1$ disjoint to Y_0. By assumption above (i.e., clause **(a)** fails) there is $b \in \mathbf{B}'$ such that $\bigwedge_{i \in Y_0} b_i \leq b$, $\bigwedge_{i \in Y_1} b_i \leq (1 - b)$. If $b \in \dot{\mathscr{I}}$ we get contradiction to the choice of Y_0, if not then $1_{\mathbf{B}} - b \in \dot{\mathscr{I}}$ contradicts the choice of Y_1. Hence the observation holds and hence the Observation 3.15 is proven. Hence Claim 3.14 is proven. $\square_{1.13}$ {5.18}

{5.19A}

{5.19} *Remark* 3.16. In other words 3.15 says

(∗) If κ is strong limit, $\mathrm{cf}(\kappa) = \aleph_0$, $\mathbf{B}$ is a Boolean algebra of cardinality $\geq \kappa$ with $\aleph_1$-separation (i.e., **(a)** of the observation fails) then $|\mathbf{B}| \geq 2^\kappa$.

§ 4. Using subtrees of $({}^{\omega\geq}2, \triangleleft)$ and theories unstable in $\aleph_0$

{3a1t3}

Theorem 4.1. *Suppose $T \subseteq T_1$ are first order theories, T_1 is countable, T is complete, superstable but $\aleph_0$–unstable.* *Then* *for $\lambda > \aleph_0$ we have*

$$\dot{I}\dot{E}(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, \beth_2\}.$$

Remark 4.2. The reader is not required to know anything on superstable theories, just to believe a result quoted below. So we can just assume $(*)$ from the proof.

Proof. The assumption that the theory is superstable and not totally transcendental (= $\aleph_0$-stable) is used to obtain $\mathrm{m_a}, \mathrm{m_b} < \omega$ and a countable set of definable (without parameters) equivalence relations $\{\varphi_n(\bar{x};\bar{y}) : n < \omega\} \subseteq \mathbb{L}(T)$ such that (we may write $\bar{x}\varphi_n\bar{y}$ instead $\varphi_n(\bar{x},\bar{y})$):

- $(*)$ (i) $\mathrm{lg}(\bar{x}) = \mathrm{lg}(\bar{y}) = \mathrm{m_a} + \mathrm{m_b}$,
  - (ii) if M is a model of $T, \bar{a} \in {}^{\mathrm{m_a}}|M|$, then the set $\{\bar{a}\hat{\ }\bar{b}/\varphi_n : \bar{b} \in {}^{(\mathrm{m_b})}|M|\}$ is finite,
  - (iii) if for $\ell = 1, 2, \mathrm{lg}(\bar{a}_\ell) = \mathrm{m_a}, \mathrm{lg}(\bar{b}_\ell) = \mathrm{m_b}$, and $(\bar{a}_1\hat{\ }\bar{b}_1)\varphi_n(\bar{a}_2\hat{\ }\bar{b}_2)$ then $\bar{a}_1 = \bar{a}_2$,
  - (iv) φ_{n+1} refines φ_n, i.e., for every $n < \omega$, $\bar{x}\varphi_{n+1}\bar{y}$ implies $\bar{x}\varphi_n\bar{y}$,
  - (v) there are (in some model M of T) $\bar{c}_\eta$ for $\eta \in {}^{\omega>}2$ such that:

$$[\mathrm{lg}(\eta) \geq n \ \&\ \mathrm{lg}(\nu) \geq n \ \text{ implies } \ \bar{c}_\eta\ \varphi_n\ \bar{c}_\nu \iff \eta \restriction n = \nu \restriction n],$$
$$\bar{c}_\eta \restriction m_a = \bar{c}_\nu \restriction \mathrm{m_a}, \quad \mathrm{lg}(\bar{c}_\eta) = \mathrm{m_a} + \mathrm{m_b}.$$

The existence of this set of equivalence relations was proved in Lemmas III 5.1, 5.2 III 5.3 of [Sh:a],[Sh:c].

Clearly without loss of generality we may expand the theory T_1. Let $\{c_\ell : \ell < m_1\} \cup \{c_{\eta,\ell} : \ell \in [m_1, m_1 + \mathrm{m_b}]$ and $\eta \in {}^{\omega>}2\}$ be new constants in T_1, we let $\bar{c}_\eta = \langle c_\ell : \ell < \mathrm{m_a}\rangle\hat{\ }\langle c_{\eta,\ell} : \ell \in [\mathrm{m_a}, \mathrm{m_a} + \mathrm{m_b})$ and suppose

$$T_1 \supseteq \{(\bar{c}_\eta\varphi_n\bar{c}_\nu) : \eta \restriction n = \nu \restriction n,\ \mathrm{lg}(\eta), \mathrm{lg}(\nu) \geq n\} \cup$$
$$\{\neg(\bar{c}_\eta\varphi_n\bar{c}_\nu) : \eta \restriction \neq \nu \restriction n, \mathrm{lg}(\eta), \mathrm{lg}(\nu) \geq n\}.$$

Also without loss of generality suppose that T_1 has Skolem functions (so the axioms saying it has Skolem functions belong to T_1).

We will use the following fact. [For a sequence $\bar{\eta}$ let $\bar{\eta} = \langle \bar{\eta}[\ell] : \ell < \mathrm{lg}(\bar{\eta})\rangle$ and $\bar{a}_{\bar{\eta}} = \bar{a}_{\bar{\eta}[0]}\hat{\ }\bar{a}_{\bar{\eta}[1]}\hat{\ }\bar{a}_{\bar{\eta}[2]}\ldots$.] $\square_{4.1}$

{3.1A}

Fact 4.3. 1) Suppose

- (a) $T \subseteq T_1$ are first order theories, T complete and superstable, unstable in $|T_1|$, $\tau = \tau(T)$, $\tau_1 = \tau(T_1)$,
- (b) τ_1 is countable or at least MA_μ for $\mu =: |T_1|$ holds,
- (c) φ_n $(n < \omega)$, $\mathrm{m_a}, \mathrm{m_b}$ are as in $(*)$ above,
- (d) $\varphi_n \in \tau$ is a $(2m_*)$–place predicate, $\Delta = \{\emptyset_n : n < \omega\}$, $\tau_1^+ = \tau_1 \cup \{d_n : n < \omega\}$, $\tau \subseteq \tau_1$, $|\tau_1| \leq \mu < 2^{\aleph_0}$.

Then there are M_1, $\bar{a}_\eta$ $(\eta \in {}^\omega 2)$ such that:

(α) M_1 is a model of $T_1, \varphi_n^{M_1}$ is an equivalence relation, $\varphi_{n+1}^{M_1}$ refines $\varphi_n^{M_1}$,

(β) $\tau(M_1) = \tau_1^+$, $\{\bar a_\eta : \eta \in {}^\omega 2\} \subseteq {}^{m_*}|M_1|$, and $\lg(\eta) \geq n$ & $\lg(\nu) \geq n \quad \Rightarrow$ $[\eta \restriction n = \nu \restriction n \iff (\bar a_\eta \varphi_n \bar a_\nu)]$,

$(\beta)_1$ $\bar a_\eta \restriction \mathrm{m_a} = \bar a_\nu \restriction \mathrm{m_a} = \langle c_\ell^{M_1} : \ell < m_a \rangle$, $\lg(\bar a_\eta) = m_* = \mathrm{m_a} + \mathrm{m_b}$ and if $n < \omega$, $\lg(\bar a) = m_a < m_*$ <u>then</u> $|\{\bar a \hat{} \bar b/\emptyset_m : \bar b \in {}^{m_b}(M_1)\}|$ is $< k_m$,

(γ) for every formula $\varphi(\bar x)$ (from τ_1) there is $n = \eta_\varphi$ such that for $n \in [\eta_\varphi, \omega)$:

$(*)^1_{\varphi,n}$ if $\bar\eta$, $\bar\nu \in {}^m({}^\omega 2)$, $\lg(\bar\eta) = \lg(\bar\nu) = \lg(\bar x)/(\mathrm{m_a} + \mathrm{m_b})$, (so $\lg(\bar a_{\bar\eta}) = \lg(\bar x)$) and $\langle \eta_\ell \restriction \eta : \ell < \lg(\bar\eta) \rangle = \langle \nu_\ell \restriction n : \ell < \lg(\bar\nu) \rangle$ is with no repetitions, then $M_1 \models \varphi[\bar a_{\bar\eta}] = \varphi[\bar a_{\bar\nu}]$,

(δ) in $M_1 \restriction \tau_1$, $\langle d_n : n < \omega \rangle$ is an indiscernible sequence over $\{\bar a_\eta : \eta \in {}^\omega 2\}$

$(\delta)^+$ $d_n \neq d_m$ for $n \neq m$.

2) If $M_1, \tau_1, \tau, \mathrm{m_a}, \mathrm{m_b}, \langle \varphi_n : n < \omega \rangle$ are as in (α), (β), $(\beta)_1$, (γ), (δ) above, $\mu = \aleph_0$ or at least MA_μ, <u>then</u> replacing ${}^\omega 2$ by a subtree, replacing $\langle \varphi_n : n < \omega \rangle$ by a sub-sequence and renaming, decreasing M_1 we can add to part (1):

$(\gamma)^+$ for every sequence of terms $\bar\sigma(\bar x)$ from τ_1^+, if $m \times (\mathrm{m_a} + \mathrm{m_b}) = \lg(\bar x)$, $\mathrm{m_a} + \mathrm{m_b} = \lg(\bar\sigma)$, $\bar\sigma(\bar x) \restriction \mathrm{m_a} = (\bar\sigma \restriction \mathrm{m_a})(\bar x \restriction \mathrm{m_d})$, $\mathrm{m_d} = \mathrm{m_e} \times (\mathrm{m_a} + \mathrm{m_b})$, [i.e., for $\bar\eta \in {}^m({}^\omega 2)$, $\bar\sigma(\bar a_{\bar\eta}) \restriction \mathrm{m_a} = (\bar\sigma \restriction \mathrm{m_a})(\bar a_{\eta \restriction \mathrm{m_e}})$], <u>then</u> there exists $n_{\bar\sigma} < \omega$ such that:

(A) for $n \geq \eta_{\bar\sigma}$ and $\bar\eta, \bar\nu \in {}^m({}^\omega 2)$ with no repetitions, $\bar\eta \restriction \mathrm{m_e} = \bar\nu \restriction \mathrm{m_e}$, we have: if $\ell \neq k \Rightarrow \bar\eta[\ell] \restriction n \neq \bar\eta[k] \restriction n$ and $(\forall \ell < m)\, [\bar\eta[\ell] \restriction n = \bar\nu[\ell] \restriction n]$ <u>then</u> for every $\bar\rho \in {}^m({}^\omega 2) : \bar\rho \restriction \mathrm{m_e} = \bar\eta \restriction \mathrm{m_e}$ implies:

$$(\bar\sigma(\bar a_{\bar\eta}) \varphi_n \bar\sigma(\bar a_{\bar\rho})) \iff (\bar\sigma(\bar a_\nu) \varphi_n \bar\sigma(\bar a_{\bar\rho})).$$

(B) For $n \geq \eta_{\bar\sigma}$, and $\bar\eta, \bar\nu \in {}^m({}^n 2)$ each with no repetition, $\bar\eta \restriction \mathrm{m_e} = \bar\nu \restriction \mathrm{m_e}$ we have, <u>iff</u> there are $k \geq n$ and $\bar\eta_1, \bar\nu_1 \in {}^m({}^\omega 2)$ such that $\neg\varphi_k(\bar\sigma(\bar a_{\bar\eta_1}), \bar\sigma(\bar a_{\bar\nu_1}))$, for $\ell < m$, $\bar\eta_1[\ell] \restriction n = \bar\eta[\ell]$, $\bar\nu_1[\ell] \restriction n = \bar\nu[\ell]$, and

$$(\forall \ell, i < m)\, [\bar\eta_1[\ell] = \bar\nu_1[i] \iff \bar\eta[\ell] = \bar\nu[i]]\,,$$

<u>then</u> for every $\bar\eta^*, \bar\nu^* \in {}^m({}^\omega 2)$ satisfying $\bar\eta^*[\ell] \restriction n = \bar\eta[\ell]$, $\bar\nu^*[\ell] \restriction n = \bar\nu[\ell]$ (for each $\ell < m$) and

$$(\forall \ell, i < m)\, [\bar\eta^*[\ell] = \bar\nu^*[i] \iff \bar\eta[\ell] = \bar\nu[i]]$$

we have $\neg(\bar\sigma(\bar a_{\bar\eta^*}) \varphi_n \bar\sigma(\bar a_{\bar\nu^*}))$.

Remark 4.4. 1) This is the only place where countability (or $\mathrm{MA}_{|\tau_1|)}$) is used.
{3.10} 2) For alternative proof see 4.13.

Proof. 1) If we ignore $(\delta)^+$ (so can have $d_n = d_0$) use Theorem [Sh:a, Ch.VII,3.7]. In general use [Sh:a, Ch.VII,Ex.3.1]. What if T_1 is uncountable but MA_μ? (The
{3.10} reader may ignore this proof or see the proof of 4.13.)

Let $\mathbb{P}$ be the forcing notion of adding $\lambda = \beth_{(2^\mu)^+}$ Cohen reals, $\langle \underset{\sim}{\eta}_i : i < \lambda \rangle$, $\underset{\sim}{\eta}_i \in {}^\omega 2$. Let $\chi = (2^\lambda)^+$ and let $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{M}$ is a model of T_1, the Skolem hull of $\{x_i : i < \lambda\}$, $\bar x_i \varphi_m \bar c_{\underset{\sim}{\eta}_i \restriction m}$". By the omitting type theorem (see, e.g., [Sh:c, Ch.VII,§5]) there are $\mathfrak{B}_1 \prec \mathfrak{B}_2$, $\mathfrak{B}_1 \prec (\mathscr{H}(\chi), \in, <^*_\chi)$, $\|\mathfrak{B}_1\| = \mu$, T_1, P, $\underset{\sim}{M}$, $\langle x_i : i < \lambda \rangle$ belong to $\mathfrak{B}_1$, in $\mathfrak{B}_2$, $\langle a_\rho : \rho \in {}^\omega 2\rangle$ is an indiscernible sequence over $\mathfrak{B}_1$, $\mathfrak{B}_2 \models$ "$\underset{\sim}{a}_i$ is an ordinal $= \lambda$".

Note that what $\mathfrak{B}_2$ consider a maximal antichain of $\mathbb{P}^{\mathfrak{B}_2}$, is really so. Now we can naturally apply MA_μ.
2) Satisfy requirement (A) by letting $\varphi^\ell_n(\bar{x}\char94\bar{z}) := E_n(\bar{x}\char94\bar{z}, F_\ell(\bar{z})\char94\bar{z})$ for $\ell < \ell^*_n < \omega$; where $F_\ell \in \tau^+$ are such that $\{F_\ell(\bar{x}) : \ell < \ell^*_n\}$ is a complete set of representatives for $\{\bar{x}\char94\bar{z}/\varphi_n : \bar{x}\}$, possibly with repetitions. (Remember T_1 has Skolem functions and there is ℓ^*_n which does not depend on $\bar{z}$ by compactness). Requirement (B) is fulfilled by trimming the perfect tree and renaming. $\square_{4.3}$

{3.2}

Claim 4.5. *For M_1, $\bar{a}_\eta$ $(\eta \in {}^\omega 2)$, φ_n as in the conclusion of 4.3 we can conclude:* {3.1A}

> $\otimes$ *iff $\nu \neq \rho$ are from ${}^\omega 2$, $\bar{\eta}_\nu = \langle \eta_{\nu,\ell} : \ell < \ell(*)\rangle$, $\bar{\eta}_\rho = \langle \eta_{\rho,\ell} : \ell < \ell(*)\rangle$, $\bar{x} = \langle x_\ell : \ell < \ell(*)\rangle$, $\bar{\sigma}(\bar{x}) = \langle \sigma_m(\bar{x}) : m < m(*)\rangle$, $\nu \restriction k = \rho \restriction k$, $\eta_{\nu,\ell} \restriction k = \eta_{\rho,\ell} \restriction k$, $\langle \eta_{\nu,\ell} : \ell < \ell(*)\rangle$ with no repetitions, $k > n_{\bar{\sigma}}$, and*
>
> $$\bigwedge_{n<\omega} [\bar{a}_\nu \varphi_n \bar{a}_\rho \iff \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\varphi_n \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})],$$
>
> *moreover the Δ-type of $\bar{a}_\nu\char94\bar{a}_\rho$ and $\bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\char94\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})$ (in M) are equal for every n, then* $\lg(\nu \cap \rho) \in \{\lg(\eta_{\nu,\ell} \cap \eta_{\rho,\ell}) : \ell < \ell(*)\}$.

Proof. Assume not.

Let $n = \lg(\rho \cap \nu)$. Then $\varphi_n(\bar{a}_\rho, \bar{a}_\nu) \wedge \neg\varphi_{n+1}(\bar{a}_\rho, \bar{a}_\nu)$. We suppose first for didactic reasons for the sake of contradiction that for every $\ell < n_0$ we have

$$\bar{\eta}_\nu[\ell] \neq \bar{\eta}_\rho[\ell] \quad \Rightarrow \quad \lg(\bar{\eta}_\nu[\ell] \cap \bar{\eta}_\rho[\ell]) < n.$$

By the equality of types $\neg\varphi_{n+1}\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho}), \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\big)$, now we can deduce by Fact 4.3(2) {3.1A} and the assumption that the conclusion of $(\otimes)$ fails, that $\neg\varphi_{n+1}\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho}), \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\big)$. Again, by the equality of types $\neg\varphi_n(\bar{a}_\rho, \bar{a}_\nu)$, a contradiction to $\varphi_n(\bar{a}_\rho, \bar{a}_\nu)$.

Now we deal with the general case, i.e., we assume

$$(*) \qquad (\forall \ell < n_0)(\lg(\bar{\eta}_\nu[\ell] \cap \bar{\eta}_\rho[\ell]) \neq n).$$

We shall derive a contradiction.

Define $\bar{\eta} \in {}^{n_0}({}^\omega 2)$:

$$\bar{\eta}[\ell] = \begin{cases} \bar{\eta}_\rho[\ell] & \text{if } \bar{\eta}_\nu[\ell] \restriction n \neq \bar{\eta}_\rho[\ell] \restriction n, \\ \bar{\eta}_\nu[\ell] & \text{otherwise.} \end{cases}$$

Clearly $\bar{\sigma}(\bar{a}_\eta) \restriction \mathrm{m_a} = \bar{\sigma}(\bar{a}_{\eta_\rho}) \restriction \mathrm{m_a} = \bar{\sigma}(\bar{a}_{\eta_\nu}) \restriction \mathrm{m_a}$ and $\bar{\eta} \restriction \mathrm{m_e} = \bar{\eta}_\nu \restriction \mathrm{m_e} = \bar{\eta}_\rho \restriction \mathrm{m_e}$ and also $\bar{\eta}$ is with no repetition and $\langle \bar{\eta}[\ell] \restriction n : \ell < n_0\rangle$ are pairwise distinct.

Since, by the definition of $\bar{\eta}$, for which $\bar{\eta}[\ell]{\restriction}n = \bar{\eta}_\rho[\ell]{\restriction}n$, using $(*)$ we obtain

$$\eta[\ell]{\restriction}(n+1) = \eta_\rho[\ell]{\restriction}(n+1).$$

Let $\bar{b} = \bar{\sigma}(\bar{a}_{\bar{\eta}})$. By reflexivity of the equivalence relation we have

$$\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\varphi_{n+1}\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big).$$

By Fact 4.3(1) $\big(\bar{\sigma}(\bar{a}_{\bar{\eta}})\varphi_{n+1}\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$, i.e., $\big(\bar{b}\varphi_{n+1}\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$. Finally (as $\neg\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\varphi_{n+1}\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$) {3.1A} using transitivity of the equivalence relation we have $\neg\varphi_{n+1}\big(\bar{b}, \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$.

By the definition of $\bar\eta$ for every $\ell < n_0$:

$$\bar\eta[\ell] = \bar\eta_\nu[\ell] \quad \text{or} \quad \lg(\bar\eta[\ell] \cap \eta_\nu[\ell]) < n.$$

But since $n > k_0$, clearly $|\{\bar\eta[\ell] \restriction k_0 : \ell < n_0\}| = n_0$, and $|\{\bar\eta_\nu[\ell] \restriction k_0 : \ell < n_0\}| = n_0$.

{3.1A} So by Fact 4.3(2) as $\neg\big(\bar b\varphi_{n+1}\bar\sigma(\bar a_{\bar\eta_\nu})\big)$ (see above) we have $\neg\big(\bar b\varphi_n\bar\sigma(\bar a_{\bar\eta_\nu})\big)$. But $\big(\bar b\varphi_n\bar\sigma(\bar a_{\bar\eta_\rho})\big)$ (see above) and $\big(\bar\sigma(\bar a_{\bar\eta_\rho})\varphi_n\bar\sigma(\bar a_{\bar\eta_\rho})\big)$, a contradiction. $\square_{4.5}$

{3.1} {3.4bis} So for proving theorem 4.1 we can assume

{3.1A} {3.2} {3.3} **Hypothesis 4.6.** $M_1, \langle\varphi_n : n < \omega\rangle, \bar a_\eta$ $(\eta \in {}^\omega 2)$ are as in $(\beta) + (\gamma)$ of 4.3(1) and $(\otimes)$ of 4.5.

Lemma 4.7. *Assume $\mu < \lambda \le 2^{\aleph_0}$. We can find $S_\xi \subseteq {}^\omega 2$ for $\xi < 2^{\aleph_0}$, pairwise disjoint, each of cardinality λ such that*

$\otimes$ *if $\xi < 2^{\aleph_0}$, and $f : S_\xi \longrightarrow {}^{\omega >}\big(\mathscr{M}_{\mu,\omega}(\bigcup_{\zeta\neq\xi} S_\zeta)\big)$ and $\mathbf{n}$ is a function, $\mathbf{n} : \{\bar\sigma :$ for some $\bar x$, $\bar\sigma = \langle\sigma_\ell(\bar x) : \ell < \ell^*\rangle$, σ_ℓ a term of $\mathbb{L}_{\mu,\omega(\tau)}\} \to \omega$ and τ is the vocabulary of $\mathscr{M}_{\mu,\omega}(\bigcup_{\zeta\neq\xi} S_\xi)$, <u>then</u> we can find m^* (see below) $S^* \subseteq S_\zeta$, $k_0 < \omega$, $n_0 = \mathrm{m_a} + \mathrm{m_b} < \omega$, sequence $\bar\sigma(\bar x) = \langle\sigma_\ell(\bar x) : \ell < \lg(\bar\sigma)\rangle$, with $\lg(\bar x) = n_0$, $\langle\bar\eta_\nu : \nu \in S^*\rangle$ and $\bar\eta_0 \in {}^{n_0}({}^\omega 2)$ with the following properties: letting $\eta_{\nu,\ell} = \bar\eta_\nu[\ell]$*

- (A) *$\eta \neq \nu \in S^* \quad\Rightarrow\quad \lg(\eta\cap\nu) > k_0$,*
- (B) *for $\nu \in S^*$ we have $\lg(\bar\eta_\nu) = n_0$ and $(\forall \ell < n_0)\,[\bar\eta_{\nu,\ell} \restriction k_0 = \bar\eta_{0,\ell} \restriction k_0]$ and $\{\bar\eta_{\nu,\ell} \restriction k_0 : \ell < n_0\} \cup \{\nu \restriction k_0\}$ are pairwise distinct,*
- (C) *$k_0 > \mathbf{n}(\bar\sigma)$,*
- (D) *for each $\ell < n_0$ either $\{\bar\eta_{\nu,\ell} : \nu \in S^*\} = \{\bar\eta_{0,\ell}\}$ or $\{\bar\eta_{\nu,\ell} : \nu \in S^*\}$ are pairwise distinct,*
- (E) *the sets*
$$\{\lg(\nu_1 \cap \nu_2) : \nu_1 \neq \nu_2 \text{ from } S^*\}$$
and
$$\{\lg(\eta_{\nu_1,\ell_1} \cap \eta_{\nu_2,\ell_2}) : \nu_1, \nu_2 \in S^* \text{ and } \ell_1, \ell_2 < n_0\}$$
are disjoint,
- (F) *for every $\nu \in S^*$, $f(\nu) = \bar\sigma(\bar\eta_\nu)$ (i.e., $\langle\sigma_\ell(\langle\eta_{\nu,n} : n < n_0\rangle) : l < m^*\rangle$),*
- (G) *for $\nu_1 \neq \nu_2 \in S^*$ we have*
$$\eta_{\nu_1,\ell} = \eta_{\nu_2,\ell} \iff \ell < \mathrm{m_a} \iff \eta_{\nu_1,\ell} = \eta_{0,\ell}$$
- (H) *S^* is μ^+-large where: we say that $S \subseteq {}^\omega 2$ is χ-large <u>iff</u> for every $n < \omega$ and $\nu \in S$ we have $|\{\rho \in S : \rho \restriction n = \nu \restriction n\}| \ge \chi$. We can replace μ^+-large by λ-large if $\mathrm{cf}(\lambda) > \aleph_0$*
- (I) *$\nu_1, \nu_2 \in S^* \,\&\, \eta_{\nu_1,\ell_1} = \eta_{\nu_2,\ell_2}$ implies $\ell_1 = \ell_2$*
- (J) *for $\eta \in \bigcup_\xi S_\xi$ let $\xi(\nu)$ be the unique ξ such that $\eta \in S_\xi$ <u>then</u> $\xi(\eta_{\nu_1,\ell_1}) = \xi(\nu_{2,\ell}), \nu_1 \neq \nu_2 \in S^*$ implies, $\nu \in S^* \Rightarrow \xi(\eta_{\nu_1\ell_1}) = \xi(\eta_{\nu,\ell_1}) = \xi(\eta_{\nu,\ell_2})$.*

{5.5Anew}

Remark 4.8. 1) This claim is a version of the "unembeddability" results (see Definitions in [Sh:331, 2], results for example in VI, and here in §1 for the tree ${}^{\omega \geq}2$); well, they are necessary somewhat weaker as in §1 here.
2) Of course, we can replace $\bigcup_{\zeta \neq \xi} S_\zeta$ by $\sum_{\zeta \neq \xi} S_\zeta$.

For proving 4.7 we will use the following combinatorial Fact which is slightly stronger than Sierpiński's lemma on almost disjoint sets of integers: {3.3}

{3.3B}

Fact 4.9. There are $W(*)$, $\{W_\eta \subseteq \omega : \eta \in {}^\omega 2\}$, and $\{U_\eta : \eta \in {}^\omega 2\}$ such that for all $\eta \in {}^\omega 2$:

- (A) $W(*), W_\eta$ are infinite subsets of ω
- (B) U_η is a perfect tree, i.e., $U_\eta \subseteq {}^{\omega >}2$ is downward closed, $\langle\rangle \in U_\eta$,
$$(\forall \rho \in U_\eta)(\exists \nu \in U_\eta)\,[\rho = \nu \restriction \lg(\nu) \;\&\; \nu\hat{\ }\langle 0\rangle \in U_\eta \;\&\; \nu\hat{\ }\langle 1\rangle \in U_\eta]\,,$$
- (C) if $\rho, \nu \in U_\eta$, $\rho \neq \nu$ and $\lg(\rho) = \ell(\nu)$ <u>then</u> $\lg(\rho \cap \nu) \in W_\eta$, where $(\rho \cap \nu)$ is the largest common initial segment of ρ and ν, i.e.,
$$\lg(\rho \cap \nu) := \mathrm{Max}\{n < \omega : \rho \restriction n = \nu \restriction n\},$$
- (D) for all $\eta_1 \neq \eta_2 \in {}^\omega 2$ and every $\rho \in U_{\eta_1}$, $\nu \in U_{\eta_2}$ there are three possibilities:
  - (a) $\lg(\rho \cap \nu) \in W_{\eta_1} \cap W_{\eta_2}$,
  - (b) $\lg(\rho \cap \nu) \in W(*)$ and $(\forall \ell < \lg(\rho \cap \nu))[\ell \in W_{\eta_1} \equiv \ell \in W_{\eta_2}]$
  - (c) $\rho \trianglelefteq \nu$ or $\nu \triangleleft \rho$,
- (E) $W(*) \cap W_\eta = \emptyset$,
- (F) for distinct η, ν from ${}^\omega 2$, we have
  - (a) $W_\eta \cap W_\nu$ is finite in fact an initial segment of both and
  - (b) if $\ell \in W(*)$ is above $W_\eta \cap W_\nu$ <u>then</u> $U_\eta \cap U_\nu$ is finite, $\subseteq {}^{\ell' >}2$ if $\ell < \ell' \in W_\eta \cup W_\nu$ and has no splitting of level $\geq \ell$ i.e. $[\rho\neg(\exists \rho \in {}^{\omega >}2)(\lg \rho) \geq \ell \;\&\; \{\eta\hat{\ }\langle 0\rangle, \eta\hat{\ }\langle 1\rangle\} \subseteq U_\eta \cap U_\nu]$, and
  - (c) if $\ell \in W(*)$ is $< \sup(W_\eta \cap W_\nu)$ <u>then</u> $U_\eta \cap {}^{\ell \geq}2 = U_\nu \cap {}^{\ell \geq}2)$).

Proof. By induction on n define: $k(n) = k_n < \omega$ and the set $W_n(*) \subseteq k(n)$ and for $\eta \in {}^n 2$ the sets $U_\eta \subseteq {}^{k(n)\geq}2$, $W_\eta \subseteq k(n)$, such that in the end (this imposes natural restrictions on them):

$$\eta \in {}^\omega 2 \;\Rightarrow\; W_\eta \cap k_n = W_{\eta \restriction n},\; U_\eta \cap {}^{k(n)\geq}2 = U_{\eta \restriction n},\; W(*) \cap k(n) = W_n(*).$$

For $n = 0$ let $k_0 = 0, W_n(*) = \emptyset$ and $W_\eta = \emptyset, U_\eta = \emptyset$ for $\eta \in {}^n 2$. For the induction step, choose $k^1(n) = k(n) + n + 1$ and for $\eta \in {}^n 2$ let $U^1_\eta = U_\eta \cup \{\nu\hat{\ }(\eta \restriction \ell) : \nu \in U_\eta \cap {}^{k(n)}2,\ \ell \leq n\}$.

Thus

$$\big(\forall \nu \in {}^{k(n)}2 \cap U_\eta\big)\big(\exists! \rho \in U^{1 k^1(n)}_\eta 2 \cap \big)[\nu \trianglelefteq \rho].$$

Define $W_{n+1}(*) = W_n(*) \cup [k(n), k^1(n))$. Fix an enumeration $\{\eta_k : k < 2^{n+1}\}$ of ${}^{n+1}2$. Let $k(n+1) := k^1(n) + 2^{n+1}$. For $\eta \in {}^{n+1}2$, there is unique $k < 2^n$ such that $\eta = \eta_k$, let

$$U_{\eta_k} := U^1_{\eta_k \restriction n} \cup \Big\{\nu : \begin{array}{l} \nu \in {}^{k(n+1)\geq}2,\ \nu \restriction k^1(n) \in U^1_{\eta_k \restriction n}, \text{ and for } \ell < 2^n \text{ we have} \\ k^1(n) + \ell < \lg(\nu) \& (\ell \neq 2k+1) \ \Rightarrow\ \nu(k^1(n) +! \ell) = 0\}, \end{array}$$

$W_{\eta_k} = W_{\eta_k} \cup \{k^1(n) + 2k + 1\}$. It is easy to verify that the construction provides a family of sets as required. $\square_{4.9}$

{3.3B} Proof of Lemma 4.7: Let $W(*)$, U_η, W_η be as in 4.9. Fix an enumeration $\{\eta_\xi : \xi < 2^{\aleph_0}\} = {}^\omega 2$ and let $W^\xi =: W_{\eta_\xi}$.

Let

$$S_\xi \subseteq \lim(U_{\eta_\xi}) \quad (= \{\rho \in {}^\omega 2 : (\forall n < \omega)[\rho \restriction n \in U_{\eta_\xi}]\})$$

be of cardinality λ. Fix $\{\rho^\xi_i : i < \lambda\} = S_\xi$ and with out loss of generality S_ξ is χ-large [Recall that we say $S \subseteq {}^\omega 2$ is χ-large if for every $n < \omega$ and $\nu \in S : |\{\rho \in S : \rho \restriction n = \nu \restriction n\}| \geq \chi$; if $\chi \geq (|\tau_1| + \aleph_0)^+$ we may omit it].

Note that for every $S \subseteq {}^\omega 2$ of cardinality $> \mu$ for some $S_1 \subseteq S$, $|S_1| \leq \mu$ and
{3.3B} $S \setminus S_1$ is μ^+–large. Let $U^\zeta = U_{\eta_\zeta}$; as note that by 4.9 clauses (B)+(D) the $S_\xi \setminus S$ are pairwise disjoint.

So let $\xi, f, \mathbf{n}$ be as in the assemption of $(\otimes)$. For $\nu \in S_\xi$ let $f(\nu) = \bar\sigma_\nu(\bar\eta_\nu)$ where $\bar\sigma_\nu$ is a finite sequence of terms, $\bar\eta_\nu$ is a finite sequence of members of $\bigcup_{\zeta \neq \xi} S_\zeta$ with no repetitions. So there are $S^* \subseteq S_\xi$ which is μ^+–large, and $\bar\sigma$, and an integer n_0 such that

$$\nu \in S^* \quad \Rightarrow \quad \bar\sigma_\nu = \bar\sigma \wedge \lg(\bar\eta_\nu) = n_0,$$

and without loss of generality for some $m_a \leq m_b < \omega$ we have $\bar\sigma(\bar\eta_\nu) \restriction \mathrm{m_a} = \bar\eta^*$ and

$$\{\eta^*_\ell : \ell < \mathrm{m_a}\} \cup \{\eta_{\nu,\ell} : \nu \in S^* \text{ and } \ell \in [\mathrm{m_a}, \mathrm{m_b})\}$$

is with no repetition (possible by the Δ-system argument).

As $S_\xi \cap \bigcup_{\zeta \neq \xi} S_\zeta = \emptyset$, clearly for $\nu \in S^*$ the sequence $\langle \nu \rangle \hat{\ } \bar\eta_{\bar\nu}$ is without repetitions. So for some $k = k_\nu < \omega$ large enough we have:

(i) $\langle \nu \restriction k \rangle \hat{\ } \langle \eta_{\nu,\ell} \restriction k : \ell < \ell(*) \rangle$ is without repetitions,

(ii) letting $\eta_{\nu,\ell} \in S_{\zeta(\nu,\ell)}$ we have $W^\xi \cap W^{\zeta(\eta_\nu,\ell)} \subseteq \{0, \ldots, k_\nu - 1\}$ moreover
{3.3} $k_\nu > \mathrm{Min}(W^\xi \setminus W^{\zeta(\eta_\nu,\ell)})$ (remember clause (F) of 4.7).

As we can shrink S^* as long as it is μ^+-large without loss of generality for some k

(iii) $\nu_1 \neq \nu_2 \in S^* \quad \Rightarrow \quad \lg(\nu_1 \cap \nu_2) > k$,

(iv) $\nu \in S^* \quad \Rightarrow \quad k_\nu < k < \omega$.

So for $\nu_1 \neq \nu_2 \in S^*$ on the one hand $\lg(\nu_1 \cap \nu_2) \in W^\xi \setminus k$ (as $\nu_1, \nu_2 \in S_\xi \subseteq \lim(U_{\eta_\xi})$
{3.3B} clause (iii) above and clause (C) of 4.9) and

$$\lg(\eta_{\nu_1 \cap \ell}, \eta_{\nu_2, \ell}) \in W(*) \cup U^{\zeta(\nu_1, \ell)} \cup U^{\zeta(\nu_2, \ell)}$$

which is disjoint to $W^\xi \setminus k$. So we have proved clause (E) of 4.7; the other clauses can be checked. $\square_{1.13}$ {3.3}

{3.7}

Claim 4.10. *If clauses (β), (γ), (δ) of 4.3(1) holds and $(\otimes)$ of 4.5 <u>then</u> for $\lambda \leq 2^{\aleph_0}$:* {3.2A}

- $(*)_\lambda$ *there is a family $\mathscr{P}$ of subsets of ${}^\omega 2$ each of cardinality λ (even their union has cardinality λ), $|\mathscr{P}| = 2^\lambda$ such that:*
  - $(*)$ *letting N^1_S be the Skolem Hull of $\{\bar{a}_\eta : \eta \in S\}$ for $S \in \mathscr{P}$, we have*
  - $(**)$ *for $Y_1 \neq Y_2$ from $\mathscr{P}$, $N^1_{Y_1}$ has no Δ-embedding into $N^1_{Y_2}$; $\|N^1_Y\| = \lambda$.*

Proof. For $X \subseteq \lambda$ let M^1_X be the Skolem Hull of $\{\bar{a}_\eta : \eta \in \bigcup_{\xi \in X} S_\xi\}$, $M_X := M^1_X \restriction L(T)$. In order to prove the theorem it is enough to assume X, $Y \subseteq \lambda$, $X \not\subseteq Y$ and show there does not exist an elementary embedding f from M_X into M_Y. Let $\xi \in X \setminus Y$. For the sake of contradiction suppose $f : M_X \to M_Y$ is an elementary embedding or just one preserving the satisfaction of $\emptyset_n, \neg\emptyset_n$.

We can represent M_Y in $\mathscr{M}_{\mu,\omega}(\bigcup_{\zeta \neq \xi} S_\zeta)$, and let us define $f' : S_\xi \longrightarrow \mathscr{M}_{\mu,\omega}(\bigcup_{\zeta \neq \xi} S_\zeta)$ be $f'(\nu) = f(\bar{a}_\nu)$, let $\mathbf{n}$ be essentially as in 4.3 translated. Apply lemma 4.7 to f', {3.3A} $\mathbf{n}$ and get S^*, k_0, n_0, $\mathrm{m_a}$, $\mathrm{m_b}$, $\bar{\sigma}$, $\langle \bar{\eta}_\nu : \nu \in S^* \rangle$ as there. Of course n_0, $\mathrm{m_a}$, $\mathrm{m_b}$ are predetermined as in 4.3. {3.1A}

So we are done proving 4.10. $\square_{4.10}$ {3.7}

Proof. <u>Proof of Theorem 4.1</u>

When $\lambda \leq 2^{\aleph_0}$ the result follows from 4.10 by 4.5 {3.2}

So the Proof of Theorem 4.1 for the case $\lambda \leq 2^{\aleph_0}$ is completed. How to deal {3.1} with the case $\lambda > 2^{\aleph_0}$? We just need to use $(\delta)^+$ i.e. use 4.12 (and Definition 4.11) {3.8} below. $\square_{4.1}$

{3.8}

Definition 4.11. For any cardinal κ and M_1 as in 4.3(1) (β), (γ), (δ), $(\delta)^+$ we {3.1A} define a model $M_{1,\kappa}$ as follows: it is a τ_1-model generated by $\{\bar{a}_\eta : \eta \in {}^\omega 2\} \cup \{d_i : i < \kappa\}$ such that for every $n < \omega$, $i_1 < \ldots$ in $< \kappa$, and $\eta_1, \ldots, \eta_m \in {}^\omega 2$, the quantifier free type of $\bar{a}_{\eta_1}{}^\frown \ldots {}^\frown \bar{a}_{\eta_m}{}^\frown \langle d_{i_1}, \ldots, d_{i_n} \rangle$ in $M_{1,\kappa}$ is equal to the quantifier free type of $\bar{a}_{\eta_1}{}^\frown \ldots {}^\frown \bar{a}_{\eta_m}{}^\frown \langle d_1, \ldots, d_n \rangle$ in M_1. (So if M_1 has Skolem function then $M_1 = M_{1,\mu}$ and they realize the same types.)

{3.9}

Claim 4.12. *If clauses $(\beta), (\gamma), (\delta), (\delta)^+$ of 4.3(1) holds and $(\otimes)$ of 4.5 <u>then</u> for* {3.2A} *$\lambda \geq 2^{\aleph_0}$:*

- $(*)_\lambda$ *there is a family $\mathscr{P}$ of subsets of ${}^\omega 2$ each of cardinality $2^{\aleph_0}$, $|\mathscr{P}| = \beth_2$ such that:*
  - $(*)$ *letting N^λ_S be the Skolem Hull of $\{\bar{a}_\eta : \eta \in S\} \cup \{d_i : i < \kappa\}$ in $M_{1,\lambda}$ (for $S \in \mathscr{P}$) (so $\|N^\lambda_S\| = \lambda$) we have*
  - $(**)$ *for $Y_1 \neq Y_2$ from $\mathscr{P}$, $N^1_{Y_1}$ has no Δ-embedding into $N^1_{Y_2}$ (i.e., no function from $N^1_{Y_1}$ into $N^1_{Y_2}$ preserves all the relations $\pm\varphi_n$).*

We may consider using relations $\emptyset_n$ which are not equivalence relations, and we may like to give another proof when $\mu > \aleph_0$ but still MA_μ holds.

{3.10} **Claim 4.13.** $[MA_\mu]$ *Suppose* M_1, τ_1, $\langle \bar a_\eta : \eta \in {}^\omega 2\rangle$, φ_n $(n < \omega)$, $\langle d_n : n < \omega\rangle$
{3.1A} *satisfies clauses (a),(b),(β), (γ), (δ) of 4.3 and* M_1 *is a* τ_1*-model of the complete first order theory* T_1. *Also suppose* $\bar a_\eta \in {}^k(M_1)$ *for* $\eta \in {}^{\omega >}2$ *are* such that*: if* $n < m < \omega, \eta, \nu \in {}^m 2$ *then* $\eta \restriction n = \nu \restriction n \Leftrightarrow M_1 \models \bar a_\eta \varphi_n \bar a_\nu$ *(so* φ_n *not necessarily an equivalence relation,* $|\tau_1| = \mu$ *not necessary countable).*

1) If we replaced ${}^{\omega \geq}2$ *by a perfect subtree (splitting determined by level only) and replacing* $\langle \emptyset_n : n < \omega\rangle$ *by a subsequence, we can add to the assumptions the statement*
{3.2} *(⊗) of 4.5.*
{3.1A} *2) So the conclusion of 4.10 holds, and if we further assume* $(\delta)^+$ *of 4.3 also the*
{3.9} *conclusion of 4.12 holds.*

Proof. We use Carlson and Simpson [CS84].

Let W^* be the set of ω-sequences η from $\{0,1\} \cup \{x_i : i < \omega\}$ such that each x_i appear infinitely ofter. For $\eta \in W^*$, let $W_\eta = \{\nu \in W^*$: if $\eta(\ell) \in \{0,1\}$ then $\nu(\ell) = \eta(\ell)$ and if $\eta(\ell_1) = \eta(\ell_2)$ then $\nu(\ell_1) = \nu(\ell_2)\}$. As set $W \subseteq W^*$ is large if it contains some W_η. Let $I_W = \{\nu \in {}^{\omega>}2$: for some $\eta \in W$ we have, for every $\ell, \ell_1, \ell_2 < \lg(\nu), \eta(\ell) \in \{0,1\} \Rightarrow \nu(\ell) = \eta(\ell)$ and $\eta(\ell_1) = \eta(\ell_2) \Rightarrow \nu(\ell_1) = \nu(\ell_1)$. Let $\mathrm{lev}(W) = \{k$: for some $\eta \in W, \eta(\ell) \notin \{0,1\}$ but $\eta(0), \ldots, \eta(\ell-1) \in \{0,1\}\}$, and $W_1 \subseteq^* W_2$ if for some n, $\{\nu \restriction [n,\omega) : \nu \in W_1\} \supseteq \{\nu \restriction (n,\omega) : \nu \in W_2\}$. By MA_μ if $\langle W_i : i < \delta \leq \mu\rangle$ is $\subseteq^*$-decreasing sequence then there is W such that $\bigwedge_i W_i \subseteq^* W$.

By the partition theorem there, if $n < \omega$, $\eta_1, \ldots, \eta_k \in {}^n 2$ are pairwise distinct, $\bar\sigma^1$, $\bar\sigma^2$ are τ_1^+-terms <u>then</u> we can find large $W_1 \subseteq W$, $W_1 \restriction n = W \restriction n$, and:

$\circledast^n_{W_1,\bar\sigma}$ if $n < m \in \mathrm{lev}(W_1), \rho^\nu_\ell \in \mathscr{T}_{W_1} \cap {}^m 2$, for $\ell = 1, \ldots, k$ and $\nu_\ell = \eta_\ell{}^\frown \rho_2 \restriction [n,\omega)$ <u>then</u> the truth value of $\bar\sigma^1(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}) \emptyset_t \bar\sigma^2(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k})$ is constant.

Repeating it we can get W_1 such that $\circledast^n_{W_1,\bar\sigma}$ for every n

(i) either g is constant $< \min(\mathrm{lev}(W_1) \setminus n)$ or

$$n \in \mathrm{lev}(W_1) \quad \Rightarrow \quad [g(n), n) \cap \mathrm{lev}(W_1) = \emptyset,$$

(ii) if $n < m \in \mathrm{lev}(W_1)$, $\eta_\ell \lhd \nu_\ell \in \mathscr{T}_{W_1} \cap {}^m 2$, then

$$\min\{i : \neg\sigma^1(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k})\varphi_i \sigma^2(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k})\} = g(m).$$

We apply such reasoning to: given $\eta_1, \ldots, \eta_k \in \mathscr{T}_W \cap {}^n 2$ pairwise distinct $n < m \in \mathrm{lev}(W_1)$, $\eta_\ell \lhd \nu^i_\ell \in \mathscr{T}_{W_1} \cap {}^n 2$ is $\bar\sigma(\bar a_{\nu^0_1}, \ldots, \bar a_{\nu^0_1})\varphi_\ell \bar\sigma(\bar a_{\nu^1_1}, \ldots, \bar a_{\nu^1_k})$. We get: this depends just on $\lg(\nu^0_\ell \cap \nu^1_\ell)$, $\nu^i_{\ell_1}(\lg(\nu^0_\ell \cap \nu^1_\ell))$; $\square_{4.13}$

{5.14} **Discussion 4.14.** The parallel (for a module $\dot{\mathbb{M}}$) concerning "a surgery at" is extending the ring $\dot{\mathbf{R}}$ to $\dot{\mathbf{R}}^+$, e.g. by $\{x_t : t \in I\}$ freely except some equation involving x and the x_i's and "below x" is replaced by the ideal x general.

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