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ON CON($\mathfrak{d}_\lambda >$ COV$_\lambda$(MEAGRE))

SAHARON SHELAH

ABSTRACT. We prove the consistency of: for suitable strongly inaccessible cardinal λ the dominating number, i.e., the cofinality of ${}^\lambda\lambda$, is strictly bigger than cov_λ(meagre), i.e., the minimal number of nowhere dense subsets of ${}^\lambda 2$ needed to cover it. This answers a question of Matet.

INTRODUCTION

Cardinal characteristics were defined, historically, over the continuum. See celebrated Van Dowen [vD84] for the general topologist perspective and the excellent survey Blass [Bla], and Bartoszyński [Bar10] for the set theoretic perspective. In recent years there have been many results concerning generalized cardinal characteristics. The idea is to imitate the definition of a given characteristic over the continuum by translating it to uncountable cardinals.

It is reasonable to distinguish regular cardinals and singular cardinals. Among the regular cardinals, it makes sense to distinguish limit cardinals from successor cardinals. In this paper we focus on strongly inaccessible cardinals. These cardinals and their characteristics behave, in many cases, much like $\aleph_0$, but certainly not always. See Landver [Lan92], Cummings-Shelah [CS95], and Matet-Shelah [MS]. Our main result is the consistency of $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ at a supercompact cardinal λ, and we begin with the following definitions.

We shall define three cardinal invariants (but the paper deals, actually, just with two of them).

Definition 0.1 (The bounding and dominating numbers)**.** Let λ be an inaccessible cardinal.

Let $f, g \in {}^\lambda\lambda$.

(a) $f \leq^* g$ if $|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \lambda$
(b) $A \subseteq {}^\lambda\lambda$ is unbounded if there is no $h \in {}^\lambda\lambda$ so that $f \in A \Rightarrow f \leq^* h$
(c) $A \subseteq {}^\lambda\lambda$ is dominating when for every $f \in {}^\lambda\lambda$ there exists $g \in A$ so that $f \leq^* g$
(d) the bounding number for λ, denoted by $\mathfrak{b}_\lambda$, is $\min\{|A| : A$ is unbounded in ${}^\lambda\lambda\}$
(e) the dominating number for λ, denoted by $\mathfrak{d}_\lambda$, is $\min\{|A| : A$ is dominating in ${}^\lambda\lambda\}$.

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Notice that the usual definitions of $\mathfrak{b}$ and $\mathfrak{d}$ are $\mathfrak{b}_{\aleph_0}$ and $\mathfrak{d}_{\aleph_0}$ according to Definition 0.1. The definition of $\mathrm{cov}_\lambda(\text{meagre})$ involves some topology.

Definition 0.2 (The meagre covering number)**.**
Let λ be a regular cardinal.

(a) ${}^\lambda 2$ is the space of functions from λ into 2
(b) $({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$, for $\nu \in {}^{\lambda>}2 := \bigcup\limits_{\alpha<\lambda} {}^\alpha 2$
(c) $\mathscr{U} \subseteq {}^\lambda 2$ is open in the topology $({}^\lambda 2)_{<\lambda}$, iff for every $\eta \in \mathscr{U}$ there exists $i < \lambda$ so that $({}^\lambda 2)^{[\eta \restriction i]} \subseteq \mathscr{U}$
(d) $\mathrm{cov}_\lambda(\text{meagre})$ is the minimal cardinality of a family of meagre subsets of $({}^\lambda 2)_{<\lambda}$, which covers this space.

This paper deals with the relationship between $\mathfrak{d}_\lambda$ and $\mathrm{cov}_\lambda(\text{meagre})$. If λ is a successor cardinal, then $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{d}_\lambda$ is consistent (see (b) below). Matet asked (a personal communication) whether $\mathfrak{d}_\lambda \leq \mathrm{cov}_\lambda(\text{meagre})$ is provable in ZFC, where λ is strongly inaccessible. We give here a negative answer.

For λ a supercompact cardinal and $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mu^\lambda$, we force large $\mathfrak{d}_\lambda$ i.e., $\mathfrak{d}_\lambda = \mu$, and small covering number (i.e., $\mathrm{cov}_\lambda(\text{meagre}) = \kappa$). A similar result should hold also for a wider class of cardinals, and we intend to return elsewhere to this subject.

Let us sketch some known results. These results are related to the inequality number and the covering number for category. Recall:

Definition 0.3 (The inequality number)**.** Let κ be an infinite cardinal. The inequality number of $\kappa, \mathfrak{e}_\kappa$ is the minimal cardinal λ satisfying that there is a set $\mathscr{F} \subseteq {}^\kappa\kappa$ of cardinality λ such that there is no $g \in {}^\kappa\kappa$ satisfying $(\forall f \in \mathscr{F})(\exists^\kappa \alpha < \kappa)(f(\alpha) = g(\alpha))$.

For $\kappa = \aleph_0, \mathfrak{e}_\kappa = \mathrm{cov}_{\aleph_0}(\text{meagre})$, see Bartoszyński ([Bar87]) and Miller ([Mil82]). Now

(a) the statement $\mathfrak{e}_\kappa = \mathrm{cov}_\kappa(\text{meagre})$ is valid for $\kappa > \aleph_0$ in the case that κ is strongly inaccessible, by [Lan92]. But if κ is a successor cardinal, it may fail
(b) if $\kappa < \kappa^{<\kappa}$, then $\mathrm{cov}_\kappa(\text{meagre}) = \kappa^+$. This is due to Landver ([Lan92]).

We intend also to address:

Problem 0.4. Can we replace "supercompact" by "strongly inaccessible"?

Problem 0.5. (1) Can we prove the consistency of $\mathrm{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda$?

(2) For λ strongly inaccessible (or just Laver indestructible supercompact), is there a non-trivial λ^+-c.c. $(<\lambda)$-strategically complete forcing notion $\mathbb{Q}$ which is ${}^\lambda\lambda$-bounding?

A point which in a previous version was just a step along the way, the referee asked to be justified fully and was analyzed to be serious. This was done but eventually was separated to [Sheb]. A posteriori the point is that in the parallel case for $\lambda = \aleph_0$, for full memory FS iteration such a claim is true. In fact, by Judah-Shelah [IHJS88], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*)\rangle$ is FS iteration of the Suslin-c.c.c. forcing notion, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$, and for notational transparency, its definition is with no parameter, and $\underset{\sim}{\zeta} : \beta(*) \to \alpha(*)$ is increasing, and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \leq \beta(*), \beta < \beta(*)\rangle$ is FS iteration, but $\underset{\sim}{\mathbb{Q}}'_\beta$ is defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ but now in

$\mathbf{V}^{\mathbb{P}'_\beta}$ rather than in $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$, then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \underset{\sim}{\eta}_{\zeta(\beta)} : \beta < \beta(*)\rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$".

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration: to use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete subforcing it generates; see [Sheb].

We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$ for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon$, ζ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \le q$ means that q gives more information in forcing notions. The symbol $\triangleleft$ is preserved for "being an initial segment". Also recall that ${}^B A = \{f : f \text{ a function from } B \text{ to } A\}$ and let ${}^{\alpha>}A = \cup\{{}^\beta A : \beta < \alpha\}$; some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, $J^{\rm bd}_\lambda$ denotes the ideal of the bounded subsets of λ.

The picture of cardinal invariants related to uncountable λ is related but usually quite different from the one for $\aleph_0$. They are more similar if κ is "large" enough, mainly strongly inaccessible.

1. Preliminaries

Definition 1.1. Let λ be supercompact. We say that $h : \lambda \to \mathscr{H}(\lambda)$ is a Laver diamond (for λ) when for every $x \in \mathbf{V}$ there are a normal fine ultrafilter D over $I = [\mathscr{H}(\chi)]^{<\lambda}$ for some χ such that $x \in \mathscr{H}(\chi)$ and the Mostowski collapse $\mathbf{j}$ on $\mathbf{V}^I/D$ maps $\langle h(\sup(u \cap \kappa)) : u \in I\rangle/D$ to x (we can use elementary embeddings instead of an ultrafilter).

Notation 1.2. If $\mathbb{P}$ is a forcing notion in $\mathbf{V}$, then $\mathbf{V}^{\mathbb{P}}$ denotes $\mathbf{V}[\mathbf{G}]$ for $\mathbf{G} \subseteq \mathbb{P}$ generic over $\mathbf{V}$; we may write $\mathbf{V}[\mathbb{P}]$ instead.

The most straightforward way to increase $\mathfrak{b}_\lambda$ in the classical case of $\aleph_0$ is Hechler forcing = dominating real forcing. A condition is a function $f_p : \omega \to \omega$ which is separated into a finite stem η_p and the rest of the function. Formally, $p = (\eta_p, f_p)$ where $\eta_p \trianglelefteq f_p$.

If p, q are conditions, then $p \le q$ iff $\eta_p \trianglelefteq \eta_q$ and $f_q(n) \ge f_p(n)$ for every $n \notin \mathrm{dom}(\eta_p)$, hence for every n. A generic object adds a function $g : \omega \to \omega$ which dominates the functions from the ground model. By iterating Hechler reals, one increases the bounding number $\mathfrak{b}$.

If $\lambda = \lambda^{<\lambda}$, then one can define the generalized Hechler forcing $\mathbb{D}_\lambda$ by replacing ω by λ. The basic step is λ-complete and λ^+-c.c. and actually λ-centered. Hence one can iterate and increase $\mathfrak{b}_\lambda$.

In [She92, §§1 and 2] and then Goldstern-Shelah [GS93] and Kellner-Shelah [KS12] consider other invariants. Consider two functions $f, g : \omega \to (\omega \backslash \{0\})$ going to infinity such that $f \ge g$ and ask about:

- $\mathfrak{c}^+_{f,g} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \prod\limits_i [f(i)]^{g(i)}$ and $(\forall \eta \in \prod\limits_i f((i))(\exists g \in \mathscr{F})[\bigwedge\limits_i \eta(i) \in g(i)]$.
- $\mathfrak{c}^-_{f,g} = \min\{\mathscr{F} : \mathscr{F} \subseteq \prod\limits_i f(i)$ and for no $g \in \prod\limits_i [f(i)]^{g(i)}$ do we have $(\forall \eta \in \mathscr{F})(\forall^\infty i)(\eta(i) \in g(i))$.

There are relevant forcing notions; we shall use a λ^+-c.c. notion as in c.c.c. creature forcing (see [RS97],[HS]).

For transparency:

Convention 1.3. Below $\lambda, \bar{\theta}$ are as in Definition 1.4.

Definition 1.4. Let λ be inaccessible, and let $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.

(1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$ by

- (α) $p \in \mathbb{Q}$ iff
  - (a) $p = (\eta, f) = (\eta^p, f^p)$
  - (b) $\eta \in \prod_{\zeta < \varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p)
  - (c) $f \in \prod_{\zeta < \lambda} \theta_\zeta$
  - (d) $\eta \triangleleft f$
- (β) $p \leq_{\mathbb{Q}} q$ iff
  - (a) $\eta^p \trianglelefteq \eta^q$
  - (b) $f^p \leq f^q$, i.e., $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$
  - (c) if $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$, then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$ actually follows.

(2) The generic is $\eta = \cup \{\eta^p : p \in \mathbf{G}_{\mathbb{Q}_{\bar{\theta}}}\}$.

The new forcing defined above is not λ-complete anymore. By fixing a stem η one can define a short increasing sequence of conditions which goes up to some θ_ζ at the ζ-th coordinate and hence has no upper bound in $\prod_{\zeta < \varepsilon} \theta_\zeta$. However, this forcing is $(< \lambda)$-strategically complete since the completeness (=COM) player can increase the stem at each move.

Remark 1.5. The forcing is parallel to the creature forcing from [She92, §§1, 2], [KS12], but they are ${}^\omega\omega$-bounding.

Recall the following.

Definition 1.6.

(1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete when for each $p \in \mathbb{P}$ in the following game $\partial_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.

A play lasts α moves. In the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$, and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

(2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete when it is α-strategically complete for every $\alpha < \lambda$.

Basic properties of $\mathbb{Q}_{\bar{\theta}}$ are summarized and proved in [GS12, §2].

The following fact describes some immediate connections between various concepts of completeness:

Fact 1.7.

- (a) If $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is a $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions, then $\mathbb{P}_\delta$ is also $(< \lambda)$-strategically complete (see e.g. [She00]).
- (b) If $\mathbb{P}$ is a $(< \lambda)$-strategically complete forcing notion, then $({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}} = ({}^{\lambda >}\mathrm{Ord})^{\mathbf{V}^{\mathbb{P}}}$, and consequently λ is strongly inaccessible in $\mathbf{V}^{\mathbb{P}}$.

(c) Like (a), replacing $(<\lambda)$-strategically complete" by "$(<\lambda)$-complete".
(d) If $\mathbb{P}$ is $(<\lambda)$-complete, then $\mathbb{P}$ is λ-strategically complete.

Definition 1.8. For an ordinal $\alpha_* = \alpha(*)$ let $\mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$ be the class of quadruples $\mathbf{q} = (\bar u, \bar{\mathbb{P}}, \bar{\underset{\sim}{\mathbb{Q}}}, \bar\eta)$ consisting of (omitting α_* means for some α_* and $\ell g(\mathbf{q}) = \alpha_{\mathbf{q}} = \alpha_*$):

(a) $\bar u = \langle u_\alpha : \alpha < \alpha_*\rangle$ and $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ where $\mathscr{P}_\alpha \subseteq [u_\alpha]^{\le\lambda}, u_\alpha \subseteq \alpha$, without loss of generality $\mathscr{P}_\alpha$ is closed under subsets (but is not necessarily an ideal)
(b) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \le \alpha_*, \beta < \alpha_*\rangle$ is a $(<\lambda)$-support iteration, and let $\mathbb{P}_{\mathbf{q},0} = \mathbb{P}_{\mathbf{q},0,\alpha(\mathbf{q})}$
(c) each of $\mathbb{P}_\alpha$ is strategically $(<\lambda)$-complete and λ^+-c.c.
(d) $\eta_\beta \in \Pi\bar\theta$ is the generic of $\mathbb{Q}_\beta$ where $\underset{\sim}{\eta}_\beta$, the generic of $\underset{\sim}{\mathbb{Q}}_\beta$ (defined in clause (e) below), is $\cup\{\eta_p : p \in \mathbf{G}_{\mathbb{Q}_\beta}\}$
(e) if $\mathbf{G} \subseteq \mathbb{P}_\beta$ is generic over $\mathbf{V}$, then $\underset{\sim}{\eta}_\alpha[\mathbf{G}]$ in $(\Pi\bar\theta, <_{J^{\rm bd}_\lambda})$ dominate every $\nu \in \Pi\bar\theta$ from $\mathbf{V}[\langle \underset{\sim}{\eta}_\gamma : \gamma \in u\rangle]$ when $u \in \mathscr{P}_\alpha$; moreover, in $\mathbf{V}[\mathbf{G}]$
  $(*)$ $\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}]$ is the subforcing of $\mathbb{Q}_{\bar\theta}$ consisting of the $p \in \mathbb{Q}_{\bar\theta}$ such that for some $\bar s, \underset{\sim}{f}, \eta_p$ (so $\eta_p = \eta$, etc.) we have:
    (α) $p = (\eta, f) = (\eta_p, f_p)$, so $\eta \in \prod\limits_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda$
    (β) $\bar s = \langle (u_i, f_i) : i < i_*\rangle$
    (γ) $i_* < \lambda$
    (δ) for each $i < i_*$ we have $u_i \in \mathscr{P}_\beta, \eta \triangleleft f_i \in \Pi\bar\theta$ and $f_i \in \mathbf{V}[\langle \underset{\sim}{\eta}_\gamma[\mathbf{G}] : \gamma \in u_i\rangle]$
    (ε) $f = \sup\{f_i : i < i_*\}$; i.e., $\varepsilon < \lambda \Rightarrow f(\varepsilon) = \cup\{f_i(\varepsilon) : i < i_*\}$.
(f) Notation: So $u_{\mathbf{q},\alpha} = u_\alpha, \mathbb{P}_{\mathbf{q},\alpha} = \mathbb{P}_\alpha$, etc., but when $\mathbf{q}$ is clear from the context we may omit it.

Definition 1.9 (For $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar\theta,\alpha(*)}$).

(1) We let $\alpha \le \alpha_*, \mathbb{P}_{1,\alpha} = \mathbb{P}^{\mathbf{q}}_{1,\alpha}$ be essentially the completion of $\mathbb{P}_\alpha$; we express it by:

$(*)_1$ the elements of $\mathbb{P}_{1,\alpha} = \mathbb{P}_{\mathbf{q},1,\alpha}$ are of the form $\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)}$ where:
  (α) $i_* = i(*) \le \lambda$
  (β) $\gamma_i \in \mathscr{U}$ for $i < i_*$
  (γ) $\mathbf{B}$ is a λ-Borel function from ${}^{i(*)}(\Pi\bar\theta)$ into $\{0,1\} = \{\text{false}, \text{true}\}$; $\mathbf{B}$ is from $\mathbf{V}$, of course, such that $\nVdash_{\mathbb{P}_{\mathbf{q}}}$ "$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma_i}, \ldots)_{i<i(*)} = 0$".

$(*)_2$ the order is natural:
$$\mathbb{P}_{1,\alpha} \models \text{"}\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(1)} \le \mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}, \ldots)_{i<i(2)}\text{"}$$
iff $\Vdash_{\mathbb{P}_\alpha}$ "if $\mathbf{B}_2(\ldots, \underset{\sim}{\eta}_{\gamma(i,2)}[\underset{\sim}{\mathbf{G}}], \ldots)_{i<i(1)}$ is equal to 1, then so is
$$\mathbf{B}_1(\ldots, \underset{\sim}{\eta}_{\gamma(i,1)}, \ldots)_{i<i(2)}\text{"}.$$

(2) For $\mathscr{U} \subseteq \alpha_*$ let $\mathbb{P}_{\mathscr{U}} = \mathbb{P}^{\mathbf{q}}_{\mathscr{U}}$ be the subforcing of $\mathbb{P}_{1,\alpha(\mathbf{q})}$ consisting of
$$\{\mathbf{B}(\ldots, \underset{\sim}{\eta}_{\gamma(i)}, \ldots)_{i<i(*)} \in \mathbb{P}_{\alpha(\mathbf{q})} : i(*) \le \lambda$$
and $\gamma_i \in \mathscr{U}$ for every $i < i(*)\}$.

Claim 1.10.

(1) For any sequence $\langle u_\alpha, \mathscr{P}_\alpha : \alpha < \alpha_*\rangle$ as above, i.e., as in clause (a) of Definition 1.8, there is one and only one $\mathbf{q} \in \mathbb{Q}_{\lambda,\bar\theta,\alpha_*}$ with $u_{\mathbf{q},\alpha} = u_\alpha, \mathscr{P}_{\mathbf{q},\alpha} = \mathscr{P}_\alpha$ for $\alpha < \alpha_*$.

(1a) For $\alpha \le \alpha_*$ and $\mathscr{U} \subseteq \alpha_*$, the forcing notions $\mathbb{P}_{\mathbf{q},\alpha}, \mathbb{P}_{\mathbf{q},\mathscr{U}}$'s are well defined and are as demanded in Definition 1.9.

(2) For every $\alpha \le \alpha_*$ the set $\mathbb{P}^\bullet_{\mathbf{q},\alpha}$ of $p \in \mathbb{P}_{\mathbf{q},\alpha}$ satisfying the following is dense:

$(*)$ if $\beta \in \mathrm{dom}(p)$, then $q = p(\beta)$ is a $\mathbb{P}^\bullet_\beta$-name of a member of $\underset{\sim}{\mathbb{Q}}_\beta$ such that:

(a) $\eta_q, i_q, \langle u_{q,i} : i < i_q\rangle$ are objects (not just $\mathbb{P}_\alpha$-names)

(b) $\underset{\sim}{f}_q = \sup\{\underset{\sim}{f}_{q,i} : i < i_q\}$, each $\underset{\sim}{f}_{q,i}$ is a $\mathbb{P}'_\beta$-name of a member of $\Pi\bar{\theta}$

(c) each $\underset{\sim}{f}_i$ has the form $\mathbf{B}_{q,i}(\ldots, \underset{\sim}{\eta}_{\gamma(i,j)}, \ldots)_{j<j(*)\le\lambda}$ where $\{\gamma(i,j) : j < j(i)\} \subseteq u_{p,i}$ and $\mathbf{B}_q$ is a Borel function from ${}^{\partial(*)}(\Pi\bar{\theta})$ into $\Pi\bar{\theta}$

(d) $p(\beta) = (\underset{\sim}{\eta}_q, \underset{\sim}{f}_q)$.

(3) Above for every $v \subseteq \alpha$ and $j_* < \lambda$ the set of $p \in \mathbb{P}^\bullet_\alpha$ such that $v \subseteq \mathrm{dom}(p) \wedge (\forall \beta \in \mathrm{dom}(p))(\ell g(\eta_{p(\beta)}) > j_*)$ is dense.

(4) $\mathbb{P}_{\mathbf{q},0,\alpha} \lessdot \mathbb{P}_{\mathbf{q},1,\alpha}$; moreover $\mathbb{P}_{\mathbf{q},1,\alpha}$ is dense in $\mathbb{P}_{\mathbf{q},2,\alpha}$ and $\mathscr{U}_1 \subseteq \mathscr{U}_2 \subseteq \alpha_{\mathbf{q}} \Rightarrow \mathbb{P}^{\mathbf{q}}_{\mathscr{U}_1} \lessdot \mathbb{P}^{\mathbf{q}}_{\mathscr{U}_2} \lessdot \mathbb{P}_{\mathbf{q},\alpha}$, so $\mathbb{P}_{\mathbf{q},\{\beta:\beta<\alpha\}} = \mathbb{P}_{\mathbf{q},2,\alpha}$ and $|\mathbb{P}_{\mathbf{q},\mathscr{U}}| \le |\mathscr{U}|^\lambda$.

(5) If $\alpha < \alpha_*$ and $u \in \mathscr{P}_\alpha$, then $\underset{\sim}{\eta}_\alpha \in \Pi\bar{\theta}$ dominate every $\nu \in (\Pi\bar{\theta})^{\mathbf{V}[\underset{\sim}{\bar{\eta}}\restriction u]}$.

(6) Assume $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{q}}$ is generic over $\mathbf{V}, \eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$, and $\eta'_\alpha \in (\Pi\bar{\theta})^{\mathbf{V}[\mathbf{G}]}$ for $\alpha < \alpha_*$, and $\{(\alpha,\varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha$ and $\eta_\alpha(\underset{\sim}{\varepsilon}) \ne \eta'_\alpha(\underset{\sim}{\varepsilon})\}$ has cardinality $< \lambda$. Then for some (really unique) $\mathbf{G}'$ we have $\mathbf{G}' \subseteq \mathbb{P}_{\mathbf{q}}$ is generic over $\mathbf{V}$ and α.

(7) Assume $\mathbf{G} \subseteq \mathbb{P}^{\mathbf{q}}_{\mathcal{U}}$ is generic over $\mathbf{V}, \eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$, and $\eta'_\alpha \in (\Pi\bar{\theta})^{\mathbf{V}[\mathbf{G}]}$ for $\alpha < \alpha_*$, and $\{(\alpha,\varepsilon) : \alpha < \alpha_*, \varepsilon < \alpha$ and $\eta_\alpha(\underset{\sim}{\varepsilon}) \ne \eta'_\alpha(\underset{\sim}{\varepsilon})\}$ has cardinality $< \lambda$. Then for some (really unique) $\mathbf{G}'$ we have $\mathbf{G}' \subseteq \mathbb{P}^{\mathbf{q}}_{\mathcal{U}}$ is generic over $\mathbf{V}$ and α.

Proof. See [Sheb]. $\square$

Theorem 1.11. *For any ordinal α_* there is a quadruple $(\mathbf{q}, \delta_*, \mathscr{U}_*, h)$ such that:*

(A) (a) $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta}}$ *and let* $\delta_* = \ell g(\mathbf{q})$

(b) $\mathscr{U}_* \subseteq \delta_*$ *has order type* α_*

(c) h *is the order preserving function from* α_* *onto* $\mathscr{U}_*$

(d) *if* $\alpha \in \mathscr{U}_*$, *then* $\mathscr{U}_* \cap \alpha \in \mathscr{P}_{\mathbf{q},\alpha}$

(B) *if* $\mathscr{U}_1 \subseteq \mathscr{U}_*, \mathscr{U}_2 \subseteq \mathscr{U}_*, \mathrm{otp}(\mathscr{U}_1) = \mathrm{otp}(\mathscr{U}_2)$, *and g is the order preserving function from $\mathscr{U}_1$ onto $\mathscr{U}_2$, then g induces an isomorphism $\hat{g}$ from $\mathbb{P}_{\mathbf{q},\mathscr{U}_1}$ onto $\mathbb{P}_{\mathbf{q},\mathscr{U}_2}$ mapping $\underset{\sim}{\eta}_\beta$ to $\underset{\sim}{\eta}_{g(\beta)}$ for $\beta \in \mathscr{U}_1$.*

Proof. By [Sheb, §4]. $\square$

2. The forcing

In this section we prove the main result of the paper, which reads as follows:

Theorem 2.1. *Assume*

(a) λ *is supercompact*

(b) $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then for some forcing notion $\mathbb{P}$ not collapsing cardinals $\ge \lambda$, λ is still supercompact in $\mathbf{V}^{\mathbb{P}}$ and $\mathrm{cov}_\lambda(\text{meagre}) = \kappa$, $\mathfrak{d}_\lambda = \mu$.

Proof. By Lemma 2.3(1) we force $\boxdot_\lambda$ while maintaining the supercompactness of λ. By Lemma 2.7 we force $\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\text{meagre}) = \kappa$ using a forcing notion $\mathbb{P}$ which is λ^+-c.c. and $(< \lambda)$-strategically complete. Notice that λ is still supercompact in the generic extension, so we are done. $\square$

Definition 2.2. For λ supercompact we define $\boxdot_\lambda$ by:

$\boxdot_\lambda$ for any regular cardinal $\chi > \lambda$ and forcing notion $\mathbb{P} \in \mathscr{H}(\chi)$ which is $(<\lambda)$-strategically complete (see Definition 1.6(2)) the following set $\mathscr{S} = \mathscr{S}_\mathbb{P} = \mathscr{S}_{\chi,\mathbb{P}}$ is a stationary subset of $[\mathscr{H}(\chi)]^{<\lambda}$:

$\mathscr{S} = \mathscr{S}_\mathbb{P} = \mathscr{S}_{\chi,\mathbb{P}}$ is the set of N's such that for some $\lambda_N, \chi_N, \mathbf{j} = \mathbf{j}_N$, $\mathbb{A} = \mathbb{A}_N, M = M_N, \mathbf{G} = \mathbf{G}_N$ we have (and we may say that $(\lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ is a witness for $N \in \mathscr{S}_{\chi,\mathbb{P}}$ or for $(N, \mathbb{P}, \chi)$)

which means:

(a) $N \prec (\mathscr{H}(\chi)^\mathbf{V}, \in)$ and $\mathbb{P} \in N$
(b) the Mostowski collapse of N is $\mathbb{A}$, and let $\mathbf{j}_N : N \to \mathbb{A}$ be the unique isomorphism
(c) $N \cap \lambda = \lambda_N$ and ${}^{(\lambda_N)>}N \subseteq N$ and λ_N is strongly inaccessible
(d) $\mathbb{A} \subseteq M := (\mathscr{H}(\chi_N), \in), M$ is transitive as well as $\mathbb{A}$
(e) $\mathbf{G} \subseteq \mathbf{j}_N(\mathbb{P})$ is generic over $\mathbb{A}$ for the forcing notion $\mathbf{j}_N(\mathbb{P})$
(f) $M = \mathbb{A}[\mathbf{G}]$.

Our first lemma is basically close to Laver's indestructibility. It consists of two parts. In the first part we prove that one can force $\boxdot_\lambda$ at a supercompact cardinal λ while preserving its supercompactness. In the second part, we prove that this can be done in an indestructible manner. Namely, any further extension of the universe by a $(<\lambda)$-directed-closed forcing notion will preserve the principle $\boxdot_\lambda$.

Lemma 2.3.

(1) *If λ is supercompact, then after some preliminary forcing of cardinality λ, λ is still supercompact and $\boxdot_\lambda$ from Definition* 2.2.

(2) *The statement $\boxdot_\lambda$ holds in $\mathbf{V}^\mathbb{P}$ when $\mathbf{V}$ satisfies $\boxdot_\lambda$ and $\mathbb{P}$ is a $(<\lambda)$-strategically complete forcing notion and $\mathbb{P}$ is $(<\lambda)$-directed closed; but see* 2.4(2).

Remark 2.4.

(1) Recall "$\mathbb{P}$ is a $(<\lambda)$-directed closed" means:

($*$) if J is a directed partial order of cardinality $<\lambda$ and $p_s \in \mathbb{P}$ for $s \in J$ and $s \leq_J t \Rightarrow p_s \leq_\mathbb{P} p_t$, then the set $\{p_s : s \in J\}$ has an upper bound in $\mathbb{P}$.

(2) In Lemma 2.3(2) we can weaken the assumption "$\boxplus_\lambda$ and $\mathbb{P}$ is $(<\lambda)$-directed closed" to:

$\boxplus$ if $\chi > \lambda$ and $\mathbb{P} \in \mathscr{H}(\chi)$, then we have (A) $\Rightarrow$ (B) where:

(A) $N \prec (\mathscr{H}(\chi), \in)$ and $\lambda_N, \chi_N, \mathbf{j}_N, \mathbf{G}_N, \mathbb{Q}$ satisfies:
  (a) $\mathbb{P}, \mathbb{Q} \in N, N \cap \lambda = \lambda_N < \chi_N < \lambda, [N]^{<\lambda_N} \subseteq N$
  (b) $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name of $(<\lambda)$-strategically closed forcing notion
  (c) $\mathbf{j}$ is the Mostowski collapse of N; its range is $\mathbb{A}$
  (d) $\mathbf{G}$ is a subset of $\mathbf{j}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})$, generic over $\mathbb{A}$
  (e) $\mathbb{A}[\mathbb{G}] = \mathscr{H}(\chi_N)$

(B) $\{p \in \mathbb{P} \cap N$: for some $(p', \underset{\sim}{q}') \in \mathbf{G}$ we have $\mathbf{j}(p) = p'\}$ has a common upper bound in $\mathbb{P}$.

(3) We can e.g. restrict the χ to be a strong limit.

Proof. (1) This is similar to the proof in Laver [Lav78] using Laver's diamond (see Definition 1.1), but as requested we elaborate. By Laver [Lav78] without loss of generality there is a Laver diamond $h : \lambda \to \mathscr{H}(\lambda)$. Let $E = \{\theta : \theta < \lambda$ and let $\alpha < \theta \Rightarrow h(\alpha) \in \mathscr{H}(\theta)\}$, clearly a club of λ, and let $\langle \kappa_\varepsilon : \varepsilon < \lambda \rangle$ list $\{\theta \in E : \theta$ is strongly inaccessible$\}$ in increasing order.

We now define $\mathbf{q}_\varepsilon$ and $\bar{\chi}^\varepsilon$ by induction on $\varepsilon \le \lambda$ such that:

$(*)$ (a) $\mathbf{q}_\varepsilon = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \le \varepsilon, \xi < \varepsilon \rangle$ is an Easton support iteration (so $\mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi$ do not depend on ε)
(b) $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$
(c) $\bar{\chi}^\varepsilon = \langle \chi_\zeta : \zeta < \varepsilon \rangle$ where each χ_ξ is a regular cardinal $\in [\kappa_\xi, \kappa_{\xi+1})$
(d) $\underset{\sim}{\mathbb{Q}}_\xi \in \mathscr{H}(\chi_{\xi+1})$ is a $\mathbb{P}_\xi$-name of a $(<\kappa_\xi)$-strategically complete forcing notion
(e) if $h(\xi) = (\underset{\sim}{\mathbb{Q}}, \chi)$ and the pair $(\underset{\sim}{\mathbb{Q}}, \chi)$ satisfies the requirements on $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\varepsilon)$ in clauses (c),(d), then $(\underset{\sim}{\mathbb{Q}}_\xi, \chi_\xi) = h(\xi)$.

Concerning clause (b), which says "$\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa_\zeta)$", note that for ζ a limit ordinal letting $\kappa_{<\zeta} = \cup\{\kappa_\xi : \xi < \zeta\}$ we have $\kappa_{<\zeta}$ is a strong limit and:

- if $\kappa_{<\zeta}$ is regular, equivalently strongly inaccessible, then $\kappa_{<\zeta} = \kappa_\zeta$ and $\mathbb{P}_\zeta = \cup\{\mathbb{P}_\xi : \xi < \zeta\}$, and so $\mathbb{P}_\zeta \subseteq \cup\{\mathscr{H}(\kappa_\xi) : \xi < \zeta\} = \mathscr{H}(\kappa_{<\zeta}) = \mathscr{H}(\kappa_\zeta)$
- if $\kappa_{<\zeta}$ is singular, then $\mathbb{P}_\zeta \subseteq \mathscr{H}(\kappa^+_{<\zeta}) \subseteq \mathscr{H}(\kappa_\zeta)$ as κ_ζ is inaccessible $> \kappa_{<\zeta}$.

We can easily carry the induction so $\mathbf{q}_\lambda$ is well defined, $\mathbb{P}_\lambda = \cup\{\mathbb{P}_\varepsilon : \varepsilon < \lambda\} \subseteq \cup\{\mathscr{H}(\kappa_\varepsilon) : \varepsilon < \lambda\} = \mathscr{H}(\lambda)$, and "$\xi < \lambda \Rightarrow \mathbb{P}_\lambda/\mathbb{P}_\xi$ is $(<\kappa_\xi)$-strategically complete"; hence $\mathbb{P}_\lambda/\mathbb{P}_\xi$ adds no new sequence of length $<\kappa_\xi$ of ordinals. Clearly it is enough to prove that in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $\boxdot_\lambda$.

Toward a contradiction assume $\chi, \mathbb{P}, \mathscr{S} = \mathscr{S}_{\chi,\mathbb{P}}$ form a counterexample in $\mathbf{V}^{\mathbb{P}_\lambda}$; hence there are $p_* \in \mathbb{P}_\lambda$ and $\mathbb{P}_\lambda$-names $\underset{\sim}{\chi}, \underset{\sim}{\mathbb{P}}, \underset{\sim}{\mathscr{S}}, \underset{\sim}{E}$ such that $p_* \Vdash_{\mathbb{P}_\lambda}$ "$\underset{\sim}{\chi} > \lambda$ is regular, $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\underset{\sim}{\chi})$, and $\underset{\sim}{\mathscr{S}}_{\underset{\sim}{\chi},\mathbb{P}}$ is defined as in $\boxdot_\lambda$, and $\underset{\sim}{E} \subseteq [\mathscr{H}(\chi)^{\mathbf{V}[\mathbb{P}_\lambda]}]^{<\lambda}$ is a club disjoint to $\underset{\sim}{\mathscr{S}}$".

As we can increase p_*, without loss of generality, $\underset{\sim}{\chi} = \chi$ and let $x = (\chi, \underset{\sim}{\mathbb{P}})$; and as $\mathbf{V} \models$ "λ is supercompact and h is a Laver diamond" for some $(I, D, \mathbf{M}, \mathbf{j}, \mathbf{j}_0, \mathbf{j}_1)$, we have:

$(*)_1$ (a) $\mathbf{M}$ is a transitive class
(b) $\mathbf{M}$ is a model of ZFC
(c) ${}^\chi\mathbf{M} \subseteq \mathbf{M}$
(d) $\mathbf{j}$ is an elementary embedding from $\mathbf{V}$ into $\mathbf{M}$
(e) $\mathrm{crit}(\mathbf{j}) = \lambda$
(f) $\mathbf{j}(h)(\lambda) = (\chi, \underset{\sim}{\mathbb{P}})$
(g) $I = [\mathscr{H}(\chi_1)]^{<\lambda}$ and $\chi_1 > \chi$
(h) D is a fine normal ultrafilter on I
(i) $\mathbf{j}_0$ is the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^I/D$
(j) $\mathbf{M}$ is the Mostowski collapse of $\mathbf{V}^I/D$
(k) $\mathbf{j}_1$ is the canonical isomorphism from $\mathbf{V}^I/D$ onto $\mathbf{M}$
(l) $\mathbf{j} = \mathbf{j}_1 \circ \mathbf{j}_0$.

Moreover, by Definition 1.1

$(*)_2$ $x = \mathbf{j}_1(\langle f(\sup(u \cap \lambda) : u \in I\rangle/D)$.

Let $\mathbf{q} = \mathbf{j}(\mathbf{q}_\lambda)$ so $\mathbf{q} = \langle \mathbb{P}_\zeta, \underset{\sim}{\mathbb{Q}}_\xi : \zeta \le \mathbf{j}(\lambda), \xi < \mathbf{j}(\lambda)\rangle$ and $\zeta < \lambda \Rightarrow \mathbb{P}^{\mathbf{q}}_\zeta = \mathbb{P}_\zeta$, etc. So

$(*)_3$ in $\mathbf{M}$ the pair $x = (\chi, \underset{\sim}{\mathbb{P}})$ satisfies:
(a) $\chi \in (\lambda, \mathbf{j}(\lambda)), \mathbf{j}(\lambda)$ is inaccessible
(b) $\underset{\sim}{\mathbb{P}} \in \mathscr{H}(\chi)$
(c) $\underset{\sim}{\mathbb{P}}$ is a $\mathbb{P}_\lambda$-name of a $(<\lambda)$-strategically complete forcing notion.

[Why? Because $[\mathbf{M}]^\chi \subseteq \mathbf{M}$; hence $\mathscr{H}(\chi^+)^{\mathbf{V}} \subseteq \mathbf{M}$.]

Now

$(*)_4$ the following sets belong to D:

(a) $\mathscr{S}_1 = \{u \in I : x \in u$ and $(\mathscr{H}(\chi_1), \in){\restriction}u \prec (\mathscr{H}(\chi_1), \in)\}$

(b) $\mathscr{S}_2 = \{u \in \mathscr{S}_1 : u \cap \lambda$ is an inaccessible cardinal we call $\lambda_u\}$

(c) $\mathscr{S}_3 = \{u \in \mathscr{S}_2$: the Mostowski collapse N^1_u of $(\mathscr{H}(\chi_1), \in){\restriction}u$ is isomorphic to some $(\mathscr{H}(\chi^1_u), \in)\}$.

[Why? As D is a fine and normal ultrafilter on I.]

$(*)_5$ for every formula $\varphi = \varphi(-) \in \mathbb{L}(\{\in\})$ the following are equivalent:

(a) $(\mathscr{H}(\chi_1), \theta) \models \varphi[x]$

(b) $(\mathscr{H}(\chi_1), \in)^I/D \models \varphi[\langle h(u \cap \lambda) : u \in I\rangle/D]$

(c) $\mathscr{X}^1_\varphi \in D$ where $\mathscr{X}^1_\varphi = \{u \in I : x \in u$ and $(\mathcal{H}(\chi_1), \in){\restriction}u \models \varphi[x]\}$

(d) $\mathscr{X}^2_\varphi \in D$ where $\mathscr{X}^2_\varphi = \{u \in I : p_*, x \in u$ and $(\mathscr{H}(\chi_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$

(e) $\mathscr{X}^3_\varphi \in D$ where $\mathscr{X}^3_\varphi = \{u \in I : x \in u, \chi^1_u = \mathrm{otp}(\chi_u{\restriction}u)$, and $(\mathscr{H}(\chi^1_u), \in) \models \varphi[\mathbf{j}_u(x)]\}$.

[Why? We have (a) $\Leftrightarrow$ (c) as D is a fine normal ultrafilter on $I = \mathscr{H}(\chi_1)$; we have (c) $\Leftrightarrow$ (d) as j_u is an isomorphism from $(\mathscr{H}(\chi_1), \in){\restriction}u$ onto $\mathscr{H}(\chi^1_u)$; we have (d) $\Leftrightarrow$ (e) by the choice of D; lastly, (b) $\Leftrightarrow$ (c) by the Łos theorem.]

Hence

$(*)_6$ there is N as required in $\mathbf{V}^{\mathbb{P}_*}$.

[Why? Choose $u \in I$ which belongs to all the sets from D mentioned in $(*)_4 + (*)_5$. Let $\zeta = u \cap \lambda$, so it is inaccessible, even measurable, and let $\mathbf{j}_u(x) = \mathbf{j}_u(\chi, \underset{\sim}{\mathbb{P}}) = h(\zeta)$, so (by the choice of $\mathbf{q}$) $h(\zeta) = (\chi, \underset{\sim}{\mathbb{Q}}_\zeta)$ and $\mathbb{Q}_\zeta$ is a $\mathbb{P}_{\mathbf{q},\zeta}$-name.

Let $\mathbf{G}$ be a subset of $\mathbb{P}_{\mathbf{q}} = \mathbb{P}_\lambda$ to which p_* belongs, $\mathbf{G}_\zeta = \mathbf{G} \cap \mathbb{P}_{q,\zeta}$, hence is a generic subset of $\mathbb{P}_{\mathbf{q},\zeta}$ over $\mathbf{V}$, hence a generic subset of $\mathbf{j}_u(\mathbb{P}_{\mathbf{q}}) \in \mathscr{H}(\chi_\zeta)$; and let $N = ((\mathscr{H}(\chi_1), \in){\restriction}u)[\mathbf{G}], \mathbb{A} = (\mathscr{H}(\chi_\zeta)^{\mathbf{V}[\mathbf{G}_\zeta]}, \in), M = \mathbb{A}^{\underset{\sim}{\mathbb{Q}}_\zeta[\mathbf{G}_\zeta]}$. Easily N is as promised, a contradiction to the choice of p_*.

So we are done proving part (1).

(2) Let $\mathbb{Q}$ be a forcing notion in $\mathbf{V}^{\mathbb{P}}$ which is $(< \lambda)$-strategically complete and (even) $(< \lambda)$-directed closed, let $\emptyset \in \mathbb{Q}$ be the weakest condition, let χ_1 be large enough so that $\lambda, \underset{\sim}{\mathbb{Q}}, \underset{\sim}{E} \in \mathscr{H}(\chi_1)$, and it suffices to prove that in $\mathbf{V}^{\mathbb{P}}$, the set $\mathscr{S}_{\chi_1,\mathbb{Q}}$ is stationary. So let $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{E}$ be $\mathbb{P}$-names such that for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi_1)$ is $(< \lambda)$-strategically complete, $(< \lambda)$-directed closed, forcing notion, $\underset{\sim}{E}$ a club of $[\mathscr{H}(\chi_1)]^{<\lambda}$ disjoint to $\underset{\sim}{\mathscr{S}}_{\chi_1,\underset{\sim}{\mathbb{Q}}}$"; no need to use a name for χ_1 as we can increase p.

Let $\chi \gg \chi_1$. Now $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathscr{H}(\chi)$ is a $(< \lambda)$-strategically complete forcing notion, and without loss of generality code (χ_1, E). As $\boxdot_\lambda$ holds in $\mathbf{V}$ we can apply it to the forcing $\mathbb{P}_{\geq p} * \underset{\sim}{\mathbb{Q}}$, so we can find a tuple $(N, \lambda_N, \chi_N, \mathbf{j}_N, \mathbb{A}_N, M_N, \mathbf{G}_N)$ witnessing it; in particular, $(p, \emptyset) \in \mathbf{G}_N, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in N$, so $\chi_1, \underset{\sim}{E} \in N$. Let $\mathbf{G}_{\mathbb{P}}$ be a subset of $\mathbb{P}$ generic over $\mathbf{V}$ which extends $\{p' : (p', q') \in \mathbf{G}_N\}$, possible because $\mathbf{G}_N$ is in $\mathbf{V}$, a subset of $\mathbb{P}$ which has an upper bound. This is the only place we use "$\mathbb{P}$ is $(< \lambda)$-directed closed".

Next, let $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_{\mathbb{P}}], N_1 = N[\mathbf{G}_{\mathbb{P}}], E_1 = \underset{\sim}{E}[\mathbf{G}_{\mathbb{P}}], \mathbb{A}_1 = \mathbb{A}[\mathbf{j}''_N(\mathbf{G}_{\mathbb{P}} \cap N)] = \mathbb{A}[\{p' : (p', \underset{\sim}{q}') \in \mathbf{G}_N], \mathbf{G}_1 = \{\underset{\sim}{q}[\mathbf{j}''(G_{\mathbb{P}} \cap N)] : (p, \underset{\sim}{q}) \in \mathbf{G}_{\mathbb{P}}\}$.

Let $N_2 = N_1{\restriction}\mathscr{H}(\chi_1)^{\mathbf{V}[\mathbf{G}_{\mathbb{P}}]}, \mathscr{S} = \mathscr{S}_{\underset{\sim}{\mathbb{Q}}}[\mathbf{G}_{\mathbb{P}}], \mathbf{j}_1 =$ the lifting of $(\mathbf{j}{\restriction}(N \cap \mathscr{H}(\chi)))$ to mapping N_1 onto $\mathbb{A}_1$.

Now recalling that p forces $\underset{\sim}{E}$ is disjoint to $\mathscr{S}$, clearly

$(*)$ $N_2 \in E$.

Hence

$(*)$ $N_1 \notin \mathscr{S}_1$.

But easily in $\mathbf{V}_1$ we have $(\lambda_N, \chi_N, \mathbf{j}_1, \mathbb{A}_1, M_1 = M, \mathbf{G}_1)$ witness $N_1 \in \mathscr{S} \cap E_1$, a contradiction to the choice of $\underset{\sim}{E}$. $\square$

Discussion 2.5. Suppose that one wishes to force an inequality between two cardinal characteristics. There are two general approaches, which can be labeled as *Top-down* and *Bottom-up*. In the Bottom-up strategy one begins with a universe in which many characteristics are small, e.g., by assuming $2^\lambda = \lambda^+$, and then increases some of them while trying to keep the smallness of the rest. In the Top-down strategy one begins with a universe in which many characteristics are large. The forcing aims to decrease some of them while keeping the large value of the rest.

We shall use the Top-down approach, so we begin by increasing $\mathfrak{b}_\lambda$ (and $\mathfrak{d}_\lambda$) to some $\mu = \mathrm{cf}(\mu) > \lambda$. Notice that $\mathfrak{b}_\lambda$ is a relatively small characteristic and, in particular, always $\mathfrak{b}_\lambda \leq \mathfrak{d}_\lambda$. The next step will be to decrease $\mathrm{cov}_\lambda(\text{meagre})$ in such a way that maintains the fact that $\mathfrak{d}_\lambda = \mu$. We shall increase $\mathfrak{b}_\lambda$ by using the generalization to λ of Hechler forcing. This is a standard way to achieve this goal, but we spell out the proof since it demonstrates the way that we employ Lemma 2.3.

Claim 2.6. Assume that:

(a) λ is supercompact.
(b) $\lambda < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then one can force $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$ while keeping the supercompactness of λ and the principle $\boxdot_\lambda$.

Proof. Begin with the preparatory forcing of Lemma 2.3 to make λ indestructible and to force $\boxdot_\lambda$ in such a way that it will be preserved by any further $(< \lambda)$-directed-closed forcing. By Lemma 2.3 as in the applications of Laver-indestructibility we can assume that GCH holds above λ after the preparatory forcing. In particular, if $\mu = \mathrm{cf}(\mu) > \lambda$, then $\mu^\lambda = \mu$ follows.

Let $\mathbb{D}_\lambda$ be the generalized Hechler forcing. A condition $p \in \mathbb{D}_\lambda$ is a pair (η_p, f_p) such that $\eta_p \in {}^{<\lambda}\lambda$, $f_p \in {}^\lambda\lambda$, and $\eta_p \trianglelefteq f_p$. If $p, q \in \mathbb{D}_\lambda$, then $p \leq q$ iff $\eta_p \trianglelefteq \eta_q$ and $f_p(\alpha) \leq f_q(\alpha)$ for every $\alpha \in \lambda$.

Let $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \mu, \beta < \mu \rangle$ be a $(< \lambda)$-support iteration of the generalized Hechler forcing notions for λ. Explicitly, $\mathbb{Q}_\alpha$ is the $\mathbb{P}_\alpha$-name of $\mathbb{D}_\lambda$ in $\mathbf{V}^{\mathbb{P}_\alpha}$ for every $\alpha < \mu$. Denote the generic λ-Hechler for $\mathbb{Q}_\alpha$ by $\underset{\sim}{f}^*_\alpha$. So $\mathbb{P}_\mu$ is the limit, and choose a generic $\mathbf{G} \subseteq \mathbb{P}_\mu$. We claim that $\mathbf{V}[\mathbf{G}] \models$ "$\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$" as witnessed by $\langle f^*_\alpha : \alpha < \mu \rangle$. Notice that $2^\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$, so it is sufficient to prove that $\mathfrak{b}_\lambda = \mu$ in $\mathbf{V}[\mathbf{G}]$.

Since λ is regular, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is $(< \lambda)$-complete. By Fact 1.7, $\mathbb{P}_\alpha$ is $(< \lambda)$-complete as well for every $\alpha \leq \mu$. Likewise, each $\underset{\sim}{\mathbb{Q}}_\alpha$ is λ-centered, so $\mathbb{P}_\mu$ is λ^+-c.c. (see [She78] or [Shed]). It follows that $\mathbf{V}[\mathbf{G}]$ preserves cardinals and cofinalities. Moreover, no new $(< \lambda)$-sequences are introduced. Notice also that $\mathbb{P}_\mu$ is $(< \lambda)$-directed-closed, and hence $\mathbf{V}[\mathbf{G}] \models$ "λ is supercompact and $\boxdot_\lambda$ holds".

The main point is that $\{f^*_\alpha : \alpha < \mu\}$ is an unbounded family in $({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$. For this, assume that $\Vdash_{\mathbb{P}_\mu}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$". For every $\alpha < \lambda$ fix a maximal antichain

$\langle p_\alpha, i : i < i_\alpha \leq \lambda\rangle$ of conditions which force a value to $\underset{\sim}{f}(\alpha)$. Let

$$\delta = \sup(\cup\{\mathrm{dom}(p_{\alpha,i}) : \alpha < \lambda, i < i_\alpha\}).$$

Since $\lambda < \mu = \mathrm{cf}(\mu)$ we see that $\delta < \mu$, and clearly $\underset{\sim}{f}$ is a $\mathbb{P}_\delta$-name. We conclude, therefore, that $\underset{\sim}{f}$ is dominated by $\underset{\sim}{f}^*_{\delta+1}$, and hence $\{\underset{\sim}{f}^*_\alpha : \alpha < \mu\}$ exemplifies $\mathfrak{b}_\lambda = \mu$. This fact completes the proof. $\square$

Our second lemma is the main burden of the proof. The statement of the theorem requires λ to be supercompact in order to obtain the indestructibility properties given by Lemma 2.3. The combinatorial part given in Lemma 2.7 below requires only strong inaccessibility. However, we assume supercompactness in order to keep $\boxdot_\lambda$.

Lemma 2.7. *Assume that:*

(a) λ *is supercompact*
(b) $\boxdot_\lambda$ *holds*
(c) $\lambda < \kappa = \mathrm{cf}(\kappa) < \mu = \mathrm{cf}(\mu) = \mu^\lambda$.

Then there exists a λ^+-c.c. $(<\lambda)$-strategically complete forcing notion $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathfrak{d}_\lambda = \mu \wedge \mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$".

Proof. By Claim 2.6 without loss of generality $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$. In particular, λ is supercompact, and $\boxdot_\lambda$ holds in the generic extension. Let $\langle f^*_\alpha : \alpha < \mu\rangle$ witness $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \mu$, and without loss of generality $\alpha < \beta < \mu \Rightarrow f^*_\alpha <_{J^{\mathrm{bd}}_\lambda} f^*_\beta$.

Recalling Definitions 1.8, 1.9, Claim 1.10, Theorem 1.11, in $\mathbf{V}$ there are $\beta(*), \mathbf{q}, \bar{u}$, $\mathscr{U}_*, \ldots$ such that:

$(*)_1(A)$ $\mathbf{q} \in \mathbf{Q}_{\lambda,\bar{\theta},\beta(*)}$ so in particular we have (in $\mathbf{q}$):
- (a) $\langle \mathbb{P}_{0,\alpha}, \underset{\sim}{\mathbb{Q}}_{0,\beta} : \alpha \leq \beta(*), \beta < \beta(*)\rangle$ is a $(<\lambda)$-support iteration
- (b) $\bar{u} = \langle u_\beta : \beta < \beta(*)\rangle$, $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*)\rangle$
- (c) $u_\beta \subseteq \beta$, $\mathscr{P}_\beta \subseteq [u_\beta]^{\leq\lambda}$ is closed under subsets
- (d) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ has generic $\underset{\sim}{\eta}_\beta \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ and $\mathbb{P}_{1,\alpha}, \mathbb{P}_{1,\mathcal{U}}$ are as in 1.9.
- (e) $\underset{\sim}{\mathbb{Q}}_{0,\beta}$ is as in Definition 1.8(e), so is $\subseteq \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u_\beta\rangle]}$ and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\underset{\sim}{\eta}_\beta \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$" and $\bar{\underset{\sim}{\eta}} = \langle \underset{\sim}{\eta}_\beta : \beta < \beta(*)\rangle$
- (f) $\mathscr{U}_* \subseteq \beta(*)$ has order type $\gamma(*) = \kappa$, and $\langle \beta^*_i : i \leq \kappa\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order
- (g) If $\beta \in \mathscr{U}_*$, then $[\mathscr{U}_* \cap \beta]^{\leq\lambda} \subseteq \mathscr{P}_\beta$; hence $\subseteq u_\beta$ and $\Vdash_{\mathbb{P}_{0,\beta+1}}$ "if $\nu \in \mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}_* \cap \beta\rangle] \cap \prod_{\varepsilon<\lambda} \theta_\varepsilon$, then $\nu <_{J^{\mathrm{bd}}_\lambda} \underset{\sim}{\eta}_\beta$"
- (h) if $\alpha \leq \beta(*)$, then $\mathbb{P}_{0,\alpha}$ is $(<\lambda)$-strategically complete and λ^+-c.c..

(B) Letting $\mathbb{P}'_i = \mathbb{P}_{\mathbf{q},1,\{\beta^*_j : j<i\}}$ for $i \leq \kappa$ we have:
- (a) The sequence $\langle \mathbb{P}'_i : i \leq \gamma(*)\rangle$ of forcing notions is $\lessdot$-increasing and is continuous for ordinals $i \leq \gamma(*)$ of cofinality $\geq \lambda$
- (b) $\mathbb{P}'_i$ is $(<\lambda)$-strategically complete for $i \leq \gamma(*)$
- (c) $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_{\gamma(*)}]} = \cup\{(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \gamma(*)\}$.
- (d) The sequence $\langle \mathbb{P}_{1,\beta} : \beta \leq \beta(*)\rangle$ is a sequence of forcing notions, $\lessdot$-increasing; and if $\beta \leq \beta(*)$, then $\mathbb{P}_{0,\beta} \lessdot \mathbb{P}_{1,\beta}$, in fact is dense in it and if $i \leq \gamma(*)$, then $\mathbb{P}'_i \lessdot \mathbb{P}_{1,\beta^*_i}$.

We shall mention more properties later.

[Why are there such objects? We apply Theorem 1.11 and Definition 1.8 and Claim 1.10.]

Also

$(*)_2$ (a) recall $\langle \beta^*_i : i \le \gamma(*)\rangle$ lists $\mathscr{U}_* \cup \{\beta(*)\}$ in increasing order
(b) for $i \le \gamma(*) = \kappa$, for $i < \gamma(*)$ let $\underset{\sim}{g}'_i$ be $\underset{\sim}{\eta}_{\beta^*_i}$ (to avoid excessive subscripts),
(c) let $\bar{g}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$
(d) let $\underset{\sim}{g}_\alpha = \underset{\sim}{\eta}_\alpha$ for $\alpha < \beta(*)$ and $\bar{g} = \langle \underset{\sim}{g}_\beta : \beta < \beta(*)\rangle$
(e) $\mathscr{P}_\alpha = \mathscr{P}_{\mathbf{q},\alpha}$, and without loss of generality $u_\alpha = \cup\{u : u \in \mathscr{P}_\alpha\}$ for $\alpha < \beta(*)$.

$(*)_3$ If $u \in \mathscr{P}_\alpha, \alpha < \beta(*)$, then $\Vdash_{\mathbb{P}_{0,\alpha+1}}$ "$\underset{\sim}{g}_\alpha \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ dominates $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\langle \underset{\sim}{g}_\beta : \beta \in u\rangle]}$", the order being modulo $J^{\rm bd}_\lambda$.

[Why? By the choice of the forcing; see Definition 1.4.]

$(*)_4$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\underset{\sim}{\bar{g}}' = \langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and cofinal in $(\prod_{\varepsilon<\lambda}\theta_\varepsilon, <_{J^{\rm bd}_\lambda})$".

[Why? By $(*)_3$ noting that $(\prod_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\kappa]} = \cup\{(\prod_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_i]} : i < \kappa\}$, which holds by $(*)_1(B)(c)$.]

Now

$(*)_5$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathrm{cov}_\lambda(\text{meagre}) \le \kappa$".

[Why? First, notice that we can look at $\prod_{\varepsilon<\lambda}\theta_\varepsilon$ instead of ${}^\lambda 2$.

Second, for each $\varepsilon < \lambda, i < \kappa$ the set $B_{\varepsilon,i} = \{\eta \in \prod_{\xi<\lambda}\theta_\xi$: for every $\zeta \in [\varepsilon,\lambda)$ we have $\eta(\zeta) \le \underset{\sim}{g}'_i(\zeta) < \theta_\zeta\}$ is closed nowhere dense, and by $(*)_4$ we have $\mathbf{V}^{\mathbb{P}'_\kappa} \models$ "$\prod_{\zeta<\lambda}\theta_\zeta = \cup\{\underset{\sim}{B}_{\varepsilon,i} : \varepsilon < \lambda, i < \kappa\}$". In fact, $\langle B_{0,i} : i < \kappa\rangle$ suffice. Alternatively we have $\langle \underset{\sim}{g}'_i : i < \kappa\rangle$ is $<_{J^{bd}_\lambda}$-increasing cofinal in $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ and let $\mathcal{W}_{i,\zeta} := \{\eta | j : j < \lambda, \eta \in^\lambda 2$ and for every $\varepsilon \in [\zeta,\lambda)$ we have either $\eta | [\Sigma_{\xi<\varepsilon}\theta_\xi, \Sigma_{\xi\le\varepsilon}\theta_\xi)$ is constantly zero or $\min\{\alpha : \Sigma_{\xi<\varepsilon}\theta_\xi + \alpha \in \eta^{-1}(\{1\})\} < g'_i(\varepsilon)\}$. So $\mathcal{W}_{i,\zeta}$ is a closed nowhere dense subset of ${}^\lambda 2$ and $\cup\{\mathcal{W}_{i,\zeta} : i < \kappa, \zeta < \lambda\} = {}^\lambda 2$ and $\kappa \times \lambda$ has cardinality $\lambda + \kappa = \kappa$.]

$(*)_6$ $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathrm{cov}_\lambda(\text{meagre}) \ge \kappa$".

[Why? Let us define the $\mathbb{P}'_{i+1}$-name $\underset{\sim}{\eta}'_i$ of a member of ${}^\lambda 2$ by $\underset{\sim}{\eta}'_i(\varepsilon) = 0$ iff $\underset{\sim}{g}'_i(\varepsilon)$ is even. Now clearly $\Vdash_{\mathbb{P}'_{i+1}}$ "$\underset{\sim}{\eta}'_i$ is a λ-Cohen sequence over $\mathbf{V}^{\mathbb{P}'_i}$". (Let us elaborate; $\underset{\sim}{\eta}'_i$ is also a $\mathbb{P}_{\beta^*_i+1}$-name and $\Vdash_{\mathbb{P}_{\beta^*_i+1}}$ "$\underset{\sim}{\eta}'_i$ is λ-Cohen over $\mathbf{V}^{\mathbb{P}_{\beta^*_i}}$, hence over $\mathbf{V}^{\mathbb{P}'_i}$"; the last hence because $\mathbb{P}'_i \lessdot \mathbb{P}_{\beta^*_i}$. As $\mathbb{P}_{\beta^*_i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ and $\mathbb{P}'_{i+1} \lessdot \mathbb{P}_{\beta^*_{i+1}}$ we are done.)

Also every closed nowhere dense subset of ${}^\lambda 2$ from $\mathbf{V}^{\mathbb{P}'_{\gamma(*)}}$ is from $\mathbf{V}^{\mathbb{P}'_i}$ for some $i < \gamma(*)$. So if $p \Vdash$ "$\mathrm{cov}_\lambda(\text{meagre}) < \kappa$", then for some $\zeta < \kappa$ and $\underset{\sim}{A}_\varepsilon(\varepsilon < \zeta)$ we have $p \Vdash$ "$\underset{\sim}{A}_\varepsilon$ is a closed nowhere dense set subset of ${}^\lambda 2$ for $\varepsilon < \zeta$" and $p \Vdash$ "$\bigcup_i \underset{\sim}{A}_i$ is equal to the set of ${}^\lambda 2$". Without loss of generality each $\underset{\sim}{A}_\varepsilon$ is a $\mathbb{P}_{i(\varepsilon)}$-name, $i(\varepsilon) < \kappa$. Hence $i = \sup\{i(\varepsilon) : \varepsilon < \zeta\} < \kappa$, and $\underset{\sim}{\eta}'_i$ gives a contradiction to the choice of $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \zeta\rangle$, so $(*)_6$ holds indeed.]

The reader may look at some explanation in Discussion 2.9.

Now we come to the main and last point, recalling $\langle f^*_\alpha : \alpha < \mu \rangle$ from Claim 2.6:

$(*)_7$ $\Vdash_{\mathbb{P}'_{\gamma(*)}}$ "no $\underset{\sim}{f} \in ({}^\lambda\lambda)$ dominates $\{f^*_\alpha : \alpha < \mu\}$".

We shall show that it suffices to prove $(*)_7$ for proving Lemma 2.3(2) and that $(*)_7$ holds, thus finishing.

Why does this suffice? As $\langle f^*_\alpha : \alpha < \mu \rangle$ is $<_{J^{\rm bd}_\lambda}$-increasing and $\mathrm{cf}(\mu) = \mu > \lambda$, this implies that $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda \geq \mu$". Also in $\mathbf{V}, \mu^\lambda = \mu > \kappa > \lambda$ and $|\mathbb{P}'_{\gamma(*)}| = \kappa^\lambda$ by (A)(g) of Claim 1.10(4) which is $\leq \mu$ and $\mathbb{P}'_\kappa$ satisfies the λ^+-c.c.; hence $\Vdash_{\mathbb{P}'_\kappa}$ "$2^\lambda = \mu$, hence together $\Vdash_{\mathbb{P}'_\kappa}$ "$\mathfrak{d}_\lambda = \mu$". Also by $(*)_1(B)(b)$, "$\mathbb{P}'_{\gamma(*)}$ is $(<\lambda)$-strategically complete $+\lambda^+$-c.c.", and by $(*)_5 + (*)_6$ we know that "$\mathrm{cov}_\lambda(\mathrm{meagre}) = \kappa$", so we are done; hence $(*)_7$ is really the last piece missing. The rest of the proof is dedicated to proving that $(*)_7$ holds.

We shall use further nice properties of $\mathbb{P}'_j, \underset{\sim}{g}'_i (j \leq \gamma(*), i < \gamma(*))$ which hold by $(*)_1 + (*)_2$ (and $(*)_3, (*)_4$) and their proof, i.e., Claim 1.10, Theorem 1.11.

$\boxplus_1$ (a) (α) $\langle \underset{\sim}{g}'_\gamma : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$; i.e., if $\mathbf{G}$ is a subset of $\mathbb{P}'_{\gamma(*)}$ generic over $\mathbf{V}$ and $g'_i = \underset{\sim}{g}'_i[\mathbf{G}]$, then $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle g'_i : i < \gamma(*) \rangle]$

(β) if $\nu \in ({}^\lambda\lambda)^{\mathbf{V}[\mathbf{G}]}$, then for some $\rho \in ({}^\lambda\gamma(*))^{\mathbf{V}}$ and λ-Borel function $\mathbf{B} \in \mathbf{V}$ we have $\nu = \mathbf{B}(\langle g'_{\rho(\varepsilon)} : \varepsilon < \lambda \rangle)$

(b) if in $\mathbf{V}[\mathbf{G}], g''_\gamma \in \prod_{\zeta<\lambda} \theta_\zeta$ for $\gamma < \gamma(*)$ and the set $\{(\gamma, \zeta) : \gamma < \gamma(*)$ and $\zeta < \lambda$ and $g''_\gamma(\zeta) \neq g'_\gamma(\zeta)\}$ has cardinality $< \lambda$, then $\bar{g}'' = \langle g''_\gamma : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ and $\mathbf{V}[\bar{g}''] = \mathbf{V}[\bar{g}']$

(c) $\Vdash_{\mathbb{P}'_\gamma}$ "$\underset{\sim}{g}'_\gamma$ dominates $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}'_\gamma]}$"

(d) if $\langle \zeta(\gamma) : \gamma < \gamma(*) \rangle$ is an increasing sequence of ordinals $< \gamma(*)$ (from $\mathbf{V}$!), then $\langle g'_{\zeta(\gamma)} : \gamma < \gamma(*) \rangle$ is generic for $\mathbb{P}'_{\gamma(*)}$ (over $\mathbf{V}$)

(e) if $\gamma \leq \gamma(*)$, then $\mathbb{P}'_\gamma$ is $(<\lambda)$-strategically complete and satisfies the λ^+-c.c.

We shall use $\boxplus_1$ freely.

To prove $(*)_7$ assume toward contradiction that this fails, and hence for some condition $p^* \in \mathbb{P}'_{\gamma(*)}$ and $\mathbb{P}'_{\gamma(*)}$-name $\underset{\sim}{f}$ and λ-Borel function $\mathbf{B}$ and $\rho \in {}^\lambda\gamma(*)$ we have

$\circledast_0$ $p^* \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\underset{\sim}{f} \in {}^\lambda\lambda$ and dominates $({}^\lambda\lambda)^{\mathbf{V}}$" and $\underset{\sim}{f} = \mathbf{B}(\langle \underset{\sim}{g}'_{\rho(i)} : i < \lambda \rangle)$.

Now let χ be regular large enough and choose $\bar{N} = \langle N_\varepsilon : \varepsilon < \lambda \rangle$ such that

$\circledast_1$ (a) N_ε is as in $\boxdot_\lambda$ for the forcing notion $\mathbb{P}'_{\gamma(*)}, N_\varepsilon \in \mathscr{S}_{\chi, \mathbb{P}'_{\gamma(*)}}$; see $\boxdot_\lambda$ of Lemma 2.3

(b) $\bar{N} \restriction \varepsilon \in N_\varepsilon$ and $\mathrm{otp}(N_\varepsilon \cap \kappa) < \theta_{(\lambda_\varepsilon)}$; hence $\bigcup_{\zeta<\varepsilon} N_\zeta \subseteq N_\varepsilon$ and $\lambda_\varepsilon := \mathrm{otp}(N_\varepsilon \cap \lambda) > \lambda^-_\varepsilon := \Sigma\{\|N_\zeta\| : \zeta < \varepsilon\} \geq \Sigma\{\lambda_\zeta : \zeta < \varepsilon\}$

(c) $\bar{\theta}, \mathbf{q}, p^*, \mathcal{U}_*, \underset{\sim}{f}, \mathbf{B}, \rho$ belong to N_ε.

Next choose $f^* \in {}^\lambda\lambda$, i.e., $\in ({}^\lambda\lambda)^{\mathbf{V}}$, such that

$\circledast_2$ for arbitrarily large $\varepsilon < \lambda$ for some $\zeta \in [\lambda^-_\varepsilon, \lambda_\varepsilon)$ we have $f^*(\zeta) > \lambda_\varepsilon$ [we can demand more].

For $\varepsilon < \lambda$ let $(\lambda_\varepsilon, \chi_\varepsilon, \mathbf{j}_\varepsilon, M_\varepsilon, \mathbb{A}_\varepsilon, \mathbf{G}_\varepsilon)$ be a witness for $(N_\varepsilon, \mathbb{P}'_{\gamma(*)}, \chi)$ recalling $\boxdot_\lambda$ from Definition 2.2 so $\lambda_\varepsilon \in (\varepsilon, \lambda)$ is strongly inaccessible and $\varepsilon < \zeta < \lambda \Rightarrow \lambda_\varepsilon < \lambda^-_\zeta < \lambda_\zeta$, recalling $\circledast_1$ and noting $\langle \lambda^-_\varepsilon : \varepsilon < \lambda \rangle$ is an increasing and a continuous sequence of cardinals below λ.

Let

$\circledast_3$ (a) $v_\varepsilon = N_\varepsilon \cap \gamma(*)$

(b) $i_\varepsilon = i(\varepsilon) = \mathrm{otp}(v_\varepsilon)$, and so $i(\varepsilon) = \mathbf{j}_\varepsilon(\gamma(*))$, etc.

(c) $\bar{\gamma}^\varepsilon = \langle \gamma_i(\varepsilon) : i < i(\varepsilon) \rangle$ list v_ε in increasing order

(d) For $i < \mathrm{otp}(v_\varepsilon)$, equivalently $i < \mathbf{j}_\varepsilon(\gamma(*))$, let $\eta^\varepsilon_i = (\mathbf{j}_\varepsilon(\underset{\sim}{g}'_{\gamma_i(\varepsilon)}))^{\mathbb{A}'_\varepsilon[\mathbf{G}_\varepsilon]} \in \prod\limits_{\zeta < \lambda_\varepsilon} \theta_\zeta$ and let $\bar{\eta}^\varepsilon = \langle \eta^\varepsilon_i : i < i_\varepsilon \rangle$.

Note that clearly

$\circledast_4$ (a) $\bar{\eta}^\varepsilon$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$; moreover

(b) for each $\varepsilon < \lambda$, if we change $\eta^\varepsilon_i(\zeta)$ (legally, i.e., to an ordinal $< \theta_\zeta$) for $< \lambda_\varepsilon$ pairs $(i, \zeta) \in \mathrm{otp}(v_\varepsilon) \times \lambda_\varepsilon$ and get $\bar{\eta}'$, then also $\bar{\eta}'$ is generic for $(\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}))$ and $\mathbb{A}_\varepsilon[\bar{\eta}'] = M_\varepsilon$

(c) like $\boxplus_1$ with $\mathbf{V}, \mathbb{P}'_{\gamma(*)}, \lambda$ there standing for $\mathbb{A}_\varepsilon, \mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)}), \lambda_\varepsilon$ here.

Hence clearly

$\circledast'_4$ for $\varepsilon < \lambda$,

(a) let $\Xi_\varepsilon := \{\bar{\nu} : \bar{\nu} = \langle \nu_i : i < i(*) \rangle$ and for some $\mathbf{G} \subseteq \mathbb{P}'_{\gamma(*)} \cap N_\varepsilon$ generic over N_ε we have $\nu_i \in \Pi_{\xi < \lambda_\varepsilon} \theta_\xi$ satisfies $\xi < \lambda_\varepsilon \Rightarrow$ some $\psi \in \mathbf{G}$ forces $g'_{\gamma_i(\varepsilon)} | \xi = \nu_i | \xi\}$

(b) if $\bar{\nu} \in \Xi_\varepsilon$ then there is one and only one $\mathbf{G} = \mathbf{G}_\nu$ as above

(c) we choose $\bar{p}^*_\varepsilon = \langle p^*_{\varepsilon,\psi} : \psi \in \mathbb{P}'_{\gamma(*)} \cap N_\varepsilon$ such that

(α) $p^*_{\varepsilon,\psi} \in \mathbb{P}_{\beta(*)}$ moreover, if $j \leq \sigma$ and $\psi \in \mathbb{P}'_{\gamma_j(\varepsilon)}$ then $p^*_{\varepsilon,psi} \in \mathbb{P}_{\gamma_j(\varepsilon)}$

(β) $\mathbb{P}_{\beta(*)} \vDash \psi \leq p^*_{\varepsilon,\psi}$ moreover, if $\mathbb{P}'_{\beta(*)} \Vdash$ "$\psi \leq \varphi$" then $p^*_{\varphi,\psi}, \varphi$ are compatible in $\mathbb{P}_{\gamma(*)}$

(γ) $\langle \mathrm{dom}(p^*_{\varepsilon,\psi}) \setminus \mathcal{U}_* : \psi \in \mathbb{P}'_{\gamma(*)} \rangle$ is a sequence of pairwise disjoint sets

(d) if $\bar{\nu} \in \Xi_\varepsilon$ then

(α) there is $q \in \mathbb{P}_{\beta(*)}$ which is an upper bound of $\{p^*_{\varepsilon,\psi} : \psi \in \mathbf{G}_{\bar{\nu}}\}$

(β) then q is $(N_\varepsilon, \mathbb{P}'_{\gamma(*)})$-generic naturally and $q \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$\mathbf{j}_\varepsilon$ can be extended naturally to an isomorphism from

$$N_\varepsilon[\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}] = N_\varepsilon[\langle \underset{\sim}{g}'_\gamma : \gamma \in v_\varepsilon \rangle]$$

onto $\mathbb{A}_\varepsilon[\bar{\eta}^\varepsilon]$".

[Why? See [Sheb].]

By the assumption toward contradiction, $\circledast_0$, and $\mathbb{P}'_{\gamma(*)}$ being $(< \lambda)$-strategically complete, recalling $\boxplus_1$, there are $\zeta(*), p^{**}$, and p^+ such that (recall that $p^* \in \mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$):

$\circledast_5$ (a) $p^* \leq p^{**} \in \mathbb{P}'_{\gamma(*)}$ and $p^+ \in \mathbb{P}_{0,\beta(*)}$ such that $\mathbb{P}_{1,\gamma(*)} \models$ "$p^{**} \leq p^+$"

(b) $\zeta(*) < \lambda$

(c) $p^{**} \Vdash_{\mathbb{P}'_{\gamma(*)}}$ "$f^*(\zeta) < \underset{\sim}{f}(\zeta)$ whenever $\zeta(*) \leq \zeta < \lambda$" where f^* is from $\circledast_2$

(d) if $\gamma \in \mathrm{Dom}(p^+)$ then $\eta^{p^+(\gamma)}$ is an object (not just a $\mathbb{P}_{0,\gamma}$-name) and has length $\geq \zeta(*)$ (recall that $\eta^{p^+(\gamma)}$ is the trunk of the condition $p^+(\gamma)$; see clause (α)(b) of Definition 1.4(1)).

Note that possibly $\mathrm{Dom}(p^+) \nsubseteq \cup\{v_\varepsilon : \varepsilon < \lambda\}$. Choose $\varepsilon(*) < \lambda$ such that $\lambda_{\varepsilon(*)} > \zeta(*) + |\mathrm{Dom}(p^+)|$ and $\gamma \in \mathrm{Dom}(p^+) \Rightarrow \varepsilon(*) > \ell g(\eta^{p^+(\gamma)})$ recalling clause (d) of $\circledast_5$ and $|\mathrm{Dom}(p^+)| < \lambda$ as $p^+ \in \mathbb{P}_{0,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is the limit of a $(<\lambda)$-support iteration.

By $\circledast_2$ we can add $(\exists\zeta)[\lambda^-_{\varepsilon(*)} \le \zeta < \lambda_{\varepsilon(*)} < f^*(\zeta)]$. Our intention is to find $q \in \mathbb{P}_{0,\beta(*)}$ above p^+, which (in $\mathbb{P}_{1,\beta(*)}$) is above some $q' \in \mathbb{P}'_{\gamma(*)}$, which is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic and forces $\underset{\sim}{\mathbf{G}}_{\mathbb{P}'_{\gamma(*)}}$ to include a generic subset of $(\mathbb{P}'_{\gamma(*)})^{N_{\varepsilon(*)}}$ hence is induced by some $\bar\nu$ as in $\circledast'_4$, recalling $\circledast_4$(b). Toward this in $\circledast_6$ below the intention is that $p^+_{i(*)}$ will serve as q.

Let $i(*) = i(\varepsilon(*))$ and let γ_i for $i < i(*)$ be such that[1] $\langle\gamma_i : i < i(*)\rangle$ list $\{\beta^*_i : i \in v_{\varepsilon(*)}\} \subseteq \mathscr{U}_* = N_{\varepsilon(*)} \cap \mathcal{U}_*$ in increasing order; recall that $\mathscr{U}_* = \{\beta^*_i : i < \gamma(*)\}$ and $i < j < \gamma(*) \Rightarrow \beta^*_i < \beta^*_j$ and $v_{\varepsilon(*)} \subseteq \gamma(*)$ has order type $i(\varepsilon(*))$. Next let $\gamma_{i(*)} = \gamma(*)$ so $\{\mathbf{j}_{\varepsilon(*)}(\gamma) : \gamma \in v_{\varepsilon(*)}\} = i(*) = \mathbf{j}_{\varepsilon(*)}(\gamma(*))$. Recall that $\gamma(*) = \kappa = \mathrm{cf}(\kappa) > \lambda, \mathrm{otp}(v_{\varepsilon(*)}) = \mathrm{otp}(N_{\varepsilon(*)} \cap \gamma(*)) = \mathrm{otp}(N_{\varepsilon(*)} \cap \kappa)$; hence $N_{\varepsilon(*)} \models$ "$i(*)$ is a regular cardinal $> \lambda_\varepsilon$"; hence $i(*)$ is really a regular cardinal, so call it σ. Now we define a game ∂ as follows:[2]

$\boxplus_2$ (A) Each play lasts $i(*) + 1 = \sigma + 1$ moves and in the i-th move:
- (a) if $i = j + 1$ the antagonist player chooses $\xi_j = \xi(j) < \sigma$ such that $j_1 < j \Rightarrow \xi(j_1) < \xi(j)$
- (b) then, if $i = j + 1$ the protagonist chooses $\zeta_j = \zeta(j) \in (\xi(j), \sigma)$, but there are more restrictions implicit in $\boxplus_3$ below
- (c) in any case the protagonist also chooses $p^+_i, \bar\nu^i$ such that $\boxplus_3$ below holds.

(B) In the end of the play the protagonist wins the play iff it always has a legal move and in the end p^+_σ is $(\mathbb{P}'_{\gamma(*)}, N)$-generic and $\{\zeta(i) : i \le i(*)\} \in \mathbb{A}_{\varepsilon(*)}$; note that trivially it belongs to $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}]$; see $\circledast_3(d)$

where

$\boxplus_3$ (a) $p^+_i \in \mathbb{P}_{0,\gamma_i}$ and $p^+_i, \langle\nu_j : j < i\rangle, \mathbf{G}_{\varepsilon,i} \subseteq \mathbb{P}'_{\gamma_i} \cap N$ are as in $\circledast'_4$
- (b) if $j < i$, then $\mathbb{P}_{0,\gamma_i} \models$ "$p^+_j \le p^+_i$"
- (c) if $\gamma \in \cup\{\mathrm{Dom}(p^+_j) : j < i\}$, then $p^+_i \restriction \gamma \Vdash_{\mathbb{P}_{0,\gamma_i}}$ "$\underset{\sim}{\eta}^{p^+_i(\gamma)}$ has length $\ge i(*)$ and $\ge \lambda_{\varepsilon(*)}$" moreover $\underset{\sim}{\eta}^{p^+_i(\gamma)}$ is an object, $\eta^{p^+_i(\gamma)}$
- (d) $\mathbb{P}_{0,\gamma_i} \models$ "$p^+ \restriction \gamma_i \le p^+_i$"
- (e) $\bar\nu^i = \langle\nu_{\gamma_j} : j < i\rangle$ and $\nu_{\gamma_j} \in \prod\limits_{\iota < \lambda_{\varepsilon(*)}} \theta_\iota p^+_\sigma$ is an upper bound of $\Omega_{\varepsilon(*),\bar\nu^i} = \{p^*_{\varepsilon(*),\psi} | \gamma_i : \bar\nu \in \Xi_{\varepsilon(*)}$ satisfies $\bar\nu^I \lhd \bar\nu$ and $\psi \in \mathbf{G}_{\bar\nu}\}$
- (f) for $j < i$ we have $\nu_{\gamma_j} \trianglelefteq \eta^{p^+_i(\gamma_j)}$ so $p^+_i \restriction \gamma_j \Vdash$ "$\nu_{\gamma_j} \lhd \underset{\sim}{g}'_{\gamma_j}$" recalling $\boxplus_1$
- (g) for $j < i$ we have (recall $\bar\eta^\varepsilon$ from $\circledast_3(d)$)
  - (α) $\nu_{\gamma_j} = \eta^{\varepsilon(*)}_{\gamma_{\zeta(j)}}$ recalling $\eta^{\varepsilon(*)}_{\gamma_j}$ is from $\circledast_3(d)$ or
  - (β) $\gamma_j \in \mathrm{Dom}(p^+)$ and $\{\iota < \lambda_{\varepsilon(*)} : \eta^{\varepsilon(*)}_{\zeta(j)}(\iota) \ne \nu_{\gamma_j}(\iota)\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$.

[1] This is used in $\boxplus_3$ and the proof of $(*)_6$. Not to be confused with $\bar\gamma^\varepsilon$ of $\circledast_3$(c).

[2] The idea is to scatter the $\eta^{\varepsilon(*)}_{\gamma_i}$'s. Why not use the original places, as then we shall have a problem in $\circledast_6$; the scattering is helpful because we are relying on Claim 1.10 and Theorem 1.11.

We shall prove

$\circledast_6$ in the game $\Game$
 (a) the antagonist has no winning strategy
 (b) at stage i, if $\langle \zeta(j) : j < i \rangle \in N$ then the protagonist has a legal move; moreover for any $\zeta(i) \in (\xi(i), \sigma)$ large enough the protagonist can choose it.

Why does $\circledast_6$ suffice?

By clause (a) of $\circledast_6$ we can choose a play $\langle (\xi(i), \zeta(i), p_i^+, \bar{\nu}^i) : i \leq \sigma \rangle$ in which the protagonist wins. Recalling $\mathbb{P}'_{\gamma(*)} \lessdot \mathbb{P}_{1,\beta(*)}$ and $\mathbb{P}_{0,\beta(*)}$ is a dense subforcing of $\mathbb{P}_{1,\beta(*)}$, clearly

$\circledast_7$ there is p such that
 (a) $p \in \mathbb{P}'_{\gamma(*)}$
 (b) if $\mathbb{P}'_{\gamma(*)} \models$ "$p \leq p'$" hence $p' \in \mathbb{P}'_{\beta(*)}$, then p', p_σ^+ are compatible in $\mathbb{P}_{1,\beta(*)}$
 (c) p is above p^{**} and it forces $\underset{\sim}{g}'_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$. Hence, $\mathbf{G}_{\mathbb{P}'_{\beta(*)}} \cap N = \mathbf{G}_{\langle \nu_{\gamma_i} : i < \gamma \rangle}$.

Then on the one hand

$\circledast'_7$ $p \in \mathbb{P}'_{\gamma(*)}$ being above p^{**} forces $f^* \restriction [\zeta(*), \lambda) < \underset{\sim}{f} \restriction [\zeta(*), \lambda)$, hence $f^* \restriction [\zeta(*), \lambda_{\varepsilon(*)}) < \underset{\sim}{f} \restriction [\zeta(*), \lambda_{\varepsilon(*)})$ recalling that $\zeta(*) < \lambda_{\varepsilon(*)}$; see $\circledast_5$ and the choice of $\varepsilon(*)$ immediately after $\circledast_5$.

On the other hand,

$\circledast''_7$ p is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic.

[Why? As it forces $\underset{\sim}{\eta}_{\gamma_i} \restriction \lambda_{\varepsilon(*)} = \nu_{\gamma_i}$ for $i < i(*)$ and $\langle \nu_{\gamma_i} : i < i(*) \rangle$ is (see $\boxplus_3(g)$ recalling $\mathrm{Dom}(p^{**})$ has cardinality $< \lambda_{\varepsilon(*)}$) "almost equal" to $\langle \eta^{\varepsilon(*)}_{\zeta(i)} : i < i(*) \rangle$ which is a subsequence of the sequence from $\circledast_3$. That is $\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}, i < i(*) = \sigma$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} \subseteq \cup \{\{(i, \iota) : \iota < \lambda_{\varepsilon(*)}$ and $\nu_{\gamma_i}(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(i)}(\iota)\} : \gamma \in v_{\varepsilon(*)} \cap \mathrm{Dom}(p^{**})\}$ so is the union of $\leq |\mathrm{Dom}(p_\sigma^+)| < \lambda_{\varepsilon(*)}$ sets each of cardinality $< \lambda_{\varepsilon(*)}$, hence is of cardinality $< \lambda_{\varepsilon(*)}$. Hence by $\circledast_4(c) + \boxplus_1(d)$ the sequence $\bar{\nu}^{i(*)}$ is generic for $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$. By $\boxplus_2$ and the choice of p_σ^+ above, it is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma(*)})$-generic, by $\circledast_7(b)$ also p is]

As $\underset{\sim}{f} \in N_{\varepsilon(*)}$ it follows from $\circledast''_7$ that

$\circledast'''_7$ $p \Vdash$ "$\underset{\sim}{f} \restriction \lambda_{\varepsilon(*)}$ is a function from $\lambda_{\varepsilon(*)}$ to $\lambda_{\varepsilon(*)}$".

Together $\circledast'_7 + \circledast'''_7$ gives a contradiction by the choice of f^* in $\circledast_2$ and of $\varepsilon(*)$ above; hence $\circledast_6$ is enough. In Lemma 2.8 below we show that $\circledast_6$ is true, so we are done. $\square$

Lemma 2.8. *The statement $\circledast_6$ holds true.*

Proof. Let us prove $\circledast_6$. First, assuming clause (b) which is proved below, for clause (a) choose any strategy $\mathbf{st}$ for the antagonist and fix a partial strategy $\mathbf{st}'$ for the protagonist choosing $(p_i^+, \bar{\nu}^i)$ depending on the previous choices and $\zeta(i) < i_{\varepsilon(*)}$ such that it is a legal move if relevant and possible. So the only freedom left for the protagonist is to choose the $\zeta(i)$. So (recalling $\boxplus_2(A)$(a)) we have in $\mathbf{V}$ a function

$F : {}^{\sigma>}\sigma \to \sigma$ (so F uses **st** ...) such that:

$(*)_F$ playing the game such that the antagonist uses **st** and the protagonist uses **st**$'$, arriving at the i-th move, $\bar\zeta = \langle \zeta(j) : j < i\rangle$ is well defined, and for the protagonist any choice $\zeta_i \in (F(\bar\zeta), \sigma) \cap \mathscr{U}_{**}$ is legal.

Now we have to find an increasing sequence $\bar\zeta = \langle \zeta(i) : i < \sigma\rangle$ from $\mathbb{A}_{\varepsilon(*)}$ not just from $M_{\varepsilon(*)} = \mathscr{H}(\chi_{\varepsilon(*)})^{\mathbf{V}}$ such that $F(\bar\zeta\restriction i) < \zeta(i) < \sigma$ and $\bar\zeta \in \mathbb{A}_{\varepsilon(*)}$. As $F \in \mathscr{H}(\chi_{\varepsilon(*)})$ and $\mathscr{H}(\chi_{\varepsilon(*)}) = \mathbb{A}_{\varepsilon(*)}[\mathbf{G}_{\varepsilon(*)}]$ where $\mathbf{G}_{\varepsilon(*)}$ is a subset of $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}'_{\gamma(*)}) \in \mathbb{A}_{\varepsilon(*)}$ generic over $\mathbb{A}_{\varepsilon(*)}$ and $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c. and $\sigma = \mathrm{cf}(\sigma) > \lambda_{\varepsilon(*)}$ this is possible. That is, there is a $\mathbf{j}_{\varepsilon(*)}(\mathbb{P}_{0,\beta(*)})$-name $\underset{\sim}{F}_* \in \mathbb{A}_{\varepsilon(*)}$ such that $F = \underset{\sim}{F}_*[\mathbf{G}_{\varepsilon(*)}]$, and we define in $\mathbb{A}_{\varepsilon(*)}$ the function $F' : {}^{\sigma>}\sigma \to \sigma$ by $F'(\langle \zeta(j) : j < i\rangle) = \sup\{\xi + 1 : \xi \in \{\zeta(j) + 1 : j < i\}$ or $\xi < \sigma$ and $\nVdash_{\mathbf{j}(\mathbb{P}_{0,\beta(*)})}$ "$\underset{\sim}{F}(\langle \zeta(j) : j < i\rangle) \neq \xi\}$"; clearly this is O.K.

We are left with proving $\circledast_6$(b).

Case 1 ($i = 0$). Let $p^+_0 = p^{**} \restriction \gamma_0$.

Case 2 (i limit). By clauses (b) and (c) of $\boxplus_3$, there is $p^+_i \in \mathbb{P}_{0,\gamma_i}$ which is an upper bound (even l.u.b.) of $\{p^+_j : j < i\} \bigcup \Omega_{\varepsilon(*),\bar\nu^i}$ and it is easily as required. Also $\bar\nu^i$ is well defined and as required.

Case 3 ($i = j+1$ and $\gamma_j \notin \mathrm{Dom}(p^{**})$). Clearly γ_i is in $\mathscr{U}_*$, the successor of γ_j, and $(\exists\iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$. As in Case 4 below but easier by the properties of the iteration.

Case 4 ($i = j+1$ and $\gamma_j \in \mathrm{Dom}(p^{**})$). Again γ_i is in $\mathscr{U}_*$, the successor of γ_j, and $(\exists\iota)(\gamma_j = \beta^*_\iota \wedge \iota \in v_{\varepsilon(*)})$.

First we find p'_j such that:

$\circledast_8$ (a) $p^+_j \le p'_j \in \mathbb{P}_{0,\gamma_j}$ and p^+_j is an upper bound of $\Omega_{\varepsilon(*),\bar\nu^i}$
(b) if $\gamma \in \mathrm{Dom}(p^+_j)$, then $p'_j \restriction \gamma \Vdash$ "$\ell g(\eta^{p'_j(\gamma)}) > i(*)$" (see $\boxplus_3(c)$)
(c) p'_j forces[3] a value to the pair $(\eta^{p^+(\gamma_j)}, \underset{\sim}{f}^{p^+(\gamma_j)} \restriction \lambda_{\varepsilon(*)})$; we call this pair $q_j = (\eta^{q_j}, f^{q_j})$.

This should be clear.

Second,

$\circledast_9$ p^+_j, hence p'_j, is $(N_{\varepsilon(*)}, \mathbb{P}'_{\gamma_j})$-generic and $\langle \nu_{\gamma_{j(1)}} : j(1) < j\rangle$ induces the generic.

[Why? As in the proof of $\circledast''_7$ of Lemma 2.7 when we assume that we have carried the induction, by $\boxplus_2$, clause (g), and $\circledast_4$.]

Now

$\circledast_{10}$ (a) $f^{q_j} \in (\prod_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbb{A}_{\varepsilon(*)}[\bar\nu^j]}$
(b) for every large enough $\zeta \in (\xi(i), \sigma)$ we have
- $\bullet_1$ $f^{q_j} \le \eta^{\varepsilon(*)}_\zeta \mod J^{\mathrm{bd}}_{\lambda_\varepsilon}$
- $\bullet_2$ $f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta]$
- $\bullet_3$ $\langle \zeta(j_1) : j_1 < j\rangle \in \mathbb{A}_{\varepsilon(*)}[\bar\eta^{\varepsilon(*)}\restriction\zeta]$

(c) $\eta^{q_j} \triangleleft f^{q_j}$.

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[3] Recall that $\eta^{p^+(\gamma_j)}$ is an object, not a name, and p^+_j is $(N_{\varepsilon(*)}, \mathbb{P}_{\gamma_j})$-generic

[Why? Clause (a) holds because $f^{q_j} \in (\prod\limits_{\zeta<\lambda_{\varepsilon(*)}} \theta_\zeta)^{\mathbf{V}}$, hence belongs to $\mathscr{H}(\chi_{\varepsilon(*)})$, which is the universe of $M_{\varepsilon(*)}$ so $f^{q_j} \in M_{\varepsilon(*)}$. But $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)}]$, recalling that $\bar{\eta}^{\varepsilon(*)}$ is a generic for $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$. Next as $\mathbb{P}'_{\gamma(*)}$ satisfies the λ^+-c.c. and $\lambda < \kappa = \mathrm{cf}(\gamma(*))$ so $\mathbf{j}_\varepsilon(\mathbb{P}'_{\gamma(*)})$ satisfies the $\lambda^+_{\varepsilon(*)}$-c.c.; hence for some $\zeta_1 < \sigma, f^{q_j} \in \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)}\restriction\zeta_1]$. Similarly for some $\zeta_2 < \sigma$ we have $\langle \zeta(j_2) : j_2 < j\rangle$ belongs to $N'_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)}\restriction\zeta_2]$. Letting $\zeta \geq \max\{\zeta_1,\zeta_2\}$ clearly clauses $\bullet_2, \bullet_3$ of $\circledast_{10}$(b) holds; also $\bullet_1$ there holds by $\boxplus_1(c)$ (and $\mathbf{j}_\varepsilon$, etc.).

$\circledast_{11}$ $M_{\varepsilon(*)} = \mathbb{A}_{\varepsilon(*)}[\bar{\eta}^{\varepsilon(*)}]$ satisfies: for every $\rho \in {}^{(\lambda_{\varepsilon(*)})>}(\lambda_{\varepsilon(*)})$, in particular, $\rho = \eta^{q_j}$, the set $\{\zeta < \sigma : \rho \triangleleft \eta^{\varepsilon(*)}_\zeta\}$ is unbounded in σ.

[Why? Because the iteration $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ is with support $< \lambda$ and similarly $\mathbf{j}_\varepsilon(\mathbf{q})$.

Lastly, $\circledast_{10}$(c) holds by $\circledast_8$(c).]

Now we choose $\zeta(j)$ as in clause (b) of $\circledast_{10}$ and $\nu_j \in \prod\limits_{\varepsilon<\lambda_{\varepsilon(*)}} \theta_\varepsilon$ such that $\eta^{q_j} \triangleleft \nu_j, \eta^{p^+(j)} \triangleleft \nu_j, f^{q_j} \leq \nu_j$, and $\{\iota < \lambda_{\varepsilon(*)} : \nu_j(\iota) \neq \eta^{\varepsilon(*)}_{\zeta(j)}\}$ is a bounded subset of $\lambda_{\varepsilon(*)}$. Next choose $p^+_i \in \mathbb{P}'_{\gamma(*)}$ such that $p^+_i\restriction\gamma_j = p'_j, \eta^{p^+_i(\gamma_i)} = \nu_j$ and $f^{p^+_i(\gamma_i)}\restriction[\lambda_\varepsilon,\lambda) = f^{p^+(\gamma)}\restriction[\lambda_\varepsilon,\lambda)$.

We have carried the induction, hence proved $\circledast_6(b)$, so we are done proving Lemma 2.8. $\square$

Discussion 2.9. (1) The reader may justly wonder why we use $\mathbf{V}' = \mathbf{V}[\underset{\sim}{\bar{g}}'] = \mathbf{V}[\underset{\sim}{\bar{g}}\restriction\mathscr{U}_*]$ rather than simply $\mathbf{V}[\underset{\sim}{\bar{g}}]$. Of course, nothing is lost by it, but why the extra complication?

(2) The answer is that during the proof we shall use: if $\zeta(i) \in \mathscr{U}_*$ is increasing with $i < \gamma(*)$, then also $\langle \underset{\sim}{g}_{\zeta(i)} : i < \kappa\rangle$ is generic over $\mathbf{V}$ for the subforcing of $\mathbb{P}_{1,\beta(*)}$ generated by $\underset{\sim}{\bar{g}}\restriction\mathscr{U}_*$; see $\circledast''_7$ inside the proof of $\circledast_6$ inside Lemma 2.8. But using $\mathscr{U}_* = \beta(*)$, we do not know this.

(3) Now in the parallel case for $\lambda = \aleph_0$ with FS iteration with full memory, such claim is true; see the Introduction.

(4) But we do not know the parallel of (3) for λ, so we use a substitute using $\mathscr{U}_*$, i.e., $\mathbb{P}'_\kappa$.

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Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel; and Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019

Email address: shelah@math.huji.ac.il

URL: http://shelah.logic.at