http://dx.doi.org/10.1090/conm/087/995276

Contemporary Mathematics
Volume 87 (1989)

E–Transitive Groups in L

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Abstract. A torsion–free reduced abelian group G is E–transitive (resp. strongly homogeneous) if $\mathrm{End}(G)$ (resp. $\mathrm{Aut}(G)$) operates transitively on the pure rank 1 subgroups of G. Assuming Gödel's axiom of constructibility $V = L$ of set theory, we construct E–transitive and not strongly homogeneous groups of infinite rank as well as strongly homogeneous groups where we prescribe the centers of their endomorphism rings. Any cotorsion–free strongly homogeneous PID occurs as the center of the endomorphism ring of such a group. We use this to answer some questions raised by J. Hausen.

Introduction. All groups in this note are torsion–free abelian groups. Undefined notations are standard as in [**F**]. Many papers have been written in recent years constructing abelian groups with prescribed rings of endomorphisms. A feature common to all these constructions is that the endomorphism ring has smaller cardinality than the group it acts on. Therefore all these groups are *not* E–transitive, i.e. the endomorphism ring does not act transitively on the pure rank 1 subgroups. Following [**H1**] we call an abelian group G **E–transitive** (resp. **strongly homogeneous**) if the endomorphism ring $\mathrm{End}(G)$ (resp. the automorphism group $\mathrm{Aut}(G)$) acts transitively on the pure rank 1 subgroups of G. We call an R–module M **separable** if each finite subset of M is contained in a free finite rank summand of M. A torsion–free abelian group is called separable if A is a module over some subring R of $\mathbf{Q}$ and A is separable as an R–module. It is easy to see that the class of separable groups is contained in the class of strongly homogeneous groups which again is contained in the class of E–transitive groups. We will show that all these inclusions are proper. To show the latter, we will utilize the diamond principle $\diamondsuit$, a consequence of $V = L$, [**J**]. Let $\mathbf{Z}_p$ be the ring of integers localized at the prime p. The cartesian product $\mathbf{Z}^{\aleph_1}$ is separable while $\mathbf{Z}_p^{\aleph_1}/\mathbf{Z}^{[\aleph_1]}$ is not separable — use the slenderness of $\mathbf{Z}$ and [**FII**, Thm. 94.4] — but strongly homogeneous where $\mathbf{Z}_p^{[\aleph_1]} = \{f \in \mathbf{Z}_p^{\aleph_1} \mid |\{\alpha < \aleph_1 \mid f(\alpha) \neq 0\}| < \aleph_1\}$, cf. [**DH**].

A principal ideal domain (PID) S is strongly homogeneous if each element of S is an integer multiple of a unit. Each separable $\aleph_1$–free module over such a ring is strongly homogeneous as an abelian group. The converse is false as follows from our main

Partially supported by NSF Grant No. DMS 8701074
1980 *Mathematics subject classifications* (1985 *Revision*). Primary 20K30, 20K20.

THEOREM ($V = L$). *Let κ be a regular, not weakly compact cardinal and S a cotorsion–free, strongly homogeneous PID with $|S| < \kappa$. Then*

(a) There exists an E–transitive cotorsion–free abelian group G, $\aleph_1$–free and indecomposable as S–module with Cent(End G) $= S$ and G is not strongly homogeneous. Moreover $Aut(G) = U(X)$, the group of units of S and $|G| = \kappa > |S|$.

(b) There exists a strongly homogeneous cotorsion–free abelian group A of cardinality κ, $\aleph_1$–free and indecomposable as S–module, with Cent(End A) $= S$.

Recall that a group G is cotorsion–free if 0 is its only cotorsion subgroup and a ring R is cotorsion–free if its additive group is cotorsion–free.

We use (b) to answer Problem 1 in [**H2**]: Let $S = \mathbf{Z}$. Then the group A in (b) is indecomposable and the construction shows that the only PID's embeddable in End(A) have cardinality $< |A|$. Thus A is strongly homogeneous but not a separable module over any PID.

Let $S' = \mathbf{Z}_p[x]$ be the polynomial ring with coefficients in $\mathbf{Z}_p$ and $I = pS'$. Then I is a prime ideal of S' and the localization $S = S'_I$ of S' at I is a strongly homogeneous PID. On the other hand, S is not an E–ring as the substitution $x \mapsto x^2$ shows (this substitution induces an endomorphism of S not representable as a multiplication). Since S is countable, torsion–free and reduced, S is cotorsion–free and (b) implies the existence of a strongly homogeneous group G with Cent(End A) $= S$ not an E–ring. This answers Problem 2 in [**H2**] in the negative. Part (a) of the Theorem finally answers Problem 1 in [**H1**]: There are E–transitive, not strongly homogeneous groups (at least in $V = L$).

The endomorphism rings of the groups constructed in (a) and (b) are as follows: For S the PID in the Theorem, κ a cardinal, let R_κ^+ be the polynomial ring over S in κ–many non–commuting variables x_α, $\alpha < \kappa$. Thus R_κ^+ is the ring freely generated by S and $\{x_\alpha \mid \alpha < \kappa\}$ with the only relations being $sx_\alpha = x_\alpha s$ for all $\alpha < \kappa$, $s \in S$. For the groups G constructed in (a) we have End(G) $\cong \widetilde{(R_\kappa^+)}$ as rings and for each regular, not weakly compact cardinal we have 2^κ such groups. Here $\widetilde{(R_\kappa^+)}$ is the completion of R_κ^+ in some topology. We refer to Chapter 2 for details. We would like to mention the result in [**H1**] that two E–transitive groups of equal type are isomorphic if and only if their endomorphism rings are topologically isomorphic with respect to the finite topology.

Let R_κ^- be the subring of the quotient field of R_κ^+ generated by R_κ^+ and $\{x_\alpha^{-1} \mid \alpha < \kappa\}$. The groups A constructed in part (b) of the Theorem have endomorphism ring isomorphic to $\widetilde{(R_\kappa^-)}$ and the group generated by $\{x_\alpha \mid \alpha < \kappa\}$ acts transitively on the pure rank 1 subgroups of A.

We don't know much about the rings $\widetilde{(R_\kappa^+)}$ and $\widetilde{(R_\kappa^-)}$. In order to prove the theorem, we show that the group of units $U\widetilde{(R_\kappa^+)}$ coincides with $U(S)$ and $\widetilde{(R_\kappa^-)}$ has only the trivial idempotents. We don't know, for instance, if $\widetilde{(S_\kappa^+)}$ is the completion of R_κ^+ in the finite topology.

In contrast to other diamond, weak diamond or "Black Box" constructions we have to use epimorphic images in our transfinite induction instead of just embeddings. This seems to make it hard to utilize a "Black Box". To keep the paper readable we restrict ourselves to diamonds rather than going for the most general result (we plan to do this in a forthcoming paper).

1. Algebraic Preliminaries. In all that follows S is a cotorsion–free, strongly homogeneous PID as defined in the introduction. Let S_0 be the pure subring of S generated by 1. Let $P_0 = \{p \text{ prime} \mid S_0 \text{ is } p\text{–divisible}\}$. Let N be a subsemigroup of $\mathbf{Z}(+)$. We will be interested only in the cases $N = \mathbf{Z}$ or $N = \omega = \{z \in \mathbf{Z} \mid z \geq 0\}$. We introduce non commuting variables x_μ, $\mu < \alpha$, α an ordinal and set

$$R_\alpha = R_{\alpha,N} = S\langle x_\mu \mid \mu < \alpha\rangle_N,$$

the polynomial ring over S in x_μ ($\mu < \alpha$) with exponents in N. Thus each element $f \in R_\alpha$ has a unique representation $f = \sum_{m\in M} s_m m$ with $s_m \in S$ and monomials m, i.e. $m = x_{\alpha_1}^{e_1} x_{\alpha_2}^{e_2} \cdots x_{\alpha_n}^{e_n}$, $0 \neq e_n \in N$, $\alpha_i < \alpha$ and $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \cdots \neq \alpha_n$. Let $M_{\alpha+1}$ be the set of all monomials in $R_{\alpha+1}$ and ${}_\alpha M_{\alpha+1}$ the set of all monomials in $M_{\alpha+1}$ with $\alpha_1 \neq \alpha$.

Each monomial $m = x_{\alpha_1}^{e_1} \cdots x_{\alpha_n}^{e_n} \in R_{\alpha+1}$ has a unique representation with $e_i \neq 0$ and $\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_n$. We define $\ell(m) = n$ to be the **length** of m. We define a linear preorder $\prec$ on $M_{\alpha+1}$ by $m \prec m'$ if either $\ell(m) < \ell(m')$ or $\ell(m) = \ell(m')$ and $e_1 < e_1'$ for $m = x_{\alpha_1}^{e_1} \ldots$ and $m' = x_{\alpha_1'}^{e_1'} \ldots$.

Next we observe:

(1.1). *$R_{\alpha+1}$ is free as a (right or left) R_α–module.*

PROOF: The set of monomials m with $\ell(m) \geq \ell(x_\alpha m)$ is together with 1 a basis of $R_{\alpha+1}$ as left R_α–module. A similar argument works on the right. ∎

Let G be a right R_α–module.

(1.2). *The tensor product $H = G \bigotimes_{R_\alpha} R_{\alpha+1}$ is isomorphic to a direct sum of copies of G as S–module and the canonical map $g \mapsto g \otimes 1$ embeds G into H. The S–module H is a right $R_{\alpha+1}$–module in the natural way.*

REMARK: Each element $h \in H$ has a unique representation $h = \sum h_{i,m} \otimes x_\alpha^i m + h_0 \otimes 1$ where $h_{i,m}$, $h_0 \in G$, $i \geq 1$ and $m \in M_{\alpha+1}$ are monomials with $\ell(m) + 1 = \ell(x_\alpha m)$. We sometimes refer to $h_{i,m} \otimes x_\alpha^i m$ as the "$x_\alpha^i m$–coordinate of h" or to $h_{i,m}$ as the entry in the $x_\alpha^i m$–coordinate of h.

We fix elements g, $h \in G$ such that the S–submodules gS and hS are pure in G. Set $K = (g \otimes x_\alpha - h \otimes 1)R_{\alpha+1}$.

(1.3). *$K = (g \otimes x_\alpha - h \otimes 1)R_{\alpha+1}$ is pure in $H = G \bigotimes_{R_\alpha} R_{\alpha+1}$.*

PROOF: Let z be a non-zero integer and $k \in K \cap zH$, $k = (g \otimes x_\alpha - h \otimes 1)\sum_{m\in M} s_m m$ with $s_m \in S \setminus \{0\}$ and $M \subseteq R_{\alpha+1}$ a finite set of monomials. Pick an element $m \in M$, maximal with respect to $\prec$. This implies that the entry in the $x_\alpha m$–coordinate of k is gs_m, provided $\ell(x_\alpha m) \geq \ell(m)$. Therefore $gs_m \in zG \cap gS = gzS$ and $s_m \in zS$ follows. If $\ell(x_\alpha m) < \ell(m)$, then $m = x_\alpha^{-1} x_{\alpha_2}^{e_2} \cdots x_{\alpha_n}^{e_n}$. This implies that the entry in the m–coordinate of k is $-hs_m \otimes m$. Thus $s_m \in zS$. Induction over $|M|$ shows the rest. ∎

(1.4). $K \cap G \otimes 1 = 0$.

PROOF: Let $k = (g \otimes x_\alpha - h \otimes 1)\sum_{m\in M} s_m m$ be as in the proof of 1.3. For $m \in M$ maximal with respect to $\prec$ we obtain $gs_m \otimes x_\alpha m = 0$ in case $\ell(x_\alpha m) \geq \ell(m)$. Thus $s_m = 0$, a contradiction to the choice of m. If $\ell(x_\alpha m) < \ell(m)$, then $m = x_\alpha^{-1} x_{\alpha_2}^{e_2} \cdots x_{\alpha_n}^{e_n}$. This implies that the entry in the m–coordinate of k is $-hs_m \otimes m = 0$. Again, $s_m = 0$, a contradiction. ∎

(1.5). *$G = G \otimes 1$ is pure in H/K.*

PROOF: Let $a \otimes 1 \in G$, $k \in K$ and suppose $a \otimes 1 + k \in zH$, z an integer. Let $k = (g \otimes x_\alpha - h \otimes 1)\sum_{m \in M} s_m m$. We may assume that $s_m \notin zS$ for all $m \in M$. A similar line of argument as above leads to some $m \in M$ with $s_m \in zS$. This implies $k = 0$ and $a \otimes 1 \in zH$. The submodule $G \otimes 1$ is (as S–module) a summand of H and therefore pure. This implies $a \in zG$ and $k \in zH \cap K = zK$. This shows $G \otimes 1$ pure in H/K. ∎

(1.6). *Let L be a pure S–submodule of G and U a finite set of monomials in $R_{\alpha+1}$ with the following properties:*

(α) $g, h \in L$

(β) If $x_\alpha^i m \in U$, then $m \in U$ if $\ell(m) < \ell(x_\alpha m)$

(γ) If $a x_\alpha^i b \in U$, with $i \neq 0$, $a \neq 1$ a monomial in R_α, $\ell(x_\alpha b) > \ell(b)$ then $La \subseteq L$ and $x_\alpha^i b \in U$.

Then $F + K$ is pure in H where $F = \sum_{m \in U} L \otimes m$.

PROOF: Let $f = \sum_{u \in U} \ell_u \otimes u \in F$ and $k = (g \otimes x_\alpha - h \otimes 1)\sum_{m \in M} s_m m \in K$. Assume that $f + k \in zH$, $k \notin zK = zH \cap K$ and $1 \leq |M|$ minimal with that property. Let $m \in M$ be maximal with respect to $\prec$.

Case I: $\ell(x_\alpha m) > \ell(m)$.

Then $g s_m \otimes x_\alpha m$ is the $x_\alpha m$–coordinate of k and we conclude $\ell_{x_\alpha m} - g s_m \in zG$. By the choice of k, $s_m \notin zK$ and $\ell_{x_\alpha m} \neq 0$ and $x_\alpha m \in U$. Therefore $m \in U$ and $g \otimes x_\alpha m$, $h \otimes m \in L$. This contradicts the minimality of $|M|$ (observe that F is pure in H because of (γ) and L being pure in G).

Case II: $\ell(x_\alpha m) = \ell(m)$.

This implies $m = x_\alpha^{e_1} x_{\alpha_2}^{e_2} \dots$ with $e_1 \neq 0, -1$. If $e_1 > 0$, $g s_m \otimes x_\alpha m$ is the $x_\alpha m$–coordinate of k. Thus $s_m \in zS$ and we may argue as in Case I. If $e_1 < 0$, then $-h s_m \otimes m$ is the entry in the m–coordinate of k. Thus $\ell_m \otimes m - h s_m \otimes m \in zH$, $\ell_m - h s_m \in zG$, $h s_m \notin zG$ which implies $m \in U$, $\ell_m \neq 0$. If $x_\alpha m \in U$, we may continue as in Case I. If $x_\alpha m \notin U$, we consider what contributes to the $x_\alpha m$–coordinate of k. For $m' \in M$ with $x_\alpha m = x_\alpha m'$ we have $m = m'$ and if $x_\alpha m = m'$, $m' \in M$ we have a contradiction to the maximality of m. Thus the $x_\alpha m$–entry of k is $g s_m$ and since $x_\alpha m \notin U$ we conclude $s_m \in zS$, a contradiction to the minimality of $|M|$.

Case III: $\ell(x_\alpha m) < \ell(m)$.

Here $\alpha_1 = \alpha$, $e_1 = -1$ and $h s_m \otimes m$ is the m–coordinate of k and $\ell_m - h s_m \in zG$. Thus $\ell_m \neq 0$, $m \in U$ and $h s_m \otimes m \in F$. Since in this case $m \in U$ implies $x_\alpha m \in U$ we have $g s_m \otimes s_\alpha m - h s_m \otimes m \in F$, a contradiction to the minimality $|M|$. ∎

Note that each finite set of monomials U is contained in a finite set of monomials U' satisfying the conditions in (β) and (γ).

Now 1.6 implies:

(1.7). *If G is $\aleph_1$–free as S–module and $g, h \in G$ as above with gS and hS pure in G, then G is a pure subgroup of the $\aleph_1$–free S–module $\bar{H} = (G \bigotimes_{R_\alpha} R_{\alpha+1})/K$.*

Note that H is a right $R_{\alpha+1}$–module. Observe that since S is a PID we have that finitely generated S–modules are direct sums of cyclics. K is pure in H

and S is strongly homogeneous which implies that $\bar{H} = H/K$ is a torsion–free S–module. Moreover Pontryagin's Theorem [**FI**, Theorem 19.1] holds since S is a PID.

(1.8). *Let G be $\aleph_1$–free as S–module. Then $G(+)$ is cotorsion–free.*

PROOF: Since S is cotorsion–free, $G(+)$ is torsion–free and reduced. Suppose there is $0 \neq f : J_p \to G$, J_p the additive group of p–adic integers. If $f(1) = g$ is p–divisible, then g is divisible which contradicts G reduced. W.l.o.g. we may assume $f(1) \neq 0$ and $g = f(1) \in G - pG$. Let $\pi \in J_p$, $\pi = \lim_{n\to\infty} \pi_n$, $\pi_n \in \mathbf{Z}$. Let F be a pure, free S–submodule of G with $f(\pi)$, $g \in F$. Then $F = \langle g\rangle_* S \bigoplus C$ with $\langle g\rangle_*$ the purification of $\langle g\rangle$, $f(\pi) = a + c$, $a \in \langle g\rangle_* S$, $c \in C$. We have $p^n/\pi - \pi_n$ and therefore $p^n/f(\pi) - g\pi_n$ and $p^n/(a - g\pi_n) + c$ in F.

Since there are no p–divisible elements $\neq 0$ in G (if there were any in the $\aleph_1$–free S–module G, S would be p–divisible and reduced, thus $\mathrm{Hom}(J_p, G) = 0$) we conclude $c = 0$ and $f(J_p) \subseteq \langle g\rangle_* S \cong S$. Since S is cotorsion–free, we conclude $f = 0$. ∎

We assume from now on that G is $\aleph_1$–free as S–module.

(1.9). *G is closed in $\bar{H}$ with respect to the p–adic topology, p any prime $\notin P_0$.*

This is an immediate consequence of

(1.10). *$\bar{H}/G$ is an $\aleph_1$–free S–module.*

PROOF: Let $h_1, \ldots, h_n \in H$. Then there exists a finite set of monomials U and a pure finite rank submodule L of G such that the hypothesis of (1.6) is satisfied and $h_i \in F = \sum_{m\in M} L \otimes m$, $1 \leq i \leq n$. Therefore (1.6) implies $(F + K)/K$ is pure in $\bar{H}$ and finitely generated as an S–module. Since S is a PID and $\bar{H}$ torsion–free, $(F + K)/K$ is free. Now Pontryagin's Theorem implies that $\bar{H}/G$ is $\aleph_1$–free. ∎

(1.11). *Let $G_0 \subset G_1 \subset G_2$ be S–modules with G_1/G_0 and G_2/G_1 $\aleph_1$–free S–modules. Then G_2/G_0 is an $\aleph_1$–free S–module.*

PROOF: Let F be a pure, finite rank submodule of G_2/G_0. The module $F/\big(F \cap (G_1/G_0)\big) \cong \big(F + (G_1/G_0)\big)/(G_1/G_0)$ is a finite rank submodule of G_2/G_1. Thus $F \cap (G_1/G_0)$ is a direct summand of F with free complement. Moreover, $F \cap (G_1/G_0)$ is free since G_1/G_0 is $\aleph_1$–free and F is free. ∎

(1.12). *Let $R_\alpha = R_{\alpha,\omega}$. Then $\bar{H} = H/(K + G\otimes 1)$ is a torsion–free R_α–module, i.e. if $0 \neq h \in H$ and $0 \neq r \in R_\alpha$, then $hr \neq 0$.*

PROOF: Let $r = \sum_{m\in M_\alpha} s_m m \in R_\alpha \setminus \{0\}$ and $u \in h + (K + G \otimes 1) \neq 0$, $u = \sum u_{i,m} \otimes x^i_\alpha m$ where $i \geq 1$, $m \in {}_\alpha M_{\alpha+1}$. We pick u such that the following is satisfied:

(I) For $M_u = \{x^i_\alpha m \mid m \in {}_\alpha M_{\alpha+1},\ u_{i,m} \neq 0\}$ let M^*_u be the subset of M_u of all maximal elements with respect to $\prec$. The equivalence class of M^*_u with respect to $\prec$ is minimal and $|M^*_u|$ is minimal for all choices $u \in h + (K + G \otimes 1)$.

Let $x^i_\alpha u \in M^*_u$ such that if $u = x^{f_1}_{\beta_1} \ldots x^{f_s}_{\beta_s}$, then f_s is maximal. Let $n \in M_\alpha$ be maximal with $s_n \neq 0$ in the unique representation of $r = \sum_{m\in M_\alpha} s_m m$. From those $n = x^{e_1}_{\alpha_1} \ldots x^{e_t}_{\alpha_t}$ we pick one in which the e_1 is largest. This implies that the $x^i_\alpha um$–coordinate of ur is $u_{i,n}s_m$. Now suppose that $ur = k + y \otimes 1 \in K + G \otimes 1$, $k = (g \otimes x_\alpha - h \otimes 1)\sum_{m \in A_{\alpha+1}} t_m m$, $t_m \in S$. Since $ur = k + y \otimes 1$, the coordinate $x^i_\alpha um$ is maximal in the representation of k, i.e. $x^i_\alpha um \in M^*_u$. Therefore the $x^i_\alpha um$–entry of k is gt with $t = t_{x^{i-1}_\alpha um} \in S\backslash\{0\}$. Thus $u_{i,n}s_m = gt$ and therefore $u_{i,n} = gt_0$ for some $t_0 \in S \setminus \{0\}$. Since gS is a pure free S–submodule of G, $u' = u - (g \otimes x_\alpha - h \otimes 1)t_0 x^{i-1}_\alpha um \in h + K + G \otimes 1$ and $M_{u'} \subseteq M_u - \{x^i_\alpha um\}$. (Observe that $x^{i-1}_\alpha um \prec x^i_\alpha um$). This contradicts the choice $x^i_\alpha u$ in (I). This implies $u \in G \otimes 1$ and we have the contradiction $h + (K + G \otimes 1) = 0$. ∎

(1.13). *Let $H = G \otimes_{R^\pm_\alpha} R_{\alpha+1}$ and $r \in R_\alpha$. If $Hr = 0$, then $r = 0$.*

PROOF: Let $r = \sum_{m \in M_\alpha} s_m m$. For $m,\ m' \in M_\alpha$, $x_\alpha m = x_\alpha m'$ iff $m = m'$. This implies $(h \otimes x_\alpha)r = \sum_{m \in M_\alpha} hs_m \otimes x_\alpha m$. If $(h \otimes x_\alpha)r = 0$, then $hs_m = 0$ for all $m \in M_\alpha$, $s_m \in S$. Since $0 \neq h \in G$, a torsion–free S–module, we conclude $s_m = 0$ for all $m \in M_\alpha$. ∎

2. The Construction. Let κ be a regular cardinal and $E \subsetneq \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \omega\}$ a stationary subset of κ. Let $A = \cup_{\alpha<\kappa} A_\alpha$ be a κ–filtration of a set of cardinality κ, (cf. [**E**]) and S a strongly homogeneous, cotorsion–free PID of cardinality $< \kappa$. We may assume $|S| = |A_0|$ and $|A_\alpha| = |S||\alpha| = |A_{\alpha+1} - A_\alpha|$ for all $\alpha < \kappa$. We assume that $\diamondsuit_\kappa(E)$ holds and $\{\phi_\alpha : A_\alpha \to A_\alpha \mid \alpha \in E\}$ is a set of Jensen–functions, (cf. [**J**] or [**E**]).

Let $R_\alpha = S\langle x_\nu \mid \nu \in \alpha \setminus E\rangle$ be the ring of polynomials in non commuting variables x_ν over S and exponents in $N \in \{\omega, \mathbf{Z}\}$ as mentioned in the introduction. Observe that R_κ has no idempotents other than 0 and 1.

Let $\{(g_{\alpha+1},\ h_{\alpha+1}) \mid \alpha \in E\}$ be list of all pairs of elements of A. W.l.o.g. we may assume $g_{\alpha+1},\ h_{\alpha+1} \in A_{\alpha+1}$ for all $\alpha \in E$. We define an R_κ–module structure on A such that:

(i) A is an $\aleph_1$–free S–module.

(ii) For each $\phi \in \mathrm{End}_{\mathbf{Z}}(A)$ there is a stationary subset $E_\phi \subseteq E$ with $\phi(A_\alpha) \subseteq A_\alpha$ for all $\alpha \in E_\phi$ and $\phi|A_\alpha = r_\alpha$ for some $r_\alpha \in R_\alpha$. This means that each $\phi \in \mathrm{End}_{\mathbf{Z}}(A)$ can be represented by a sequence $(r_\alpha)_{\alpha \in E_\phi}$ with $A_\alpha(r_\alpha - r_\beta) = 0$ for $\alpha \leq \beta$. On the other hand, each such sequence induces an element in $\mathrm{End}_{\mathbf{Z}}(A)$.

(iii) $g_{\alpha+1}x_{\alpha+1} = h_{\alpha+1}$ for all $\alpha \in E$ whenever the S–submodule of A generated by $g_{\alpha+1}$ and $h_{\alpha+1}$ is a pure S–submodule of A.

An immediate consequence of (iii) is that A is E–transitive if $N = \omega$ and A is strongly homogeneous if $N = \mathbf{Z}$. The chain of A_α's will satisfy:

(1) If $\lambda \leq \kappa$ is a limit, $A_\lambda = \cup_{\alpha<\lambda} A_\alpha$ as modules.

(2) A_α is a right R_α–module and $\aleph_1$–free as S–module.

(3) If $\beta \notin E$, then A_α / A_β is an $\aleph_1$–free S–module for all $\beta \leq \alpha < \kappa$.

(This implies that for $\beta \notin E$, A_β is p–adically closed in A_α for each prime p for which S is p–reduced).

(4) If $\beta \leq \alpha$, then A_β is an R_β–submodule of the R_β–module A_α where the R_β–module structure of A_α is induced by the R_α–module structure of A_α.

(5) If $\alpha \in E$ and $g_{\alpha+1},\ h_{\alpha+1}$ both generate pure S–submodules of $A_{\alpha+1}$, we define $A_{\alpha+2} = (A_{\alpha+1} \bigotimes_{R_\alpha} R_{\alpha+2})/(g_{\alpha+1} \otimes x_{\alpha+1} - h_{\alpha+1} \otimes 1)R_{\alpha+2}$ and $A_{\alpha+2} =$

$A_{\alpha+1} \bigotimes_{R_\alpha} R_{\alpha+2}$ otherwise.

The results in §1 guarantee that (2), (3) and (4) are preserved for $A_{\alpha+2}$, $\alpha \in E$.

(6) If $\alpha \neq \beta+1$, $\beta+2$ for each $\beta \in E$, set $A_{\alpha+1} = A_\alpha \bigotimes_{R_\alpha} R_{\alpha+1}$.

(7) Let p be a prime such that S is p–reduced and let $\alpha \in E$. If

(7a) $\phi_\alpha : A_\alpha \to A_\alpha$ is a $\mathbf{Z}$–homomorphism with $\phi_\alpha \notin R_\alpha$, we will define $A_{\alpha+1}$ to be an R_α–module with $A_{\alpha+1}/A_\alpha$ p–divisible as S–module and ϕ_α does not lift to $A_{\alpha+1}$.

(7b) If the hypothesis in (7a) is not satisfied, set $A_{\alpha+1} = A_\alpha \bigoplus R_\alpha$.

If $\alpha \in E$ and $\beta < \alpha$, $\beta \notin E$ then $A_{\alpha+1}/A_\beta$ is an $\aleph_1$–free S–module, since $A_{\alpha+2}/A_{\alpha+1}$ is $\aleph_1$–free and $A_{\alpha+2}/A_\beta$ is $\aleph_1$–free (1.11).

We have to show how to define $A_{\alpha+1}$, $\alpha \in E$, to satisfy (7a), (3) and (4).

If $\alpha \in E$, α is a limit ordinal with countable cofinality. Pick a sequence of successor ordinals $\alpha_0 < \cdots < \alpha_n < \alpha_{n+1} < \dots$ with $\sup\{\alpha_n \mid n < \omega\} = \alpha$ and for each n, $\alpha_n \neq \beta+1$, $\beta+2$ for each $\beta \in E$. Let p be a prime such that S is p–reduced. Choose $a_n \in A_{\alpha_n-1} \setminus pA_{\alpha_n-1}$ and set $k_n = a_n \otimes x_{\alpha_n-1} \in A_{\alpha_n}$. Note that A_{α_n-1} is a direct summand of the S–module A_{α_n} and k_n is in a complementary summand.

Let $y = \sum_{n=0}^{\infty} k_n p^{e_n}$, $0 \leq e_0 < e_1 < \cdots < e_n < e_{n+1} < \dots$ a sequence of integers with $\ell_n = e_{n+1} - e_n > 0$ a strictly increasing sequence.

Consider $M = \langle A_\alpha, y\rangle^*_{R_\alpha}$, the p–pure R_α–submodule of the p–adic completion $\bar{A}_\alpha$ of A_α generated by A_α and y. The following is crucial in the sequel:

(8) If $r \in R_\alpha$ with $yr \in A_\alpha$, then $r = 0$:

Suppose $yr \in A_\alpha$. Since $k_n \in A_{\alpha_n}$ and $r \in R_{\alpha_{n_0}}$ for some $n_0 < \omega$ there is a $d < \omega$ such that $r \in R_{\alpha_d-1}$ and $m_d = \sum_{i=d}^{\infty} k_i p^{e_i} r \in A_{\alpha_d-1}$. Thus $m_d \otimes 1 - k_d p^{e_d} r \in p^{e_{d+1}} A_\alpha \cap A_{\alpha_d} = p^{e_{d+1}} A_{\alpha_d}$ and $p^{e_{d+1}} / m_d \otimes 1 - k_d p^{e_d} r = m_d \otimes 1 - a_d \otimes x_{\alpha_d-1} p^{e_d} r$. Since $A_{\alpha_d} = A_{\alpha_d-1} \bigotimes_{R_{\alpha_d-1}} R_{\alpha_d}$ we conclude $p^{e_{d+1}}/p^{e_d} r$ or $p^{e_{d+1}-e_d} = p^{\ell_d}/r$. Since $\ell_d \to \infty$ we conclude $r = 0$.

We have to show that M is an $\aleph_1$–free S–module. Let $y_m = \sum_{i=m}^{\infty} k_i p^{e_i - e_m}$ and consider $L_m = y_m R_{\alpha_m-1} \bigoplus A_{\alpha_m-1}$. Note that $M = \cup_{m<\omega} L_m$. We shall show that L_m is a pure S–submodule of M. Let $r \in R_{\alpha_m}$ and $a \in A_{\alpha_m-1}$ and consider the equation $px = y_m r + a$. W.l.o.g. $x = y_\ell s + b$, $b \in A_\alpha$, $s \in R_\alpha$, $\ell \geq m$. Thus $y_\ell ps - y_m r = a - pb$, i.e. $p^{e_\ell + e_m}(a - pb) = p^{e_m}(y_1 p^{e_\ell})ps - p^{e_\ell}(y_m p^{e_m})r = p^{e_m+1} s(y - \sum_{i=0}^{\ell-1} k_i p^{e_i}) - p^{e_\ell}(y - \sum_{i=0}^{m-1} k_i p^{e_i})r$.

By the above, this implies $p^{e_m+1} s = p^{e_\ell} r$ or $ps = p^{e_\ell - e_m} r$. Now we obtain $a - pb = y_\ell p^{e_\ell - e_m} r - y_m r = (y_m - \sum_{i=m}^{\ell-1} k_i p^{e_i - e_m}) r - y_m r = -\sum_{i=m}^{\ell-1} k_i p^{e_i - e_m} r \in A_{\alpha_m-1} + pA_\alpha$. If $\ell = m$ we conclude $a = pb$ and $x \in L_m$. If $\ell > m$, we have $k_m r = a_m \otimes x_{\alpha_m-1} r \in (A_{\alpha_m-1} + pA_\alpha) \cap A_{\alpha_m} = A_{\alpha_m-1} + (pA_\alpha \cap A_{\alpha_m}) = A_{\alpha_m-1} + pA_{\alpha_m}$ and $k_m r$ lies in a complement of A_{α_m-1} in A_{α_m}. Thus $k_m r \in pA_{\alpha_m}$ and therefore $r \in pR_{\alpha_m-1}$ and $a \in A_{\alpha_m-1} \cap pM = pA_{\alpha_m-1}$. Thus $y_m r + a \in pL_m$ and L_m is p–pure in M. To show that L_m is q–pure in M for $p \neq q$ is easier and left to the reader. The S–module L_m is obviously $\aleph_1$–free and therefore $M = \cup_{m<\omega} L_m$ is $\aleph_1$–free since S is a PID and Pontryagin's Theorem holds.

Let $\beta < \alpha$, $\beta \notin E$. We want to show that M/A_β is $\aleph_1$–free as S–module: for some $n < \omega$ we have $A_\beta \subseteq A_{\alpha_{m-1}}$. Thus $M/A_\beta = (\cup_{m<\omega} L_m)/A_\beta = \cup_{m>n}(y_m R_{\alpha_m-1} \bigoplus A_{\alpha_m-1}/A_\beta)$ is $\aleph_1$–free.

Now suppose that ϕ_α lifts to M. This implies $p^n \phi_\alpha(y) = yr + a$ for some

$n < \omega$, $r \in R_\alpha$. If $p^n\phi_\alpha = r \in R_\alpha$ we have $r = p^n r'$ for some $r' \in R_\alpha$ and $p^n(\phi_\alpha - r') = 0$ implies $\phi_\alpha = r'$. Therefore there is some $a \in A_\alpha$ with $p^n\phi_\alpha(a) \neq ar$. Since A_α is cotorsion–free (1.8) we may choose a p–adic number π with $(p^n\phi_\alpha(a) - ar)\pi \notin A_\alpha$.

Set $y' = y + a\pi$ and let M' be the p–pure submodule of $\bar{A}_\alpha$ generated by A_α and y'. Mutatis mutandis M' has the same properties as the module M. Suppose ϕ_α lifts to M' as well. Then there is $s \in R_\alpha$, $b \in A_\alpha$ and $m < \omega$ with $p^m\phi_\alpha(y') = y's + b$.

Since $p^n\phi_\alpha(y) = yr + a$, we subtract and obtain

$$p^{n+m}\phi_\alpha(a\pi) = y(sp^n - rp^m) + a\pi sp^n + bp^n - ap^m.$$

Hence $y(sp^n - rp^m)$ is in the p–adic closure of some A_{α_k}, $k < \omega$. The subgroup A_{α_k} is closed in A_α and M since M/A_{α_k} is $\aleph_1$–free as S–module as shown above. Thus $y(sp^n - rp^m) \in A_\alpha$ and therefore $sp^n = rp^m$. This implies $p^{n+m}\phi_\alpha(a) - arp^m)\pi \in A_\alpha$ and $(p^n\phi_\alpha(a) - ar)\pi \in A_{\alpha'}$, a contradiction to the choice of a and π. This shows that if $\phi_\alpha \notin R_\alpha$ then ϕ_α does not lift to M or M'. We define $A_{\alpha+1}$ to be M or M' ensuring that ϕ_α does not lift to $A_{\alpha+1}$ if $\phi_\alpha \notin R_\alpha$.

Now let $\phi : A \to A$ be a $\mathbf{Z}$–homomorphism. Because of $\diamondsuit_\kappa(E)$, the set $E_\phi = \{\alpha \in E : \phi|A_\alpha = \phi_\alpha\}$ is stationary and $A_{\alpha+1}$ is the p–adic closure of A_α in A and $\phi|A_\alpha = \phi_\alpha$ lifts to $A_{\alpha+1}$ for all $\alpha \in E_\phi$. By the definition of $A_{\alpha+1}$, $\phi|A_\alpha = r_\alpha$ for some $r \in R_\alpha$ if $\alpha \in E_\phi$. For $\alpha, \beta \in E_\phi$, $\alpha < \beta$, we obtain $A_\alpha(r_\alpha - r_\beta) = 0$ and ϕ can be represented as in (ii).

(9) *If $N = \omega$, A is not strongly homogeneous and* $\mathrm{Cent}(\mathrm{End}_{\mathbf{Z}}(A)) = S$.

PROOF: Let $\phi = (r_\alpha)_{\alpha \in E_\phi}$ be an element of $\mathrm{Aut}_{\mathbf{Z}}(A)$. For $\alpha \in E_\phi$ we have $\phi(A_\alpha) \subseteq A_\alpha$. Since ϕ is an automorphism, $D = \{\alpha < \kappa \mid \phi(A_\alpha) = A_\alpha\}$ is a cub and we may assume $E_\phi \subseteq D$. Since $r_\alpha \in R_\alpha^+$ and α is a limit ordinal, there is some $\alpha_0 < \alpha$ and $r_\alpha \in R_{\alpha_0+1} \setminus R_{\alpha_0}$ in case $r_\alpha \notin S$. Let $\beta = \alpha_0 + 3$. Then $A_\beta r_\alpha \subseteq A_\beta$ and (1.12) and (8) imply that for any $a \in A_\beta$ with $br_\alpha = a$ we have $b \in A_\beta$. Thus $A_\beta r_\alpha = A_\beta = A_{\beta-1} \bigotimes_{R_{\beta-1}} R_\beta$. This implies $r_\alpha \in S$ and r_α is a unit in S. Since A is a torsion–free S–module we conclude $\phi = s$ for some unit $s \in S$. Obviously, $\mathrm{Cent}(\mathrm{End}_{\mathbf{Z}}(A)) \supseteq S$. To prove equality, let $\phi = (r_\alpha)_{\alpha \in E\phi} \in \mathrm{End}_{\mathbf{Z}}(A)$. If ϕ is in the center, $\phi r = r\phi$ for all $r \in R_\kappa$. This implies $r_\alpha r = rr_\alpha$ for all $\alpha \in E\phi$, $\alpha > \beta$ with $r \in R_{\beta+1} \setminus R_\beta$. This implies $r_\alpha \in S$ for all $\alpha \in E\phi$. Thus $\phi \in S$. The same argument shows the last part of

(10) *If $N = \mathbf{Z}$, A is strongly homogeneous and A is indecomposable as abelian group. Moreover* $S = \mathrm{Cent}(\mathrm{End}_{\mathbf{Z}}(A))$.

PROOF: We only have to show that A is indecomposable. So assume $\phi = (r_\alpha)_{\alpha \in E_\phi}$ is idempotent. This implies $A_{\beta+3}r_\alpha(1 - r_\alpha) = 0$ for some $\beta < \alpha$, $r_\alpha \in R_{\beta+2}^-$. By (1.13) this implies $r_\alpha(1 - r_\alpha) = 0$ and r_α is idempotent in R_α^-. The ring R_α^+ is a domain and R_α^- is a subring of its field of quotients. Thus R_α^- is a domain and therefore $r_\alpha = 0$ or $1 - r_\alpha = 0$. This implies $\phi \in \{0, 1\}$ and A is indecomposable. This ends the proof of our Theorem.

In order to obtain 2^κ many such modules A, we may use almost disjoint stationary sets E as in [**E**].

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