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Polish topologies for graph products of groups

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Abstract

We give strong necessary conditions on the admissibility of a Polish group topology for an arbitrary graph product of groups $G(\Gamma, G_a)$, and use them to give a characterization modulo a finite set of nodes. As a corollary, we give a complete characterization in case all the factor groups G_a are countable.

1. *Introduction*

Definition 1. Let $\Gamma = (V, E)$ be a graph and $\{G_a : a \in \Gamma\}$ a set of non-trivial groups each presented with its multiplication table presentation and such that for $a \neq b \in \Gamma$ we have $e_{G_a} = e = e_{G_b}$ and $G_a \cap G_b = \{e\}$. We define the *graph product of the groups* $\{G_a : a \in \Gamma\}$ *over* Γ , denoted $G(\Gamma, G_a)$, via the following presentation:

$$\text{generators: } \bigcup_{a \in V} \{g : g \in G_a\},$$

$$\text{relations: } \bigcup_{a \in V} \{\text{the relations for } G_a\} \cup \bigcup_{\{a,b\} \in E} \{gg' = g'g : g \in G_a \text{ and } g' \in G_b\}.$$

This paper is the sixth in a series of paper written by the authors which address the following problems:

Problem 2. Characterize the graph products of groups $G(\Gamma, G_a)$ admitting a Polish group topology (respectively, a non-Archimedean Polish group topology).

Problem 3. Determine which graph products of groups $G(\Gamma, G_a)$ are embeddable into a Polish group (respectively, into a non-Archimedean Polish group).

The beginning of the story is the following question[†]: can a Polish group be an uncountable free group? This was settled in the negative by Shelah in [10], in the case the Polish group was assumed to be non-Archimedean, and in general in [11]. Later this negative result has been extended by the authors to the class of so-called right-angled Artin groups [6]. After the authors wrote [6], they discovered that the impossibility results therein follow from an old important result of Dudley [2]. In fact, Dudley's work proves more strongly that any homomorphism from a Polish group G into a right-angled Artin group H is continuous with respect to the discrete topology on H. The setting of [6] has then been further generalized by the authors in [8] to the class of graph products of groups $G(\Gamma, G_a)$ in which all the factor

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[†]The non-Archimedean version of this question was originally formulated by David Evans.

$$\begin{tikzcd}
{[1121]} \\
{[1115]} \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & {[1117]} \\
{[1112]} \arrow[u, Rightarrow] \\
{[771]} \arrow[u, Rightarrow] \\
{[744]} \arrow[u, Rightarrow]
\end{tikzcd}$$

FIGURE 1. *Logical structure of the references.*

groups G_a are cyclic, or, equivalently, cyclic of order a power of prime or infinity. In this case the situation is substantially more complicated, and the solution of the problem establishes that $G = G(\Gamma, G_a)$ admits a Polish group topology if and only if it admits a non-Archimedean Polish group topology if and only if $G = G_1 \oplus G_2$ with G_1 a countable graph product of cyclic groups and G_2 a direct sum of finitely many continuum-sized vector spaces over a finite field. Concerning Problem 3, in [7] the authors give a complete solution in the case all the G_a are cyclic, proving that $G(\Gamma, G_a)$ is embeddable into a Polish group if and only if it is embeddable into a non-Archimedean Polish group if and only if Γ admits a metric which induces a separable topology in which E_Γ is closed. We hope to conclude this series of studies with an answer to Problem 3 at the same level of generality of this paper. The logical structure of the references just mentioned (plus the present paper) is illustrated in Figure 1, where we use the numbering of Shelah's publication list, and one-direction arrows mean generalization and two-direction arrows mean solutions to Problem 2/Problem 3 at the same level of generality.

In the present study we focus on Problem 2, proving the following theorems:

NOTATION 4. (1) We denote by $\mathbb{Q} = G^*_\infty$ the rational numbers, by $\mathbb{Z}^\infty_p = G^*_p$ the divisible abelian p-group of rank 1 (for p a prime), and by $\mathbb{Z}_{p^k} = G^*_{(p,k)}$ the finite cyclic group of order p^k (for p a prime and $k \geqslant 1$).

(2) We let $S_* = \{(p, k) : p \text{ prime and } k \geqslant 1\} \cup \{\infty\}$ and $S_{**} = S_* \cup \{p : p \text{ prime}\}$;

(3) For $s \in S_{**}$ and λ a cardinal, we let $G^*_{s,\lambda}$ be the direct sum of λ copies of G^*_s.

THEOREM 5. *Let $G = G(\Gamma, G_a)$ and suppose that G admits a Polish group topology. Then for some countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ we have*

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*
(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_*\}$ (cf. Notation 4);*
(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*
(d) *if in addition $A = \emptyset$, then for every $s \in S_*$ we have that $\sum\{\lambda_{a,s} : a \in \Gamma\}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

The following more involved theorems give more information on the possible graph products decompositions of a group G admitting a Polish group topology, and it can be seen as a solution modulo a finite set of nodes to Problem 2.

THEOREM 6. (1) *Let $G = G(\Gamma, G_a)$. If G admits a Polish group topology, then there is $\bar{A} = (A_0, A_5, A_6, A_7, A_8, A_9)$ such that*

(a) *$\bar{A}$ is a partition of Γ;*
(b) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A_0$, a is adjacent to b;*
(c) *A_5 and A_6 are finite;*
(d) *A_0, A_7 and A_8 are countable,*
(e) *for each $a \in A_0$, G_a is countable;*
(f) *if $a \in A_7 \cup A_8$, then $G_a = H_a \oplus \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_{**}\}$, for some countable $H_a \leqslant G_a$;*
(g) *if $a \in A_9$, then $G_a = \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_{**}\}$;*
(h) *for each $s \in S_{**} - S_*$, $\sum\{\lambda_{a,s} : a \in A_7 \cup A_8 \cup A_9\} \leqslant \aleph_0$;*
(i) *for each $s \in S_*$, $\sum\{\lambda_{a,s} : a \in A_7 \cup A_8 \cup A_9\}$ is $\leqslant 2^{\aleph_0}$;*
(j) *for some $1 \leqslant n < \omega$ we have $\sum\{\lambda_{a,(p,k)} : a \in A_7 \cup A_8 \cup A_9\} > \aleph_0 \Rightarrow p^k \mid n$;*
(k) *we can define explicitly the functions A_i from $\{G_a : a \in \Gamma\}$.*

(2) *Furthermore, if we assume CH and we let $\bar{A} = (A_0, A_5, A_6, A_7, A_8, A_9)$ be as above and $A = A_0 \cup A_7 \cup A_8 \cup A_9$, then $G(\Gamma \restriction A, G_a)$ admits a non-Archimedean Polish group topology.*

THEOREM 7. (1) *For given $G = G(\Gamma, G_a)$ the following conditions are equivalent:*

(a) *for some finite $B_1 \subseteq \Gamma$, for every finite B_2 such that $B_1 \subseteq B_2 \subseteq \Gamma$, $G(\Gamma \restriction \Gamma - B_2, G_a)$ admits a Polish group topology;*
(b) *there is $\bar{A}$ as in Theorem 6 and for some finite $B \supseteq A_5 \cup A_6$, for every $s \in S_*$ the cardinal $\lambda^B_s = \sum\{\lambda_{a,s} : a \in (A_7 \cup A_8 \cup A_9) - B\}$ is either $\aleph_0$ or $2^{\aleph_0}$.*

(2) *If $B_0 \subseteq \Gamma$ is finite, $\bar{A}$ is as in Theorem 6 for $G(\Gamma \restriction \Gamma - B_0, G_a)$ and we let $B_1 = B_0 \cup A_5 \cup A_6$ (which is a finite subset of Γ), then the following conditions on $B \subseteq \Gamma - B_1$ are equivalent:*

(a) *$G(\Gamma \restriction B)$ admits a Polish group topology;*
(b) *for every $s \in S_*$ the cardinal $\lambda^B_s = \sum\{\lambda_{a,s} : a \in B\}$ is either $\aleph_0$ or $2^{\aleph_0}$.*

REMARK 8. Let

(a) $s \in S_*$;
(b) $\aleph_0 < \lambda < 2^{\aleph_0}$;
(c) Γ a complete graph on ω_1;
(d) $G_0 = G_{s,2^{\aleph_0}} \oplus G_*$;
(e) G_* an uncountable centerless group admitting a Polish group topology;
(f) $G_\alpha = G_{s,\lambda}$, for $\alpha \in [1, \omega_1)$.

Then $G(\Gamma, G_a)$ admits a Polish group topology, but letting $\bar{A}$ be the partition from Theorem 6 we have that $\sum\{\lambda_{a,s} : a \in A_7 \cup A_8 \cup A_9\} = \lambda < 2^{\aleph_0}$, and so for $A = A_0 \cup A_7 \cup A_8 \cup A_9$, we have that $G(\Gamma \restriction A, G_a)$ does *not* admit a Polish group topology (in fact in this case $A_0 = A_6 = A_7 = A_8 = \emptyset$, $A_5 = \{0\}$ and $A_9 = [1, \omega_1)$, cf. the explicit definition of the functions A_i in the proof of Theorem 6).

From our theorems and their proofs we get the following corollaries.

COROLLARY 9. *Let $G = G(\Gamma, G_a)$ with all the G_a countable. Then G admits a Polish group topology if and only if G admits a non-Archimedean Polish group topology if and only if there exist a countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ such that*

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*
(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_*\}$;*
(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*
(d) *for every $s \in S_*$, $\sum\{\lambda_{a,s} : a \in \Gamma - A\}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

Corollary 10. *Let G be an abelian group which is a direct sum of countable groups, then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exists a countable $H \leqslant G$ and $1 \leqslant n < \omega$ such that*

$$G = H \oplus \bigoplus_{\alpha < \lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p^k | n} \bigoplus_{\alpha < \lambda_{(p,k)}} \mathbb{Z}_{p^k},$$

with λ_∞ and $\lambda_{(p,k)} \leqslant \aleph_0$ or $2^{\aleph_0}$.

Corollary 11 (Slutsky [**12**]). *If G is an uncountable group admitting a Polish group topology, then G cannot be expressed as a non-trivial free product.*

The following problem gets in the way of a complete characterization of the groups $G = G(\Gamma, G_a)$ admitting a Polish group topology in the case no further assumptions are made on the factors G_a. We have

Fact 12. Let $s_1 \neq s_2 \in S_*$ and λ a cardinal (cf. Notation 4).

(1) If $\aleph_0 < \lambda < 2^{\aleph_0}$, then $G_{s_1,\lambda} \oplus G_{s_1,2^{\aleph_0}} \cong G_{s_1,2^{\aleph_0}}$ admits a Polish group topology, but $G_{s_1,\lambda}$ does not admit one such topology.
(2) If $\aleph_0 < \lambda < 2^{\aleph_0}$, $H_1 = G_{s_1,2^{\aleph_0}} \oplus G_{s_2,\lambda}$ and $H_2 = G_{s_1,\lambda} \oplus G_{s_2,2^{\aleph_0}}$, then $H_1 \oplus H_2$ admits a Polish group topology, but neither H_1 nor H_2 admit one such topology.

Hence, a general characterization seems to depend on the failure of CH. Despite this, our impression is that CH would not help. This leads to a series of conjectures on the possible direct summands of a Polish group G:

Conjecture 13 (Polish direct summand conjecture). Let G be a group admitting a Polish group topology.

(1) If G has a direct summand isomorphic to $G^*_{s,\lambda}$, for some $\aleph_0 < \lambda \leqslant 2^{\aleph_0}$ and $s \in S_*$, then it has one of cardinality $2^{\aleph_0}$.
(2) If $G = G_1 \oplus G_2$ and $G_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$, then for some G'_1, G'_2 we have
 (i) $G_1 = G'_1 \oplus G'_2$;
 (ii) G'_1 admits a Polish group topology;
 (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda'_s} : s \in S_*\}$.
(3) If $G = G_1 \oplus G_2$, then for some G'_1, G'_2 we have
 (i) $G_1 = G'_1 \oplus G'_2$;
 (ii) G'_1 admits a Polish group topology;
 (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$.

The paper is organized as follows. In Section 2 we prove some preliminaries results to be used in later sections. In Section 3 we prove Theorem 5. In Section 4 we prove Theorems 6 and 7. In Section 5 we prove Corollaries 9–11.

In a work in preparation we deal with Conjecture 13, and mimic Theorems 5 and 6 in a weaker context, that is, the topology on G need not be Polish.

2. *Preliminaries*

In notation and basic results we follow [1]. Given $A \subseteq \Gamma$ we denote the induced subgraph of Γ on vertex set A as $\Gamma \restriction A$.

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FACT 14. Let $G = G(\Gamma, G_a)$, $A \subseteq \Gamma$ and $G_A = (\Gamma \restriction A, G_a)$. Then there exists a unique homomorphism $\mathbf{p} = \mathbf{p}_A : G \to G_A$ such that $\mathbf{p}(g) = g$ if $g \in G_A$, and $\mathbf{p}(g) = e$ if $g \in G_{\Gamma - A}$.

Proof. For arbitrary $G = G(\Gamma, G_a)$, let $\Omega_{(\Gamma, G_a)}$ be the set of equations from Definition 1 defining $G(\Gamma, G_a)$. Then for the $\Omega_{(\Gamma, G_a)}$ of the statement of the fact we have $\Omega_{(\Gamma, G_a)} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

(a) $\Omega_1 = \Omega_{(\Gamma \restriction A, G_a)}$;
(b) $\Omega_2 = \Omega_{(\Gamma \restriction \Gamma - A, G_a)}$;
(c) $\Omega_3 = \{bc = cb : bE_\Gamma c \text{ and } \{b, c\} \not\subseteq A\}$.

Note now that $\mathbf{p}$ maps each equation in Ω_1 to itself and each equation in $\Omega_2 \cup \Omega_3$ to a trivial equation, and so p is a homomorphism (clearly unique). □

DEFINITION 15. A word in $G(\Gamma, G_a)$ is either e (the empty word) or a formal product $g_1 \cdots g_n$ with each $g_i \in G_{a_i}$ for some $a_i \in \Gamma$. The elements g_i are called the syllables of the word. The length of the word $g_1 \cdots g_n$ is $|g_1 \cdots g_n| = n$, with the length of the empty word defined to be 0. If $g \in G(\Gamma, G_a)$ satisfies $G(\Gamma, G_a) \models g = g_1 \cdots g_n$, then we say that the word $g_1 \cdots g_n$ represents (or spells) g. We will abuse notation and do not distinguish between a word and the element of G that it represents.

DEFINITION 16. The word $g_1 \cdots g_n$ is a normal form if it cannot be changed into a shorter word by applying a sequence of moves of the following type:

(M_1) delete the syllable $g_i = e$;
(M_2) if $g_i, g_{i+1} \in G_a$, replace the two syllables g_i and g_{i+1} by the single syllable $g_i g_{i+1} \in G_a$;
(M_3) if $g_i \in G_a$, $g_{i+1} \in G_b$ and $aE_\Gamma b$, exchange g_i and g_{i+1}.

FACT 17 (Green [4] for (1) and Hermiller and Meier [5] for (2)).

(1) If a word in $G(\Gamma, G_a)$ is a normal form and it represents the identity element, then it is the empty word.
(2) If w_1 and w_2 are two words representing the same element $g \in G(\Gamma, G_a)$, then w_1 and w_2 can be reduced to identical normal forms using moves $(M_1) - (M_3)$.

DEFINITION 18. Let $g \in G(\Gamma, G_a)$. We define

(1) $sp(g) = \{a \in \Gamma : g_i \text{ is a syllable of a normal form for } g \text{ and } g_i \in G_a - \{e\}\}$;
(2) $lg(g) = |w|$, for w a normal form for g;
(3) $F(g) = \{g_1 : g_1 \cdots g_n \text{ is a normal form for } g\}$;
(4) $L(g) = \{g_n : g_1 \cdots g_n \text{ is a normal form for } g\}$;
(5) $\hat{L}(g) = \{g_n^{-1} : g_n \in L(g)\}$.

(Here F and L stand for 'first' and 'last', respectively.)

DEFINITION 19. (1) We say that the word w is *weakly cyclically reduced* when

$$F(w) \cap \hat{L}(w) = \emptyset.$$

(2) We say that the word $g_1 \cdots g_n$ is *cyclically reduced* if no combination of moves $(M_1) - (M_4)$ results in a shorter word, where $(M_1) - (M_3)$ are as in Definition 16 and the move (M_4) is as follows:

(M_4) replace $g_1 \cdots g_n$ by either $g_2 \cdots g_n g_1$ or $g_n g_1 \cdots g_{n-1}$.

(3) We say that $g \in G(\Gamma, G_a)$ is (a,b)-cyclically reduced (or (G_a, G_b)-cyclically reduced) when $g \neq e$, $F(g) \subseteq G_a - \{e\}$ and $L(g) \subseteq G_b - \{e\}$.

Observation 20. Note that if $g \in G(\Gamma, G_a)$ is spelled by a cyclically reduced (respectively, a weakly cyclically reduced) normal form, then any of the normal forms spelling g is cyclically reduced (respectively, weakly cyclically reduced).

Definition 21. Recalling Observation 20, we say that $g \in G(\Gamma, G_a)$ is cyclically reduced (respectively, weakly cyclically reduced) if any of the normal forms spelling g is cyclically reduced (respectively, weakly cyclically reduced).

Remark 22. (1) Note that if g is cyclically reduced, then g cannot be written as a normal form $h_1 h_2 \cdots h_{n-1} h_n$ with $h_1, h_n \in G_a$ for some $a \in \Gamma$, since otherwise $lg(h_2 \cdots h_{n-1} h_n h_1) < lg(h_1 h_2 \cdots h_{n-1} h_n)$.

(2) Note that if g is weakly cyclically reduced, then g cannot be written as a normal form $h_1 h_2 \cdots h_{n-1} h_1^{-1}$, since otherwise $F(w) \cap \hat{L}(w) \neq \emptyset$.

(3) Hence, if g is cyclically reduced and spelled by the normal form $h_1 h_2 \cdots h_{n-1} h_n$, then, unless $n = 1$ and $h_1 = h_1^{-1}$, we have that g is weakly cyclically reduced.

Proposition 23. *Let $a \neq b \in \Gamma$, $\{a,b\} \notin E_\Gamma$ and $g_1 u g_2 \in G(\Gamma, G_a)$. Assume that $F(g_1), F(u), F(g_2) \subseteq G_a - \{e\}$, $L(g_1), L(u), L(g_2) \subseteq G_b - \{e\}$ and $p \geqslant 2$, then*

(a) *$g_1 u^p g_2$ is (a,b)-cyclically reduced;*
(b) *if g_1, u, g_2 are written as normal forms, then $g_1 \underbrace{u \cdots u}_{p} g_2$ is a normal form;*
(c) *$lg(g_1 u^p g_2) = lg(g_1) + p lg(u) + lg(g_2) > lg(g_1 u g_2) > lg(u)$.*

Proof. Clear. □

Convention 24. Given a sequence of words $w_1, \ldots, w_k$ with some of them possibly empty, we say that the word $w_1 \cdots w_k$ is a normal form (respectively, a (weakly) cyclically reduced normal form) if after deleting the empty words the resulting word is a normal form (respectively, a (weakly) cyclically reduced normal form).

Fact 25 [1, Corollary 24]. Any element $g \in G(\Gamma, G_a)$ can be written in the form $w_1 w_2 w_3 w_2' w_1^{-1}$, where

(1) $w_1 w_2 w_3 w_2' w_1^{-1}$ is a normal form;
(2) the element $w_3 w_2' w_2$ is cyclically reduced (cf. Observation 20 and Definition 21);
(3) $sp(w_2) = sp(w_2')$;
(4) if $w_2 \neq e$, then $\Gamma \restriction sp(w_2)$ is a complete graph;
(5) $F(w_2) \cap \hat{L}(w_2') = \emptyset$.

(Note that by (5) if $w_2 \neq e$ then $w_2 w_3 w_2'$ is weakly cyclically reduced).

Definition 26. Let $g \in G(\Gamma, G_a)$ and $g = w_1 w_2 w_3 w_2' w_1^{-1}$ as in Fact 25. We let

(1) $csp(g) = sp(w_2 w_3 w_2')$;
(2) $clg(g) = lg(w_2 w_3 w_2')$.

(Inspection of the proof of Fact 25 from [1] shows that this is well defined).

Proposition 27. *Let $G = G(\Gamma, G_a)$, with $\Gamma = \{a_1, a_2, b_1, b_2\}$ and $|\{a_1, a_2, b_1, b_2\}| = 4$. Suppose also that, for $i = 1, 2$, we have that a_i and b_i are not adjacent. Then*

(1) *if $g \in G$ has finite order, then $csp(g)$ is a complete graph (and so $|csp(g)| \leqslant 2$);*
(2) *let $q < p$ be primes, $g_i \in G_{a_i} - \{e\}$ and $h_i \in G_{b_i} - \{e\}$ $(i = 1, 2)$, and $g = (g_1 g_2 h_1 h_2)^p$. Then for every $d \in G$ such that $csp(g)$ is a complete graph (and so $|csp(g)| \leqslant 2$) we have that $dg \in G$ does not have a qth root.*

Proof. This can be proved using the canonical representation of $d \in G$ that we get from Fact 25, and analyzing the possible cancellations occurring in the word dg, in the style, for example, of the proof of Proposition 29. The details are omitted. □

Notation 28. We denote the free product of two group H_1 and H_2 as $H_1 * H_2$. Note that $H_1 * H_2$ is $G(\Gamma, G_a)$ for Γ a discrete graph (that is, no edges) on two vertices a and b, and $G_a = H_1$ and $G_b = H_2$. Thus, when we use $lg(g)$, $sp(g)$, etc., for $g \in H_1 * H_2$, we mean with respect to the corresponding $G(\Gamma, G_a)$.

Proposition 29. *Let $k_* \geqslant 2$ be even and $p >> k_*$ (for example, as an overkill, we might let $p = 36k_* + 100$). Then (A) implies (B), where*

(A) (a) *$H = H_1 * H_2$;*
 (b) *$g_* \in H_1 - \{e\}$;*
 (c) *$h_{(\ell,i)} \in H_2 - \{e\}$, for $\ell < k_*$ and $i = 1, 2$;*
 (d) *$(h_{(\ell,i)} : \ell < k_*$ and $i = 1, 2)$ is with no repetitions;*
 (e) *$(h_{(0,2)})^{-1} \neq h_{(k_*-1,1)} \neq (h_{(1,2)})^{-1}$;*
 (f) *for $i \in \{1, 2\}$ and $0 < \ell < k_*$ we have $h_{(0,2)} \neq (h_{(\ell,2)})^{-1}$;*
 (g) *for $i \in \{1, 2\}$ and $0 \leqslant \ell < k_* - 1$ we have $h_{(k_*-1,1)} \neq (h_{(\ell,1)})^{-1}$;*
 (h) *$g_i = h_{(0,i)} g_* h_{(1,i)} g_*^{-1} \cdots h_{(k_*-2,i)} g_* h_{(k_*-1,i)} g_*^{-1}$, for $i = 1, 2$;*

(B) *for every $u \in H$, at least one of the following holds:*
 (a) *$lg(g_1 u^p g_2) > lg(u)$;*
 (b) *$clg(u) \leqslant 1$, $lg(g_1 u^p g_2) \geqslant 2k_*$ and $g_1 u^p g_2$ is (H_2, H_1)-cyclically reduced;*
 (c) *$clg(u) \leqslant 1$, $lg(g_1 u^p g_2) \geqslant 2k_*$ and $lg(g_1 u^p g_2) = lg(u)$.*

Proof. Let $u \in H$, write $u = w_1 w_2 w_3 w_2' w_1^{-1}$ as in Fact 25 and set $w_2 w_3 w_2' = w_0$. Clearly the element $g_1 u^p g_2$ is spelled by the following word (thinking of g_i as a word (cf. its definition)):

$$w_* = g_1 w_1 \underbrace{w_0 \cdots w_0}_{p} w_1^{-1} g_2.$$

Case 1. $lg(w_0) \geqslant 2$ and $lg(w_0)$ is even.

Note that in this case the word

$$w_1 \underbrace{w_0 \cdots w_0}_{p} w_1^{-1}$$

is a normal form for u^p, and so the only places where cancellations (that is, consecutive applications of moves (M_1) and (M_2) as in Definition 16) may occur in w_* are at the junction of g_1 and w_1 and at the junction of w_1^{-1} and g_2. Since by assumption $lg(g_i) = 2k_*$ $(i = 1, 2)$ and $p >> 4k_*$, we get that $lg(g_1 u^p g_2) > lg(u)$. Thus, clause $B(a)$ is true.

Case 2. $lg(w_0) \geqslant 3$ and $lg(w_0)$ is odd.

In this case, for some $\ell \in \{1, 2\}$, $F(w_0), L(w_0) \in H_{3-\ell} - \{e\}$, $w_2, w'_2 \in H_\ell - \{e\}$ and $w'_2 w_2 \neq e$, and so, letting w'_0 stand for a normal form for $w_3 w'_2 w_2$ (that is, $w'_0 = w_3(w'_2 w_2)$), we have that $lg(w'_0) \geqslant 2$. Thus, the word

$$w_1 w_2 \underbrace{w'_0 \cdots w'_0}_{p-1} w_3 w'_2 w_1^{-1}$$

is a normal form for u^p. Hence, arguing as in Case 1, we see that $lg(g_1 u^p g_2) > lg(u)$. Thus, clause $B(a)$ is true.

Case 3. $lg(w_0) = 1$, $(w_0)^p \neq e$, and $w_1 = e = w_1^{-1}$.
This case is clear by assumption (A)(e). Clearly in this case clause $B(a)$ is true.

Case 4. $lg(w_0) = 1$, $(w_0)^p \neq e$ and $w_1 \neq e \neq w_1^{-1}$.
If this is the case, then $(w_0)^p = (w_3)^p = g^p$ for some $g \in H_1 \cup H_2$ (and $w_2 = e = w'_2$). Note crucially that in $w_* = g_1 w_1 (w_0)^p w_1^{-1} g_2$ if a cancellation occurs at the junction of g_1 and w_1 (respectively, of w_1^{-1} and g_2), then it cannot occur at the junction of w_1^{-1} and g_2 (respectively, of g_1 and w_1), since for $i = 1, 2$ we have $F(g_i) \subseteq H_2$ and $L(g_i) \subseteq H_1$, whereas $e \neq F(w_1) = \hat{L}(w_1^{-1}) \neq e$.

Case 4.1. No cancellation occurs at the junction of g_1 and w_1.
Let m_* be the number of cancellations occurring at the junction of w_1^{-1} and g_2.

Case 4.1.1. $2lg(w_1) + 1 > 2k_*$.
Clearly $m_* \leqslant 2k_*$ and so we have

$$\begin{aligned} lg(g_1 u^p g_2) &\geqslant 2k_* + 2lg(w_1) + 1 - m_* \\ &\geqslant 2lg(w_1) + 1 \\ &= lg(u), \end{aligned}$$

and so either clause (B)(a) or B(c) is true.

Case 4.1.2. $2lg(w_1) + 1 \leqslant 2k_*$.
First of all, necessarily $2lg(w_1) + 1 < 2k_*$. Furthermore, note crucially that $m_* < 2lg(w_1) + 1$, because otherwise we would have

$$h_{0,2} = \hat{L}(w_1^{-1}) \text{ and } h_{lg(w_1),2} = (F(w_1))^{-1},$$

contradicting assumption (A)(f).
Hence, g_1 is an initial segment of a normal form spelling $g_1 u^p g_2$ and so we have:

$$\begin{aligned} lg(u) &= 2lg(w_1) + 1 \\ &< 2k_* \\ &\leqslant lg(g_1 u^p g_2), \end{aligned}$$

and so clause $B(a)$ is true.

Case 4.2. No cancellation occurs at the junction of g_2 and w_1^{-1}.
Let m_* be the number of cancellations occurring at the junction of w_1^{-1} and g_2.

Case 4.2.1. $2lg(w_1) + 1 > 2k_*$.
As in Case 4.1.1.

Case 4.2.2. $2lg(w_1) + 1 \leqslant 2k_*$.
Similar to Case 4.1.2, using assumption (A)(g).

Case 5. $lg(w_0) = 0$, or $lg(w_0) = 1$ and $(w_0)^p = e$.

If either of these cases happen, then $w_* = g_1 g_2$ is a normal form of length $4k_*$, and so clearly $lg(g_1 u^p g_2) \geqslant 2k_*$ and $g_1 u^p g_2$ is (H_2, H_1)-cyclically reduced. Thus, clause $B(b)$ is true. □

PROPOSITION 30. *The set of equations Ω has no solution in H, when*

(a) $k(n) \geqslant 2$ *is even and* $p(n) >> k(n)$, *for* $n < \omega$;
(b) $n < m < \omega$ *implies* $k(n) < k(m)$;
(c) $H = H_1 * H_2$;
(d) *for every* $n < \omega$ *we have*
  (d.1) $g_{(n,*)} \in H_1 - \{e\}$;
  (d.2) $h_{(n,\ell,i)} \in H_2 - \{e\}$, *for* $\ell < k(n)$ *and* $i = 1, 2$;
  (d.3) $(h_{(n,\ell,i)} : \ell < k(n)$ *and* $i = 1, 2)$ *is with no repetitions;*
  (d.4) $(h_{(n,0,2)})^{-1} \neq h_{(n,k(n)-1,1)} \neq (h_{(n,1,2)})^{-1}$;
  (d.5) *for* $i \in \{1, 2\}$ *and* $0 < \ell < k(n)$ *we have* $h_{(n,0,2)} \neq (h_{(n,\ell,2)})^{-1}$;
  (d.6) *for* $i \in \{1, 2\}$ *and* $0 \leqslant \ell < k(n) - 1$ *we have* $h_{(n,k(n)-1,1)} \neq (h_{(n,\ell,1)})^{-1}$;
  (d.7) *for* $i = 1, 2$ *we have*

$$g_{(n,i)} = h_{(n,0,i)} g_{(n,*)} h_{(n,1,i)} g_{(n,*)}^{-1} \cdots h_{(n,k(n)-2,i)} g_{(n,*)} h_{(n,k(n)-1,i)} g_{(n,*)}^{-1};$$

(e) $\Omega = \{x_n = g_{(n,1)}(x_{n+1})^{p(n)} g_{(n,2)} : n < \omega\}$.

Proof. Let $(t_n : n < \omega)$ witness the solvability of Ω in H. Note that

$$\exists n_0^* \text{ such that } n \geqslant n_0^* \text{ implies } t_n \text{ is not } (H_2, H_1)\text{-cyclically reduced.} \tag{1}$$

(Why? Let $n_0^* = lg(t_0) + 1$, and, toward contradiction, assume that $n \geqslant n_0^*$ and t_n is (H_2, H_1)-cyclically reduced. By downward induction on $\ell \leqslant n$ we can prove that t_ℓ is (H_2, H_1)-cyclically reduced and $lg(t_\ell) \geqslant lg(t_n) + n - \ell$. For $\ell = n$, this is clear. For $\ell < n$, by the inductive hypothesis we have that $t_{\ell+1}$ is (H_2, H_1)-cyclically reduced and $lg(t_{\ell+1}) \geqslant lg(t_n) + n - (\ell + 1)$. Now, by Proposition 23 applied to $(g_1, u, g_2) = (g_{(\ell,1)}, t_{\ell+1}, g_{(\ell,2)})$, we have that $g_{\ell,1}(t_{\ell+1})^{p(\ell)} g_{\ell,2} = t_\ell$ is (H_2, H_1)-cyclically reduced and $lg(t_\ell) > lg(t_{\ell+1})$, from which it follows that $lg(t_\ell) \geqslant lg(t_n) + n - \ell$, as wanted. Hence, letting $\ell = 0$ we have that $lg(t_0) \geqslant lg(t_n) + n \geqslant n_0^* > lg(t_0)$, a contradiction.)

Thus, we have

$$\text{for } n \geqslant n_0^* \text{ we have } lg(t_n) > lg(t_{n+1}) \text{ or } lg(t_n) = lg(t_{n+1}) \wedge lg(t_n) \geqslant 2k(n). \tag{2}$$

(Why? By Proposition 29(B) applied to $(g_1, u, g_2) = (g_{n,1}, t_{n+1}, g_{n,2})$, as case (B)(b) of Proposition 29 is excluded by (1).) Now, by (2), we get

$$(lg(t_n) : n \geqslant n_*) \text{ is non-increasing.} \tag{3}$$

Thus, by (3), we get

$$(lg(t_n) : n \geqslant n_*) \text{ is eventually constant.} \tag{4}$$

Hence, by the second half of (2) and (4), we contradict assumption (b). □

We will also need the following results of abelian group theory. We follow [**3**].

DEFINITION 31. Let G be an abelian group.

(1) For $1 \leqslant n < \omega$, we denote by $Tor_n(G)$ the set of $g \in G$ such that $ng = 0$ (in [**3**] this is denoted as $G[n]$, cf. p. 4).
(2) For $1 \leqslant n < \omega$, we say that G is n-bounded if $Tor_n(G) = G$ (cf. [**3**, p. 25]).
(3) We say that G is bounded if it is n-bounded for some $1 \leqslant n < \omega$ (cf. [**3**, p. 25]).

(4) We say that G is divisible if for every $g \in H$ and $n < \omega$ there exists $h \in G$ such that $nh = g$ (cf. [3, p. 98]).
(5) We say that G is reduced if it has no divisible subgroups other than 0 (cf. [3, p. 200]).

FACT 32 [3, Theorem 23.1]. Let G be a divisible abelian group and $P = \{p : p \text{ prime}\}$. Then

$$G \cong \bigoplus_{\alpha<\lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p\in P} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p^\infty.$$

FACT 33 [3, Theorem 17.2]. Let G be a bounded abelian group. Then G is a direct sum of cyclic groups.

FACT 34. Let G be an abelian group and $1 \leqslant n < \omega$. Then $Tor_n(G)$ is the direct sum of finite cyclic groups or order divisible by n.

Proof. This is an immediate consequence of Fact 33. □

DEFINITION 35. Let G be an abelian group and $P = \{p : p \text{ prime}\}$.

(1) For $1 \leqslant n < \omega$, we say that G is n-bounded-divisible when

$$G \cong \bigoplus_{\alpha<\lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p\in P} \bigoplus_{\alpha<\lambda_p} \mathbb{Z}_p^\infty \oplus \bigoplus_{p^m|n} \bigoplus_{\alpha<\lambda_{p,m}} \mathbb{Z}_{p^m}.$$

(2) We say that G is bounded-divisible if it is n-bounded-divisible for some $1 \leqslant n < \omega$.

FACT 36 [3, p. 200]. Let G be an abelian group. Then for some $H \leqslant G$ (unique up to isomorphism) we have

(1) G has a unique maximal divisible subgroup $Div(G)$;
(2) $G = Div(G) \oplus H$;
(3) H is reduced.

FACT 37. Let G be an abelian group and $1 \leqslant n < \omega$. If for every $g \in G$ there exists a divisible $K \leqslant G$ such that $g \in K + Tor_n(G)$, then G is n-bounded-divisible.

Proof. This is an immediate consequence of Facts 32, 34 and 36. □

FACT 38. Let G be a group, $1 \leqslant n < \omega$ and (for ease of notation) $G' = Cent(G)$. Suppose that both G/G' and $G'/(Div(G') + Tor_n(G'))$ are countable. Then $G = K \oplus M$, with K countable and M bounded-divisible.

Proof. By Fact 36, $G' = Div(G') \oplus H$, with H reduced. Furthermore, by assumption, $G'/(Div(G') + Tor_n(G'))$ is countable. So we can find a sequence $(g_i : i < \theta \leqslant \aleph_0)$ of members of G' such that G' is the union of $(g_i + (Tor_n(G) + Div(G')) : i < \theta)$. Thus, since also G/G' is countable, we can find $K \leqslant G$ such that

(a) K is countable;
(b) $G = \bigcup\{G'h : h \in K\}$;
(c) K includes $\{g_i : i < \theta\}$.

Now, by Facts 32 and 34, $L := Div(G') + Tor_n(G')$ can be represented as $\bigoplus_{i<\lambda} G_i$ with each $G_i \cong \mathbb{Q}$ or $G_i \cong \mathbb{Z}_{p^\ell}$ (with $p^\ell \mid n$, for some $1 \leqslant n < \omega$). Without loss of generality, for some countable $\mathcal{U} \subseteq \lambda$ we have $K \cap L = \bigoplus\{G_i : i \in \mathcal{U}\}$. Let $M = \bigoplus\{G_i : i \in \lambda - \mathcal{U}\}$, and note that

(1) K and M commute (since $M \subseteq G'$);
(2) $K + M = G$;
(3) $K \cap M = \{e\}$.

Hence, $G = K \oplus M$ and so we are done. $\square$

Finally, we will make a crucial use of the following special case of [**11**, 3.1].

FACT 39 [**11**]. Let $G = (G, \mathfrak{d})$ be a Polish group and $\bar{g} = (\bar{g}_n : n < \omega)$, with $\bar{g}_n \in G^{\ell(n)}$ and $\ell(n) < \omega$.

(1) For every non-decreasing $f \in \omega^\omega$ with $f(n) \geqslant 1$ and $(\varepsilon_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ there is a sequence $(\zeta_n)_{n<\omega}$ (which we call an f-continuity sequence for $(G, \mathfrak{d}, \bar{g})$, or simply an f-continuity sequence) satisfying the following conditions:
- (A) for every $n < \omega$:
  - (a) $\zeta_n \in (0,1)_{\mathbb{R}}$ and $\zeta_n < \varepsilon_n$;
  - (b) $\zeta_{n+1} < \zeta_n/2$;
- (B) for every $n < \omega$, group term $\sigma(x_0, \ldots, x_{m-1}, \bar{y}_n)$ and $(h_{(\ell,1)})_{\ell<m}, (h_{(\ell,2)})_{\ell<m} \in G^m$, the $\mathfrak{d}$-distance from $\sigma(h_{(0,1)}, \ldots, h_{(m-1,1)}, \bar{g}_n)$ to $\sigma(h_{(0,2)}, \ldots, h_{(m-1,2)}, \bar{g}_n)$ is $< \zeta_n$, when
  - (a) $m \leqslant n + 1$;
  - (c) $\sigma(x_0, \ldots, x_{m-1}, \bar{y}_n)$ has length $\leqslant f(n) + 1$;
  - (c) $h_{(\ell,1)}, h_{(\ell,2)} \in Ball(e; \zeta_{n+1})$;
  - (d) $G \models \sigma(e, \ldots, e, \bar{g}_n) = e$.

(2) The set of equations $\Gamma = \{x_n = d_{(n,1)}(x_{n+1})^{k(n)} d_{(n,2)} : n < \omega\}$ is solvable in G when for every $n < \omega$:
- (a) $f \in \omega^\omega$ is non-decreasing and $f(n) \geqslant 1$;
- (b) $1 \leqslant k(n) < f(n)$;
- (c) $(\zeta_n)_{n<\omega}$ is an f-continuity sequence;
- (d) $\mathfrak{d}(d_{(n,\ell)}, e) < \zeta_{n+1}$, for $\ell = 1, 2$.

CONVENTION 40. If we apply Fact 39(1) without mentioning $\bar{g}$ it means that we apply Fact 39(1) for $\bar{g}_n = \emptyset$, for every $n < \omega$.

We shall use the following observation freely throughout the paper.

OBSERVATION 41. Suppose that $(G, \mathfrak{d})$ is Polish, $A \subseteq G^k$ is uncountable, $1 \leqslant k < \omega$ and $\zeta > 0$. Then for some $(g_{1,\ell} : \ell < k) = \bar{g}_1 \neq \bar{g}_2 = (g_{2,\ell} : \ell < k) \in A$ we have $\mathfrak{d}((g_{1,\ell})^{-1} g_{2,\ell}, e) < \zeta$, for every $\ell < k$.

Proof. We give a proof for $k = 1$, the general case is similar. First of all, note that we can find $g_1 \in A$ such that g_1 is an accumulation point of A, because otherwise we contradict the separability of $(G, \mathfrak{d})$. Furthermore, the function $(x, y) \mapsto x^{-1}y$ is continuous and so for every $(x_1, y_1) \in G^2$ and $\zeta > 0$ there is $\delta > 0$ such that, for every $(x_2, y_2) \in G^2$, if $\mathfrak{d}(x_1, x_2), \mathfrak{d}(y_1, y_2) < \delta$, then $\mathfrak{d}((x_1)^{-1}y_1, (x_2)^{-1}y_2) < \zeta$. Let now $g_2 \in Ball(g_1; \delta) \cap A - \{g_1\}$, then $\mathfrak{d}((g_1)^{-1}g_2, (g_1)^{-1}g_1) = \mathfrak{d}((g_1)^{-1}g_2, e) < \zeta$. $\square$

3. *First venue*

In this section we prove Theorem 5. We will prove a series of lemmas from which the theorem follows.

Lemma 42. *Let Γ be such that either of the following cases happens:*

(i) *in Γ there are $\{a_i : i < \omega_1\}$ and $\{b_i : i < \omega_1\}$ such that if $i < j < \omega_1$, then $a_i \neq a_j$, $b_i \neq b_j$, $|\{a_i, a_j, b_i, b_j\}| = 4$ and a_i is not adjacent to b_i;*
(ii) *in Γ there are a_* and $\{b_i : i < \omega_1\}$ such that if $i < j < \omega_1$, then $|\{a_*, b_i, b_j\}| = 3$ and a_* is not adjacent to b_i.*

Then $G(\Gamma, G_a)$ does not admit a Polish group topology.

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish.

Case 1. There are $\{(a_i, b_i) : i < \omega_1\}$ as in (i) above.

Let $(\zeta_n)_{n<\omega} \in (0,1)^{\omega}_{\mathbb{R}}$ be as in Fact 39 for $f \in \omega^{\omega}$, for example, constantly 30 (recall Convention 40). Using Observation 41, by induction on $n < \omega$, choose $(i(n), j(n))$, $(g_{i(n)}, g_{j(n)})$ and $(h_{i(n)}, h_{j(n)})$ such that

(a) if $m < n$, then $j_m < i_n$;
(b) $i_n < j_n < \omega_1$;
(c) $g_{i(n)} \in G_{a_{i(n)}} - \{e\}$ and $g_{j(n)} \in G_{a_{j(n)}} - \{e\}$;
(d) $h_{i(n)} \in G_{b_{i(n)}} - \{e\}$ and $h_{j(n)} \in G_{b_{j(n)}} - \{e\}$;
(e) $\mathfrak{d}((g_{i(n)})^{-1} g_{j(n)}, e), \mathfrak{d}((h_{i(n)})^{-1} h_{j(n)}, e) < \zeta_{n+4}$.

Consider now the following set of equations:

$$\Omega = \{x_n = (x_{n+1})^2 (t_n)^{-1} : n < \omega\},$$

where $t_n = ((g_{i(n)})^{-1} g_{j(n)} (h_{i(n)})^{-1} h_{j(n)})^3$. By (e) above and Fact 39(1)(B) we have $\mathfrak{d}(t_n, e) < \zeta_{n+1}$, and so by Fact 39(2) the set Ω is solvable in G. Let $(d'_n)_{n<\omega}$ witness this. Now $sp(d'_0)$ is finite, and so we can find $0 < n < \omega$ such that $sp(d'_0) \cap \{a_{i(n)}, a_{j(n)}, b_{i(n)}, b_{j(n)}\} = \emptyset$. Let now $A = \{a_{i(n)}, a_{j(n)}, b_{i(n)}, b_{j(n)}\}$, $\mathbf{p} = \mathbf{p}_A$ the corresponding homomorphism from Fact 14 and let $\mathbf{p}(d'_m) = d_m$. Then we have

(A) $d_0 = e$;
(B) $m < n \Rightarrow d_m = (d_{m+1})^2 \mathbf{p}(t_m) = (d_{m+1})^2$.

Thus, $(d_n)^{2^n} = e$. Hence, by Proposition 27(1), we have that $csp(d_n)$ is a complete graph (and so $|csp(d_n))| \leqslant 2$). Furthermore, we have

$$(d_{n+1})^2 = d_n t_n.$$

Hence, we reach a contradiction with Proposition 27(2).

Case 2. There is a_* and $\{b_i : i < \omega_1\}$ as in (ii) above.

Let $k(n)$ and $p(n)$ be as in Proposition 30, $g_* \in G_{a_*} - \{e\}$ and let $(\zeta_n)_{n<\omega} \in (0,1)^{\omega}_{\mathbb{R}}$ be as in Fact 39 for $f \in \omega^{\omega}$ such that $f(n) = p(n) + 4k(n) + 4$ and $\bar{g}_n = (g_*)$ (and so in particular $\ell(n) = 1$). Using Observation 41, by induction on $n < \omega$, choose $(i(n), j(n)) = (i_n, j_n)$ and $(h_{i(n)}, h_{j(n)})$ such that

(a) if $m < n$, then $j_m < i_n$;
(b) $i_n < j_n < \omega_1$;
(c) $h_{i(n)} \in G_{b_{i(n)}} - \{e\}$ and $h_{j(n)} \in G_{b_{j(n)}} - \{e\}$;
(d) $\mathfrak{d}((h_{i(n)})^{-1} h_{j(n)}, e) < \zeta_{n+2k(n)+2}$.

Let $g_{(n,*)} = g_*$, $h_n = (h_{i(n)})^{-1} h_{j(n)}$, $h_{(n,\ell,1)} = h_{n!+2\ell}$ and $h_{(n,\ell,2)} = h_{n!+(2\ell+1)}$, for $\ell < k(n)$. Let then $g_{(n,i)}$ and Ω be as in Proposition 30. By (e) above and Fact 39(1)(B) we have $\mathfrak{d}(g_{(n,i)}, e) < \zeta_{n+1}$. Thus, by Fact 39(2) the set Ω is solvable in G. Let now $A = \{a_*\} \cup \{b_{i(n)}, b_{j(n)} : n < \omega\}$ and $\mathbf{p} = \mathbf{p}_A$ be the corresponding homomorphism from Fact 14. Then projecting onto $\mathbf{p}(G) =$

$G(\Gamma \restriction A, G_a)$ and using Proposition 30 we get a contradiction, since, for every $n < \omega$, a_* is adjacent to neither $b_{i(n)}$ nor $b_{j(n)}$, and so $G(\Gamma \restriction A, G_a) = G_{a_*} * G(\Gamma \restriction A - \{a_*\}, G_a)$. $\square$

As a corollary of the previous lemma we get

Corollary 43. *Let $G = G(\Gamma, G_a)$. If G admits a Polish group topology, then there exists a countable $A_1 \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A_1$, a is adjacent to b.*

Lemma 44. *If the set*

$$A_2 = \{a \in \Gamma : G_a \text{ is not abelian}\}$$

is uncountable, then $G(\Gamma, G_a)$ does not admit a Polish group topology.

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish, and let $A_1 \subseteq \Gamma$ be as in Corollary 43 (recall that A_1 is countable). By induction on n, choose (a_n, g_n, t_n), (b_n, d_n, z_n), $(h_n, h_{<n})$ and $(\zeta_n^\ell : \ell = 1, \ldots, 4)$ such that

(a) $a_n \neq b_n \in A_2 - (A_1 \cup \{a_\ell, b_\ell : \ell < n\})$;
(b) $g_n, t_n \in G_{a_n}$ and they do not commute;
(c) $d_n, z_n \in G_{b_n}$ and they do not commute;
(d) $\mathfrak{d}((g_n)^{-1} d_n), e), \mathfrak{d}((t_n)^{-1} z_n, e) < \zeta_n^4$;
(e) $h_n = (g_n)^{-1} d_n$ and $h_{<n} = h_0 \cdots h_{n-1}$;
(f) $\zeta_n^\ell \in (0,1)_{\mathbb{R}}$, $\frac{1}{4}\zeta_n^\ell \geqslant \zeta_n^{\ell+1}$ and $\frac{1}{4}\zeta_n^4 \geqslant \zeta_{n+1}^1$;
(g) if $n = m+1$ and $g \in Ball(h_{<n}; \zeta_n^2)$, then g and $(t_m)^{-1} z_m$ do not commute;
(h) if $n = m+1$, and $g \in Ball(e; \zeta_n^3)$, then $\mathfrak{d}(h_{<n} g, h_{<n}) \leqslant \zeta_m^2$.

(How? For $n = 0$, let $\zeta_n^\ell = \frac{1}{4^{\ell+1}}$, and choose (a_0, g_0, t_0), (b_0, d_0, z_0), $(h_0, h_{<0})$ as needed (where we let $h_{<0} = e$). So assume $n = m+1$, and let $\zeta_n^1 = \frac{1}{4}\zeta_m^4$. Now, (g_m, d_m) are well defined, and so $h_{<n} = h_{<m} h_m$ is well defined. Furthermore, $h_{<n}$ does not commute with $(t_m)^{-1} z_m$, that is, $h_{<n} (t_m)^{-1} z_m (h_{<n})^{-1} (z_m)^{-1} t_m \neq e$. Thus, there is $\zeta_n^2 \in (0, \frac{1}{4}\zeta_n^1)_{\mathbb{R}}$ such that

$$g \in Ball(h_{<n}, \zeta_n^2) \Rightarrow g(t_m)^{-1} z_m g^{-1} (z_m)^{-1} t_m \neq e.$$

Also, let $\zeta_n^3 \in (0, \frac{1}{4}\zeta_n^2)_{\mathbb{R}}$ be as in Fact 39(1)(B) with $(\zeta_n^2, \zeta_n^3, 4)$ here standing for $(\zeta_n, \zeta_{n+1}, f(n))$ there. Similarly, choose $\zeta_n^4 \in (0, \frac{1}{4}\zeta_n^3)_{\mathbb{R}}$. Finally, we show how to choose (a_n, g_n, t_n) and (b_n, d_n, z_n). For every $a \in A_2 - (A_1 \cup \{a_\ell, b_\ell : \ell < n\})$ we have that G_a is not abelian, and so we can find $g_n^a, p_n^a \in G_a$ which do not commute. Since A_2 is uncountable whereas $A_1 \cup \{a_\ell, b_\ell : \ell < n\}$ is countable and $(G, \mathfrak{d})$ is separable, we can find uncountable $A_n' \subseteq A_2 - (A_1 \cup \{a_\ell, b_\ell : \ell < n\})$ and g_n^* such that $\{g_n^a : a \in A_n'\} \subseteq Ball(g_n^*, \zeta_n^4/2)$. Similarly, we can find uncountable $A_n'' \subseteq A_n'$ and p_n^* such that $\{p_n^a : a \in A_n'\} \subseteq Ball(p_n^*, \zeta_n^4/2)$. Chose $a_n \neq b_n \in A_n''$ and let

$$g_n = g_n^a,\ t_n = p_n^a,\ g_n^b = d_n \text{ and } p_n^b = z_n.$$

Then (a_n, g_n, t_n), (b_n, d_n, z_n), $(h_n, h_{<n})$ and $(\zeta_n^\ell : \ell = 1, \ldots, 4)$ are as wanted.) Then we have

(A) $(h_{<n} : n < \omega)$ is Cauchy, let its limit be h_∞;
(B) $\mathfrak{d}(h_\infty, h_{<n+1}) < \zeta_n^1$;
(C) h_∞ and $(t_n)^{-1} z_n$ do not commute.

(Why? By clause (d) above, for each n we have $\mathfrak{d}((g_{n+1})^{-1} d_{n+1}, e) < \zeta_{n+1}^4 < \zeta_{n+1}^3$, and so by clause (h) we have $\mathfrak{d}(h_{<n+1}, h_{<n}) \leqslant \zeta_n^2$. Furthermore, $\zeta_{n+1}^2 < \zeta_{n+1}^1 \leqslant \frac{1}{4}\zeta_n^4 < \frac{1}{4}\zeta_n^2$. Thus, clearly the sequence $(h_{<n} : n < \omega)$ is Cauchy. Moreover, we have:

$$\mathfrak{d}(h_\infty, h_{<n+1}) \leqslant \sum \{\zeta_k^2 : k \geqslant n\} \leqslant 2\zeta_n^2 < \zeta_n^1,$$

so clause (B) is satisfied. Finally, clause (C) follows by (B) and clause (g) above.) Let $n < \omega$ be such that $\{a_n, b_n\} \cap sp(h_\infty) = \emptyset$. Then h_∞ and $(t_n)^{-1} z_n$ commute (cf. the choice of A_1), contradicting (C). $\square$

LEMMA 45. *Let $G = G(\Gamma, G_a)$ and $A_1, A_2 \subseteq \Gamma$ be as in Corollary 43 and Lemma 44. For $n < \omega$, $a \in \Gamma - (A_1 \cup A_2)$ and $g \in G_a$ we write $\varphi_n(g, G_a)$ to mean that for no divisible $K \leqslant G_a$ we have $g \in K + Tor_n(G_a)$ (cf. Definition 31). If for every $n < \omega$ the set*

$$A_3(n) = \{a \in \Gamma - (A_1 \cup A_2) : \exists g \in G_a \text{ such that } \varphi_n(g, G_a)\}$$

is uncountable, then G does not admit a Polish group topology.

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish, and let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ be as in Fact 39 for $f \in \omega^\omega$ such that $f(n) = n + 4$. By induction on $n < \omega$, choose $(a(n), b(n))$ and $(g_{a(n)}, g_{b(n)})$ such that

(a) $a(n) \neq b(n) \in \Gamma - (A_1 \cup A_2 \cup \{a(\ell), b(\ell) : \ell < n\})$;
(b) $g_{a(n)} \in G_{a(n)} - \{e\}$ and $g_{b(n)} \in G_{b(n)} - \{e\}$;
(c) for no divisible $K \leqslant G_{a(n)}$ we have $g_{a(n)} \in K + Tor_{n!}(G_{a(n)})$;
(d) $\mathfrak{d}((g_{b(n)})^{-1} g_{a(n)}, e) < \zeta_{n+1}$.

Consider now the following set of equations:

$$\Omega = \{x_n = (x_{n+1})^{n+1} h_n : n < \omega\},$$

where $h_n = (g_{b(n)})^{-1} g_{a(n)}$. By (d) above we have $\mathfrak{d}(h_n, e) < \zeta_{n+1}$, and so by Fact 39(2) the set Ω is solvable in G. Let $(d'_n)_{n<\omega}$ witness this. Let then $0 < n < \omega$ be such that $sp(d'_0) \cap \{a(n), b(n)\} = \emptyset$. Let now $A = \{a_n\}$, $\mathbf{p} = \mathbf{p}_A$ the corresponding homomorphism from Fact 14 and let $\mathbf{p}(d'_n) = d_n$. Then we have (in additive notation):

(i) $d_0 = e$;
(ii) $m \neq n \Rightarrow d_m = (m+1)d_{m+1} + \mathbf{p}(h_m) = (m+1)d_{m+1}$;
(iiii) $d_n = (n+1)d_{n+1} + \mathbf{p}(h_n) = (n+1)d_{n+1} + g_{a(n)}$.

Thus, by (ii) for $m < n$ we have $n!d_n = 0$, that is,

$$d_n \in Tor_{n!}(G_{a(n)}). \tag{5}$$

Furthermore, by (ii) for $m > n$ the subgroup K of $G_{a(n)}$ generated by $\{d_{n+1}, d_{n+2}, \ldots\}$ is divisible. Hence, by (iii) and (5) we have

$$g_{a(n)} = -(n+1)d_{n+1} + d_n \in K + Tor_{n!}(G_{a(n)}),$$

which contradicts the choice of $g_{a(n)}$. $\square$

DEFINITION 46. Let $G = G(\Gamma, G_a)$. We define (recalling the notation of Lemma 45):

(1) $n(G) = min\{m \geqslant 2 :$ for all but $\leqslant \aleph_0$ many $a \in \Gamma, \forall g \in G_a(\neg\varphi_m(g, G_a))\}$;
(2) $A_3 = \{a \in \Gamma : G_a$ is abelian and $\exists g \in G_a(\varphi_{n(G)}(g, G_a))\}$.

COROLLARY 47. *Let $G = G(\Gamma, G_a)$, and suppose that G admits a Polish group topology. Then*

(1) *the natural number $n = n(G)$ from Definition 46(1) is well defined;*
(2) *the set A_3 from Definition 46(2) is countable;*
(3) *the set $A_4 = \{a \in \Gamma : G_a$ is abelian and not n-bounded-divisible$\}$ is countable.*

Proof. This follows from Lemma 45 and Fact 37. $\square$

LEMMA 48. *Suppose that $G = G(\Gamma, G_a)$ admits a Polish group topology and let $A_1, \ldots, A_4$ be as Corollary 43, Lemma 44, Definition 46 and Corollary 47. Then there exists a countable $A \subseteq \Gamma$ and $n < \omega$ such that*

(a) $A_1 \cup \cdots \cup A_4 \subseteq A$;
(b) *if $a \in \Gamma - A$, then G_a is n-bounded-divisible.*

Proof. This is because of Corollaries 43 and 47, and Lemmas 44 and 45. $\square$

LEMMA 49. *Let $G = G' \oplus G''$, with $G'' = \bigoplus_{\alpha<\lambda} G_\alpha$, $\lambda > \aleph_0$ and $G_\alpha \cong \mathbb{Z}_p^\infty$ (for $\mathbb{Z}_p^\infty$ cf. Notation 4). Then G does not admit a Polish group topology.*

Proof. Suppose that $G = (G, \mathfrak{d})$ is Polish, and that $G = G' \oplus G''$ is as in the assumptions of the lemma. Let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ be as in Fact 39 for $f \in \omega^\omega$ such that $f(n) = p^{k(n)} + 1$, $k(n) > n$ and $2nk(n) < k(n+1)$. For every $n < \omega$, choose $(\alpha(n), \beta(n))$ and (g_n, h_n) such that

(a) $\alpha(n) < \beta(n) < \lambda$ and $\alpha(n), \beta(n) \notin \{\alpha(\ell), \beta(\ell) : \ell < n\}$;
(b) $g_n \in G_{\alpha(n)}$ and $h_n \in G_{\beta(n)}$;
(c) g_n and h_n have order $p^{nk(n)}$ (so $g_0 = e = h_0$);
(d) $\mathfrak{d}((g_n)^{-1}h_n, e) < \zeta_{n+1}$.

Consider now the following set of equations:

$$\Omega = \{x_n = (x_{n+1})^{p^{k(n)}} t_n : n < \omega\},$$

where $t_n = (g_n)^{-1}h_n$. By (d) above we have $\mathfrak{d}(t_n, e) < \zeta_{n+1}$, and so by Fact 39(2) the set Ω is solvable in G. Let $(d'_n)_{n<\omega}$ witness this. Let then $\mathbf{p}$ be the natural projection from G onto $G_* = \bigoplus_{n<\omega} G_{\beta_n}$ (cf. Fact 14), and set $d_n = \mathbf{p}(d'_n)$. Hence, for every $n < \omega$, we have (in additive notation):

$$G_* \models d_n = p^{k(n)} d_{n+1} + h_n,$$

and so

$$G_* \models d_0 = h_0 + p^{k(0)} h_1 + p^{k(0)+k(1)} h_2 + \cdots + p^{\sum_{\ell<n} k(\ell)} h_n + p^{\sum_{\ell\leqslant n} k(\ell)} h_{n+1}. \tag{6}$$

Thus, multiplying both sides of (6) by $p^{nk(n)}$, we get

$$G_* \models p^{nk(n)} d_0 = p^{\sum_{\ell\leqslant n} k(\ell)} p^{nk(n)} h_{n+1}, \tag{7}$$

since, for $\ell \leqslant n$, $h(\ell)$ has order $p^{\ell k(\ell)}$ and $\ell k(\ell) \leqslant nk(n)$, and so we have $p^{nk(n)} h_\ell = 0$. Note now that that the right side of (7) is $\neq 0$, since $p^{\sum_{\ell\leqslant n} k(\ell)} p^{nk(n)}$ divides $p^{nk(n)} p^{nk(n)} = p^{2nk(n)}$, $2nk(n) < k(n+1) < (n+1)k(n+1)$ and the order of h_{n+1} is $p^{(n+1)k(n+1)}$. Hence, also the left side of (7) is $\neq 0$, but this is contradictory, since G_* is an abelian p-group and $k(n) > n$, for every $n < \omega$. $\square$

The next lemma is stronger than what needed for the proof of Theorem 5, we need this formulation for the proof of Theorem 6.

LEMMA 50. *Suppose that G admits a Polish group topology, $G = G_1 \oplus G_2$, G_1 is countable and $G_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$ (cf. Notation 4). Then for every $s \in S_*$ we have that λ_s is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

Proof. Let $G = (G, \mathfrak{d})$ be Polish and $G = G_1 \oplus G_2$ be as in the assumptions of the lemma. Then $G_2 \cong \bigoplus\{G_t : t \in I\}$, where for each $t \in I$ we have $G_t \cong G^*_s$ for some $s \in S_*$. For $s \in S_*$, let $I_s = \{t \in I : G_t \cong G^*_s\}$. So $(I_s : s \in S_*)$ is a partition of I. We want to show that for each

$s \in S_*$ we have that $|I_s| \leqslant \aleph_0$ or $|I_s| = 2^{\aleph_0}$. Since S_* is countable, $|I_s| \leqslant |G|$ and $(G, \mathfrak{d})$ is Polish, it suffices to show that $|I_s| > \aleph_0$ implies $|I_s| = 2^{\aleph_0}$. Note that the case $s = (p, n)$ is actually taken care of by [**8**, Lemma 18 and Observation 19], but for completeness of exposition we give a direct proof also in the case $s = (p, n)$.

For $s \in S_*$ and $t \in I_s$, let $g_t \in G_t - \{e\}$ be such that g_t satisfies no further demands in the case $s = \infty$, and g_t generates G_t in the case $s = (p, n)$. Now, fix $s \in S_*$ and, using Observation 41, by induction on $n < \omega$, choose

$$(a(n), b(n), g_{a(n)}, g_{b(n)}, (h_{\mathcal{U}} : \mathcal{U} \subseteq n), h_n, \zeta^1_n, \zeta^2_n),$$

such that

(a) $h_{\mathcal{U}} = \prod_{\ell \in \mathcal{U}} h_\ell$;
(b) $0 < \zeta^1_n < \zeta^2_n < 1$;
(c) if $\mathcal{U} \subseteq n$ and $g \in Ball(e; \zeta^2_n)$, then $\mathfrak{d}(h_{\mathcal{U}} g, h_{\mathcal{U}}) < \zeta^1_n$;
(d) $a(n) \neq b(n) \in I_s - \{a(\ell), b(\ell) : \ell < n\}$;
(e) $h_n = (g_{a(n)})^{-1} g_{b(n)}$;
(f) $\mathfrak{d}(h_n, e) < \zeta^2_n$;
(g) $\zeta^2_{n+1} < \frac{1}{2}\zeta^1_n$.

Then for $\mathcal{U} \subseteq \omega$ we have that $(h_{\mathcal{U} \cap n} : n < \omega)$ is a Cauchy sequence. Let $h_{\mathcal{U}}$ be its limit.

Case 1. $s = \infty$.
Let:

$$E_\infty = \{(\mathcal{U}_1, \mathcal{U}_2) : \mathcal{U}_1, \mathcal{U}_2 \subseteq \omega \text{ and } \exists n \geqslant 2 \text{ and } \exists g \in G_1((h_{\mathcal{U}_1}(h_{\mathcal{U}_2})^{-1})^n g^{-1} = e)\}.$$

Note that

(i) E_∞ is an equivalence relation on $\mathcal{P}(\omega)$;
(ii) E_∞ is analytic (actually even Borel, recalling G_1 is countable);
(iii) $\mathcal{U}_1, \mathcal{U}_2 \subseteq \omega$ and $\mathcal{U}_2 - \mathcal{U}_1 = \{m\}$, then $\neg(\mathcal{U}_1 E_\infty \mathcal{U}_2)$.

Hence, by [**9**, Lemma 13], we get $(\mathcal{U}_\alpha : \alpha < 2^{\aleph_0})$ such that the functions $h_{\mathcal{U}_\alpha}$ are pairwise non-E_∞-equivalent. Note now that $\bigoplus\{G_t : t \notin I_\infty\}$ is torsion, while the functions $h_{\mathcal{U}_\alpha}$ have infinite order. Furthermore, by the choice of E_∞ we have that $\alpha < \beta < 2^{\aleph_0}$ implies that for every $n \geqslant 2$ we have $((h_{\mathcal{U}_\alpha}(h_{\mathcal{U}_\beta})^{-1})^n \notin G_1$. It follows that

$$G/(\bigoplus\{G_t : t \notin I_\infty\} \oplus G_1)$$

has cardinality $2^{\aleph_0}$, and so $|I_\infty| = 2^{\aleph_0}$, as wanted.

Case 2. $s = (p, n)$.
Let

$$E_{(p,n)} = \{(\mathcal{U}_1, \mathcal{U}_2) : \mathcal{U}_1, \mathcal{U}_2 \subseteq \omega \text{ and } (h_{\mathcal{U}_1}(h_{\mathcal{U}_2})^{-1})^{p^{n-1}} \in G_1 + pG_2)\}.$$

Note that

(i) $E_{(p,n)}$ is an equivalence relation on $\mathcal{P}(\omega)$;
(ii) $E_{(p,n)}$ is analytic (actually even Borel, recalling G_1 is countable);
(iii) $\mathcal{U}_1, \mathcal{U}_2 \subseteq \omega$ and $\mathcal{U}_2 - \mathcal{U}_1 = \{m\}$, then $\neg(\mathcal{U}_1 E_{(p,n)} \mathcal{U}_2)$.

Hence, by [**9**, Lemma 13], we get $(\mathcal{U}_\alpha : \alpha < 2^{\aleph_0})$ such that the functions $h_{\mathcal{U}_\alpha}$ are pairwise non-$E_{(p,n)}$-equivalent. Note now that

$$(h_{\mathcal{U}_a})^{p^n} = e \text{ and } (h_{\mathcal{U}_a})^{p^{n-1}} \neq e. \tag{8}$$

Furthermore, by the choice of $E_{(p,n)}$ we have that

$$\alpha < \beta < 2^{\aleph_0} \text{ implies } (h_{\mathcal{U}_\alpha}(h_{\mathcal{U}_\beta})^{-1})^{p^{n-1}} \notin G_1 + pG_2. \tag{9}$$

Let $\mathbf{p}$ be the projection of G onto G_2 (cf. Fact 14), and for $\alpha < 2^{\aleph_0}$ let $\mathbf{p}(h_{\mathcal{U}_\alpha}) = h'_\alpha$. Thus, by (9), we get

$$\alpha < \beta < 2^{\aleph_0} \text{ implies } (h'_\alpha(h'_\beta)^{-1})^{p^{n-1}} \neq e. \tag{10}$$

Thus, from (8) and (10) it follows that

$$Tor_{p^n}(G_2)/(Tor_{p^{n-1}}(G_2) + pG_2)$$

has cardinality $2^{\aleph_0}$, and so $|I_{(p,n)}| = 2^{\aleph_0}$, as wanted. □

Proof of Theorem 5. This follows directly from Lemma 48 (cf. the definitions of $A_1, \ldots, A_4$ there), Lemma 49 (recalling Definition 35) and Lemma 50. □

4. *Second venue*

In this section we prove Theorem 6. As in the previous section, we will prove a series of lemmas from which the theorem follows.

LEMMA 51. *If $G = G(\Gamma, G_a)$, $a \neq b \in \Gamma$, $\{a,b\} \notin E_\Gamma$ and G_b is uncountable, then G does not admit a Polish group topology.*

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish, and let $a \neq b \in \Gamma$ be as in the assumptions of the lemma. Let $k(n)$ and $p(n)$ be as in Proposition 30, $g_* \in G_a - \{e\}$ and let $(\zeta_n)_{n<\omega} \in (0,1)^\omega_{\mathbb{R}}$ be as in Fact 39 for $f \in \omega^\omega$ such that $f(n) = p(n) + 4k(n) + 4$ and $\bar{g}_n = (g_*)$ (and so in particular $\ell(n) = 1$). Using Observation 41, by induction on $n < \omega$, choose h_n such that

(a) $e \neq h_n \in G_b - \{h_\ell : \ell < n\}$;
(b) $\mathfrak{d}(h_n, e) < \zeta_{n+2k(n)+2}$.

Let $g_{(n,*)} = g_*$, $h_{(n,\ell,1)} = h_{n!+2\ell}$ and $h_{(n,\ell,2)} = h_{n!+(2\ell+1)}$, for $\ell < k(n)$. Let then $g_{(n,i)}$ and Ω be as in Proposition 30. By (b) above and Fact 39(1)(B) we have $\mathfrak{d}(g_{(n,i)}, e) < \zeta_{n+1}$, and so by Fact 39(2) the set Ω is solvable in G. Let now $A = \{a,b\}$ and $\mathbf{p} = \mathbf{p}_A$ be the corresponding homomorphism from Fact 14. Then projecting onto $\mathbf{p}(G) = G(\Gamma \restriction A, G_a)$ and using Proposition 30 we get a contradiction, since a is not adjacent to b, and so $G(\Gamma \restriction A, G_a) = G_a * G_b$. □

DEFINITION 52. For Γ a graph, let

$$A_0 = A_0(\Gamma) = \{a \in \Gamma : \text{for some } b \in \Gamma - \{a\} \text{ we have } \{a,b\} \notin E_\Gamma\}.$$

LEMMA 53. *If the set*

$$A_5 = \{a \in \Gamma - A_0 : G_a \textit{ is not abelian and } [G_a : Cent(G_a)] \textit{ is uncountable}\}$$

is infinite, then $G(\Gamma, G_a)$ does not admit a Polish group topology.

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish and that the set A_5 in the statement of the lemma is infinite. Let then $\{a(n) : n < \omega\}$ be an enumeration of A_5 without repetitions. First of all, note that for every $a \in \Gamma$ such that $[G_a : Cent(G_a)]$ is uncountable we have

$$\text{for every } \varepsilon \in (0,1)_{\mathbb{R}} \text{ we have } Ball(e;\varepsilon) \cap G_a \nsubseteq Cent(G_a). \tag{11}$$

Now, by induction on $n < \omega$, choose $(g_{n,1}, g_{n,2}, (h_{\mathcal{U}} : \mathcal{U} \subseteq n), \zeta_n^2, \zeta_n^1)$ such that

(a) $h_{\mathcal{U}} = \prod_{\ell \in \mathcal{U}} h_\ell$;
(b) $\zeta_n^1 < \zeta_n^2 \in (0,1)_{\mathbb{R}}$, and for $n = m+1$ we have $\zeta_n^2 < \frac{\zeta_m^1}{4}$;
(c) if $h \in Ball(e; \zeta_{n+1}^2) \cap G_{a(n)}$ and $\mathcal{U} \subseteq n$, then $\mathfrak{d}(h_{\mathcal{U}} h, h_{\mathcal{U}}) < \zeta_n^1$;
(d) $g_{n,1} \in (Ball(e;\zeta_n^2) \cap G_{a(n)}) - Cent(G_{a(n)})$, $g_{n,2} \in G_{a(n)}$ and $g_{n,1}$ and $g_{n,2}$ do not commute;
(e) if $h \in Ball(g_{n,1}; \zeta_n^1) \cap G_{a(n)}$, then $h \in Ball(e;\zeta_n^2) \cap G_{a(n)}$, and h and $g_{n,2}$ do not commute;
(f) $h_n = g_{n,1}$.

(How? First choose ζ_n^2 satisfying clauses (b) and (c). Then, using (11), choose $g_{n,1} = h_n$ as in clause (d). Finally, choose $\zeta_n^1 \in (0, \zeta_n^2)_{\mathbb{R}}$ as in clause (e).) For $n < \omega$, let $h_{<n} = h_0 \cdots h_{n-1}$. Then $(h_{<n} : n < \omega)$ is Cauchy, let its limit be h_∞. Note now that because of Lemma 51 without loss of generality we can assume that $n < m < \omega$ implies $\{a(n), a(m)\} \in E_\Gamma$, and also that if $b \in \Gamma - \{a(n)\}$ then $a(n) E_\Gamma b$. For $n < m$, let $h_{n,m} = h_n \cdots h_m$ and $h_{n,\infty} = lim(h_{n,m} : n < m < \omega)$. Let now $n < \omega$ be such that $sp(h_\infty) \cap \{a(n)\} = \emptyset$. Then we have

(a′) $g_{n,2}$ and h_n do not commute;
(b′) $g_{n,2}$ commutes with $h_0, \ldots, h_{n-1}$ and with $h_{n+1,\infty}$;
(c′) $h_\infty = h_0 \cdots h_{n-1} h_n h_{n+1,\infty}$;
(d′) h_∞ and $g_{n,2}$ do not commute.

(Why? Clause (a′) is by the inductive choices (a)–(f). Clause (b′) is because for $\ell < n$ we have $a(\ell) E_\Gamma a(n)$. Clause (c′) is easy. Clause (d′) is an immediate consequence of (a′), (b′) and (c′).] Thus, by (d′) we get a contradiction, since $sp(h_\infty) \cap \{a(n)\} = \emptyset$, $g_{n,2} \in G_{a(n)}$ and $b \in \Gamma - \{a(n)\}$ implies $a(n) E_\Gamma b$. □

Lemma 54. *For G a group, we write $\psi(G)$ to mean that $[G : Cent(G)]$ is countable, and (for ease of notation) we let $G' = Cent(G)$. If for every $n < \omega$ the set (recalling Fact 36 and Definition 31):*

$$A_6(n) = \{a \in \Gamma - A_0 : \psi(G_a) \text{ and } G'_a/(Div(G'_a) + Tor_n(G'_a)) \text{ is uncountable}\}$$

is infinite, then $G(\Gamma, G_a)$ does not admit a Polish group topology.

Proof. Suppose that $G = G(\Gamma, G_a) = (G, \mathfrak{d})$ is Polish, and let $A_6^* = \bigcup_{n<\omega} A_6(n)$. Note now that

(a) $a \in A_6^*$ implies $a \notin A_0(\Gamma)$ (cf. Definition 52);
(b) $(Cent(G), \mathfrak{d} \restriction Cent(G))$ is a Polish group;
(a) $Cent(G) \subseteq G(\Gamma \restriction B, G_a)$, where $B = \Gamma - A_0(\Gamma)$;
(b) $G(\Gamma \restriction B, G_a) = \bigoplus_{a \in B} G_a$;
(c) $Cent(\bigoplus_{a \in B} G_a) = \bigoplus_{a \in B} Cent(G_a) = G(\Gamma \restriction B, Cent(G_a))$.

(Why? (a) is because of Lemma 51. (b) is because the commutator function is continuous and a closed subgroup of a Polish group is Polish. The rest is clear.) Hence it suffices to prove the lemma for the abelian case, that is, assume that Γ is complete and all the factors groups G_a are abelian. Let then $(\zeta_n)_{n<\omega} \in (0,1)_{\mathbb{R}}^\omega$ be as in Fact 39 for $f \in \omega^\omega$ such that $f(n) = n+4$. Toward contradiction, assume that for every $n < \omega$ the set $A_6(n)$ is infinite. Then we can choose $a(n) \in \Gamma - \{a(\ell) : \ell < n\}$ such that $a(n) \in A_6(n!)$, by induction on n. So we can find $g_{n,\alpha} \in G_{a(n)} - \{e\}$, for $\alpha < \omega_1$, such that

$$(g_{n,\alpha} + (Div(G_{a(n)}) + Tor_{n!}(G_{a(n)})) : \alpha < \omega_1) \text{ are pairwise distinct.} \tag{12}$$

By induction on $n < \omega$, choose $\alpha(n) < \beta(n) < \omega_1$ such that $\mathfrak{d}((g_{n,\alpha(n)})^{-1} g_{n,\beta(n)}, e) < \zeta_{n+1}$. Then $h_n = (g_{n,\beta(n)})^{-1} g_{n,\alpha(n)} \in G_{a(n)}$ satisfies

(a) $\mathfrak{d}(h_n, e) < \zeta_{n+1}$;
(b) $h_n \notin Div(G_{a(n)}) + Tor_{n!}(G_{a(n)})$.

(Why? Clause (a) is clear. Clause (b) is by (12).) Consider now the following set of equations:

$$\Omega = \{x_n = (x_{n+1})^{n+1} h_n : n < \omega\}.$$

By (a) above and Fact 39(2) the set Ω is solvable in G. Let $(d'_n)_{n<\omega}$ witness this. Let then $0 < n < \omega$ be such that $sp(d'_0) \cap \{a(n)\} = \emptyset$. Let now $A = \{a(n)\}$, $\mathbf{p} = \mathbf{p}_A$ the corresponding homomorphism from Fact 14 and let $\mathbf{p}(d'_n) = d_n$. Then we have (in additive notation):

(i) $d_0 = e$;
(ii) $m \neq n \Rightarrow d_m = (m+1)d_{m+1} + \mathbf{p}(h_m) = (m+1)d_{m+1}$;
(iii) $d_n = (n+1)d_{n+1} + \mathbf{p}(h_n) = (n+1)d_{n+1} + h_n$.

Thus, by (ii) for $m < n$ we have $n!d_n = 0$, that is,

$$d_n \in Tor_{n!}(G_{a(n)}). \tag{13}$$

Furthermore, by (ii) for $m > n$ the subgroup K of $G_{a(n)}$ generated by $\{d_{n+1}, d_{n+2}, \ldots\}$ is divisible. Hence, by (iii) and (13) we have

$$h(n) = -(n+1)d_{n+1} + d_n \in K + Tor_{n!}(G_{a(n)}),$$

which contradicts (b) above. □

We now have all the ingredients for proving Theorem 6.

Proof of Theorem 6. Suppose that $G = G(\Gamma, G_a)$ admits a Polish group topology, and let n be minimal such that $A_6(n)$ is finite (cf. Lemma 54). We define (note that A_6 below is in fact $A_6(n)$):

(i) $A_0 = \{a \in \Gamma : \text{for some } b \in \Gamma - \{a\} \text{ we have } \{a, b\} \notin E_\Gamma\}$;
(ii) $A_5 = \{a \in \Gamma : G_a \text{ is not abelian and } [G_a : Cent(G_a)] \text{ is uncountable}\}$;
(iii) $A_6 = \{a \in \Gamma : \psi(G_a) \text{ and } G'_a/(Div(G'_a) + Tor_n(G'_a)) \text{ is uncountable}\}$;
(iv) $A_7 = \{a \in \Gamma : a \notin A_0 \cup A_5 \cup A_6 \text{ and } G_a \text{ is not abelian}\}$;
(v) $A_8 = \{a \in \Gamma : a \notin A_0 \cup A_5 \cup A_6 \text{ and } G_a \text{ is abelian and not bounded-divisible}\}$;
(vi) $A_9 = \{a \in \Gamma : a \notin A_0 \cup A_5 \cup A_6 \text{ and } G_a \text{ is abelian and bounded-divisible}\}$.

We claim that $\bar{A} = (A_0, A_5, A_6, A_7, A_8, A_9)$ is as wanted, that is, we verify clauses (1a)–(1k) of the statement of the theorem. Clauses (1a), (1b) and (1k) are clear. Clause (c) is by Lemmas 53 and 54. Clause (1d) for A_0 is by Lemma 42, for A_7 is by Lemma 44 and for A_8 is by Corollary 47. Clause (1e) is by Lemma 51. Clause (1f) is by Fact 38. Clause (1g) is by Definition 35. Clause (1h) is by Lemma 49. Clause (1j) is by Lemma 48, modulo renaming the factor groups G_a (if necessary).

Finally, we want to show that assuming CH and letting $A = A_0 \cup A_7 \cup A_8 \cup A_9$ we have that $G_A = G(\Gamma \restriction A, G_a)$ admits a non-Archimedean Polish group topology. By clauses (1a)–(1k) of the statement of the theorem we have

$$G_A \cong H \oplus \bigoplus_{\alpha < \lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p^n | n_*} \bigoplus_{\alpha < \lambda_{(p,n)}} \mathbb{Z}_{p^n},$$

for some countable H and $\lambda_\infty, \lambda_{(p,n)} \in \{0, 2^{\aleph_0}\}$. Since finite sums of groups admitting a non-Archimedean Polish group topology admit a non-Archimedean Polish group topology, it suffices to show that $H_1 = \bigoplus_{\alpha<2^{\aleph_0}} \mathbb{Q} \cong \mathbb{Q}^\omega$ and $H_2 = \bigoplus_{\alpha<2^{\aleph_0}} \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^n}^\omega$ admit one such topology. Let K be either $\mathbb{Q}$ or $\mathbb{Z}_{p^n}$, and let A be a countable first-order structure such that $Aut(A) = K$. Let B be the disjoint union of $\aleph_0$ copies of A, then $K^\omega \cong Aut(B)$, and so we are done. □

Proof of Theorem 7. The fact that (1)(a) (respectively, (2)(a)) implies (1)(b) (respectively, (2)(b)) is clear. Concerning the other implications, argue as in the proof of Theorem 6. □

5. *Third venue*

In this section we prove Corollaries 9–11.

Proof of Corollary 9. By Theorem 5 and Lemma 50 the necessity of the conditions is clear. Concerning the sufficiency, argue as in the proof of Theorem 6. □

Proof of Corollary 10. This is an immediate consequence of Corollary 9. □

Proof of Corollary 11. This is a consequence of Corollary 43 and Lemma 51. □

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