Weak compactness and the structure of Ext(A,$\mathbb{Z}$) *

by
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In this paper we address a conjecture of the second author, namely that in L, for any cardinal $\kappa \geq \aleph_1$ and any prescription of cardinals $\lambda p \leq \kappa^+$ to the primes p, there exists an abelian group A for which $\nu_0(A) = \kappa^+$ = the rank of the torsion free part of Ext(A,$\mathbb{Z}$), and $\lambda p = \nu_p(A)$ = the rank of the p-part of Ext(A,$\mathbb{Z}$).

It is well known that these cardinals characterize the divisible group Ext(A,$\mathbb{Z}$) for torsion free A. The conjecture is false for countable A, where it has been shown by C. Jensen [11] that $\nu_p(A)$ is either finite or $2^{\aleph_0}$ and $\nu_p(A) \leq \nu_0(A)$. Similarly Hulanicki [8,9] has shown that for divisible abelian groups which admit a compact topology $\nu_p(A) \leq \nu_0(A)$ and $\nu_p(A)$ is finite or of the form 2^λ, λ infinite. However we have shown that the conjecture is true for $\kappa = \aleph_1 = |A|$ under ZFC + GCH alone, see [12], and Eklof and Huber this volume, [4]. Just using the fact that $\mathrm{Ext}_p(\oplus A_i,\mathbb{Z}) = \Pi \mathrm{Ext}_p(A_i,\mathbb{Z})$ it is now easy to see that for any cardinal $\kappa \geq \aleph_1$ and successor cardinals $\lambda p \leq \kappa^+$ there exists an abelian group for which $|A| = \kappa$ and $\nu_p(A) = \lambda p$.

The question remains whether we can have $\nu_p(A) = |A|$ or $\nu_p(A)$ singular or $\nu_p(A)$ inaccessible. We show that the conjecture is not true in all generality by proving that

THEOREM. If A is a torsion free abelian group of weakly compact cardinality κ and $\nu_p(A) \geq \kappa$, then $\nu_p(A) = 2^\kappa$.

Since weak compactness is consistent with V = L, provided it is consistent with ZFC, the above theorem displays some restriction of the conjecture.

There are a number of equivalent definitions of weak compactness [1,10]; a suitable one for a non logician is:

*This work has been partially supported by The National Science Foundation (grant No. 710646).

DEFINITION. (i) A cardinal κ has the tree property iff, for every tree, T, of height κ with levels of cardinality $<\kappa$ has a branch of length κ. (ii) κ is weakly compact iff it is inaccessible and has the tree property.

Weak compactness was originally introduced in relation to compactness of certain infinitary languages which in turn can be readily related to the following equivalent property for inaccessible κ:

Any κ-complete filter D in a κ-complete field B of subsets of κ can be extended to a κ-complete prime filter in B.

As for the tree property: $\aleph_0$ has the tree property by Köenigs lemma and it is also a well known result of Aronszajn that $\aleph_1$ does not have the tree property; moreover singular cardinals, and with GCH, also successor to regular cardinals do not have the tree property.

The treatment of $\mathrm{Ext}_p(A,\mathbb{Z})$ is based on the following theorem.

DEFINITION. Let $H: \mathrm{Hom}(A,\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{Z}/p\mathbb{Z})$ be the natural homomorphism defined by:

$$[H(h)](x) = h(x)/p\mathbb{Z}, \quad h \in \mathrm{Hom}(A,\mathbb{Z}), \ x \in A, \ p \text{ a prime.}$$

THEOREM. For abelian torsion free A

$$\mathrm{Ext}_p(A,\mathbb{Z}) \cong \mathrm{Hom}(A,\mathbb{Z}/p\mathbb{Z})/H[\mathrm{Hom}\ A,\mathbb{Z}] .$$

Proof: The exact sequence $0 \to p\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/p\mathbb{Z} \to 0$, α the identity embedding, β natural, induces the long exact sequence

$$0 \to \mathrm{Hom}(A,p\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{Z}) \to \mathrm{Hom}(A,\mathbb{Z}/p\mathbb{Z})$$

$$\xrightarrow{E_*} \mathrm{Ext}(A,p\mathbb{Z}) \xrightarrow{\alpha_*} \mathrm{Ext}(A,\mathbb{Z}) \xrightarrow{\beta_*} \mathrm{Ext}(A,\mathbb{Z}/p\mathbb{Z}) \to 0$$

(see Fuchs [6]). Since the sequence is exact,

$$J = \mathrm{Hom}(A,\mathbb{Z}/p\mathbb{Z})/H[\mathrm{Hom}(A,\mathbb{Z})] \cong \mathrm{Ker}(A_*) = \mathrm{Im}(E_*)$$

$\mathbb{Z}$, $p\mathbb{Z}$ are isomorphic; hence also $\mathrm{Ext}(A,\mathbb{Z})$, $\mathrm{Ext}(A,p\mathbb{Z})$; in particular elements of order p of $\mathrm{Ext}(A,\mathbb{Z})$, are represented by elements of order p in $\mathrm{Ext}(A,p\mathbb{Z})$. All elements of J are of order p. Hence it suffices to show that all extension $E \in \mathrm{Ext}(A,p\mathbb{Z})$ of order p are mapped to 0 by α_*. Let $E \in \mathrm{Ext}(A,p\mathbb{Z})$, $pE = 0$, be represented by a factor set $f: A \times A \to p\mathbb{Z}$. Thus for some function $g: A \to p\mathbb{Z}$ with $g(o) = 0$, $pf(x,y) = g(x) + g(y) - g(x+y) \in \mathrm{Trans}(A,\mathbb{Z})$, $\forall\, x,y \in A$.

Since α is an injection, $\alpha_*(E)$ can be represented by the same f. Now since $A, \mathbb{Z}$ are torsion free, there is a unique $g': A \to \mathbb{Z}$ such that $pg'(x) = g(x)$, $\forall\, x \in A$. Therefore $f(x,y) = g'(x) + g'(y) - g'(x+y)$, hence also $\alpha^*(E) = 0$. □

THEOREM (ZFC). If G is a group of weakly compact cardinality, κ, for which $\nu_p(G) \geq \kappa$, then $\nu_p(G) = 2^\kappa$.

Proof: Let G be a group of weakly compact cardinality κ with $\nu_p(G) \geq \kappa$. We shall show that $\nu_p(G) = 2^\kappa$. This is done by constructing a filtration $\langle G_\alpha : \alpha \leq \kappa\rangle$ of G, and a tree of homomorphisms $h_\eta : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$, $\eta \in {}^\alpha 2$, ordered by inclusion ($\eta \subset \eta' \Rightarrow h_\eta \subset h_{\eta'}$) and continuous, i.e. if $C = \{h_{\eta_\beta} : \beta < \alpha,\ h_{\eta_\beta} \subseteq \operatorname{Hom}(G_\beta, \mathbb{Z}/p\mathbb{Z})\}$ is a chain with α limit and $\eta = \bigcup_{\alpha<\beta} \eta_\beta$, then $h_\eta = \bigcup_{\beta<\alpha} h_{\eta_\beta}$. We will construct the tree T, such that at each level α, $\{h_\eta : \eta \in {}^\alpha 2\} = T_\alpha$ are independent homomorphisms mod $H[\operatorname{Hom}(G_\alpha, \mathbb{Z})]$, where H is the operation described above. It is easy to see that this property will be preserved at limit ordinals.

We first exploit the tree property of κ to obtain the following lemma:

LEMMA. If $D \subset \operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ are independent mod $H[\operatorname{Hom}(G,\mathbb{Z})]$ and $|D| < \kappa$, then for any subgroup $G' \subseteq G$, $|G'| < \kappa$ there exists a subgroup G'', $G' < G'' < G$, $|G''| < \kappa$ such that $\{h \restriction G'' : h \in D\}$ are independent mod $H[\operatorname{Hom}(G,\mathbb{Z})]$.

Proof: If not, then there exists a continuous strictly increasing sequence of subgroups of G, $\langle G_\alpha\rangle_{\alpha<\kappa}$ with $|G_\alpha| < \kappa$ and $G_0 = G'$, $G_\kappa = G$, such that $\forall \alpha < \kappa$, $h', h'' \in D, h' \neq h''$ and $h' \restriction G_\alpha = h'' \restriction G_\alpha \bmod (H[\operatorname{Hom}(G_\alpha, \mathbb{Z})])$.

Notation. Henceforth we denote for $h', h'' \in \operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$, $h' \equiv_\alpha h''$ iff $h' \equiv h'' \bmod (H[\operatorname{Hom}(G\ ,\mathbb{Z})])$.

Now for $\beta < \alpha$, $h' \restriction G_\alpha \equiv_\alpha h'' \restriction G_\alpha \Rightarrow h' \restriction G_\beta \equiv_\beta h'' \restriction G_\beta$. Moreover if $h \in \operatorname{Hom}(G_\alpha, \mathbb{Z})$ is such that $h' \restriction G_\alpha = h'' \restriction G_\alpha + h/p\mathbb{Z}$, then $h' \restriction G_\beta = h'' \restriction G_\beta + h \restriction G_\beta/p\mathbb{Z}$. Clearly if $h', h'' \in D$, $\alpha < \kappa$ and $h' \restriction G_\alpha \not\equiv h'' \restriction G_\alpha$, then $h' \restriction G_\beta \not\equiv_\beta h'' \restriction G_\beta, \forall \beta$, $\alpha \leq \beta \leq \kappa$. Therefore,

$\exists h^*, h^{**} \in D$ such that $\forall \alpha < \kappa$ $h^* \upharpoonright G_\alpha \equiv_\alpha h^{**} \upharpoonright G_\alpha$. For these fixed h^*, h^{**} let $K_\alpha = \{h \in \mathrm{Hom}(G_\alpha, \mathbb{Z}) : h^* \upharpoonright G_\alpha = h^{**} \upharpoonright G_\alpha + h/p\mathbb{Z}\}$

Clearly $\{h \upharpoonright G_\beta : h \in K_\alpha\} \subseteq K_\beta$, $\beta < \alpha$ and $|K_\alpha| < \kappa$. If $K = K_\alpha$ and K is partially ordered by extension, $\prec$, then $\langle K, \prec \rangle$ is a tree of height κ with levels of cardinality less than κ. Thus by weak compactness there exists a branch b of length κ, $b = \{h_\alpha : \alpha < \kappa\}$. Then $\bigcup_{\alpha<\kappa} h_\alpha = h \in \mathrm{Hom}(G, \mathbb{Z})$ and $h^* = h^{**} + h/p\mathbb{Z}$.

This is a contradiction. □

<u>Construction of the filtration and respective tree, T, of homomorphisms.</u>

For $\alpha = 0$, we set $G_0 = 0$, $T_0 = \emptyset$. For $\alpha = 1$, we choose $G_1 \subset G$ of cardinality $<\kappa$ such that $\exists\, h_0, h_1 \in \mathrm{Hom}(G_1, \mathbb{Z}/p\mathbb{Z})$ which are independent mod $H[\mathrm{Hom}(G_1, \mathbb{Z})]$. Such h_0, h_1 exist by the lemma. We set $T_1 = \{h_0, h_1\}$. For α limit we just take unions, i.e. $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$, and $T_\alpha = \{\bigcup_{\eta \in b} h_\eta : b$ is a branch through $\bigcup_{\beta<\alpha} T_\beta\}$. For $\alpha = \beta + 1$ successor, we first choose $G_{\beta+1} > G_\beta$ such that

$\nu_p(G_{\beta+1}) > (2^{|G_\beta|})^+$. This is again possible by our lemma. For every $\eta \in {}^\beta 2$ let h_{η_0} be any extension of h_η in $\mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$. These will be independent mod $H[\mathrm{Hom}(G_\alpha, \mathbb{Z})]$ since the $\{h_\eta : \eta \in {}^\beta 2\}$ are independent mod $H[\mathrm{Hom}(G^\beta, \mathbb{Z})]$. We must choose the h_{η_1} so that they

are all similarly independent. By our choice of $G_{\beta+1}$ we can find a family of $(2^{|G_\beta|})^+$ homomorphisms containing $\{h_{\eta_0} : \eta \in {}^\beta 2\}$ which are independent mod $H[\mathrm{Hom}(G_\alpha, \mathbb{Z})]$. Again from our cardinality assumptions, we can choose from these, $|G_\beta|$ distinct disjoint pairs of homomorphisms (h', h'') such that $h' \upharpoonright G_\beta = h'' \upharpoonright G_\beta$. Thus we can assign from these to every $\eta \in {}^\beta 2$ a distinct pair (h'_η, h''_η) and set $h_{\eta_1} = h_{\eta_0} + (h'_\eta - h''_\eta)$. Thus we clearly have

$h_{\eta_0} \upharpoonright G_\beta = h_{\eta_1} \upharpoonright G_\beta = h_\eta$: and the $(T_\alpha =)$ $\{h_{\eta_0}, h_{\eta_1} : \eta \in {}^\beta 2\}$ are independent mod $H[\mathrm{Hom}(G_{\beta+1}, \mathbb{Z})]$.

Since $G = G_\kappa$, $T = T_\kappa \subseteq \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) - H[\mathrm{Hom}(G, \mathbb{Z})]$ and $|T_\kappa| = 2^\kappa$, we have $\nu_p(G) = 2^\kappa$. □

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