FUNDAMENTA
MATHEMATICAE
247 (2019)

Reconstructing structures with the strong small index property up to bi-definability

by

Gianluca Paolini (Torino) and
Saharon Shelah (Jerusalem and Piscataway, NJ)

Abstract. Let $\mathbf{K}$ be the class of countable structures M with the strong small index property and locally finite algebraicity, and $\mathbf{K}_*$ the class of $M \in \mathbf{K}$ such that $\mathrm{acl}_M(\{a\}) = \{a\}$ for every $a \in M$. For homogeneous $M \in \mathbf{K}$, we introduce what we call the expanded group of automorphisms of M, and show that it is second-order definable in $\mathrm{Aut}(M)$. We use this to prove that for $M, N \in \mathbf{K}_*$, $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as abstract groups if and only if $(\mathrm{Aut}(M), M)$ and $(\mathrm{Aut}(N), N)$ are isomorphic as permutation groups. In particular, we deduce that for $\aleph_0$-categorical structures the combination of the strong small index property and no algebraicity implies reconstruction up to bi-definability, in analogy with Rubin's (1994) well-known $\forall\exists$-interpretation technique. Finally, we show that every finite group can be realized as the outer automorphism group of $\mathrm{Aut}(M)$ for some countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity.

1. Introduction. Reconstruction theory deals with the problem of reconstruction of countable structures from their automorphism groups. The first degree of reconstruction that is usually dealt with is the so-called *reconstruction up to bi-interpretability*. The second and stronger degree of reconstruction is known as *reconstruction up to bi-definability*. In group-theoretic terms, the first degree of reconstruction corresponds to reconstruction of *topological group isomorphisms* from abstract group isomorphisms, while the second degree of reconstruction corresponds to reconstruction of *permutation group isomorphisms* from abstract group isomorphisms. Two independent techniques lead the scene in this field: the (strong) small index property (see e.g. [4]) and Rubin's $\forall\exists$-interpretation [9].

2010 *Mathematics Subject Classification*: 20B27, 03C35, 03C15.
Key words and phrases: reconstruction theory, automorphism groups, $\aleph_0$-categorical structures, small index property, outer automorphisms.
Received 2 April 2017; revised 21 March 2018 and 26 September 2018.
Published online 12 April 2019.

DOI: 10.4064/fm640-9-2018

On the reconstruction up to bi-interpretability side, the cornerstones of the theory are the following two results:

THEOREM (Rubin [9]). *Let M and N be countable $\aleph_0$-categorical structures and suppose that M has a $\forall\exists$-interpretation. Then* $\mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ *if and only if M and N are bi-interpretable.*

THEOREM (Lascar [5]). *Let M and N be countable $\aleph_0$-categorical structures and suppose that M has the small index property. Then* $\mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ *if and only if M and N are bi-interpretable.*

On the reconstruction up to bi-definability side, all the known results are based on the following theorem of Rubin:

THEOREM (Rubin [9]). *Let M and N be countable $\aleph_0$-categorical structures with no algebraicity and suppose that M has a $\forall\exists$-interpretation. Then* $\mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ *if and only if M and N are bi-definable.*

In particular, on the small index property side there is no result that pairs with the last cited result of Rubin. In this paper we fill this gap proving the following:

THEOREM 1.1. *Let $\mathbf{K}_*$ be the class of countable structures M satisfying:*

(1) *M has the strong small index property;*
(2) *for every finite $A \subseteq M$, $\mathrm{acl}_M(A)$ is finite;*
(3) *for every $a \in M$, $\mathrm{acl}_M(\{a\}) = \{a\}$.*

Then for $M, N \in \mathbf{K}_*$, $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as abstract groups if and only if $(\mathrm{Aut}(M), M)$ and $(\mathrm{Aut}(N), N)$ are isomorphic as permutation groups. Moreover, if $\pi : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ is an abstract group isomorphism, then there is a bijection $f : M \to N$ witnessing that $(\mathrm{Aut}(M), M)$ and $(\mathrm{Aut}(N), N)$ are isomorphic as permutation groups and such that $\pi(\alpha) = f \alpha f^{-1}$.

Thus we deduce an analog of Rubin's result on reconstruction up to bi-definability:

COROLLARY 1.2. *Let M and N be countable $\aleph_0$-categorical structures with the strong small index property and no algebraicity. Then $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as abstract groups if and only if M and N are bi-definable. Moreover, if $\pi : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ is an abstract group isomorphism, then there is a bijection $f : M \to N$ witnessing the bi-definability of M and N such that $\pi(\alpha) = f \alpha f^{-1}$.*

For a structure M satisfying the conclusion of Theorem 1.1 it is easy to determine the outer automorphism group of $\mathrm{Aut}(M)$: in fact any $f \in \mathrm{Aut}(\mathrm{Aut}(M))$ is induced by a permutation of M. For example, as already noted by Rubin in [9], using this fact it is easy to see that for R_n the

n-coloured random graph ($n \geq 2$) we have $\mathrm{Out}(\mathrm{Aut}(R_n)) \cong \mathrm{Sym}(n)$. Similarly, but in a different direction, one easily sees that for M_n the K_n-free random graph ($n \geq 3$), $\mathrm{Aut}(M_n)$ is complete. We show here that in this setting any finite group can occur:

THEOREM 1.3. *Let K be a finite group. Then there exists a countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity such that $K \cong \mathrm{Out}(\mathrm{Aut}(M))$.*

Our main technical tool is what we call the *expanded group of automorphisms* of a homogeneous structure M with the strong small index property and locally finite algebraicity. This powerful object encodes the combinatorics of $\mathrm{Aut}(M)$-stabilizers of M, and it is a crucial ingredient of our proof of Theorem 1.1. In Theorem 2.13 we show that the expanded group of automorphisms is second-order orbit-definable in $\mathrm{Aut}(M)$ (cf. Definition 2.12), a fact of essential importance.

The results of this paper pair with those of [8] and [7], where sufficient conditions for the strong small index property are isolated and applied in the concrete case of the group of automorphisms of Hall's universal locally finite group. Finally, we would like to mention another recent result of ours [6] in this area, a powerful non-reconstruction theorem: no algebraic or topological property of $\mathrm{Aut}(M)$ can detect any form of stability of the countable structure M.

2. The expanded group of automorphisms. In this section we introduce the expanded group of automorphisms of M (for certain M), and show that it is second-order definable in $\mathrm{Aut}(M)$.

Given a structure M and $A \subseteq M$, and considering $\mathrm{Aut}(M) = G$ in its natural action on M, we denote the pointwise (resp. setwise) stabilizer of A under this action by $G_{(A)}$ (resp. $G_{\{A\}}$). Also, we denote the subgroup relation by $\leq$.

DEFINITION 2.1. Let M be a structure and $G = \mathrm{Aut}(M)$.

(1) We say that a is *algebraic* (resp. *definable*) over $A \subseteq M$ in M if the orbit of a under $G_{(A)}$ is finite (resp. trivial).
(2) The *algebraic closure* of $A \subseteq M$ in M, denoted by $\mathrm{acl}_M(A)$, is the set of elements of M which are algebraic over A.
(3) The *definable closure* of $A \subseteq M$ in M, denoted by $\mathrm{dcl}_M(A)$, is the set of elements of M which are definable over A.

DEFINITION 2.2. Let M be a countable structure and $G = \mathrm{Aut}(M)$.

(1) We say that M (or G) has the *small index property* (SIP) if every subgroup of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ contains the pointwise stabilizer of a finite set $A \subseteq M$.

(2) We say that M (or G) has the *strong small index property* (SSIP) if every subgroup of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ lies between the pointwise and the setwise stabilizer of a finite set $A \subseteq M$.

HYPOTHESIS 2.3. Throughout this section, let M be a countable homogeneous structure with the strong small index property and locally finite algebraicity, i.e. for every finite $A \subseteq M$ we have $|\mathrm{acl}_M(A)| < \omega$.

REMARK 2.4. Notice that all ω-categorical structures have locally finite algebraicity.

NOTATION 2.5. We let

$$\mathbf{A}(M) = \{\mathrm{acl}_M(B) : B \subseteq_{\mathrm{fin}} M\},$$
$$\mathbf{EA}(M) = \{(K, L) : K \in \mathbf{A}(M) \text{ and } L \leq \mathrm{Aut}(K)\}.$$

DEFINITION 2.6. For $(K, L) \in \mathbf{EA}(M)$, we define

$$G_{(K,L)} = \{f \in \mathrm{Aut}(M) : f{\restriction}K \in L\}.$$

Notice that if $L = \{\mathrm{id}_K\}$, then $G_{(K,L)} = G_{(K)}$, i.e. it equals the pointwise stabilizer of K, and that if $L = \mathrm{Aut}(K)$, then $G_{(K,L)} = G_{\{K\}}$, i.e. it equals the setwise stabilizer of K. We then let

$$\mathcal{PS}(M) = \{G_{(K)} : K \in \mathbf{A}(M)\}, \quad \mathcal{SS}(M) = \{G_{(K,L)} : (K, L) \in \mathbf{EA}(M)\}.$$

The crucial point is the following:

LEMMA 2.7. *Let $\mathcal{G} = \{H \leq G : [G : H] < 2^\omega\}$. Then $\mathcal{G} = \mathcal{SS}(M)$.*

Proof. The containment from right to left is trivial. Let then $H \leq G$ with $[G : H] < 2^\omega$. By the strong small index property, there is a finite $K \subseteq M$ such that $G_{(K)} \leq H \leq G_{\{K\}}$. It follows that $G_{(\mathrm{acl}_M(K))} \leq H \leq G_{\{\mathrm{acl}_M(K)\}}$, and so without loss of generality we can assume that $K \in \mathbf{A}(M)$. First of all we claim that $G_{(K)} \trianglelefteq G_{\{K\}}$. In fact, for $g \in G_{\{K\}}$, $h \in G_{(K)}$ and $a \in K$, we have $ghg^{-1}(a) = gg^{-1}(a) = a$, since $g^{-1}(a) \in K$ and $h \in G_{(K)}$. Furthermore, for $g, h \in G_{\{K\}}$, we have $g^{-1}h \in G_{(K)}$ iff $g{\restriction}K = h{\restriction}K$. Hence, the map $f : gG_{(K)} \mapsto g{\restriction}K$, for $g \in G_{\{K\}}$, is such that

$$(\star) \qquad f : G_{\{K\}}/G_{(K)} \cong \mathrm{Aut}(K),$$

since every $f \in \mathrm{Aut}(K)$ extends to an automorphism of M. Thus, by the fourth isomorphism theorem we have $H = G_{(K,L)}$ for $L = \{f{\restriction}K : f \in H\}$. ∎

PROPOSITION 2.8. *Let $H_1, H_2 \in \mathcal{SS}(M)$. The following conditions are equivalent:*

(1) $H_1 \trianglelefteq H_2$ *and* $[H_2 : H_1] < \omega$;
(2) *there are* $K \in \mathbf{A}(M)$ *and* $L_1 \trianglelefteq L_2 \leq \mathrm{Aut}(K)$ *such that* $H_i = G_{(K,L_i)}$ *for* $i = 1, 2$.

Proof. The proof of (2)⇒(1) is immediate, since by the normality of L_1 in L_2 we know that, for $g \in G_{(K,L_2)}$ and $h \in G_{(K,L_1)}$, $ghg^{-1}\restriction K \in L_1$, while the fact that $[H_2 : H_1] < \omega$ follows from the proof of Lemma 2.7.

We show that (1) implies (2) via statements $(*)_i$ below (each is followed by its proof). By assumption, $H_i = G_{(K_i,L_i)}$ for $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$.

$(*)_1$ $K_2 \subseteq K_1$.

Suppose not, and let $a \in K_2 - K_1$ witness this. Then we can find $f \in G$ such that $f\restriction K_1 = \mathrm{id}_{K_1}$ and $f(a) \notin K_2$. It follows that $f \in H_1 - H_2$, a contradiction.

$(*)_2$ $K_1 \subseteq K_2$.

Suppose not, and let $f_n \in G$, for $n < \omega$, be such that $f_n\restriction K_2 = \mathrm{id}_{K_2}$, and in addition $\{f_n(K_1 - K_2) : n < \omega\}$ are pairwise disjoint. Then clearly, for every $n < \omega$, $f_n \in H_2$ and $\{f_n H_1 : n < \omega\}$ are distinct, contradicting the assumption $[H_2 : H_1] < \omega$.

$(*)_3$ $L_1 \leq L_2$.

Suppose not, and let $h \in L_1 - L_2$. Then h extends to an automorphism f of M. Clearly $f \in H_1 - H_2$, a contradiction.

$(*)_4$ $L_1 \trianglelefteq L_2$.

Suppose not, and let $g_i \in L_i$ $(i = 1, 2)$ be such that $g_2 g_1 g_2^{-1} \notin L_1$. Then g_i extends to an automorphism f_i of M $(i = 1, 2)$. Clearly $f_i \in H_i$ $(i = 1, 2)$, and $f_2 f_1 f_2^{-1} \notin H_1$, a contradiction. ∎

Proposition 2.9. *Let $\mathcal{G} = \{H \in \mathcal{SS}(M)$: there is no $H' \in \mathcal{SS}(M)$ with $H' \subsetneq H$, $H' \trianglelefteq H$ and $[H : H'] < \omega\}$. Then $\mathcal{PS}(M) = \mathcal{G}$.*

Proof. First we show the containment from left to right. Let $H_2 \in \mathcal{PS}(M)$ and assume that there exists $H_1 \in \mathcal{SS}(M)$ such that $H_1 \subsetneq H_2$, $H_1 \trianglelefteq H_2$ and $[H_2 : H_1] < \omega$. By Proposition 2.8, $H_i = G_{(K_i,L_i)}$ for $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$ and $K_1 = K = K_2$. Now, as $H_2 \in \mathcal{PS}(M)$, we have $L_2 = \{\mathrm{id}_K\}$. Hence, $L_1 = L_2$, and so $H_1 = H_2$, a contradiction.

We now show the containment from right to left. Let $H \in \mathcal{G}$; then $H = G_{(K,L)}$ for $(K, L) \in \mathbf{EA}(M)$. If $L \neq \{\mathrm{id}_K\}$ then letting $H' = G_{(K,\{\mathrm{id}_K\})}$ we have $H' \subsetneq H$, $H' \trianglelefteq H$ and $[H : H'] < \omega$, a contradiction. ∎

Let $\mathbf{L}(M)$ be a set of finite groups such that for every $K \in \mathbf{A}(M)$ there is a unique $L \in \mathbf{L}(M)$ such that $L \cong \mathrm{Aut}(K)$.

Proposition 2.10. *Let $L \in \mathbf{L}(M)$ and $H \in \mathcal{SS}(M)$. The following conditions are equivalent:*

(1) *$H = G_{(K)} \in \mathcal{PS}(M)$ and* $\mathrm{Aut}(K) \cong L$;
(2) *there is $H' \in \mathcal{SS}(M)$ such that $H \trianglelefteq H'$, $[H' : H] < \omega$, H' is maximal under these conditions and $H'/H \cong L$.*

Proof. For (1)⇒(2), let $H' = G_{\{K\}}$; then, by Proposition 2.8 and $(\star)$ in the proof of Lemma 2.7, we find that H' is as wanted. For (2)⇒(1), if H and H' are as in (2), then, by Proposition 2.8 and $(\star)$, it must be the case that $H' = G_{\{K\}}$ and $H = G_{(K)}$ for some $K \in \mathbf{A}(M)$ such that $\mathrm{Aut}(K) \cong L$. ∎

DEFINITION 2.11. We define the structure $\mathrm{Ex\,Aut}(M)$, the *expanded group of automorphisms* of M, as follows:

(1) $\mathrm{Ex\,Aut}(M)$ is a two-sorted structure;
(2) the first sort has set of elements $\mathrm{Aut}(M) = G$;
(3) the second sort has set of elements $\mathbf{EA}(M)$;
(4) we identify $\{(K, \{\mathrm{id}_K\}) : K \in \mathbf{A}(M)\}$ with $\mathbf{A}(M)$;
(5) the relations are:
   (a) $P_{\mathbf{A}(M)} = \{K \in \mathbf{A}(M)\}$ (recalling the above identification);
   (b) for $L \in \mathbf{L}(M)$, $P_{L(M)} = \{K \in \mathbf{A}(M) : \mathrm{Aut}(K) \cong L\}$;
   (c) $\leq_{\mathbf{EA}(M)} = \{((K_1, L_1), (K_2, L_2)) : (K_i, L_i) \in \mathbf{EA}(M)\ (i = 1, 2),\ K_1 \leq K_2$ and $L_2 \restriction K_1 \leq L_1\}$;
   (d) $\leq_{\mathbf{A}(M)} = \{(K_1, K_2) : K_i \in \mathbf{A}(M)\ (i = 1, 2)$ and $K_1 \leq K_2\}$;
   (e) $P^{\min}_{\mathbf{A}(M)} = \{K \in \mathbf{A}(M) : \mathrm{acl}(\emptyset) \neq K \in \mathbf{A}(M)$ is minimal in $(\mathbf{A}(M), \subseteq)\}$;
(6) the operations are:
   (f) composition on $\mathrm{Aut}(M)$;
   (g) for $f \in \mathrm{Aut}(M)$ and $K \in \mathbf{A}(M)$, $\mathrm{Op}(f, K) = f(K)$;
   (h) for $f \in \mathrm{Aut}(M)$ and $(K_1, L_1) \in \mathbf{EA}(M)$, $\mathrm{Op}(f, (K_1, L_1)) = (K_2, L_2)$ iff $f(K_1) = K_2$ and $L_2 = \{f \restriction K_1 \pi f^{-1} \restriction K_2 : \pi \in L_1\}$.

DEFINITION 2.12. We say that a set of subsets of a structure N is *invariant* if it is preserved by automorphisms of N. We say that a structure M is *second-order orbit-definable* in a structure N if there is a injective map $\mathbf{j}$ mapping $\emptyset$-definable subsets of M to invariant sets of subsets of N.

THEOREM 2.13. *Let M and N be as in Hypothesis* 2.3, *and let* $G = \mathrm{Aut}(M)$. *Then:*

(1) *The map* $\mathbf{j}_M = \mathbf{j} : (f, (K, L)) \mapsto (\{f\}, G_{(K,L)})$ *witnesses second-order orbit-definability of* $\mathrm{Ex\,Aut}(M)$ *in* $\mathrm{Aut}(M)$.
(2) *Every* $F : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ *has an extension* $\hat{F} : \mathrm{Ex\,Aut}(M) \cong \mathrm{Ex\,Aut}(N)$.

Proof. We prove (1) by establishing statements $(*)_i$ below.

$(*)_1$ The map $(f, (K, L)) \mapsto (\{f\}, G_{(K,L)})$ is one-to-one.

Suppose that $(K_1, L_1) \neq (K_2, L_2) \in \mathbf{EA}(M)$; we want to show that $G_{(K_1,L_1)} \neq G_{(K_2,L_2)}$. Suppose that $K_1 \neq K_2$. By symmetry, we can assume that $K_1 \not\subseteq K_2$. Then there is $f \in G$ such that $f \restriction K_2 = \mathrm{id}_{K_2}$ and $f(K_1) \neq K_1$, since K_2 is algebraically closed. Thus, $f \in G_{(K_2,L_2)} - G_{(K_1,L_1)}$. Suppose now that $K_1 = K = K_2$ and $L_1 \neq L_2$. By symmetry, we can assume that $L_1 \not\subseteq L_2$. Let $g \in L_1 - L_2$; then g extends to an automorphism f of M. Thus, $f \in G_{(K,L_1)} - G_{(K,L_2)}$. Finally, notice that it is not possible that $\{f\} = G_{(K,L)}$, and so we are done.

$(*)_2$ The range $\mathbf{j}(\mathbf{EA}(M)) = \mathcal{SS}(M)$ is mapped onto itself by any $F \in \mathrm{Aut}(G)$.

By Lemma 2.7.

$(*)_3$ The range $\mathbf{j}(P_{\mathbf{A}(M)})) = \mathcal{PS}(M)$ is mapped onto itself by any $F \in \mathrm{Aut}(G)$.

By Proposition 2.9.

$(*)_4$ For $L \in \mathbf{L}(M)$, the range $\mathbf{j}(P_{L(M)}) = \{G_{(K)} : \mathrm{Aut}(K) \cong L\}$ is mapped onto itself by any $F \in \mathrm{Aut}(G)$.

By Proposition 2.10.

$(*)_5$ The range $\mathbf{j}(\leq_{\mathbf{EA}(M)}) = \{(G_{(K_1,L_1)}, G_{(K_2,L_2)}) : G_{(K_1,L_1)} \supseteq G_{(K_2,L_2)}$, $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$, $K_1 \leq K_2$ and $L_2 \restriction K_1 \leq L_1\}$ is preserved by any $F \in \mathrm{Aut}(G)$.

For $(K_i, L_i) \in \mathbf{EA}(M)$ $(i = 1, 2)$ and $F \in \mathrm{Aut}(G)$, we obviously have $\mathbf{j}(K_1, L_1) \supseteq \mathbf{j}(K_2, L_2)$ if and only if $F(\mathbf{j}(K_1, L_1)) \supseteq F(\mathbf{j}(K_2, L_2))$, since F induces an automorphism of $(\mathcal{P}(\mathrm{Aut}(G)), \subseteq)$.

$(*)_6$ The range $\mathbf{j}(\leq_{\mathbf{A}(M)}) = \{(G_{(K_1)}, G_{(K_2)}) : G_{(K_1)} \supseteq G_{(K_2)}$, $K_1, K_2 \in \mathbf{A}(M)$, $K_1 \leq K_2\}$ is preserved by any $F \in \mathrm{Aut}(G)$.

As in $(*)_5$, i.e. because any $F \in \mathrm{Aut}(G)$ induces an automorphism of $(\mathcal{P}(\mathrm{Aut}(G)), \subseteq)$.

$(*)_7$ The range

$$\mathbf{j}(P^{\min}_{\mathbf{A}(M)}) = \{H \in \mathcal{PS}(M) : G \neq H \text{ is maximal in } (\mathcal{PS}(M), \subseteq)\}$$

is preserved by any $F \in \mathrm{Aut}(G)$.

As in $(*)_5$, i.e. because any $F \in \mathrm{Aut}(G)$ induces an automorphism of $(\mathcal{P}(\mathrm{Aut}(G)), \subseteq)$.

$(*)_8$ For any $F \in \mathrm{Aut}(G)$, $F(\{gh\}) = F(\{g\})F(\{h\})$.

Obvious.

$(*)_9$ $\mathbf{j}(\mathrm{Op}(f, K)) = \mathbf{j}(f(K)) = G_{(f(K))} = fG_{(K)}f^{-1}$ and

$$F(\mathbf{j}((\mathrm{Op}(f, K)))) = \mathrm{Op}(F(f), F(\mathbf{j}(K))) \quad \text{for any } F \in \mathrm{Aut}(G).$$

Observe that

$$F(\mathbf{j}((\mathrm{Op}(f,K)))) = F(fG_{(K)}f^{-1}) = F(f)F(G_{(K)})(F(f))^{-1}$$
$$= F(f)(F(\mathbf{j}(K)) = \mathrm{Op}(F(f), F(\mathbf{j}(K))),$$

since by $(*)_3$, $\mathcal{PS}(M)$ is mapped onto itself by any $F \in \mathrm{Aut}(G)$.

$(*)_{10}$ $\mathbf{j}(\mathrm{Op}(f,(K_1,L_1))) = (fG_{(K_1)}f^{-1}, fG_{(K_1,L_1)}f^{-1})$ and

$$F(\mathbf{j}((\mathrm{Op}(f,(K_1,L_1))))) = \mathrm{Op}(F(f), F(\mathbf{j}((K_1,L_1))))$$
$$\text{for any } F \in \mathrm{Aut}(G).$$

Similar to $(*)_9$.

This concludes the proof of (1). Finally, (2) follows directly from (1), in fact for $F : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$ letting $\hat{F} = \mathbf{j}_N^{-1} F \mathbf{j}_M$ we have $\hat{F} : \mathrm{Ex\,Aut}(M) \cong \mathrm{Ex\,Aut}(N)$. ■

3. Reconstruction and outer automorphisms. In this section we prove the theorems stated in the introduction.

Definition 3.1. Let $\mathbf{K}_*$ be the class of countable structures M satisfying:

(1) M has the strong small index property;
(2) for every finite $A \subseteq M$, $\mathrm{acl}_M(A)$ is finite;
(3) for every $a \in M$, $\mathrm{acl}_M(\{a\}) = \{a\}$;

As in the previous section, we let $G = \mathrm{Aut}(M)$. We denote $G_{(\{a\})}$ simply as $G_{(a)}$. The crucial point in asking the additional condition of Definition 3.1(3) is the following:

Proposition 3.2. *Let $M \in \mathbf{K}_*$ be homogeneous, and define*

$$\mathcal{M} = \{G_{(a)} : a \in M\}.$$

Then $\mathbf{j}_M(P^{\min}_{\mathbf{A}(M)}) = \mathcal{M}$ (*recall Definition* 2.11 *and Theorem* 2.13).

Proof. Notice that by (3) of Definition 3.1 we have $P^{\min}_{\mathbf{A}(M)} = \{\{a\} : a \in M\}$, and so directly by the definition of the interpretation $\mathbf{j}_M$ (Theorem 2.13) we obtain $\mathbf{j}_M(P^{\min}_{\mathbf{A}(M)}) = \mathcal{M}$. ■

We will use the suggestive notation $\mathcal{M} = \{G_{(a)} : a \in M\}$ also below.

Definition 3.3. Let M and N be structures and consider $\mathrm{Aut}(M)$ (resp. $\mathrm{Aut}(N)$) as acting naturally on M (resp. N). We say that $(\mathrm{Aut}(M), M)$ and $(\mathrm{Aut}(N), N)$ are *isomorphic as permutation groups* if there exists a bijection $f : M \to N$ such that the map $h \mapsto fhf^{-1}$ is an isomorphism from $\mathrm{Aut}(M)$ onto $\mathrm{Aut}(N)$.

We recall the statement of Theorem 1.1 (without the "moreover" part) and prove it.

THEOREM 1.1. *Let* $\mathbf{K}_*$ *be the class of countable structures* M *satisfying:*

(1) M *has the strong small index property;*
(2) *for every finite* $A \subseteq M$, $\mathrm{acl}_M(A)$ *is finite;*
(3) *for every* $a \in M$, $\mathrm{acl}_M(\{a\}) = \{a\}$.

Then for $M, N \in \mathbf{K}_*$, $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *are isomorphic as abstract groups if and only if* $(\mathrm{Aut}(M), M)$ *and* $(\mathrm{Aut}(N), N)$ *are isomorphic as permutation groups.*

Proof of Theorem 1.1. Let $M, N \in \mathbf{K}_*$, and suppose that $F : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$. Passing to canonical relational structures (see [2, p. 26]), i.e. adding a relation symbol R of arity n for every n-ary $\mathrm{Aut}(M)$-orbit $\Omega \subseteq M^n$, we can assume without loss of generality that M and N are homogeneous. Now, by Theorem 2.13(2), the isomorphism F induces the isomorphism

$$\hat{F} = \mathbf{j}_N^{-1} F \, \mathbf{j}_M : \mathrm{Ex}\,\mathrm{Aut}(M) \cong \mathrm{Ex}\,\mathrm{Aut}(N).$$

In particular, $\hat{F}$ maps $P^{\min}_{\mathbf{A}(M)}$ onto $P^{\min}_{\mathbf{A}(N)}$. Furthermore, by Proposition 3.2,

$$\mathbf{j}_M(P^{\min}_{\mathbf{A}(M)}) = \mathcal{M} \quad \text{and} \quad \mathbf{j}_N(P^{\min}_{\mathbf{A}(N)}) = \mathcal{N}.$$

Hence, $\hat{F}$ induces a bijection $f : M \to N$ such that (recall that $\hat{F} = \mathbf{j}_N^{-1} F \, \mathbf{j}_M$)

$$F(\mathrm{Aut}(M)_{(a)}) = \mathrm{Aut}(N)_{f(a)} \in \mathcal{N}.$$

Let $G : h \mapsto f h f^{-1}$ for $h \in \mathrm{Aut}(M)$. We claim that $G = F$. Let in fact $h \in \mathrm{Aut}(M)$, $a, b \in M$ and suppose that $F(h)(f(a)) = f(b)$. Then

$$\begin{aligned} F(h)(f(a)) = f(b) &\iff F(h)\,\mathrm{Aut}(N)_{(f(a))}(F(h))^{-1} = \mathrm{Aut}(N)_{(f(b))} \\ &\iff h\,\mathrm{Aut}(M)_{(a)}h^{-1} = \mathrm{Aut}(M)_{(b)} \\ &\iff h(a) = b. \end{aligned}$$

So, $fhf^{-1}(f(a)) = fh(a) = f(b)$, as wanted. Hence, $f : M \to N$ witnesses that $(\mathrm{Aut}(M), M)$ and $(\mathrm{Aut}(N), N)$ are isomorphic as permutation groups. Notice that the "moreover" part of the theorem is clear from the proof (since $G = F$). ∎

DEFINITION 3.4. We say that two structures M and N are *bi-definable* if there is a bijection $f : M \to N$ such that for every $A \subseteq M^n$, A is $\emptyset$-definable in M if and only if $f(A)$ is $\emptyset$-definable in N.

FACT 3.5 ([9, Proposition 1.3]). *Let* M *and* N *be countable* $\aleph_0$*-categorical structures. Then the following are equivalent:*

(1) $(\mathrm{Aut}(M), M) \cong (\mathrm{Aut}(N), N)$;
(2) M *and* N *are bi-definable.*

Proof of Corollary 1.2. Let $M, N \in \mathbf{K}_*$, and suppose that $\mathrm{Aut}(M) \cong \mathrm{Aut}(N)$. As before, passing to canonical relational structures, we can assume that M and N are homogeneous. Furthermore, since M and N are $\aleph_0$-categorical, this passage preserves definability. Now, since M and N are $\aleph_0$-categorical and with no algebraicity, the conditions of Theorem 1.1 are met (see Remark 2.4), and so $(\mathrm{Aut}(M), M) \cong (\mathrm{Aut}(N), N)$ as permutation groups. Hence, by Fact 3.5, we are done. Notice that the "moreover" part of the corollary is taken care of by the "moreover" part of Theorem 1.1. ∎

We now pass to the proof of Theorem 1.3.

FACT 3.6 (Frucht's Theorem [3]). *Every finite group is the group of automorphisms of a finite graph.*

Proof of Theorem 1.3. Let Γ be a finite graph on the vertex set $\{0, \dots, n-1\}$ and

$$L_\Gamma = \{P_\ell : \ell < n\} \cup \big\{R_{\ell,k} : \ell < k < n \text{ and } \{\ell, k\} \in E_\Gamma\big\}$$

be such that the P_ℓ are unary predicates and the $R_{\ell,k}$ are binary relations. Let $\mathbf{K}_\Gamma$ be the class of finite L_Γ-models M such that

(1) $(P_\ell^M : \ell < n)$ is a partition of M;
(2) $R_{\ell,k}^M$ is a symmetric irreflexive relation on $P_\ell \times P_k$.

Notice that $\mathbf{K}_\Gamma$ is a free amalgamation class (see [8, Definition 4]). Let M_Γ be the corresponding countable homogeneous structure. By [8, Corollary 2], M_Γ has the strong small index property, and obviously M_Γ is $\aleph_0$-categorical and has no algebraicity. Using Corollary 1.2 it is now easy to see that

$$\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Aut}(M_\Gamma))/\mathrm{Inn}(\mathrm{Aut}(M_\Gamma)) = \mathrm{Out}(\mathrm{Aut}(M_\Gamma)).$$

Thus, by Fact 3.6 we are done. ∎

Acknowledgments. Research partially supported by European Research Council grant 338821. This is no. 1109 on Shelah's publication list. The present paper was written while the first-named author was a post-doc research fellow at the Einstein Institute of Mathematics of the Hebrew University of Jerusalem.

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Gianluca Paolini
Department of Mathematics "Giuseppe Peano"
University of Torino
Via Carlo Alberto 10
10123 Torino, Italy
E-mail: gianluca.paolini@unito.it

Saharon Shelah
Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Edmond J. Safra Campus
Givat Ram, 9190401, Jerusalem
and
Department of Mathematics
Rutgers University
The State University of New Jersey
Hill Center–Busch Campus 110
Frelinghuysen Road
Piscataway, NJ 08854-8019, U.S.A.
E-mail: shelah@math.huji.ac.il

Sh:1109