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THE CONSISTENCY OF ZFC + $2^{\aleph_0} > \aleph_\omega + \mathscr{I}(\aleph_2) = \mathscr{I}(\aleph_\omega)$

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§1. Introduction. Let κ be an uncountable cardinal and the edges of a complete graph with κ vertices be colored with $\aleph_0$ colors. For $\kappa > 2^{\aleph_0}$ the Erdős-Rado theorem implies that there is an infinite monochromatic subgraph. However, if $\kappa \leq 2^{\aleph_0}$, then it may be impossible to find a monochromatic triangle. This paper is concerned with the latter situation. We consider the types of colorings of finite subgraphs that must occur when $\kappa \leq 2^{\aleph_0}$. In particular, we are concerned with the case $\aleph_1 \leq \kappa \leq \aleph_\omega$.

The study of these color patterns (known as identities) has a history that involves the existence of compactness theorems for two cardinal models [4]. When the graph being colored has size $\aleph_1$, the identities that must occur ($\mathscr{I}(\aleph_1)$) have been classified by Shelah [6]. If the graph has size greater than or equal to $\aleph_\omega$ the identities that must occur ($\mathscr{I}(\aleph_\omega)$) have also been classified in [5]. This leaves open the question of how the sets $\mathscr{I}(\aleph_m)$ $(2 \leq m < \omega)$ fit between $\mathscr{I}(\aleph_1)$ and $\subseteq \mathscr{I}(\aleph_\omega)$. Some progress in this direction has been made in the paper [2]. It is there shown that if ZFC is consistent then so is ZFC $+ \mathscr{I}(\aleph_{m+1}) \supsetneqq \mathscr{I}(\aleph_m)$ for each $m < \omega$. The number of colors is fixed at $\aleph_0$ as it is the natural place to start and the results here can be generalized to more colors. We first give some definitions and establish some notation.

An *ω-coloring* is a pair $\langle f, B\rangle$ where $f\colon [B]^2 \to \omega$. The set B is the *field* of f and denoted $\mathrm{fld}(f)$.

DEFINITION 1.1. Let f, g be ω-colorings. We say that f *realizes* the coloring g if there is a one-to-one function $k\colon \mathrm{fld}(g) \to \mathrm{fld}(f)$ such that for all $\{x, y\}$, $\{u, v\} \in \mathrm{dom}(g)$

$$f(\{k(x), k(y)\}) \neq f(\{k(u), k(v)\}) \implies g(\{x, y\}) \neq g(\{u, v\}).$$

We write $f \simeq g$ if f realizes g and g realizes f. It should be clear that $\simeq$ induces an equivalence relation on the class of ω-colorings. We call the $\simeq$-classes *identities*.

If f, g, h, k are ω-colorings, with $f \simeq g$ and $h \simeq k$, then f realizes h if and only if g realizes k. Thus without risk of confusion we may speak of identities realizing colorings and of identities realizing other identities. We say that an identity I is of *size* r if $|\mathrm{fld}(f)| = r$ for some (all) $f \in I$. In the following we will consider only identities of finite size.

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Let κ be a cardinal and $f\colon [\kappa]^2 \to \omega$. We define $\mathscr{I}(f)$ to be the collection of identities realized by f and $\mathscr{I}(\kappa)$ to be

$$\bigcap \{\, \mathscr{I}(f) \mid f\colon [\kappa]^2 \to \omega \,\}.$$

We now define a specific collection of identities. Let $h\colon {}^{<\omega}2 \to \omega$ be one-to-one. Define $f\colon [2^\omega]^2 \to \omega$ by $f(\{\alpha, \beta\}) = h(\alpha \cap \beta)$. We define $\mathscr{J} = \mathscr{I}(f)$. Note that $\mathscr{J}$ is independent of the choice of h. In [5], the second author proved that $2^{\aleph_0} > \aleph_\omega$ implies $\mathscr{I}(\aleph_\omega) = \mathscr{J}$.

In [2], was shown consistency of ZFC $+ \mathscr{I}(\aleph_2) \neq \mathscr{I}(\aleph_\omega)$. Here we will show

MAIN THEOREM. *If* ZFC *is consistent then*

$$\text{ZFC} + 2^{\aleph_0} > \aleph_\omega + \mathscr{I}(\aleph_2) = \mathscr{I}(\aleph_\omega)$$

is consistent.

This is accomplished by adding $\nu > \aleph_\omega$ random reals to a model of GCH. As $2^{\aleph_0} > \aleph_\omega$ holds in the resulting model we need only show that $\mathscr{I}(\aleph_2) \supseteq \mathscr{J}$ is true.

§2. The partial order.

We establish the notation necessary to add many random reals to a model of ZFC. For a more detailed explanation see [3]. Let $\nu > \aleph_\omega$ be a cardinal. Let $\Omega = {}^\nu\{0, 1\}$. Let T be the set of functions t from a finite subset of ν into $\{0, 1\}$. For each $t \in T$, let $S_t = \{\, f \in \Omega : t \subset f \,\}$ and let $\mathcal{S}$ be the σ-algebra generated by $\{\, S_t : t \in T \,\}$. The product measure m on $\mathcal{S}$ is the unique measure so that $m(S_t) = 1/2^{|t|}$. We define $\mathscr{B}_1$ to be the boolean algebra $\mathcal{S}/J$ where J is the ideal of all $X \in \mathcal{S}$ of measure 0. We define a partial order $\langle \mathbb{P}, < \rangle$ by letting $\mathbb{P} = \mathscr{B}_1 \setminus J$ and the order be inclusion modulo J. The following two theorems can be found in [3].

THEOREM 2.1. *$\mathbb{P}$ is c.c.c.*

THEOREM 2.2. *Let M be a model of set theory and G be $\mathbb{P}$-generic. Then $M[G]$ satisfies $2^{\aleph_0} \geq \nu$.*

Let $Y = \{\, y_\alpha : \alpha < \nu \,\}$. Let Γ denote the collection of all $\tau(\bar{y})$ where $\bar{y}$ is a tuple from Y and $\tau(\bar{x})$ is a boolean term with free variables $\bar{x}$. For $\alpha < \nu$ denote by $t_\alpha \in T$ the function whose domain is $\{\alpha\}$ such that $t_\alpha(\alpha) = 0$. There is an obvious embedding of Γ into $\mathcal{S}$ which extends the map $y_\alpha \mapsto S_{t_\alpha}$ and respects the boolean operations. We denote by $\mathscr{B}_0$ the image of Γ in $\mathcal{S}$. It should be clear that $\mathscr{B}_0$ is a boolean algebra. We call the elements of Y *generators*. Elements of $\mathscr{B}_0$ are denoted by their preimage in Γ. The following theorem should be clear.

THEOREM 2.3. *For $p \in \mathcal{S}$ and $\varepsilon > 0$ there exists a finite $u \subset Y$ and a boolean formula $\tau(\bar{x})$ such that $\mu\big(\tau(\bar{u}) \,\Delta\, p\big) < \varepsilon$.*

§3. A combinatorial statement.

Here we formulate a combinatorial statement

$$[I, \kappa, \lambda, g, f]$$

which will play a crucial role in the proof of the main result. We require some preliminary definitions. Let Y, $\mathcal{S}$, $\mathscr{B}_0$, $\mathscr{B}_1$, μ and $\mathbb{P}$ be as in the previous section.

Let $g, f : \omega \to \omega$. For each $L < \omega$ let $\mathcal{T}_L$ be a finite set of boolean terms $\tau(\bar{x})$ where $\bar{x} = (x_1, \dots, x_{f(L)})$ which is complete in the sense that for any boolean term $\sigma(\bar{x})$ there is some $\tau(\bar{x}) \in \mathcal{T}_L$ such that $\sigma(\bar{x}) = \tau(\bar{x})$ is a valid formula of the theory of boolean algebras. Let $\mathcal{T} = \bigcup\{ \mathcal{T}_L : L < \omega \}$. In the following we work only with boolean formulas in $\mathcal{T}$. List $\mathcal{T}_L$ as $\{ \tau^L_i : i \leq h(L) \}$. For $L < \omega$ define $\mathbb{T}_L = (\mathcal{T}_L)^{g(L)}$. For $w \in [\kappa]^2$ and $L < \omega$ define

$$\mathbb{T}_{w,L} = \{ \langle \tau_1(\bar{x}^{w,t}_L), \dots, \tau_{g(L)}(\bar{x}^{w,t}_L) \rangle : t = \langle \tau_1, \dots, \tau_{g(L)} \rangle \in \mathbb{T}_L \}$$

where $\bar{x}^{w,t}_L = \langle x^{w,t}_{L,1}, \dots, x^{w,t}_{L,f(L)} \rangle$ is a sequence of distinct variables for each triple (w, t, L), and where

$$\bar{x}^{w,t}_L \cap \bar{x}^{v,u}_m \neq \emptyset \implies (t = u \wedge w = v \wedge L = M).$$

Let X denote

$$\bigcup\{ \bar{x}^{w,t}_L : t \in \mathbb{T}_L, L < \omega, w \in [\kappa]^2], \}.$$

Let $\mathcal{C}(P, L)$ denote

$$\{ c : c \text{ is a mapping of } [P]^2 \text{ into } \{1, \dots, g(L)\} \}.$$

Definition 3.1. Let k, $m < \omega$ and $\langle \tau_n(\bar{x}) : n \leq k \rangle$ be a sequence of m-ary boolean formulas. Let $\bar{u}$ be an m-tuple from Y. Then $\langle \tau_n(\bar{u}) : n \leq k \rangle$ is called a *partition sequence* if

$$\mu\big(\tau_m(\bar{u}) \cap \tau_n(\bar{u})\big) = 0$$

for all m, n with $m \neq n$, and

$$\mu\Big(\bigcup\{ \tau_n(\bar{u}) : n \leq k \}\Big) = 1.$$

The combinatorial statement will now be defined.

Definition 3.2. Let I be an r-identity, $\lambda \leq \omega$ and κ a cardinal. We say that $[I, \kappa, \lambda, g, f]$ *holds* if the following is true: there exist $\bar{u}_{w,L}$, $\tau^w_{L,m}$ ($w \in [\kappa]^2$, $L < \lambda$, $1 \leq m \leq g(L)$) such that for all $w \in [\kappa]^2$, $L < \lambda$ and $P \in [\kappa]^r$

(C1) $\bar{u}_{w,L}$ is a tuple in Y of length $f(L)$
(C2) $\tau^w_{L,m} \in \mathcal{T}_L$, $\langle \tau^w_{L,1}, \dots, \tau^w_{L,g(L)} \rangle \in \mathbb{T}_L$
(C3) $\langle \tau^w_{L,m}(\bar{u}_{w,L}) : 1 \leq m \leq g(L) \rangle$ is a partition sequence
(C4) for $N \leq L$,

$$\mu\Big(\bigcup\{ \tau^w_{N,m}(\bar{u}_{w,N}) \cap \tau^w_{L,m}(\bar{u}_{w,L}) : m \leq g(N) \}\Big) \geq 1 - 1/2^N$$

(C5) the measure of

$$\bigcup\Big\{ \bigcap\{ \tau^z_{L,c(z)}(\bar{u}_{z,L}) : z \in [P]^2 \} : c \in \mathcal{C}(P, L) \wedge c \text{ realizes } I \Big\}$$

is less than $1/L$.

§4. Proof of the main theorem. The theorem follows from the following three lemmas which will be proved later.

Lemma 4.1. *Let $I \in \mathscr{J}$. For no $g, f\colon \omega \to \omega$ and $\kappa > \aleph_\omega$ do we have $[I, \kappa, \omega, g, f]$.*

Lemma 4.2. *Let $I \in \mathscr{J}$, $\kappa \geq \aleph_0$ and $g, f\colon \omega \to \omega$ be such that $[I, \kappa, \omega, g, f]$ fails. Then there exists $m < \omega$ such that $[I, m, m, g, f]$ fails.*

Lemma 4.3. *Let $I \in \mathscr{J}$ and M be a model of set theory satisfying* GCH. *Let G be $\mathbb{P}$-generic over M. If it is true in $M[G]$ that $I \notin \mathscr{J}(\aleph_2)$, then in M there exists $g, f\colon \omega \to \omega$ such that $[I, m, m, g, f]$ holds for all $m < \omega$.*

We suppose that these lemmas are true and prove the main result. Let M be a model of ZFC + GCH. Let $I \in \mathscr{J}$ and towards a contradiction suppose that $I \notin \mathscr{J}(\aleph_2)$ in $M[G]$ where G is $\mathbb{P}$-generic over M. By Lemma 4.3 in M there exist $g, f\colon \omega \to \omega$ such that $[I, m, m, g, f]$ holds for all $m < \omega$. But from Lemma 4.1, $[I, (\aleph_\omega)^+, \omega, g, f]$ fails, and so by Lemma 4.2 there exists $m < \omega$ such that $[I, m, m, g, f]$ fails, contradiction.

4.1. Proof of the first lemma. Assume that the conclusion of the lemma fails. Let $\kappa > \aleph_\omega$. Let $g; f\colon \omega \to \omega$ be such that $[I, \kappa, \omega, g, f]$ holds. We force with the partial order $\mathbb{P}$, where $\mathbb{P}$ is as defined as in the second section with $\nu = \kappa$. Let $G \subseteq \mathbb{P}$ be a generic set. For $L < \omega$ we define $c_L\colon [\kappa]^2 \to \omega$ by $c_L(w) = m$ if $\tau^w_{L,m}(\bar{u}_{w,L})/J \in G$.

Proposition 4.4. *For all $w \in [\kappa]^2$ there exists $N < \omega$, $m < \omega$ such that $c_L(w) = m$ for all $L > N$.*

Proof. For $w \in [\kappa]^2$ define

$$D_w = \big\{\, p \in \mathbb{P} : p \Vdash \exists N\, \exists m\, \big(c_L(w) = m \text{ for all } L > N\big) \,\big\}.$$

We claim that D_w is dense in $\mathbb{P}$. To this end choose $p^* \in \mathbb{P}$ and let $p \in \mathcal{S}$ be such that $p/J = p^*$. Let $\mu(p) = \delta$. As $\delta > 0$ we can choose N such that $\sum_{L>N} 1/2^L < \delta/3$. By (C4) of the definition of $[I, \kappa, \omega, g, f]$,

$$\mu\Big(\bigcup\Big\{\bigcap\{\, \tau^w_{L,m}(\bar{u}_{w,L}) : L > N \,\} : m \leq g(N) \Big\}\Big) > 1 - (\delta/3).$$

Thus

$$\mu\Big(\bigcup\Big\{\bigcap\{\, \tau^w_{L,m}(\bar{u}_{w,L}) : L > N \,\} : m \leq g(N) \Big\} \cap p\Big) > \delta/3.$$

There is thus an $m \leq g(N)$ such that $\mu(q) > 0$, where

$$q = \bigcap\{\, \tau^w_{L,m}(\bar{u}_{w,L}) : L > N \,\} \cap p.$$

Clearly $q/J \Vdash c_L(w) = m$ for all $L > N$. Thus the proposition is proved.

We now continue with the proof of the lemma. Define $c\colon [\kappa]^2 \to \omega$ by $c(w) = \lim_{L\to\omega} c_L(w)$. Fix $P \in [\kappa]^r$. By property (C5) of $[I, \kappa, \omega, g, f]$,

$$\sup\{\, \mu(p) : p/J \Vdash \text{“}c_L \text{ realizes } I \text{ on } P\text{”} \,\} < 1/L.$$

Thus

$$\sup\{\, \mu(p) : p/J \Vdash \text{“}c \text{ realizes } I \text{ on } P\text{”} \,\} < 1/L$$

for all sufficiently large $L < \omega$. This set has measure 0 and so it is true that c does not induce I in any generic extension. A contradiction occurs as $\kappa > \aleph_\omega$ and by [5] every coloring $c\colon [\kappa]^2 \to \omega$ must realize I. Thus the lemma is proved. $\dashv$

4.2. Proof of the second lemma. The proof of Lemma 4.2 is accomplished by showing that it is possible to represent the statement $[I, \kappa, \omega, g, f]$ by a theory in a language of propositional constants when the propositional constants are assigned suitable meanings. The compactness theorem is then used to show that the failure of $[I, \kappa, \omega, g, f]$ implies the failure of $[I, m, m, g, f]$ for all sufficiently large m in ω.

Throughout this section fix $g, f\colon \omega \to \omega$. Let $\mathcal{B}_0$ and μ be as previously defined. Let I be an r-identity for some $r < \omega$. Consider X, the collection of free variables previously defined. Define $\mathcal{L} = \{\, p_w : w \in [X]^2 \,\}$ to be a collection of propositional constants. For each partition $\mathcal{P}$ of X let $\sim_{\mathcal{P}}$ denote the associated equivalence relation. Let

$$\mathcal{A}\colon [\kappa]^2 \times \{\, (L, m) : L < \omega \wedge 1 \leq m \leq g(L) \,\} \to \mathcal{T}$$

be such that $\mathcal{A}(w, L, m) \in \mathcal{T}_L$ for all $w \in [\kappa]^2$ and $1 \leq m \leq g(L)$. Let

$$\mathcal{Q} = \{\, q^w_{L,m,i} : w \in [\kappa]^2,\ L < \omega,\ 1 \leq m \leq g(L),\ i \leq h(L) \,\}$$

be a collection of propostional constants. Denote $\mathcal{R} = \mathcal{L} \cup \mathcal{Q}$. For each $\mathcal{P}$ a partition of X and function $\mathcal{A}$ define a truth valuation $V_{\mathcal{P},\mathcal{A}}\colon \mathcal{R} \to \{\mathbf{T}, \mathbf{F}\}$ by $V_{\mathcal{P},\mathcal{A}}(p_w) = \mathbf{T}$ if and only if $w = \{i, j\} \wedge i \sim_{\mathcal{P}} j$ and $V_{\mathcal{P},\mathcal{A}}(q^w_{L,m,i}) = \mathbf{T}$ if and only if $\mathcal{A}(w, L, m) = \tau^L_i$. There is a propositional theory T_0 such that a truth valuation V models T_0 if and only if $V = V_{\mathcal{P},\mathcal{A}}$ for some function $\mathcal{A}$ and partition $\mathcal{P}$.

Let V be a truth valuation that models the theory T_0. Denote by $\mathcal{P}_V$ the partition of X defined by $x_1 \sim_{\mathcal{P}_V} x_2 \iff V(p_{\{x_1,x_2\}}) = \mathbf{T}$. Fix a mapping $v_V\colon X \to Y$ such that $v_V(x) = v_V(y) \iff x \sim_{\mathcal{P}_V} y$. For $L < \omega$, $1 \leq m \leq g(L)$ and $w \in [\kappa]^2$ define $\tau^{V,w}_{L,m}$ to be τ^L_i if $V(q^w_{L,m,i}) = \mathbf{T}$. Let $t = t^{V,w}_L$ denote $\langle \tau^{V,w}_{L,1}, \dots, \tau^{V,w}_{L,g(L)} \rangle \in \mathbb{T}_L$. For each such sequence let $\bar{x}^{V,w,t}_L$ denote $\bar{x}^{w,t}_L$ and write $\tau^{V,w}_{L,m}(\bar{u}^{V,w}_L)$ for the $\mathcal{B}_0$-term obtained from $\tau^{V,w}_{L,m}(\bar{x}^{V,w,t}_L)$ by substituting the variables $\bar{x}^{V,w,t}_L$ by their image under v_V. Note that since $\mathbb{T}_L$ is finite, for each $L < \omega$ and $w \in [\kappa]^2$,

$$X^w_L =_{\mathrm{def}} \bigcup \{\, \bar{x}^{V,w,t}_L : t = t^{V,w}_L \in \mathbb{T}_L \wedge V \text{ models } T_0 \,\}$$

is finite.

Lemma 4.5. *Let $k < \omega$ and $\sigma(x_1, \dots, x_k)$ be a boolean term. For $1 \leq i \leq k$ let $L_i < \omega$, $1 \leq m_i \leq g(L_i)$ and $w_i \in [\kappa]^2$. Let $\theta(y)$ be a statement of one of the forms $\mu(y) < 1/n$, $\mu(y) > 1/n$ or $\mu(y) = 0$, where y runs through $\mathcal{B}_0$. There exists a propositional formula χ such that for all valuations V modelling T_0, V models χ if and only if*

$$\theta\Bigl(\sigma\bigl(\tau^{V,w_1}_{L_1,m_1}(\bar{u}^{V,w_1}_{L_1}), \dots, \tau^{V,w_k}_{L_k,m_k}(\bar{u}^{V,w_k}_{L_k})\bigr)\Bigr).$$

Proof. Let $W = \bigcup \{\, X^{w_i}_{L_i} : 1 \leq i \leq k \,\}$. Define

$$\mathcal{V} = \{\, V : V \text{ is a truth valuation modelling } T_0 \,\}.$$

Since $\mathscr{T}_{L_i}$ is finite for all $1 \leq i \leq k$ the collection

$$S = \{ \langle \tau^{V,w_i}_{L_i,m_i} : 1 \leq i \leq k \rangle : V \in \mathscr{V} \}$$

is a finite set. For each $s \in S$ define

$$\mathscr{V}_s = \{ V \in \mathscr{V} : \langle \tau^{V,w_i}_{L_i} : 1 \leq i \leq k \rangle = s \}.$$

For the moment fix $s \in S$. Each $V \in \mathscr{V}_s$ induces a partition, $\mathscr{P}_{V_s}$ of X and thus of W. Since every permutation of Y induces an automorphism of $\mathscr{B}_0$ which preserves the measure, for $V_1, V_2 \in \mathscr{V}_s$, $\mathscr{P}_{V_1} \restriction W = \mathscr{P}_{V_2} \restriction W$ implies

$$\mu\Big(\sigma\big(\tau^{V_1,w_1}_{L_1,m_1}(\bar{u}^{V_1,w_1}_{L_1}), \ldots, \tau^{V_1,w_k}_{L_k,m_k}(\bar{u}^{V_1,w_k}_{L_k})\big)\Big) \\ = \mu\Big(\sigma\big(\tau^{V_2,w_1}_{L_1,m_1}(\bar{u}^{V_2,w_1}_{L_1}), \ldots, \tau^{V_2,w_k}_{L_k,m_k}(\bar{u}^{V_2,w_k}_{L_k})\big)\Big).$$

As there are only finitely many partitions of W there is a formula χ_s that chooses those partitions in $\{ \mathscr{P}_V : V \in \mathscr{V}_s \}$ that produce the desired measure. We define

$$\chi = \bigvee_{s \in S} (\eta_s \implies \chi_s),$$

where η_s is a formula such that $V \in \mathscr{V}$ implies $s = \langle \tau^{V,w_i}_{L_i,m_i} : 1 \leq i \leq k \rangle$ if and only if $V(\eta_s) = \mathbf{T}$. $\dashv$

LEMMA 4.6. *There is a propositional theory T such that T is consistent if and only if $[I, \kappa, \omega, g, f]$ holds.*

PROOF. By the previous lemma, for each triple (w, L, P) where $w \in [\kappa]^2$, $L < \omega$ and $P \in [\kappa]^r$ there exists a formula $\chi_{w,L,P}$ such that a truth valuation V models $T_0 \cup \{\chi_{w,L,P}\}$ implies (C1)–(C5) hold for w, L, P and the sequences of boolean terms and generators defined by the valuation. We define T to be

$$T_0 \cup \{ \chi_{w,L,P} : w \in [\kappa]^2, L < \omega \text{ and } P \in [\kappa]^r \}.$$

It is easily seen that the consistency of T implies that $[I, \kappa, \omega, g, f]$ holds. In this regard one should observe that Y is large enough to realize any desired partition.

Now suppose that $[I, \kappa, \omega, g, f]$ holds. The existence of the sequences of terms $t^w_L = \langle \tau^w_{L,1}, \ldots, \tau^w_{L,g(L)} \rangle$ and generators $\bar{u}_{w,L} = \langle u_{w,L,1}, \ldots, u_{w,L,f(L)} \rangle$ defines a function $\mathscr{A}$ and partition $\mathscr{P}$ in the following manner. Let $\mathscr{A}(w, L, m) = \tau^L_i$ if $\tau^w_{L,m} = \tau^L_i$. A partition $\mathscr{P}'$ of

$$\bigcup \{ \bar{x}^{w,t}_L : t = t^w_L,\ w \in [\kappa]^2,\ L < \omega \}$$

is first defined by setting $x^{w,t}_{L,i} \sim_{\mathscr{P}'} x^{v,u}_{M,j}$ if $u_{w,L,i} = u_{v,M,j}$ where $t = t^w_L$ and $s = t^v_m$. We choose a partition of X which is an extension of $\mathscr{P}'$ and denote it by $\mathscr{P}$. The truth valuation $V_{\mathscr{P},\mathscr{A}}$ models the theory T. This completes the proof of Lemma 4.6. $\dashv$

Now Lemma 4.2 follows by the compactness theorem for propositional logic.

4.3. Proof of the third lemma. Towards a contradiction let I be an identity on $r < \omega$ elements, d a $\mathbb{P}$-name for a function and $p \in \mathbb{P}$ such that

$$p \Vdash \text{“}d : [\aleph_2]^2 \to \omega \wedge d \text{ does not realize } I\text{”}.$$

Without loss of generality we assume that $p = 1_{\mathbb{P}}$. For each $w \in [\aleph_2]^2$ choose a sequence $\langle b^w_n : n < \omega \rangle$ and a sequence $\langle p^w_n : n < \omega \rangle \in [\mathcal{S}]^\omega$ such that $\langle p^w_n/J : n < \omega \rangle$ is a maximal antichain in $\mathbb{P}$ and $p^w_n/J \Vdash d(w) = b^w_n$. Let $b : [\aleph_2]^2 \times \omega \to \omega$ be defined by $b(w, n) = b^w_n$.

For $w \in [\aleph_2]^2$, $L < \omega$ choose $g(w, L)$ so that

$$\sum_{n > g(w,L)} \mu(p^w_n) < \frac{1}{(2^{L+5}L)}.$$

Lemma 4.7. *There exists a function $f : [\aleph_2]^2 \times \omega \to \omega$ sequences of boolean terms $\langle \sigma^w_{L,m} : m \le g(w,L) \rangle$ and generators $\bar{v}_{w,L}$ ($w \in [\aleph_2]^2$, $L < \omega$) such that:*

(1) $\bar{v}_{w,L} = \{ y_{w,L,k} : k \le h(w,L) \}$.

(2) *For $m \le g(w,L)$ we have*

$$\mu(p^w_m \,\Delta\, \sigma^w_{L,m}(\bar{v}_{w,L})) < \frac{1}{(L2^{L+5}[g(w,L)]^{r^2+1})}.$$

Proof. This follows immediately from the results in the section where $\mathbb{P}$ is defined. ⊣

Lemma 4.8. *There exists a function $f : [\aleph_2]^2 \times \omega \to \omega$ sequences of boolean terms $\langle \rho^w_{L,m} : m \le g(w,L) \rangle$ and generators $\bar{v}_{w,L}$ ($w \in [\aleph_2]^2$, $L < \omega$) such that:*

(1) $\bar{v}_{w,L} = \{ y_{w,L,k} : k \le f(w,L) \}$.

(2) $\langle \rho^w_{L,m}(\bar{v}_{w,L}) : m \le g(w,L) \rangle$ *is a partition sequence.*

(3) *For $m < g(w,L)$ we have*

$$\mu(p^w_m \,\Delta\, \rho^w_{L,m}(\bar{v}_{w,L})) < \frac{1}{2^{L+3}L[g(w,L)]^{r^2}}.$$

(4) $\mu(p^w_{g(w,L)} \,\Delta\, \rho^w_{L,g(w,L)}(\bar{v}_{w,L})) < 1/L2^{L+3}$.

Proof. Let f, $\sigma^w_{L,m}$, and $\bar{v}_{w,L}$ satisfy the conclusion of the last lemma. For $m < g(w,L)$ define

$$\rho^w_{L,m}(\bar{v}_{w,L}) = \sigma^w_{L,m}(\bar{v}_{w,L}) \setminus \bigcup \{ \sigma^w_{L,i}(\bar{v}_{w,L}) : i < m \}.$$

Define

$$\rho^w_{L,g(w,L)}(\bar{v}_{w,L}) = 1 \setminus \bigcup \{ \sigma^w_{L,i}(\bar{v}_{w,L}) : i < g(w,L) \}.$$

Parts (1) and (2) of the conclusion clearly hold. For $m < g(w,L)$,

$$\begin{aligned}
\mu(p^w_m \,\Delta\, \rho^w_{L,m}(\bar{v}_{w,L})) &\le \sum_{i \le m} \mu(p^w_i \,\Delta\, \sigma^w_{L,i}(\bar{v}_{w,L})) \\
&\le g(w,L)/2^{L+5}L[g(w,L)]^{r^2+1} \\
&= 1/2^{L+5}L[g(w,L)]^{r^2}.
\end{aligned}$$

For $m = g(w, L)$

$$\begin{aligned}\mu\big(p^{w}_{g(w,L)}\,\Delta\,\rho^{w}_{L,g(w,L)}(\bar{v}_{w,L}) \\ &\leq \sum_{i \leq g(w,L)} \mu\big(p^{w}_{i}\,\Delta\,\sigma^{w}_{L,i}(\bar{v}_{w,L})\big) + \mu\Big(\bigcup\{\, p^{w}_{i} : i > g(w,L)\,\}\Big) \\ &\leq g(w,L)/(L2^{L+5}[g(w,L)]) + 1/L2^{L+5}.\end{aligned}$$

Lemma 4.9 (GCH). *Let $s < \omega$ and for $1 \leq i \leq s$ let $h_i \colon [\aleph_2]^2 \times \omega \to \omega$. There exists $A = \langle \alpha_i : i < \omega \rangle \in [\aleph_2]^\omega$ and for $1 \leq i \leq s$ there exist functions $\hat{h}_i \colon \omega \to \omega$ such that*

$$\forall n < \omega\, \forall m \leq n\, \forall w \in [\{\, \alpha_i : n < i < \omega \,\}]^2 \, \big(h_i(w, m) = \hat{h}_i(m)\big).$$

Proof. A standard ramification argument will show that there exists $Z_0 \subseteq \aleph_2$ of order type $\aleph_1$ such that for $\alpha < \beta < \gamma$ in Z_0, $L < \omega$, and $1 \leq i \leq s$

$$\big(h_i(\{\alpha, \beta\}, L) = h_i(\{\alpha, \gamma\}, L)\big).$$

See [8, 1] for details. For $\alpha \in Z_0$, $L < \omega$ and $1 \leq i \leq s$ define $h_{i,\alpha}(L) = h_i(\{\alpha, \beta\}, L)$ where $\beta > \alpha$ is chosen in Z_0. By cardinality considerations there exists a sequence $\langle Z_i : 1 \leq i < \omega \rangle$ of subsets of Z_0 such that for all $k < \omega$, we have $Z_{k+1} \subseteq Z_k$, $|Z_k| = \aleph_1$ and for all $\alpha, \beta \in Z_{k+1}$,

$$h_{i,\alpha} \restriction (k+1) = h_{i,\beta} \restriction (k+1).$$

We define $A = \{\, \alpha_i : i < \omega \,\}$ in the following manner. Let α_0 be minimal in Z_1 and inductively define α_i to be minimal in $Z_{i+1} \setminus \{\alpha_0, \ldots, \alpha_{i-1}\}$. We then define the functions $\hat{h}_i$ by $\hat{h}_i(k) = h_{i,\alpha_k}(k)$.

To verify the lemma let $n < \omega$ and $m \leq n$. Choose

$$w = \{\alpha_t, \alpha_v\} \in [\{\, \alpha_k : n < k < \omega \,\}]^2.$$

Then for $1 \leq i \leq s$

$$h_i(w, m) = h_i(\{\alpha_t, \alpha_v\}, m) = h_{i,\alpha_t}(m) = h_{i,\alpha_m}(m) = \hat{h}_i(m).$$

Thus the lemma is proved. ⊣

Let $b, g \colon [\aleph_2]^2 \times \omega \to \omega$ be the function chosen above and f, $\rho^{w}_{L,m}$, $\bar{v}^{w}_{L}$ satisfy the conclusion of Lemma 4.8. Let $A = \langle \alpha_i : i < \omega \rangle \in [\aleph_2]^\omega$, $\hat{b}, \hat{g}, \hat{f} \colon \omega \to \omega$ be the set of functions obtained when Lemma 4.9 is applied with $s = 3$ and $(h_1, h_2, h_3) = (b, g, f)$. We now verify that $[I, n, n, \hat{g}, \hat{f}]$ holds for all $n < \omega$. To this end fix $n < \omega$. Define $t < \omega$ to be

$$n + \max\{\, g(m) : m \leq n \,\} + 1.$$

For $w = \{i, j\} \in [n]^2$ define w^* to be $\{\, \alpha_{t+i}, \alpha_{t+j} \,\}$. Then for $w \in [n]^2$, $L < n$, $1 \leq m \leq \hat{g}(L)$ define $\tau^{w}_{L,m}$ to be $\rho^{w^*}_{L,m}$ and $\bar{u}_{w,L}$ to be $\bar{v}_{w^*,L}$.

We will now verify that (C1)–(C5) hold for these sequences of boolean terms and generators. (C1)–(C3) will follow from Lemma 4.10, (C4) from Lemma 4.11 and (C5) from Lemma 4.12.

LEMMA 4.10. *Let $\hat{g}$, $\hat{f} : \omega \to \omega$, $A \subset \aleph_2$ and $\tau^w_{L,m}$, $\bar{u}_{w,L}$, $(w \in [n]^2$, $L < n$, $1 \le m \le \hat{g}(L))$ be as defined above. Then*

(1) $\bar{u}_{w,L} = \{ y_{w,L,k} : k \le \hat{f}(L) \}$.
(2) $\langle \tau^w_{L,m}(\bar{u}_{w,L}) : m \le \hat{g}(L) \rangle$ *is a partition sequence.*
(3) *For $m < \hat{g}(L)$ we have*

$$\mu\big(p^{w^*}_m \,\Delta\, \tau^w_{L,m}(\bar{u}_{w,L})\big) < \frac{1}{2^{L+3}L[\hat{g}(L)]^{r^2}}.$$

(4) $\mu\big(p^{w^*}_{\hat{g}(L)} \,\Delta\, \tau^w_{L,\hat{g}(L)}(\bar{u}_{w,L})\big) < 1/L2^{L+3}$.

PROOF. For $w \in [n]^2$, $L < n$ $g(w^*, L) = \hat{g}(L)$ and $f(w^*, L) = \hat{f}(L)$. ⊣

LEMMA 4.11. *Let $w \in [n]^2$ and $N < L < n$. For the sequences of boolean terms defined above*

$$\Big(\mu\Big(\bigcup\{ \tau^w_{N,m}(\bar{u}_{w,N}) \cap \tau^w_{L,m}(\bar{u}_{w,L}) : m \le \hat{g}(N) \}\Big) > 1 - 1/(2^N).$$

PROOF.

$$\begin{aligned}
&\mu\Big(\bigcup\{ \tau^w_{N,m}(\bar{u}_{w,N}) \cap \tau^w_{L,m}(\bar{u}_{w,L}) : m \le \hat{g}(N) \}\Big)\\
&\quad\ge \mu\Big(\bigcup\{ \tau^w_{N,m}(\bar{u}_{w,N}) \cap \tau^w_{L,m}(\bar{u}_{w,L}) \cap p^{w^*}_m : m \le \hat{g}(N) \}\Big)\\
&\quad= 1 - \Big[\mu\Big(\Big(\bigcup\{ \tau^w_{N,m}(\bar{u}_{w,N}) \cap \tau^w_{L,m}(\bar{u}_{w,L}) \cap p^{w^*}_m : m \le \hat{g}(N) \}\Big)^c\Big)\Big]\\
&\quad\ge 1 - \Big(\sum_{n<\hat{g}(N)} \mu\big(p^{w^*}_n \,\Delta\, \tau^w_{N,n}(\bar{u}_{w,N})\big) + \sum_{n<\hat{g}(L)} \mu\big(p^{w^*}_n \,\Delta\, \tau^w_{L,n}(\bar{u}_{w,L})\big)\\
&\qquad\quad + \mu\big(p^{w^*}_{\hat{g}(N)} \,\Delta\, \tau^w_{L,\hat{g}(N)}(\bar{u}_{w,N})\big) + \mu\big(p^{w^*}_{\hat{g}(L)} \,\Delta\, \tau^w_{L,\hat{g}(L)}(\bar{u}_{w,L})\big)\\
&\qquad\quad + \mu\Big(\bigcup\{ p^{w^*}_m : m > \hat{g}(N) \}\Big)\Big)\\
&\quad\ge 1 - (3/2^{N+2})\\
&\quad\ge 1 - 1/2^N.
\end{aligned}$$

This concludes the proof of Lemma 4.11. ⊣

LEMMA 4.12. *Let $L < n$ and $P \in [n]^r$. The measure of*

$$\bigcup\Big\{ \bigcap\{ \tau^z_{L,c(z)}(\bar{u}_{z,L}) : z \in [P]^2 \} : c \in \mathscr{C}(P, L) \wedge c \textit{ realizes } I \Big\}$$

is less than $1/L$.

PROOF. First note that for $z \in [P]^2$ and $1 \le m \le \hat{g}(L)$,

$$p^{z^*}_m / J \Vdash d(z^*) = b(z^*, m).$$

Now $z^* \in [\{ \alpha_s : s \ge t \}]^2$ and $m < t$ so $b(z^*, m) = \hat{b}(m)$. Thus, for $c \in \mathscr{C}(P, L)$,

$$q =_{\text{def}} \bigcap\{ p^{z^*}_{c(z)} : z \in [P]^2 \}/J \Vdash \big(\forall z \in [P]^2\, \big(d(z^*) = b\big(c(z)\big)\big)\big)$$

if $q \neq J$. Thus if c realizes I on P and $q \neq J$ then in some generic extension, d realizes I on $P^* = \{ \alpha_{t+i} : i \in P \}$. Since we assume that d does not realize I we can conclude that $q = J$ and

$$\mu\Big(\bigcap \{ p^{z^*}_{c(z)} : z \in [P]^2 \}\Big) = 0.$$

Secondly note that $|\mathscr{C}(P, L)| < g(L)^{r^2}$.

We first examine those colorings that induce I and involve at least one color other than $g(L)$. For each such c,

$$\mu\Big(\bigcap \{ \tau^z_{L,c(z)}(\bar{u}_{z,L}) : z \in [P]^2 \}\Big) \leq \min\{ \mu(\tau^z_{L,c(z)}(\bar{u}_{z,L}) \,\Delta\, p^{z^*}_{c(z)} : z \in [P]^2 \}.$$

By Lemma 4.10 this measure is at most $1/(2L[g(L)]^{r^2})$. Thus the probability of any of the colorings under consideration inducing I is less than $1/2L$. In the case that the coloring induces I and uses only the color $g(L)$ (implying that there is only one such coloring),

$$\mu\Big(\bigcap \{ \tau^z_{L,g(L)}(\bar{u}_{z,L}) : z \in [P]^2 \}\Big) \leq \min\{ \mu\big(\tau^z_{L,g(L)}(\bar{u}_{z,L}) \,\Delta\, p^{z^*}_{g(L)} : z \in [P]^2 \}.$$

By Lemma 4.10 this value is less than $1/2L$. Thus Lemma 4.12 is proved. ⊣

This finishes the proof of Lemma 4.3. ⊣

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