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Turing invariant sets and the perfect set property

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We show that ZF + DC + "all Turing invariant sets of reals have the perfect set property" implies that all sets of reals have the perfect set property. We also show that this result generalizes to all countable analytic equivalence relations.

1 Introduction

We refer to the statement "all Turing invariant sets are determined" as *Turing Determinacy*, in short, TD. The results of this paper are motivated by the well known open problem whether TD implies AD. Woodin proved that TD + $\mathsf{DC}_{\mathbb{R}}$ + **V=L**($\mathbb{R}$) implies AD; however, whether TD alone is enough remained open. Inspired by this question, we asked the following analogous question:

Question 1.1 Let Γ be a regularity property (e.g., the perfect set property, Lebesgue measurability, etc), does ZF + DC+"all Turing invariant sets have property Γ" imply that all sets of reals have property Γ?

The main result of this paper answers the above question in the affirmative when Γ is the perfect set property. We also observe that Turing equivalence can be replaced by a more general collection of countable Borel equivalence relations. Furthermore, we provide a recursion theoretic argument due to Liang Yu, showing that the result generalizes to all countable analytic equivalence relations.

The paper is organized as follows: The main result of § 2 is an affirmative answer to Question 2 where Γ is the perfect set property. In § 3, we present Yu's argument generalizing the result of § 2 to all countable analytic equivalence relations. In § 4, we present several open problems.

2 The main result

In what follows, given a Turing machine M and a real η, we write $M(\eta)$ for the real computed from η by M (i.e., via the associated Turing functional).

Although the following proof is using DC and the recursion theoretic proof in the next section uses AC_ω, both can be eliminated by observing that if $V \models$ ZF+"all Turing invariant sets have the perfect set property" and $X \in V$ is a set of reals, then **HOD**($\mathbb{R}, X$) $\models$ ZF + DC+"all Turing invariant sets have the perfect set property".

Theorem 2.1 (ZF + DC)*: The perfect set property for all Turing invariant sets of reals implies the perfect set property for all sets of reals.*

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Proof. Let κ be large enough and fix a countable elementary submodel N of $(\mathbf{H}_\kappa, \in)$. If T is a tree, we write $\lim(T) := \{\eta : \forall n < \omega(\eta \restriction n \in T)\}$ for the set of branches through T. Fix a perfect tree T such that, for every $n < \omega$ and pairwise distinct $\eta_0, \dots, \eta_{n-1} \in \lim(T)$, $(\eta_0, \dots, \eta_{n-1})$ is N-generic for $(2^{<\omega}, \leq)^n$. Clearly, if $\eta \neq \nu \in \lim(T)$, then η and ν are not Turing equivalent. We break the proof into five claims:

Proposition 2.2 *For every pair (M_1, M_2) of Turing machines, there is a natural number $n = n(M_1, M_2)$ such that, for every $\eta \neq \nu \in \lim(T)$, if $\eta \restriction n = \nu \restriction n$ then:*

(a) $M_2(M_1(\eta)) = \eta$ if and only if $M_2(M_1(\nu)) = \nu$.
(b) If $M_1(\eta) = M_1(\nu)$, then $M_1(\eta) \in N$.
(c) $M_1(\eta) = M_2(\eta)$ if and only if $M_1(\nu) = M_2(\nu)$.
(d) If $M_2(M_1(\eta)) = \eta$, then $M_1(\eta)$ is not Turing equivalent to $M_1(\nu)$.

Proof. First we note that clause (d) follows from clause (a): Suppose towards contradiction that $M_2(M_1(\eta)) = \eta$ and $M_1(\eta)$ is Turing equivalent to $M_1(\nu)$, then $\eta = M_2(M_1(\eta))$ is Turing equivalent to $M_2(M_1(\nu))$. By clause (a), $M_2(M_1(\nu)) = \nu$, hence η is Turing equivalent to ν, a contradiction. Clause (b) follows from the mutual genericity over N of the branches in $\lim(T)$. We shall now prove clause (a), the proof of clause (c) is similar. Given $\eta \in \lim(T)$, there is some n such that $\eta \restriction n$ (as a Cohen condition) decides the truth value of "$M_2(M_1(\eta)) = \eta$" and such that for every $\eta \restriction n \leq \nu \in \lim(T)$, $M_2(M_1(\nu)) = \nu$ if and only if $M_2(M_1(\eta)) = \eta$. Denote $\eta \restriction n$ by c_η and the set of $\nu \in \lim(T)$ such that $\nu \restriction n = c_\eta$ by U_η. By compactness, there is some $k < \omega$ and $\eta_0, \dots, \eta_{k-1}$ such that $2^\omega = U_{\eta_0} \cup \dots \cup U_{\eta_{k-1}}$. Let $n = n(M_1, M_2)$ be the maximum length of $\{c_{\eta_i} : i < k\}$, then n is as required. This completes the proof of Proposition 2.2. □

Now let $((M_{n,0}, M_{n,1}) : n < \omega)$ be an enumeration of all ordered pairs of Turing machines, where $M_{0,0}$ and $M_{0,1}$ act as the identity function. For $n < \omega$, let X_n be the set of all $\eta \in \lim(T)$ such that

1. $M_{n,1}(M_{n,0}(\eta)) = \eta$, and
2. $M_{n,0}(\eta) \notin \{M_{\ell,0}(\eta) : \ell < n, M_{\ell,1}(M_{\ell,0}(\eta)) = \eta\}$.

Now, for each $n < \omega$, let $Y_n = \{M_{n,0}(\eta) : \eta \in X_n\}$.

Proposition 2.3 *For every $n < \omega$, there exists k_n such that, for every $\eta \in \lim(T)$, $\eta \restriction k_n$ determines the truth value of "$\eta \in X_n$". It follows that each X_n is closed, and hence, each Y_n is closed (being a continuous image of a compact set).*

Proof. This is similar to the proof of Proposition 2.2. Given $\eta \in \lim(T)$, there is some $n < \omega$ such that $\eta \restriction n$ decides the membership of the Cohen generic in X_n. Denote $\eta \restriction n$ by c_η and denote the set of all $\nu \in \lim(T)$ such that $\nu \restriction n = c_\eta$ by U_η. Again, by compactness, there is some $k < \omega$ and $\eta_0, \dots, \eta_{k-1}$ such that $2^\omega = U_{\eta_0} \cup \dots \cup U_{\eta_{k-1}}$, and we let k_n be the maximum length of $\{c_{\eta_i} : i < k\}$. This completes the proof of Proposition 2.3. □

Proposition 2.4 *The set $\bigcup_{n<\omega} Y_n$ is the closure of $\lim(T)$ under Turing equivalence.*

Proof. Every element of $\bigcup_{n<\omega} Y_n$ is Turing equivalent to an element of $\lim(T)$, by the definition of X_n and Y_n. Suppose that ν is Turing equivalent to some $\eta \in \lim(T)$, so there are Turing machines M_0 and M_1 such that $\nu = M_0(\eta)$ and $\eta = M_1(\nu)$. There is some $n < \omega$ such that $(M_0, M_1) = (M_{n,0}, M_{n,1})$, and therefore, $M_{n,1}(M_{n,0}(\eta)) = M_1(M_0(\eta)) = \eta$ and $M_{n,0}(\eta) = \nu$. Let $m < \omega$ be the minimal natural number with this property, then $\eta \in X_m$ and $\nu = M_{m,0}(\eta) \in Y_m$. This completes the proof of Proposition 2.4. □

Proposition 2.5 *The family $\{Y_n : n < \omega\}$ consists of pairwise disjoint sets.*

Proof. Suppose towards contradiction that there is some $\eta \in Y_n \cap Y_m$ where $m < n$. Let $\nu \in X_n$ and $\nu' \in X_m$ such that $\eta = M_{n,0}(\nu) = M_{m,0}(\nu')$, then $\nu' = M_{m,1}(M_{m,0}(\nu')) = M_{m,1}(M_{n,0}(\nu)) \in N[\nu]$. By the mutual genericity of ν and ν', it must be the case that $\nu = \nu'$, so $\eta = M_{n,0}(\nu) = M_{m,0}(\nu)$. But this contradicts the fact that $\nu \in X_n$. It follows that $Y_n \cap Y_m = \varnothing$, which completes the proof of the Proposition 2.5. □

Fix a homeomorphism $F : 2^\omega \to \lim(T)$. In order to show that every uncountable $A \subseteq 2^\omega$ contains a perfect subset, it suffices to show that every uncountable $B \subseteq \lim(T)$ contains a perfect subset: If $A \subseteq 2^\omega$ is uncountable, then $B = \{F(\eta) : \eta \in A\} \subseteq \lim(T)$ is uncountable and contains a perfect subset, and by F being a homeomorphism, so does $A = F^{-1}(B)$.

Now let $A \subseteq \lim(T)$ be uncountable, we shall find a perfect subset of A. For $n < \omega$, let $A_{1,n} = \{M_{n,0}(\eta) : \eta \in A \cap X_n\}$ and let $A_2 = \bigcup_{n<\omega} A_{1,n}$.

Proposition 2.6 *The set A_2 is Turing invariant.*

P r o o f. We shall prove that A_2 is the closure of A under Turing equivalence. Obviously, every element of A_2 is Turing equivalent to an element of A, by the definition of A_2. Suppose now that ν is Turing equivalent to some $\eta \in A$, then there is a minimal $n < \omega$ such that $M_{n,1}(M_{n,0}(\eta)) = \eta$ and $M_{n,0}(\eta) = \nu$. Therefore, $\eta \in X_n \cap A$, hence $\nu \in A_{1,n} \subseteq A_2$. This completes the proof of Proposition 2.6. □

As A_2 is Turing invariant and uncountable (recalling that it contains A), by the assumption, it contains a perfect subset P. Note that $A_{1,n} \subseteq Y_n$ for every $n < \omega$, so $P \subseteq \bigcup_{n<\omega} Y_n$. As the Y_n are closed and pairwise disjoint, we may assume without loss of generality that there is some $n^* < \omega$ such that $P \subseteq Y_{n^*}$, so $P \subseteq A_2 \cap Y_{n^*} = A_{1,n^*}$ (recalling that the Y_n are pairwise disjoint and $A_{1,n} \subseteq Y_n$). Let $A_3 = \{M_{n^*,1}(\eta) : \eta \in P\}$, then $A_3 \subseteq A$. Therefore, it suffices to show that A_3 is perfect. Note that if $\eta, \eta' \in P \subseteq Y_{n^*}$, then there are $\nu, \nu' \in X_{n^*}$ such that $\eta = M_{n^*,0}(\nu)$ and $\eta' = M_{n^*,0}(\nu')$, and therefore, if $M_{n^*,1}(\eta) = M_{n^*,1}(\eta')$, then $\nu = \nu'$ and $\eta = \eta'$, so $M_{n^*,1}$ is injective and continuous on P. Similarly, $M_{n^*,0}$ is injective on A_3: Note that if $\eta \in P \subseteq Y_{n^*}$, then $\eta = M_{n^*,0}(\nu)$ for some $\nu \in X_{n^*}$, hence $M_{n^*,1}(\eta) = \nu \in X_{n^*}$. Therefore, $A_3 \subseteq X_{n^*}$. Note that $M_{n^*,0}$ is injective on X_{n^*}, hence it follows that $M_{n^*,0}$ is injective and continuous on A_3. It's easy to verify that $M_{n^*,1}$ restricted to P is the inverse of $M_{n^*,0}$ restricted to A_3, and it follows that A_3 is perfect. This completes the proof of Theorem 2.1. □

Finally, we observe that the above results can be generalized as follows:

Definition 2.7 Let $\mathcal{F} = \{f_n : n < \omega\}$ a countable family of ground model-definable partial continuous functions from a Polish space X to itself and let $\{(g_{n,0}, g_{n,1}) : n < \omega\}$ be a fixed enumeration of all ordered pairs from $\mathcal{F}$. Let $E_{\mathcal{F}}$ be the following relation on X: $x E_{\mathcal{F}} y$ if and only if there is some $n < \omega$ such that $g_{n,0}(x) = y$ and $g_{n,1}(y) = x$. It is not hard to see that $E_{\mathcal{F}}$ is countable Borel equivalence relation on X.

Note that the only property of the Turing equivalence relation that we used in our proof is that it has the form $E_{\mathcal{F}}$ where $\mathcal{F}$ is the collection of all functions of the form $M(\eta) = \nu$ where M is a Turing machine. Therefore, we obtain the following corollary:

Corollary 2.8 *Assume* $\mathsf{ZF} + \mathsf{DC}$. *Let E be a countable Borel equivalence relation of the form $E_{\mathcal{F}}$ where $\mathcal{F}$ is as above. If all E-invariant sets of reals have the perfect set property, then all sets of reals have the perfect set property. In particular, the above result holds for $E = E_0$.*

3 A recursion theoretic proof and a generalization to all countable analytic equivalence relations

In this section, we sketch a recursion theoretic reformulation of the proof of the main result from § 2, and show how to modify the argument in order to generalize the result of § 2 to all countable analytic equivalence relations. All arguments in this section are due to Liang Yu.

We begin by sketching a recursion theoretic proof of Theorem 2.1:

P r o o f. Let $P \subset 2^\omega$ be a perfect set so that

(1) any two different reals from P have different degrees;
(2) any real in P is hyperimmune-free.

Now fix an uncountable set A of reals. Without loss of generality, we may assume that $A \subseteq P$. By Property (2), the Turing closure $[A]_{\mathrm{T}} = \{x \mid \exists y \in A(\equiv_{\mathrm{T}} x)\} = [A]_{\mathrm{tt}} = \{x \mid \exists y \in A(\equiv_{\mathrm{tt}} x)\}$, where tt stands for truth table reduction, has a perfect subset Q. Now for any pair of indexes of truth table reductions $e, i \in \omega$, let $Q_{e,i} = \{z \mid \exists x \in Q(z = \Phi_e^x \wedge x = \Phi_i^z)\}$. Then for each e, i, $Q_{e,i}$ is a closed set and $\bigcup_{e,i} Q_{e,i} = [Q]_{\mathrm{tt}} = [Q]_{\mathrm{T}}$. Also by Property (1), $Q_{e,i} \cap P \subseteq A$ and $\bigcup_{e,i}(Q_{e,i} \cap P) = A \cap [Q]_{\mathrm{T}}$ and $[Q]_{\mathrm{T}} = [[Q]_{\mathrm{T}} \cap A]_{\mathrm{T}}$. By DC (in fact AC_ω),

$[Q]_T \cap A$ is uncountable. By DC (AC_ω, again), there must be some e and i so that $Q_{e,i} \cap A$ is uncountable. Then $Q_{e,i} \cap P = Q_{e,i} \cap A$ is an uncountable closed set and so must contain a perfect subset. □

Theorem 3.1 *In Theorem 2.1, Turing equivalence can be replaced by any countable analytic equivalence relation.*

Proof. We begin with a short lemma:

Lemma 3.2 *If x is Δ^1_1-dominated, then for any $\alpha < \omega_1^{\mathrm{CK}}$, there is some $\beta < \omega_1^{\mathrm{CK}}$ so that $x^{(\alpha)} \leq x \oplus \varnothing^{(\beta)}$.*

Proof. The real $x^{(\alpha)}$ is Turing equivalent to a $\Pi^0_1(x)$-singleton $f \in \omega^\omega$. Since x is Δ^1_1-dominated, there is a hyperarithmetic function g majorizing f. Then $f \leq_T x \oplus g \leq_T x \oplus \varnothing^{(\beta)}$ for some $\beta < \omega_1^{\mathrm{CK}}$. □

We shall now return to the proof of Theorem 3.1. Suppose that E is a countable analytic equivalence relation. We may assume that E is a (lightface) Σ^1_1 countable equivalence relation. For the boldface case, we just need a relativization. By the property of E, for any pair x, y, xEy implies $x \equiv_h y$, where $\leq_h$ is hyperarithmetic reduction (this follows from the fact that if a Σ^1_1 set is countable, then all of its members are hyperarithmetic; cf., e.g., [1, Lemma 2.5.4]). let $P \subset 2^\omega$ be a perfect set so that

(1') any two different reals from P have different hyperdegrees;
(2') any real in P is Δ^1_1-dominated (i.e., for any $x \in P$ and $f \in \omega^\omega$ with $f \leq_h x$, there is hyperarithmetic function g dominating f. Note that this implies $\omega_1^x = \omega_1^{\mathrm{CK}}$).

Now fix an uncountable set A of reals. Without loss of generality, we may assume that $A \subseteq P$. Then by replacing the conditions (1), (2) and Turing reduction with (1'), (2') and hyperarithmetic reduction respectively, we may apply the same arguments as in the recursion theoretic proof of Theorem 2.1 together with Lemma 3.2 to prove that A has a perfect subset. □

4 Open problems

As noted in the introduction, it is not known whether Turing determinacy implies AD. Furthermore, it is not even known whether Turing determinacy implies weak consequences of AD such as "all sets of reals have property Γ" for a regularity property Γ. We therefore ask:

Question 4.1 Let Γ be a regularity property, does Turing determinacy imply that all sets of reals have property Γ?

Question 4.2 Does Turing determinacy imply that all Turing invariant sets of reals have the perfect set property? A positive answer to this question, combined with the results of this paper, will establish that Turing determinacy implies the perfect set property for all sets of reals, answering a question from [4] (cf. also [3]).

Question 4.3 For which countable Borel equivalence relations E do we have that "all E-invariant sets are determined" imply AD? We note that recent progress on this problem has been made in [2].

Question 4.4 For which Borel equivalence relations E and regularity properties Γ do we have that $\mathsf{ZF} + \mathsf{DC}$+"all E-invariant sets of reals have property Γ" imply "all sets of reals have property Γ"?

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