FUNDAMENTA
MATHEMATICAE
247 (2019)

Torsion-free abelian groups are consistently aΔ^1_2-complete

by

Saharon Shelah (Jerusalem) and **Douglas Ulrich** (Irvine, CA)

Abstract. Let TFAG be the theory of torsion-free abelian groups. We show that if there is no countable transitive model of ZFC$^-$ + "$\kappa(\omega)$ exists", then TFAG is aΔ^1_2-complete; in particular, this is consistent with ZFC. We define the α-ary Schröder–Bernstein property, and show that TFAG fails the α-ary Schröder–Bernstein property for every $\alpha < \kappa(\omega)$. We leave open whether or not TFAG can have the $\kappa(\omega)$-ary Schröder–Bernstein property; if it did, then it would not be aΔ^1_2-complete, and hence not Borel complete.

1. Introduction. In their seminal paper [3], Friedman and Stanley introduced *Borel complexity*, a measure of the complexity of the class of countable models of a sentence $\Phi \in \mathcal{L}_{\omega_1\omega}$. Let $\mathrm{Mod}(\Phi)$ be the set of all countable models of Φ with universe $\mathbb{N}$ (or any other fixed countable set). Then $\mathrm{Mod}(\Phi)$ can be made into a standard Borel space in a natural way.

Definition 1.1. Suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then write $\Phi \leq_{\mathrm{B}} \Psi$ (Φ is *Borel reducible* to Ψ) if there is a Borel-measurable function $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ satisfying the following: for all $M_1, M_2 \in \mathrm{Mod}(\Phi)$, $M_1 \cong M_2$ if and only if $f(M_1) \cong f(M_2)$.

Write $\Phi \sim_B \Psi$ (Φ and Ψ are *Borel bi-reducible*) if $\Phi \leq_{\mathrm{B}} \Psi$ and $\Psi \leq_{\mathrm{B}} \Phi$.

One way to think about the definition of $\leq_{\mathrm{B}}$ is that f induces an injection from $\mathrm{Mod}(\Phi)/\cong$ to $\mathrm{Mod}(\Psi)/\cong$; in other words, we are comparing the *Borel cardinality* of $\mathrm{Mod}(\Phi)/\cong$ and $\mathrm{Mod}(\Psi)/\cong$.

In [3], Friedman and Stanley showed that there is a maximal class of sentences under $\leq_{\mathrm{B}}$, namely the *Borel complete sentences.* For example, the theories of graphs, groups, rings, linear orders, and trees are all Borel complete. This provides a way to answer the question "Is it possible to classify

2010 *Mathematics Subject Classification*: Primary 03C55.
Key words and phrases: Borel complexity, torsion-free abelian groups.
Received 24 June 2018; revised 25 November 2018.
Published online 12 August 2019.

DOI: 10.4064/fm673-12-2018

the countable models of Φ" negatively in a precise sense: if Φ is Borel complete, then classifying the countable models of Φ is as hard as classifying arbitrary countable structures.

In [3], Friedman and Stanley leverage the Ulm analysis [15] to show that torsion abelian groups are far from Borel complete. They then pose the following question:

QUESTION. Let TFAG be the theory of torsion-free abelian groups. Is TFAG Borel complete?

This has attracted considerable attention, but has nonetheless remained open. The following theorem of Hjorth [7] is the best known so far, where $(\Phi_\alpha : \alpha < \omega_1)$ is the Friedman–Stanley tower:

THEOREM 1.2. $\Phi_\alpha \leq_{\mathrm{B}}$ TFAG *for every* $\alpha < \omega_1$.

This means that if TFAG is not Borel complete, then it represents a very new phenomenon. In fact, in [3], Friedman and Stanley separately described the following question as one of the basic open problems of the general theory: if Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ and if $\Phi_\alpha \leq_{\mathrm{B}} \Phi$ for each $\alpha < \omega_1$, must Φ be Borel complete?

In Section 2, we give a uniform treatment of the main currently known techniques of coding information into abelian groups. The basic idea for these codings is old, dating at least to [7] and [2]; namely, we start with a free abelian group, and then tag various subgroups by making the elements infinitely divisible by particular primes. However, to make the coding more robust we adopt an idea of [5], replacing the use of primes by an algebraically independent sequence of p-adic integers for a fixed prime p. As a first application, we show the following, where AG is the theory of abelian groups:

THEOREM 1.3. TFAG $\sim_B$ AG. *Further, if R is any countable ring, then* R-mod*, the theory of left R-modules, satisfies* R-mod $\leq_{\mathrm{B}}$ AG.

In Section 3, we expand on Hjorth's proof of Theorem 1.2. To state our results we need to introduce some more terminology.

DEFINITION 1.4. By ZFC$^-$, we mean ZFC without the power-set axiom, but where we have replacement and not only collection, and we strengthen choice to the well-ordering principle; this is as in [4].

$\kappa(\omega)$ is the least cardinal κ such that $\kappa \to (\omega)_2^{<\omega}$. This makes sense even in models of ZFC$^-$ (or less).

For example, if κ is a regular cardinal, then $H(\kappa) \models$ ZFC$^-$, where $H(\kappa)$ is the set of sets of hereditary cardinality at most κ; this is easily checked by running through the list of axioms. In particular, HC $\models$ ZFC$^-$, where HC is the set of hereditarily countable sets.

Definition 1.5. Sometimes natural reductions that arise require transfinite recursion, and thus are not Borel. A coarser notion of reduciblity that allows for this is absolute Δ^1_2 reducibility, denoted $a\Delta^1_2$. This notion has been studied, for instance, by Hjorth [8, Chapter 9]. Namely: suppose Φ, Ψ are sentences of $\mathcal{L}_{\omega_1\omega}$. Then put $\Phi \leq_{a\Delta^1_2} \Psi$ if there is some function $f : \mathrm{Mod}(\Phi) \to \mathrm{Mod}(\Psi)$ with a Δ^1_2 graph such that for all $M, N \in \mathrm{Mod}(\Phi)$, $M \cong N$ if and only if $f(M) \cong f(N)$, and such that this continues to hold in any forcing extension. Explicitly, if $\sigma(x, y)$ is the Π^1_2 definition of the graph of f, and $\tau(x, y)$ is the Σ^1_2-definition of the graph of f, and if $\mathbb{V}[G]$ is a forcing extension, then $\sigma(x, y)$ and $\tau(x, y)$ coincide on $\mathrm{Mod}(\Phi)^{\mathbb{V}[G]} \times \mathrm{Mod}(\Psi)^{\mathbb{V}[G]}$ and define the graph of a function $f^{\mathbb{V}[G]}$ such that for all $M, N \in \mathrm{Mod}(\Phi)^{\mathbb{V}[G]}$, $M \cong N$ if and only if $f^{\mathbb{V}[G]}(M) \cong f^{\mathbb{V}[G]}(N)$.

Using the basic idea of Theorem 1.2, we are able to prove the following theorem in Section 3:

Theorem 1.6. *Suppose there is no transitive model of* ZFC$^-$ + *"*$\kappa(\omega)$ *exists". Then* Graphs $\leq_{\mathrm{a}\Delta^1_2}$ TFAG.

Corollary 1.7. *It is consistent with* ZFC *that* Graphs $\leq_{\mathrm{a}\Delta^1_2}$ TFAG*, and hence that* TFAG *is* aΔ^1_2*-complete.*

It is natural to ask whether the set-theoretic hypothesis is necessary. For instance, the second author [16] can show that if $\kappa(\omega)$ exists, then a key part of the proof of Theorem 1.6 fails, namely, the conclusion of Theorem 3.3 below. This failure suggests the following question: are models of TFAG controlled by some sort of biembeddability invariants? We investigate this question in Section 4.

The Schröder–Bernstein property is the simplest way that biembeddability can control isomorphism. This notion was originally introduced by Nurmagambetov [10], [11], who defined that a complete first order theory T has the *Schröder–Bernstein property* in the class of all models if for all $M, N \models T$, if M and N are elementarily biembeddable, then $M \cong N$. Goodrick investigated this property further, including in his thesis [6] where he proves that if T has the Schröder–Bernstein property in the class of all models, then T is classifiable of depth 1, i.e. $I(T, \aleph_\alpha) \leq |\alpha + \omega|^{2^{\aleph_0}}$ for all α.

For our purposes, we want to tweak the definition in several ways. First of all, elementary embedding is somewhat awkward to deal with outside the context of complete first order theories.

Definition 1.8. Suppose M, N are $\mathcal{L}$-structures. Then $f : M \leq N$ is an *embedding* if the following holds: whenever R is a relation symbol of $\mathcal{L}$ then $f[R^M] \subseteq [R^N]$, and whenever F is a function symbol of $\mathcal{L}$ then $f \circ F^M = F^N \circ f$. Write $M \leq N$ if there is an embedding $f : M \to N$. Also, write $(M, \overline{a}) \leq (N, \overline{b})$ if there is an embedding $f : M \leq N$ with

$f(\overline{a}) = \overline{b}$. Finally, write $M \sim N$ if $M \leq N \leq M$, and write $(M, \overline{a}) \sim (N, \overline{b})$ if $(M, \overline{a}) \leq (N, \overline{b}) \leq (M, \overline{a})$.

In the context of groups, we will only want to consider injective embeddings; formally then, we add a binary predicate for $\{(a, b) : a \neq b\}$.

The following is what we mean by the Schröder–Bernstein property:

DEFINITION 1.9. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Then Φ has the *Schröder–Bernstein property* if whenever M, N are countable models of Φ, if $M \sim N$ then $M \cong N$.

This fails for TFAG, as first proved by Goodrick [6]. Recently, Calderoni and Thomas [1] have shown that the relation of biembeddability on models of TFAG is Σ^1_1-complete, which is as bad as possible.

However, the proof of Theorem 1.6 suggests a weaker property: perhaps a group $G \models$ TFAG is determined by $\{(G, a)/{\sim} : a \in G\}$. We will call this the 1-ary Schröder Bernstein property. In Section 4, we generalize this further to the α-ary Schröder–Bernstein property, for any ordinal α; the 0-ary Schröder–Bernstein property is the Schröder–Bernstein property.

The second author proves in [16]:

THEOREM 1.10. *Suppose $\kappa(\omega)$ exists, and suppose α is an ordinal. If Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ with the α-ary Schröder–Bernstein property, then Φ is not* aΔ^1_2*-complete* (*and hence not Borel complete*).

In Section 4, we prove:

THEOREM 1.11. *For every $\alpha < \kappa(\omega)$,* TFAG *fails the α-ary Schröder–Bernstein property.*

The construction breaks down at $\kappa(\omega)$, so the following remains open:

QUESTION. Does TFAG have the $\kappa(\omega)$-ary Schröder–Bernstein property?

2. Some bireducibilities with TFAG. Notation: If X is a set and G is a group we let $\bigoplus_X G$ denote the group of functions from X to G with finite support; so we consider $\bigoplus_X G \leq G^X$.

For p a prime, $\mathbb{Z}[1/p]$ is the subring of $\mathbb{Q}$ generated by $1/p$; and similarly for sets of primes. $\mathbb{Z}_{(p)}$ (read: $\mathbb{Z}$ localized at the ideal (p)) is $\mathbb{Z}[1/q : q \neq p]$. Let $\mathbb{Z}_p$ be the p-adic integers, i.e. the completion of $\mathbb{Z}_{(p)}$ under the p-adic metric. Let $\mathbb{Q}_p$ be the field completion of $\mathbb{Z}_p$.

Given groups $G \leq H$, say that G is a *pure subgroup* of H if for every $n < \omega$, $nH \cap G = nG$. If p is a prime, say that G is a *p-pure subgroup* of H if for every $n < \omega$, $p^n H \cap G = p^n G$.

The following is a generalization of Hjorth's notion of "eplag."

DEFINITION 2.1. Suppose $\mathcal{I}$ and $\mathcal{J}$ are countable index sets. Then let $\mathcal{L}_{\mathcal{I},\mathcal{J}}$ be the language extending the language of abelian groups, with a unary predicate symbol G_i for each $i \in I$, and a unary function symbol ϕ_j for each $j \in J$ (we will allow ϕ_j to be a partial function).

Let $\Omega_{\mathcal{I},\mathcal{J}}$ be the infinitary $\mathcal{L}_{\mathcal{F}}$-sentence such that $(G, +, G_i, \phi_j : i \in I, j \in J) \models \Omega_{\mathcal{I},\mathcal{J}}$ if and only if the following all hold:

- $(G, +) \equiv_{\infty\omega} \bigoplus_\omega \mathbb{Z}$;
- each G_i is a subgroup of G;
- each $\mathrm{dom}(\phi_j)$ is either equal to all of G, or else to some G_i;
- each $\phi_j : \mathrm{dom}(\phi_j) \to G$ is a homomorphism.

Let $\Omega^p_{\mathcal{I},\mathcal{J}}$ assert additionally that each G_i is a pure subgroup of G.

Some important examples: the countable models of $\Omega_{\{0\},0}$ are of the form (G, H) where G is free abelian of infinite rank (i.e., isomorphic to $\bigoplus_\omega \mathbb{Z}$) and H is a subgroup of G. The countable models of $\Omega_{0,\{0\}}$ are of the form (G, ϕ) where G is free abelian of infinite rank and $\phi : G \to G$ is a homomorphism. The countable models of $\Omega_{\omega,0}$ are of the form $(G, G_n : n < \omega)$, where G is free abelian of infinite rank and each G_n is a subgroup of G.

We will sometimes denote a model $(G, G_i, \phi_j : i \in \mathcal{I}, j \in \mathcal{J}) \models \Omega_{\mathcal{I},\mathcal{J}}$ as $\overline{G}$.

We aim to prove the following. Let AG denote the theory of abelian groups.

THEOREM 2.2. *Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets, not both empty. Then $\Omega^p_{\mathcal{I},\mathcal{J}} \sim_{\mathrm{B}} \Omega_{\mathcal{I},\mathcal{J}} \sim_{\mathrm{B}}$ TFAG $\sim_{\mathrm{B}}$ AG.*

The proof will be via many lemmas.

LEMMA 2.3. TFAG $\leq_{\mathrm{B}} \Omega^p_{\{0\},0}$ *and* AG $\leq_{\mathrm{B}} \Omega_{\{0\},0}$.

Proof. We describe the essential features of the construction, leaving it to the reader to check that it is Borel when formulated as an operation on Polish spaces. Suppose G is an (infinite) countable abelian group. Define $\phi : \bigoplus_G \mathbb{Z} \to G$ to be the augmentation map, that is given $a \in \bigoplus_G \mathbb{Z}$, let $\phi(a) = \sum_{b \in G} a(b)b$ (this is really a finite sum). Let K be the kernel of ϕ. Thus $G \mapsto (\bigoplus_G \mathbb{Z}, K)$ works, with the use of $G \cong \bigoplus_G \mathbb{Z}/K$. This shows AG $\leq_{\mathrm{B}} \Omega_{\{0\},0}$; but note that if G is torsion-free, then K will be pure, so we also get TFAG $\leq_{\mathrm{B}} \Omega^p_{\{0\},0}$. ∎

LEMMA 2.4. $\Omega_{\{0\},0} \leq_{\mathrm{B}} \Omega_{0,\{0\}}$. *Hence, whenever $\mathcal{I}, \mathcal{J}$ are not both empty, $\Omega_{\{0\},0} \leq_{\mathrm{B}} \Omega_{\mathcal{I},\mathcal{J}}$ and $\Omega^p_{\{0\},0} \leq_{\mathrm{B}} \Omega^p_{\mathcal{I},\mathcal{J}}$.*

Proof. Suppose $(G, H) \models \Omega_{\{0\},0}$ is a given countable model; so G is free abelian of infinite rank and H is a subgroup of G. Write $G' = G \times H'$, where $H' \cong H$; note that H' and hence G' is free abelian, since subgroups of free abelian groups are free. Define $\phi : G' \to G'$ via $\phi\restriction_G = 0$ and $\phi\restriction_{H'} : H' \cong H$. Then $(G, H) \mapsto (G', \phi)$ works, where we use $G = \ker(\phi)$ and $H = \mathrm{im}(\phi)$.

The second claim follows trivially (note $\Omega^p_{0,\{0\}} = \Omega_{0,\{0\}}$). ■

Lemma 2.5. *For any countable index sets $\mathcal{I}, \mathcal{J}$, we have $\Omega_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega_{\omega,0}$ and $\Omega^p_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega^p_{\omega,0}$.*

Proof. Write $\mathcal{I}' = \mathcal{I} \cup \mathcal{J} \cup \{*_0, *_1\}$ (we suppose this is a disjoint union). We show that $\Omega_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega_{\mathcal{I}',0}$ and $\Omega^p_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega^p_{\mathcal{I}',0}$.

Suppose $(G, G_i : i \in \mathcal{I},\ \phi_j : j \in \mathcal{J}) \models \Omega_{\mathcal{I},\mathcal{J}}$. Define $G' = G \times G$; for each $i \in \mathcal{I}$, define G'_i to be the copy of G_i in the first factor of G'; for each $j \in \mathcal{J}$, define G'_j to be the graph of ϕ_j; define $G'_{*_0} = G \times 0$; and finally let G'_{*_1} be the graph of the identify function $\mathrm{id}_G : G \to G$. Then $(G', G'_i : i \in \mathcal{I}') \models \Omega_{\mathcal{I}',0}$ works. Also note that if each G_i is pure, then so is each $G'_{i'}$; this is because the graph of a partial homomorphism is pure if and only if its domain is pure. ■

Lemma 2.6. $\Omega_{\omega,0} \leq_{\mathrm{B}} \Omega^p_{\omega,0}$.

Proof. By the preceding lemma, it suffices to find index sets $\mathcal{I}, \mathcal{J}$ such that $\Omega_{\omega,0} \leq_{\mathrm{B}} \Omega^p_{\mathcal{I},\mathcal{J}}$. Set $\mathcal{I} = \omega \cup \{*\}$ and $\mathcal{J} = \omega$.

Suppose $(G, G_n : n < \omega) \models \Omega_{\omega,0}$. We define $G' = G \times \bigoplus_{n<\omega}(\bigoplus_{G_n} \mathbb{Z})$. For each $n < \omega$ let $G'_n = \bigoplus_{G_n} \mathbb{Z}$; let $G'_* = G$. Finally, define $\phi_n : G'_n \to G'$ to be the augmentation map $\bigoplus_{G_n} \mathbb{Z} \to G_n$. Then clearly $(G', G'_i : i \in \mathcal{I},\ \phi_j : j \in \mathcal{J})$ works ($G = G'_*$ and $G_n = \mathrm{Im}(\phi_n)$ for all n). ■

Note that to finish the proof of Theorem 2.2, it suffices to show that $\Omega^p_{\omega,0} \leq_{\mathrm{B}}$ TFAG. Indeed, we would then know that for any countable index sets $\mathcal{I}, \mathcal{J}$ not both empty, TFAG $\leq_{\mathrm{B}} \Omega^p_{\{0\},0} \leq_{\mathrm{B}} \Omega^p_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega^p_{\omega,0} \leq_{\mathrm{B}}$ TFAG, and thus these are all equivalent; and similarly AG $\leq_{\mathrm{B}} \Omega_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega_{\omega,0} \leq_{\mathrm{B}} \Omega^p_{\omega,0} \leq_{\mathrm{B}}$ AG, and so these are also all equivalent.

This remaining reduction is more involved than the others; the basic idea for it is due to Goodrick [5]. To begin, we need the following lemma. The point is that if G is a p-pure subgroup of $\bigoplus_\omega \mathbb{Z}_p$, then the isomorphism type of $(\mathbb{Z}_pG, G)$ depends only on the isomorphism type of G, where $\mathbb{Z}_pG$ is the $\mathbb{Z}_p$-submodule of $\bigoplus_\omega \mathbb{Z}_p$ generated by G.

Lemma 2.7. *Suppose G is a p-pure subgroup of $\bigoplus_\omega \mathbb{Z}_p$. Then there is a $\mathbb{Z}_p$-module isomorphism $\phi : (\mathbb{Z}_p \otimes G)/(p^\infty(\mathbb{Z}_p \otimes G)) \to \mathbb{Z}_pG$, where $\mathbb{Z}_p \otimes G$ is the tensor product* (*over $\mathbb{Z}$*). *Further, $\phi(1 \otimes g + p^\infty(\mathbb{Z}_p \otimes G)) = g$ for each $g \in G$.*

Proof. Define $\psi(\gamma, a) = \gamma a$, going from $\mathbb{Z}_p \times G$ to $\mathbb{Z}_pG$. As ψ is clearly a $\mathbb{Z}$-bilinear map, it induces a group homomorphism $\phi_0 : \mathbb{Z}_p \otimes G \to \mathbb{Z}_pG$. Clearly ϕ_0 is 0 on $p^\infty(\mathbb{Z}_p \otimes G)$ so induces a map $\phi : (\mathbb{Z}_p \otimes G)/(p^\infty(\mathbb{Z}_p \otimes G)) \to \mathbb{Z}_pG$. We check this works. Clearly ϕ is surjective and $\phi(1 \otimes g + p^\infty(\mathbb{Z}_p \otimes G))$

$= \psi(1,g) = g$, and ϕ preserves the $\mathbb{Z}_p$-action. So it suffices to check that the kernel of ϕ_0 is $p^\infty(\mathbb{Z}_p \otimes G)$.

Given $\gamma \in \mathbb{Z}_p$ and $n < \omega$, let $\gamma\restriction_n \in \{0, \ldots, p^n - 1\}$ be the unique element with $\gamma - \gamma\restriction_n \in p^n\mathbb{Z}_p$ (recall that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ in the p-adic metric; so choose $(k_m : m < \omega)$ a sequence from $\mathbb{Z}$ converging to γ and note that $k_m \bmod p^n$ must eventually be constant).

Suppose $\sum_{i<n} \gamma_i a_i = 0$; we want to show $\sum_{i<n} \gamma_i \otimes a_i \in p^\infty(\mathbb{Z}_p \otimes G)$. Note that for each m, $\sum_{i<n} \gamma_i a_i \in p^m(\bigoplus_\omega \mathbb{Z}_p)$. Hence, for each m,

$$b_m := \sum_{i<n} \gamma_i\restriction_m a_i \in p^m G$$

(we use the fact that G is p-pure). Note that in $\mathbb{Z}_p \otimes G$, $\sum_{i<n} \gamma_i\restriction_m \otimes a_i = 1 \otimes b_m$, since we can move all the $\gamma_i\restriction_m$'s to the right-hand side; and $1 \otimes b_m \in p^m(\mathbb{Z}_p \otimes G)$. Also, $1 \otimes b_m - \sum_{i<n} \gamma_i \otimes a_i \in (p^m\mathbb{Z}_p) \otimes G$, as it is equal to $\sum_{i<n}(\gamma_i\restriction_m - \gamma_i) \otimes a_i$. Thus $\sum_{i<n} \gamma_i \otimes a_i \in p^m(\mathbb{Z}_p \otimes G)$ for all m, as desired. ∎

Finally, we have:

Lemma 2.8. $\Omega^p_{\omega,0} \leq_{\mathrm{B}}$ TFAG.

Proof. Let p be a prime. Let $(\gamma_n : 1 \leq n < \omega)$ be a sequence of algebraically independent elements of $\mathbb{Z}_p$ over $\mathbb{Q}$ such that each γ_n is a unit of $\mathbb{Z}_p$ (in particular is not divisible by p). Write $\gamma_0 = 1$. Note that $(\gamma_n : n < \omega)$ is then linearly independent over $\mathbb{Q}$.

Let $(\bigoplus_\omega \mathbb{Z}, G_n : n < \omega) \models \Omega^p_{\omega,0}$; we can suppose $G_0 = G_1 = \bigoplus_\omega \mathbb{Z}$. Let G be the p-pure subgroup of $\bigoplus_\omega \mathbb{Z}_p$ generated by $\bigcup_{n<\omega} \gamma_n G_n$ (that is, close off under addition, inverses, and division by p within $\bigoplus_\omega \mathbb{Z}_p$). We want to check that the map $\overline{G} \mapsto G$ works.

First, suppose $(\bigoplus_\omega \mathbb{Z}, G_n : n < \omega) \cong (\bigoplus_\omega \mathbb{Z}, G'_n : n < \omega)$; we want to verify that the corresponding groups G, G' are isomorphic. Let ϕ be the isomorphism. Then ϕ lifts canonically to an isomorphism $\phi^* : \bigoplus_\omega \mathbb{Z}_p \cong \bigoplus_\omega \mathbb{Z}_p$ (let $(e_i : i < \omega)$ be the standard basis of $\bigoplus_\omega \mathbb{Z}$, define $\phi^*(\sum_i \gamma_i e_i) = \sum_i \gamma_i \phi(e_i)$, where $(e_i : i < \omega)$ is the standard basis of $\bigoplus_\omega \mathbb{Z}$; more abstractly, $\phi^* = 1 \otimes \phi$ where we view $\bigoplus_\omega \mathbb{Z}_p = \mathbb{Z}_p \otimes \bigoplus_\omega \mathbb{Z}$). Then clearly $\phi^*\restriction_G$ is an isomorphism onto G'.

For the reverse it suffices, by Lemma 2.7, to show we can canonically recover each G_n from $(\mathbb{Z}_p G, G)$.

Note that every $a \in G$ can be written as $\sum_{n<\omega} \gamma_n p^{k(n)} b_n$, where $k(n) \in \mathbb{Z}$, $b_n \in G_n$ with all but finitely many $b_n = 0$, and $k(n) = 0$ whenever $b_n = 0$. (Not all such sums are in G; G contains such sums which are additionally in $\bigoplus_\omega \mathbb{Z}_p$.) We call this a representation of a if each $p \nmid b_n$. Then representations are unique: For suppose $\sum_{n<\omega} \gamma_n p^{k(n)} b_n = \sum_{n<\omega} \gamma_n p^{k'(n)} b'_n$. Let $i \in \omega$; then we have $\sum_{n<\omega}(p^{k(n)} b_n(i) - p^{k'(n)} b'_n(i))\gamma_n = 0$. By linear independence of $(\gamma_n : n < \omega)$ this implies $p^{k(n)} b_n(i) = p^{k'(n)} b'_n(i)$ for all i and n. Hence

$p^{k(n)}b_n = p^{k'(n)}b'_n$ for all n. Then by divisibility assumptions we deduce that $b_n = b'_n$ and so $k(n) = k'(n)$ for all n.

Suppose $f \in \mathbb{Z}_p G$ and let $1 \leq m < \omega$. It suffices to show that $a \in G_m$ if and only if $a \in G$ and $\gamma_m a \in G$: Left to right follows from our assumption that $\gamma_0 = 1$. For right to left: let $\sum_{n<\omega} \gamma_n p^{k(n)} b_n$ be the representation of a, and let $\sum_{n<\omega} \gamma_n p^{k'(n)} b'_n$ be the representation of $\gamma_m a$. Let $i \in \omega$. Then $\sum_{n<\omega} \gamma_m \gamma_n p^{k(n)} b_n(i) = \sum_{n<\omega} \gamma_n p^{k'(n)} b'_n(i)$. Note that the only time $\gamma_m \gamma_n = \gamma_k$ is when $n = 0$ and $k = m$. Thus by linear independence of $(\gamma_n : n < \omega)^\frown(\gamma_m \gamma_n : 1 \leq n < \omega)$ we find that $b_n = 0$ for all $n \neq 0$, and $b'_n = 0$ for all $n \neq m$. In particular, $a = p^k b$ for some $b \in G_m$. Since $\bigoplus_\omega \mathbb{Z}$ is p-pure in $\bigoplus_\omega \mathbb{Z}_p$ and since G_m is p-pure in $\bigoplus_\omega \mathbb{Z}$, we see that $a \in G_m$. ∎

Remark 2.9. It is easy to add to the list in Theorem 2.2. For instance, we can additionally insist that each ϕ_j is a pure embedding, i.e. preserves the divisibility relations.

A much stronger condition is the following: let $\Omega^*_{\mathcal{I},\mathcal{J}}$ be $\Omega_{\mathcal{I},\mathcal{J}}$ together with the second-order assertion saying, given $(G, G_i : i \in I, \phi_j : j \in J)$, that there is a basis $\mathcal{B}$ of G (as a $\mathbb{Z}$-module) such that each G_i is spanned by basis elements of $\mathcal{B}$ and each ϕ_j takes basis elements to basis elements. All of the known complexity of TFAG is also present in $\Omega^*_{\omega,\{0\}}$; see the next section.

Finally, we aim to show that whenever R is a countable ring, then R-mod (the theory of left R-modules) is Borel reducible to AG. This will not be used in the remainder of the paper.

Definition 2.10. Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets. Let $\Omega^-_{\mathcal{I},\mathcal{J}}$ be the $\mathcal{L}_{\mathcal{I},\mathcal{J}}$-theory such that $(G, +, G_i, \phi_j : i \in \mathcal{I}, j \in \mathcal{J}) \models \Omega^-_{\mathcal{I},\mathcal{J}}$ if and only if:

- $(G, +)$ is an abelian group;
- each G_i is a subgroup of G;
- each $\mathrm{dom}(\phi_j)$ is either all of G or else some G_i;
- each $\phi_j : \mathrm{dom}(\phi_j) \to G$ is a homomorphism.

So the only difference with $\Omega_{\mathcal{I},\mathcal{J}}$ is that we are no longer requiring $G \equiv_{\infty\omega} \bigoplus_\omega \mathbb{Z}$.

Theorem 2.11. *For all countable index sets $\mathcal{I}, \mathcal{J}$, we have $\Omega^-_{\mathcal{I},\mathcal{J}} \sim_{\mathrm{B}}$ AG.*

Proof. Clearly AG $\leq_{\mathrm{B}} \Omega^-_{\mathcal{I},\mathcal{J}}$. (Given $G \models$ AG, let each $G_i = G$ and let each ϕ_j be the identity of G.) Also, by exactly the same argument as before, $\Omega^-_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega^-_{\omega,0}$. So it suffices to show that $\Omega^-_{\omega,0} \leq_{\mathrm{B}} \Omega_{\omega\cup\{*\},0}$.

Given $(G, G_n : n < \omega) \models \Omega^-_{\omega,0}$ (that is, G is an abelian group and each G_n is a subgroup of G), write $G' = \bigoplus_G \mathbb{Z}$; let G'_* be the kernel of the augmentation map $G' \to G$; and finally define $G'_n = G'_* + \bigoplus_{G_n} \mathbb{Z}$. Then

$(G', G'_n : n < \omega, G'_*)$ works; we use $G \cong G'/G_*$ via an isomorphism that takes each G_n to G'_n/G_*. ∎

COROLLARY 2.12. *Suppose R is a countable ring. Then R*-mod $\leq_{\mathrm{B}}$ AG.

Proof. An R-module $(M, +, \cdot_r : r \in R)$ can be viewed as a model of $\Omega^-_{0,R}$, and this gives a reduction R-mod $\leq_{\mathrm{B}} \Omega^-_{0,R}$. ∎

3. Embedding graphs into TFAG. In this section, we prove Theorem 1.6: if there is no transitive model of $\mathrm{ZFC}^- +$ "$\kappa(\omega)$ exists", then Graphs $\leq_{\mathrm{a}\Delta^1_2}$ TFAG. To begin, we introduce some terminology for colored trees.

DEFINITION 3.1. A *colored tree* is a structure $(T, \leq, 0, c)$ where $(T, \leq)$ is a tree (of height at most ω) with root 0, and $c : T \to \omega$. We view these as model-theoretic structures; formally we can replace c with a sequence of unary predicates. Let CT be the sentence of $\mathcal{L}_{\omega_1\omega}$ describing colored trees.

As notation, whenever we say $\mathcal{T}, \mathcal{S}$, etc. is a colored tree, we will have $\mathcal{T} = (T, <_{\mathcal{T}}, 0_{\mathcal{T}}, c_{\mathcal{T}})$, $\mathcal{S} = (S, <_{\mathcal{S}}, 0_{\mathcal{S}}, c_{\mathcal{S}})$, etc., unless stated otherwise.

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two colored trees. Then say that $f : \mathcal{T} \leq \mathcal{T}'$ is an *embedding of trees* if:

- $f(0_{\mathcal{T}}) = 0_{\mathcal{S}}$;
- f preserves height;
- for all $s \in T$, $c_{\mathcal{T}'}(f(s)) = c_{\mathcal{T}}(s)$;
- for all $s, t \in T$, if $s \leq_{\mathcal{T}} t$ then $f(s) \leq_{\mathcal{T}'} f(t)$.

We do not require that f be injective.

Say that $\mathcal{T}$ and $\mathcal{T}'$ are *tree-biembeddable* ($\mathcal{T} \sim \mathcal{T}'$) if $\mathcal{T} \leq \mathcal{T}'$ and $\mathcal{T}' \leq \mathcal{T}$. These definitions agree with the definitions from the introduction, provided we add predicates for the elements of height n.

If $\mathcal{T}$ and $t \in T$ then $\mathrm{ht}(t)$ denotes its height in $\mathcal{T}$ (if there is ambiguity we will write $\mathrm{ht}_{\mathcal{T}}(t)$). Let $\mathcal{T}_{\geq t}$ denote the subtree of all elements of T bigger than or equal to t, with the induced coloring.

We will now split the proof of Theorem 1.6 into two main subtheorems.

THEOREM 3.2. *There is a Borel map $f : \mathrm{Mod}(\mathrm{CT}) \to \mathrm{Mod}(\mathrm{TFAG})$ such that for all $\mathcal{T}, \mathcal{T}' \models \mathrm{CT}$, if $\mathcal{T} \cong \mathcal{T}'$ then $f(\mathcal{T}) \cong f(\mathcal{T}')$, and if $f(\mathcal{T}) \cong f(\mathcal{T}')$ then $\mathcal{T} \sim \mathcal{T}'$. (In fact, for every $t \in T$, there is $t' \in T'$ of the same height with $\mathcal{T}_{\geq t} \sim \mathcal{T}'_{\geq t'}$, and conversely.)*

THEOREM 3.3. *Suppose there is no transitive model of* $\mathrm{ZFC}^- +$ "$\kappa(\omega)$ *exists". Then there is an absolute Δ^1_2-reduction $g : \mathrm{Mod}(\mathrm{Graphs}) \to \mathrm{Mod}(\mathrm{CT})$ such that whenever $G, G' \in \mathrm{Mod}(\mathrm{Graphs})$, if $G \ncong G'$ then $g(G) \nsim g(G')$.*

In Theorem 3.3, recall that as part of the definition of absolute Δ^1_2-reduction, we know that if $G \cong G'$ then $g(G) \cong g(G')$. We are essentially following Hjorth's proof of [7, Theorem 1.2], although Theorem 2.2 will make our life easier. The second author shows in [16] that if $\kappa(\omega)$ exists, then the conclusion of Theorem 3.3 fails.

Before proceeding, note that it suffices to establish Theorem 3.2 and Theorem 3.3. Indeed, let $h = f \circ g : \mathrm{Mod}(\mathrm{Graphs}) \to \mathrm{Mod}(\mathrm{TFAG})$. Clearly $f \circ g$ has a Δ^1_2 graph, and preserves isomorphism; we need to check this remains true in forcing extensions. Suppose $\mathbb{V}[G]$ is a forcing extension. By the definition of absolute Δ^1_2-reduction, $g^{\mathbb{V}[G]}$ still makes sense, and is a reduction from Graphs to CT. The remaining properties of f, g are preserved by Shoenfield's absoluteness theorem.

Proof of Theorem 3.2. Suppose $\mathcal{T} = (T, <_T, c_T) \models \mathrm{CT}$. We define a model $\mathcal{T} \otimes \mathbb{Z}$ of $\Omega_{\omega\times\omega,\{0\}}$ (f will be the function $\mathcal{T} \mapsto \mathcal{T} \otimes \mathbb{Z}$). Let the underlying group of $\mathcal{T} \otimes \mathbb{Z}$ be $\bigoplus_T \mathbb{Z}$; define the group homomorphism $\pi_{\mathcal{T}} : \bigoplus_T \mathbb{Z} \to \bigoplus_T \mathbb{Z}$ by $\pi_{\mathcal{T}}(a)(t) = \sum_{s \in \mathrm{succ}_{\mathcal{T}}(t)} a(s)$, where $\mathrm{succ}_{\mathcal{T}}(t)$ denotes the set of all immediate successors of s in $\mathcal{T}$. Viewing $T \subseteq \bigoplus_T \mathbb{Z}$, note that $\pi_{\mathcal{T}}(0_{\mathcal{T}}) = 0$, and for all $s \neq 0_{\mathcal{T}}$, $\pi_{\mathcal{T}}(s)$ is the immediate predecessor of s. For each $n, i < \omega$ write $G_{\mathcal{T},n,i} = \bigoplus_t \mathbb{Z}$, where the sum is over all $t \in T$ of height n and with $c_{\mathcal{T}}(t) = i$. Let $\mathcal{T} \otimes \mathbb{Z}$ be the structure $(\bigoplus_T \mathbb{Z}, G_{\mathcal{T},n,i}, \pi_{\mathcal{T}} : n, i < \omega)$.

Let $\mathrm{CT} \otimes \mathbb{Z}$ be the Σ^1_1-sentence describing the closure under isomorphism of $\{\mathcal{T} \otimes \mathbb{Z} : \mathcal{T} \models \mathrm{CT}\}$.

Note that it is obvious that if $\mathcal{T}_1 \cong \mathcal{T}_2$ then $\mathcal{T}_1 \otimes \mathbb{Z} \cong \mathcal{T}_2 \otimes \mathbb{Z}$.

Fix some countable $\mathcal{T} \models \mathrm{CT}$. We perform some analysis on $\mathcal{T} \otimes \mathbb{Z}$; write $G = \bigoplus_T \mathbb{Z}$.

For each $\bar{i} = (i_m : m < n+1) \in \omega^{n+1}$, let $G_{\mathcal{T},\bar{i}}$ be the subgroup of all $a \in G$ such that $\pi^m_{\mathcal{T}}(a) \in G_{\mathcal{T},n-m,i_{n-m}}$ for each $m \leq n$. Also let $G_{\mathcal{T},\emptyset} = 0$. Note that $\pi_{\mathcal{T}}$ takes $G_{\mathcal{T},\bar{i}}$ to $G_{\mathcal{T},\bar{i}\restriction_n}$, also G is the direct sum of the various $G_{\mathcal{T},\bar{i}}$'s. Further, $G_{\mathcal{T},\bar{i}}$ is spanned by $\{t \in T : \mathrm{ht}(t) = n, \bar{c}_{\mathcal{T}}(t) = \bar{i}\}$, where $\bar{c}_{\mathcal{T}}(t) = (c_{\mathcal{T}}(t\restriction_0), c_{\mathcal{T}}(t\restriction_1), \ldots, c_{\mathcal{T}}(t))$.

For each nonzero $a \in G_{\mathcal{T},\bar{i}}$, let T^*_a denote the set of all b such that for some $\bar{i} \subseteq \bar{j}$ (i.e. $\bar{i}$ is an initial segment of $\bar{j}$), $b \in G_{\mathcal{T},\bar{j}}$ and $\pi_{\mathcal{T}}^{\mathrm{lg}(\bar{j})-\mathrm{lg}(\bar{i})}(b) = a$. If we define $c^*_a(b) = \bar{j}(\mathrm{lg}(\bar{j}) - 1)$, and if we let $b \leq_a b'$ if and only if some $\pi^m(b')$ is b, then $(T^*_a, \leq_a, c^*_a) = \mathcal{T}^*_a$ is a colored tree.

We need to characterize the colored trees $\mathcal{T}^*_a$ up to biembeddability. This will be done in terms of products of trees:

Definition 3.4. If $(\mathcal{S}_k : k < k_*)$ are colored trees, then by the product $\prod_{k<k_*} \mathcal{S}_k$, we mean the colored tree whose elements are all sequences $(s_k : k < k_*)$, where for some $n < \omega$, each s_k has height n, and for some $(i_m : m \leq n) \in \omega^{n+1}$, we have $c_{\mathcal{S}_k}(s_k\restriction_m) = i_m$ for all $m \leq n$. Then we define the *color* of $(s_k : k < k_*)$ to be i_n.

Clearly, $\prod_{k<k_*}\mathcal{S}_k \leq \mathcal{S}_{k'}$ for each $k' < k_*$, via projection onto the k'-factor. In fact, $\mathcal{T} \leq \prod_{k<k_*}\mathcal{S}_k$ if and only if $\mathcal{T} \leq \mathcal{S}_k$ for each $k < k_*$. This is because if $\mathcal{T} \leq \prod_{k<k_*}\mathcal{S}_k$, then we can compose with the projection maps to get $\mathcal{T} \leq \mathcal{S}_k$ for each k; and if $f_k : \mathcal{T} \leq \mathcal{S}_k$ for each $k < k_*$, we can define $f : \mathcal{T} \leq \prod_{k<k_*}\mathcal{S}_k$ via $f(t) = (f_k(t) : k < k_*)$.

Claim 1. *Suppose* $a \in G_{\mathcal{T},\bar{i}}$ *is nonzero; enumerate* $\mathrm{supp}(a) = \{t_k : k < k_*\}$. (*Here, we are viewing* $a \in \bigoplus_T \mathbb{Z}$ *as a function from* T *to* $\mathbb{Z}$ *of finite support* $\mathrm{supp}(a)$.) *Then* $\mathcal{T}^*_a \sim \prod_{k<k_*}\mathcal{T}_{\geq t_k}$.

Proof. First we will define an embedding $f : \mathcal{T}^*_a \leq \prod_{k<k_*}\mathcal{T}_{\geq t_k}$. We will define $f(b)$ inductively on the height of $b \in \mathcal{T}^*_a$; our inductive hypothesis will be that $f(b) = (t_k : k < k_*)$ is a sequence from $\mathrm{supp}(b)$, and if we let $\bar{i}$ be such that $b \in G_{\mathcal{T},\bar{i}}$, then $\bar{c}_{\mathcal{T}}(t_k) = \bar{i}$.

So we are given b and $f(b) = (t_k : k < k_*)$. Suppose $i < \omega$ and $c \in G_{\mathcal{T},\bar{i}i}$ satisfies $\pi_{\mathcal{T}}(c) = b$. Then $\pi_{\mathcal{T}}[\mathrm{supp}(c)] \supseteq \mathrm{supp}(b)$, so for each $k < k_*$ we can find $s_k \in \mathrm{supp}(c)$ with $\pi_{\mathcal{T}}(s_k) = t_k$. Clearly we can then define $f(c) = (s_k : k < k_*)$, and continue.

For the reverse embedding $\prod_{k<k_*}\mathcal{T}_{\geq t_k} \leq \mathcal{T}^*_a$, write $a = \sum_{k<k_*}\lambda_k t_k$, and send $(s_k : k < k_*) \in \prod_{k<k_*}\mathcal{T}_{\geq t_k}$ to $\sum_{k<k_*}\lambda_k s_k \in \mathcal{T}^*_a$. ∎

Given an ω-labeled tree $\mathcal{S}$, let $G_{\mathcal{T},\bar{i},\mathcal{S}}$ be the set of all $a \in G_{\mathcal{T},\bar{i}}$ such that $\mathcal{S} \leq T^*_a$, along with $a = 0$. From the preceding claim it is clear that $G_{\mathcal{T},\bar{i},\mathcal{S}}$ is a subgroup of $G_{\mathcal{T},\bar{i}}$. Also, let $G_{\mathcal{T},\bar{i},>\mathcal{S}}$ be the sum of all $G_{\mathcal{T},\bar{i},\mathcal{S}'}$ for $\mathcal{S} < \mathcal{S}'$ (by this we mean that $\mathcal{S} \leq \mathcal{S}'$ and $\mathcal{S}' \not\leq \mathcal{S}$).

Note that if $a \in G_{\mathcal{T},\bar{i}}$, then always $a \in G_{\mathcal{T},\bar{i},\mathcal{T}^*_a}$, but sometimes also $a \in G_{\mathcal{T},\bar{i},>\mathcal{T}^*_a}$. Say that a is *good* if this is not the case, i.e. $a \in G_{\mathcal{T},\bar{i},\mathcal{T}^*_a} \setminus G_{\mathcal{T},\bar{i},>\mathcal{T}^*_a}$.

Claim 2. *Suppose* $a \in G_{\mathcal{T},\bar{i}}$. *Then* a *is good if and only if* a *is nonzero, and there is some* $t \in \mathrm{supp}(a)$ *such that* $\mathcal{T}^*_a \sim \mathcal{T}_{\geq t}$.

Proof. Enumerate $\mathrm{supp}(a) = \{t_k : k < k_*\}$, and write $a = \sum_{k<k_*}\lambda_k t_k$. Then by Claim 1, $\mathcal{T}^*_a \leq \prod_{k<k_*}\mathcal{T}_{\geq t_k}$, so $\mathcal{T}^*_a \leq \mathcal{T}_{\geq t_k}$ for each $k < k_*$.

If a is good, then we cannot have $\mathcal{T}^*_a < \mathcal{T}_{\geq t_k}$ for each k, so $\mathcal{T}_{\geq t_k} \sim \mathcal{T}^*_a$ for some k as desired. For the converse, suppose $t \in \mathrm{supp}(a)$ satisfies $\mathcal{T}^*_a \sim \mathcal{T}_{\geq t}$. Write $a = \sum_{i<i_*} b_i$. Then $t \in \mathrm{supp}(b_i)$ for some $i < i_*$. By Claim 1, $\mathcal{T}^*_{b_i} \leq \mathcal{T}_{\geq t}$, and thus $\mathcal{T}^*_{b_i} \not> \mathcal{T}_{\geq t} \sim \mathcal{T}^*_a$. ∎

In particular, if $a \in G_{\mathcal{T},\bar{i}}$ is good then $\mathcal{T}^*_a \sim \mathcal{T}_{\geq t}$ for some $t \in T$, and so we can recover $\{\mathcal{T}_{\geq t}/\!\sim\, : t \in T,\ \mathrm{ht}(t) = n\}$ from the isomorphism class of $\mathcal{T} \otimes \mathbb{Z}$, for each n. This concludes the proof of Theorem 3.2. ∎

Before turning to the proof of Theorem 3.3, we need some set-theoretic observations.

First, we note that various familiar facts about $\kappa(\omega)$ continue to hold when the ambient set theory is just ZFC$^-$ (less suffices as well). Recall that a cardinal κ (in a model of ZFC) is *totally indescribable* if for every n, for every sentence ϕ in the language of set theory with an extra relation symbol, and for every $R \subseteq \mathbb{V}_\kappa$ with $(\mathbb{V}_{\kappa+n}, \in, R) \models \phi$, there is an $\alpha < \kappa$ such that $(\mathbb{V}_{\alpha+n}, \in, R \cap \mathbb{V}_\alpha) \models \phi$. This is a large cardinal notion; it implies that κ is weakly compact. In fact, weak compactness is equivalent to this condition when restricted to $n = 1$ (see [9, Theorem 6.4], due to Hanf and Scott).

LEMMA 3.5. *Work in* ZFC^-.

(A) *Suppose* $\kappa \to (\omega)_2^{<\omega}$ *and N is a transitive model of* ZFC^- *containing* κ *(possibly a proper class). Then* $(\kappa \to (\omega)_2^{<\omega})^N$.
(B) *If* $\mathbb{V} = \mathbb{L}$ *(we really just need global choice), and if* $\kappa(\omega)$ *exists, then* $\kappa(\omega)$ *is inaccessible (i.e.,* $\kappa(\omega)$ *is a regular cardinal, and for all* $\alpha < \kappa(\omega)$, $\mathcal{P}(\alpha)$ *exists and has cardinality less than* $\kappa(\omega)$*). Thus,* $\mathbb{L}_{\kappa(\omega)} = \mathbb{V}_{\kappa(\omega)}$ *is a set model of* ZFC.
(C) *If* $\mathbb{V} = \mathbb{L}$ *and if* $\kappa(\omega)$ *exists, then* $\mathbb{V}_{\kappa(\omega)} \models$ *"There exist totally indescribable cardinals."*
(D) *If* $\mathbb{V} = \mathbb{L}$*, then* $\kappa(\omega)$ *is the least cardinal* κ *such that whenever* $f : [\kappa]^{<\omega} \to 2$*, there is an increasing sequence* $(\alpha_n : n < \omega)$ *from* κ *such that* $f(\alpha_0, \ldots, \alpha_{n-1}) = f(\alpha_1, \ldots, \alpha_n)$ *for all* n.
(E) *If* $\mathbb{V} = \mathbb{L}$*, then* $\kappa(\omega)$ *is the least cardinal* κ *such that there is no antichain* $(\mathcal{T}_\alpha : \alpha < \kappa(\omega))$ *of* ω*-colored trees; by an antichain we mean that for all* $\alpha < \beta < \kappa(\omega)$, $\mathcal{T}_\alpha \not\leq \mathcal{T}_\beta$ *and* $\mathcal{T}_\beta \not\leq \mathcal{T}_\alpha$*. (If* $\kappa(\omega)$ *does not exist then we just mean that for every cardinal* κ*, there is an antichain of length* κ*.)*

Note that Corollary 1.7 follows from Theorem 1.6 and (B). Moreover, (C) provides a strengthening: it is consistent with ZFC + "There is a totally indescribable cardinal" that Graphs $\leq_{\mathrm{a}\Delta^1_2}$ TFAG.

Proof of Lemma 3.5. All of these are routine modifications of the case where the ambient set theory is ZFC. In the context of ZFC, (A) and (D) are due to Silver [13]; (B) is also due to Silver [14], or see Corollary 7.6 of Kanamori [9]; (C) is due to Silver and Reinhardt, see [9, Exercise 9.18]; and (E) is due to Shelah [12]—we provide a sketch of the proof.

First suppose $\kappa < \kappa(\omega)$. Choose some $f : [\kappa]^{<\omega} \to 2$ failing (D). For each $\alpha < \kappa$, we define a colored tree $\mathcal{T}_\alpha$ as follows. Let T_α be all finite increasing sequences of ordinals from κ whose first term is α; let $<_{\mathcal{T}_\alpha}$ be initial segment. Let $c_{\mathcal{T}_\alpha}(s) = f(s)$. Let $\mathcal{S}_\alpha$ be $\mathcal{T}_\alpha$ together with the tree of descending sequences from α, with the new elements all colored 2.

Note that for all $\alpha_0 < \alpha_1 < \kappa$, $\mathcal{T}_{\alpha_0} \not\leq \mathcal{T}_{\alpha_1}$, as given an embedding $\rho : \mathcal{T}_\alpha \leq \mathcal{T}_\beta$, we can inductively find $(\alpha_n : n < \omega)$ such that for all n, $\rho(\alpha_i : i < n) = (\alpha_i : 1 \leq i \leq n+1)$; but this clearly contradicts the hypothesized property of f. From this it follows that $(\mathcal{S}_\alpha : \alpha < \kappa)$ is the desired antichain.

In the other direction, suppose $(\mathcal{T}_\alpha : \alpha < \kappa(\omega))$ is a sequence of colored trees. Write $\kappa = \kappa(\omega)$; choose an elementary substructure $H \leq (\mathbb{V}_\kappa, \ldots)$ (using $<_{\mathbb{L}}$) such that H is the Skolem hull of an infinite set $\{\alpha_n : n < \omega\}$ of indiscernible ordinals. Then it is easy to check that $\mathcal{T}_{\alpha_0} \leq \mathcal{T}_{\alpha_1}$. ∎

We can now finish.

Proof of Theorem 3.3. Suppose A is a hereditarily countable set. We describe a colored tree $\mathcal{T}_A = (T_A, <_A, c_A)$, and then show that $\mathcal{T}_A \not\sim \mathcal{T}_{A'}$ for all $A \neq A'$. Moreover, the operation $A \mapsto \mathcal{T}_A$ will be absolute for transitive models of ZFC$^-$.

Before proceeding, we indicate how we finish. Given $G \in \mathrm{Mod}(\mathrm{Graphs})$, let $g(G)$ be the $<_{\mathbb{L}[G]}$-least element of Mod(CT) which is isomorphic to $\mathcal{T}_{\mathrm{css}(G)}$, where css$(G)$ is the canonical Scott sentence of G. (Note that $\mathcal{T}_{\mathrm{css}(G)} \in (\mathrm{HC})^{\mathbb{L}[G]}$ since $(\mathrm{HC})^{\mathbb{L}[G]} \models \mathrm{ZFC}^-$, so $\mathcal{T}_{\mathrm{css}(G)}$ does have models with universe ω in $\mathbb{L}[G]$.) Clearly, for any $G, G' \models \mathrm{CT}$, if $G \cong G'$ then $\mathrm{css}(G) = \mathrm{css}(G')$ so $g(G) = g(G')$, and if $G \not\cong G'$ then $\mathrm{css}(G) \neq \mathrm{css}(G')$ and so $g(G) \not\sim g(G')$. To finish, note that g is computed correctly in any countable transitive model of ZFC$^-$. Hence g has a Δ^1_2-graph: $g(G) = \mathcal{T}$ if and only if for some or any well-founded $V \in \mathrm{Mod}(\mathrm{ZFC}^-)$ containing codes for G and $\mathcal{T}$, V believes $g(G) = \mathcal{T}$. Moreover, by Shoenfield's absoluteness theorem (or just repeating the proof), this continues to hold in every forcing extension, and hence g is an absolute Δ^1_2-reduction.

So we define $A \mapsto \mathcal{T}_A$. Let A be given, and let $\alpha = \mathrm{rnk}(A)$, where rnk is foundation rank. Let $(\mathcal{S}_\beta : \beta \leq \alpha)$ be the $<_{\mathbb{L}}$-least antichain of colored trees indexed by $\alpha + 1$. This is computed correctly in any transitive model of ZFC$^-$, since if M is any transitive model of ZFC$^-$ with $\alpha \in M$, then $\mathbb{L}^M$ does not believe that $\kappa(\omega)$ exists, and so $\mathbb{L}^M$ can find a $<_{\mathbb{L}^M}$-least sequence $(\mathcal{S}_\beta : \beta \leq \alpha)$ such that $\mathbb{L}^M \models (\mathcal{S}_\beta : \beta \leq \alpha)$ is an antichain. But the property of being an antichain of colored trees of length $\alpha+1$ is absolute to models of ZFC$^-$; thus $(\mathcal{S}_\beta : \beta \leq \alpha)$ is the $<_{\mathbb{L}}$-least antichain of colored trees indexed by $\alpha + 1$.

We define a preliminary colored tree $\mathcal{T}_{0,A} = (T_{0,A}, <_{0,A}, c_{0,A})$. First, let $(T_{0,A}, <_{0,A})$ be the tree of all nonempty finite sequences $(a_0, \ldots, a_n)$ from $\mathrm{tcl}(A \cup \{A\})$ such that $a_0 = A$ and $\mathrm{rnk}(a_0) > \mathrm{rnk}(a_1) > \cdots > \mathrm{rnk}(a_n)$. Given $(a_0, \ldots, a_n) \in T_{0,A}$, let $c_{0,A}(a_0, \ldots, a_n) = 0$ if $a_{n-1} \in a_n$, and $c_{0,A}(a_0, \ldots, a_n) = 1$ otherwise. Let $\mathcal{T}_A$ be obtained from $\mathcal{T}_{0,A}$ as follows: above each $(a_0, \ldots, a_n) \in T_{0,A}$, put a copy of $(S_\beta, <_{S_\beta})$ where β is the foundation rank of a_n; given $t \in S_\beta$, let the color of the copy of t above $(a_0, \ldots, a_n)$ be $c_{\mathcal{S}_\beta}(t) + 2$.

Suppose $\mathcal{T}_A \sim \mathcal{T}_{A'}$. Let $\alpha = \mathrm{rnk}(A)$ and $\alpha' = \mathrm{rnk}(A')$. Let $f : \mathcal{T}_A \leq \mathcal{T}_{A'}$ and $f' : \mathcal{T}_{A'} \leq \mathcal{T}_A$ witness that $\mathcal{T}_A \sim \mathcal{T}_{A'}$. Note that $f\restriction_{T_{A,0}}$ and $f'\restriction_{T_{A',0}}$ witness that $\mathcal{T}_{A,0}$ and $\mathcal{T}_{A',0}$ are biembeddable; since $\mathcal{T}_{A,0}$ is well-founded of rank α, and $\mathcal{T}_{A',0}$ is well-founded of rank α', this implies $\alpha = \alpha'$. Let $(\mathcal{S}_\beta : \beta \leq \alpha)$ be as above.

Now, consider the embedding

$$h := f' \circ f : \mathcal{T}_A \leq \mathcal{T}_A.$$

We claim that $h\restriction_{\mathcal{T}_{0,A}}$ must be the identity. This suffices, since it implies $\mathcal{T}_{0,A} \cong \mathcal{T}_{0,A'}$ and hence $A = A'$.

Suppose that $(a_0, \ldots, a_n) \in T_{0,A}$; write $\beta = \mathrm{rnk}(a_n)$ and $h(a_0, \ldots, a_n) = (b_0, \ldots, b_n)$. We show by induction on β that $a_n = b_n$; this suffices. Note that $\mathcal{S}_\beta \leq \mathcal{S}_{\mathrm{rnk}(b_n)}$, and hence $\mathrm{rnk}(b_n) = \beta$ also (this is the key point!).

If $\beta = 0$, then $a_n = b_n = \emptyset$. Suppose we have verified the claim for all $\gamma < \beta$. We show that for every $a \in \mathrm{tcl}(A \cup \{A\})$ with $\mathrm{rnk}(a) < \beta$, we have $a \in a_n$ if and only if $a \in b_n$. Indeed, suppose a is given. Write $h(a_0, \ldots, a_n, a) = (b_0, \ldots, b_n, b)$. By construction of the coloring, we have $a \in a_n$ if and only if $b \in b_n$; but by the inductive hypothesis, we have $a = b$.

4. Schröder–Bernstein properties for TFAG. We repeat a bit from the introduction.

DEFINITION 4.1. Suppose M, N are $\mathcal{L}$-structures. Then $f : M \leq N$ is an *embedding* if whenever R is a relation symbol of $\mathcal{L}$ then $f[R^M] \subseteq [R^N]$, and whenever F is a function symbol of $\mathcal{L}$ then $f \circ F^M = F^N \circ f$. Write $M \leq N$ if there is an embedding $f : M \to N$. Also, write $(M, \overline{a}) \leq (N, \overline{b})$ if there is an embedding $f : M \leq N$ with $f(\overline{a}) = \overline{b}$. Finally, write $M \sim N$ if $M \leq N \leq M$ and write $(M, \overline{a}) \sim (N, \overline{b})$ if $(M, \overline{a}) \leq (N, \overline{b}) \leq (M, \overline{a})$.

In the context of groups, we will only want to consider injective embeddings; formally, we then add a unary predicate for $\{(a, b) : a \neq b\}$.

DEFINITION 4.2. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. Then Φ has the *Schröder–Bernstein property* if whenever M, N are countable models of Φ, if $M \sim N$ then $M \cong N$.

This fails for TFAG, as first proved by Goodrick [6] and in a strong form by Calderoni and Thomas [1]. Nonetheless, the statement of Theorem 3.2 suggests a weaker property: is a group $G \models$ TFAG determined by $\{(G, a)/\!\sim\, : a \in G\}$? We will call this the 1-ary Schröder–Bernstein property. We generalize further:

DEFINITION 4.3. Suppose M, N are $\mathcal{L}$-structures, and $\overline{a}, \overline{b} \in M$ are tuples of the same length. By induction on the ordinal α we define what it means to have $(M, \overline{a}) \sim_\alpha (N, \overline{b})$.

- $(M, \overline{a}) \sim_0 (N, \overline{b})$ if and only if $(M, \overline{a}) \sim (N, \overline{b})$.
- For δ limit, $(M, \overline{a}) \sim_\delta (N, \overline{b})$ if and only if $(M, \overline{a}) \sim_\alpha (N, \overline{b})$ for all $\alpha < \delta$.

- $(M, \bar{a}) \sim_{\alpha+1} (N, \bar{b})$ if and only if for all $a \in M$ there is $b \in M$ with $(M, \bar{a}a) \sim_\alpha (N, \bar{b}b)$, and conversely.

Write $M \sim_\alpha N$ if $(M, \emptyset) \sim_\alpha (N, \emptyset)$.

Note the similarity between these clauses and those defining $\equiv_{\alpha\omega}$; the only change is in the base case.

Definition 4.4. Suppose $\alpha < \omega_1$. Then Φ has the *α-ary Schröder–Bernstein property* if for all countable models $M, N \models \Phi$, if $M \sim_\alpha N$ then $M \cong N$.

This notion can be extended to $\alpha \geq \omega_1$, with some care:

Definition 4.5. Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$. A *pinned name* for a model of Φ is a pair $(P, \dot{M})$, where P is a forcing notion, $P \Vdash \dot{M} \in \mathrm{Mod}(\check{\Phi})$, and $P \times P \Vdash \dot{M}_0 \cong \dot{M}_1$, where $\dot{M}_0$ is the copy of $\dot{M}$ in the first factor of $P \times P$, and $\dot{M}_1$ is the copy of $\dot{M}$ in the second factor of $P \times P$.

Suppose $(P, \dot{M})$ and $(Q, \dot{N})$ are pinned names for models Φ, and α is an ordinal. Then write $(P, \dot{M}) \sim_\alpha (Q, \dot{N})$ if $P \times Q \times R \Vdash \dot{M} \sim_\alpha \dot{N}$, where R is some or any forcing notion which makes $\alpha, P, Q, \dot{M}, \dot{N}$ all countable. Write $(P, \dot{M}) \cong (Q, \dot{N})$ if $P \times Q \Vdash \dot{M} \cong \dot{N}$.

Φ has the *α-ary Schröder–Bernstein property* if for all pinned names $(P, \dot{M})$, $(Q, \dot{N})$ for models of Φ, if $(P, \dot{M}) \sim_\alpha (Q, \dot{N})$ then $(P, \dot{M}) \cong (Q, \dot{N})$.

This does not conflict with the previous definition, by a downward Löwenheim–Skolem argument (see [16]). (In [16], canonical Scott sentences are used in place of pins, but this is equivalent.)

The following will serve as the only interface we need with the machinery of pins:

Lemma 4.6. *Suppose Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$, and α is an ordinal. Suppose there are $M, N \models \Phi$ such that $M \sim_\alpha N$ but $M \not\equiv_{\infty\omega} N$. Then Φ fails the α-ary Schröder–Bernstein property.*

Proof. Let P_M be the set of all finite partial functions from ω to M, and let $\dot{f}_M$ be the P_M-name for the generic surjection from ω onto $\check{M}$ added by P_M. Let $P_N, \dot{f}_N$ be defined similarly. Then $(P_M, \dot{f}_M^{-1}(\check{M}))$ and $(P_N, \dot{f}_N^{-1}(\check{N}))$ are pinned names for models of Φ, and it is easy to check that $(P_M, \dot{f}_M^{-1}(\check{M})) \sim_\alpha (P_N, \dot{f}_N^{-1}(\check{N}))$ but $(P_M, \dot{f}_M^{-1}(\check{M})) \not\cong (P_N, \dot{f}_N^{-1}(\check{N}))$. ■

Looking at the statement of Theorem 3.2, it is reasonable to ask if TFAG has the 1-ary Schröder–Bernstein property. This would have consequences for the complexity of TFAG, as the following theorem of [16] shows:

Theorem 4.7. *Suppose $\kappa(\omega)$ exists, and α is an ordinal. If Φ is a sentence of $\mathcal{L}_{\omega_1\omega}$ with the α-ary Schröder–Bernstein property, then Φ is not* aΔ^1_2*-complete (and hence not Borel complete).*

In this section, we prove Theorem 4, namely: for every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder–Bernstein property. The construction breaks down at $\kappa(\omega)$, so the following remains open:

QUESTION. Does TFAG have the $\kappa(\omega)$-ary Schröder–Bernstein property?

In the remainder of this section, we prove the following:

THEOREM 4.8. *Suppose $\kappa(\omega)$ does not exist. Then for every ordinal α,* TFAG *fails the α-ary Schröder–Bernstein property.*

Note that Theorem 4.7 follows: for every $\alpha < \kappa(\omega)$, TFAG fails the α-ary Schröder–Bernstein property. This is because we can always apply Theorem 4.8 in $\mathbb{V}_{\kappa(\omega)} = H(\kappa(\omega))$.

So, in the remainder of this section, suppose $\kappa(\omega)$ does not exist; equivalently, for every cardinal λ, there is an antichain of colored trees of length λ.

First of all, we note the following lemma:

LEMMA 4.9. *Suppose $\mathcal{I}, \mathcal{J}$ are countable index sets, not both empty; let $F : \Omega^p_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}}$ TFAG be the Borel reduction from the proof of Theorem* 2.2 (*that is, the composition of the reductions from Lemmas* 2.5 *and* 2.8). *Suppose $\overline{G}^0, \overline{G}^1 \in \mathrm{Mod}(\Omega_{\mathcal{I},\mathcal{J}})$ and $\alpha < \omega_1$. If $\overline{G}^0 \sim_{2\cdot(\omega\cdot\alpha)} \overline{G}^1$, then $F(\overline{G}^0) \sim_\alpha F(\overline{G}^1)$. Hence, if $\Omega^p_{\mathcal{I},\mathcal{J}}$ fails the α-ary Schröder–Bernstein property for every ordinal α, then so does* TFAG.

Proof. The final claim follows, since the first part continues to hold in forcing extensions.

Write $\mathcal{I}' = \mathcal{I} \cup \mathcal{J} \cup \{*_0, *_1\}$ (we suppose this is a disjoint union).

Let $F_0 : \Omega^p_{\mathcal{I},\mathcal{J}} \leq_{\mathrm{B}} \Omega^p_{\mathcal{I}',0}$ be as in Lemma 2.5 and let $F_1 : \Omega^p_{\omega,0} \leq_{\mathrm{B}}$ TFAG be as in Lemma 2.8.

First we look at F_0. We recap the definition of F_0, for the reader's convenience. Suppose $\overline{G} = (G, G_i : i \in \mathcal{I},\ \phi_j : j \in \mathcal{J}) \models \Omega^p_{\mathcal{I},\mathcal{J}}$ is countable. Define $G' = G \times G$; for each $i \in \mathcal{I}$, define G'_i to be the copy of G_i in the first factor of G'; for each $j \in \mathcal{J}$, define G'_j to be the graph of ϕ_j; define $G'_{*_0} = G \times 0$; and finally let G'_{*_1} be the graph of the identity function $\mathrm{id}_G : G \to G$. Then $F(G, G_i, \phi_j : i \in I,\ j \in J)$ is $\overline{G}' = (G', G'_{i'} : i' \in \mathcal{I}')$ (suppressing the coding that arranges everything to have universe ω).

Suppose $\overline{G}_0, \overline{G}_1 \models \Omega^p_{\mathcal{I},\mathcal{J}}$ are countable, and define $\overline{G}'_0, \overline{G}'_1$ as above. Then it is easy to check that for all $((a^0_i, a^1_i) : i < i_*)$ from $\overline{G}'_0$ and all $(b^0_i, b^1_i) : i < i_*)$ from $\overline{G}'_1$, if $f : (\overline{G}_0, (a^j_i : i < i_*,\ j < 2)) \leq (\overline{G}_1, (b^j_i : i < i_*,\ j < 2)$, then $f \times f : (\overline{G}'_0, ((a^0_i, a^1_i) : i < i_*)) \leq (\overline{G}'_1, ((b^0_i, b^1_i) : i < i_*))$. From this it follows by an easy inductive argument that for all $\beta < \omega_1$, if $(\overline{G}_0, (a^j_i : i < i_*,$

$j < 2)) \sim_{2\cdot\beta} (\overline{G}_1, (b^j_i : i < i_*, j < 2)$, then $(\overline{G}'_0, ((a^0_i, a^1_i) : i < i_*)) \sim_\beta (\overline{G}'_1, ((b^0_i, b^1_i) : i < i_*))$.

Next we look at F_1. Let $(\gamma_n : 1 \leq n < \omega)$ be as in Lemma 2.8, i.e. a sequence of algebraically independent units of $\mathbb{Q}_p$; and let $\gamma_0 = 1$. Let $\overline{G} = (\bigoplus_\omega \mathbb{Z}, G_n : n < \omega)$ be a countable model of $\Omega^p_{\omega,0}$; we only consider the case where $G_0 = G_1 = \bigoplus_\omega \mathbb{Z}$, without loss of generality. Then recall $F_1(\overline{G})$ is (isomorphic to) G, where G is the p-pure subgroup of $\bigoplus_\omega \mathbb{Z}_p$ generated by $\bigcup_n \gamma_n G_n$. Recall that every $a \in G$ can be written as a sum $\sum_{n<\omega} \gamma_n p^{k(n)} b_n$, where $k(n) \in \mathbb{Z}$ and $b_n \in G_n$ for all n, and all but finitely many $k(n), b_n$ are 0. Say that this is a *weak representation* of a (it may not be a full representation: we do not require that $p \nmid b_n$ in G_n.)

Suppose $\overline{G}^j = (\bigoplus_\omega \mathbb{Z}, G^j_n : n < \omega)$ are countable models of $\Omega^p_{\omega,0}$ for $j < 2$; let G^0, G^1 be defined from $\overline{G}^0, \overline{G}^1$ as above. Suppose $f : \overline{G}^0 \leq \overline{G}^1$. Define $f_* : \bigoplus_\omega \mathbb{Z}_p \to \bigoplus_\omega \mathbb{Z}_p$ via $f_*(\sum_n \gamma_n e_n) = \sum_n \gamma_n f(e_n)$, where $(e_n : n < \omega)$ is the standard basis. Moreover, $f_* \restriction_{G^0} : G^0 \leq G^1$, since f_* preserves the action of $\mathbb{Z}_p$.

Suppose $(a_i : i < i_*)$ is a sequence from $\bigoplus_\omega \mathbb{Z}$, and suppose $(a'_i : i < i_*)$ is a sequence from $\bigoplus_\omega \mathbb{Z}$. Suppose for each $i < i_*$, $a_i = \sum_{n \in \Gamma_i} \gamma_n p^{k_i(n)} b_{i,n}$ is a weak representation with respect to $\overline{G}^0$, and $a'_i = \sum_{n \in \Gamma_i} \gamma_n p^{k_i(n)} b'_{i,n}$ is a weak representation with respect to $\overline{G}^1$, for finite sets $\Gamma_i \subset \omega$. Suppose finally that $f : (\overline{G}^0, (b_{i,n} : n \in \Gamma_i, i < i_*)) \leq (\overline{G}^1, (b'_{i,n} : n \in \Gamma_i, i < i_*))$. Then note that each $f_*(p^{k_i(n)} b_{i,n})$ equals $p^{k_i(n)} b'_{i,n}$, hence each $f_*(a_i)$ equals a'_i, hence $f_* : (G^0, (a_i : i < i_*)) \leq (G^1, (a'_i : i < i_*))$.

From this, an easy inductive argument shows that if $(\overline{G}^0, (b_{i,n} : n \in \Gamma_i, i < i_*)) \sim_{\omega\cdot\alpha} (\overline{G}^1, (b'_{i,n} : n \in \Gamma_i, i < i_*))$, then $(G^0, (a_i : i < i_*)) \sim_\alpha (G^1, (a'_i : i < i_*))$. ∎

Thus it suffices to show that some $\Omega^p_{\mathcal{I},\mathcal{J}}$ fails the α-ary Schröder–Bernstein property for all α.

For the next lemma, we make the obvious definitions for $\Omega^p_{\mathcal{I},\mathcal{J}}$ in the case where the index sets are possibly uncountable. Recall that we are assuming $\kappa(\omega)$ does not exist; the following lemma is the only place this is used.

Lemma 4.10. *Suppose $\mathcal{I}, \mathcal{J}$ are index sets, and suppose $\overline{G}^0, \overline{G}^1 \models \Omega_{\mathcal{I},\mathcal{J}}$. Then we can find $\mathbf{F}(\overline{G}^0), \mathbf{F}(\overline{G}^1) \models \Omega^p_{\omega\times\omega\cup\{0,1\},\{0,1\}}$ such that $\overline{G}^0 \equiv_{\infty\omega} \overline{G}^1$ if and only if $\mathbf{F}(\overline{G}^0) \equiv_{\infty\omega} \mathbf{F}(\overline{G}^1)$, and for every ordinal β, if $\overline{G}^0 \sim_\beta \overline{G}^1$ then $\mathbf{F}(\overline{G}^0) \sim_\beta \mathbf{F}(\overline{G}^1)$.*

Proof. We can suppose $\mathcal{J} = \emptyset$, by applying the construction from Lemma 2.5.

Choose λ large enough so that $\mathcal{I}, \overline{G}^0, \overline{G}^1$ are all of size at most λ. We can suppose $\mathcal{I} = \lambda$.

Let $(\mathcal{T}_\gamma : \gamma < \lambda)$ be a family of pairwise-non-biembeddable colored trees. Let $\mathcal{T}$ be the colored tree such that $c_\mathcal{T}(0) = 0$ (say), and for each $\gamma < \lambda$, there are λ-many $t \in T$ of height 1 such that $\mathcal{T}_{\geq t} \cong \mathcal{T}_\gamma$, and for each $t \in T$ of height 1, $\mathcal{T}_{\geq t}$ is isomorphic to some such $\mathcal{T}_\gamma$.

Recall the definition of $\mathcal{T} \otimes \mathbb{Z} = (G_\mathcal{T}, G_{\mathcal{T},n,i}, \pi : n, i < \omega) \models \Omega^p_{\omega\times\omega,\{0\}}$ from Theorem 3.2. For each $\gamma < \lambda$, let $\mathcal{E}_\gamma$ be the set of all $t \in T$ of height 1 such that $\mathcal{T}_{\geq t} \cong \mathcal{T}_\gamma$. Let $\hat{G}_{\mathcal{T},\gamma}$ denote the subgroup of $G_\mathcal{T}$ spanned by $\mathcal{E}_\gamma$. Note that each $\hat{G}_{\mathcal{T},\gamma}$ is $\mathcal{L}_{\infty\omega}$-definable, since $(\mathcal{T}_\gamma : \gamma < \lambda)$ is an antichain, and so $g \in \hat{G}_{\mathcal{T},\gamma}$ if and only if $g = 0$ or else $\mathcal{T}_\gamma$ embeds into $\mathcal{T}^*_g$.

Let $\mathbf{F}(\overline{G}^\ell) = (G_\mathcal{T} \oplus G^\ell, G_{\mathcal{T},n,i}, H^0, H^1, \pi, \psi^\ell : n, i < \omega) \models \Omega^p_{\omega\times\omega\cup\{0,1\},\{0,1\}}$, where $H^0 = \mathcal{T} \otimes \mathbb{Z}$, $H^1 = G^\ell$, and where $\psi^\ell : G_\mathcal{T} \to G^\ell$ satisfies:

- $\psi^\ell(t) = 0$ for all $t \in T$ not of height 1,
- for every $\gamma < \lambda$, $\psi\restriction_{\mathcal{E}_\gamma} : \mathcal{E}_\gamma \to G^\ell_\gamma$ is λ-to-one.

It is easy to check that this works. ■

Thus, to finish it suffices to verify the following.

Lemma 4.11. *Suppose α_* is an ordinal. Then for some index set $\mathcal{I}$, there are $\overline{G}^0_*, \overline{G}^1_* \models \Omega^p_{\mathcal{I},\{0\}}$ with $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$ yet $\overline{G}^0_* \not\equiv_{\infty\omega} \overline{G}^1_*$.*

Our idea is the following: given $\overline{G} = (G, G_i : i \in \mathcal{I}, \phi) \models \Omega^p_{\mathcal{I},\{0\}}$, define $X^{\overline{G}} := G \backslash \bigcup_i G_i$ and define $\leq^{\overline{G}}$ to be the partial order of $X^{\overline{G}}$ given by: $a \leq^{\overline{G}} b$ if and only if $\phi^n(a) = b$ for some $n < \omega$ satisfying further the condition that $\phi^m(a) \in X^{\overline{G}}$ for all $m < n$. Then we will arrange that $(X^{\overline{G}^0_*}, \leq^{\overline{G}^0_*})$ is ill-founded, but $(X^{\overline{G}^1_*}, \leq^{\overline{G}^1_*})$ is well-founded. It turns out we can make $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$ without upsetting this.

We will be approximating $\overline{G}^0_*$ and $\overline{G}^1_*$ as a union of chains. To control the eventual behavior of $(X^{\overline{G}^i_*}, \leq^{\overline{G}^i_*})$, we will be defining upper bounds to the rank function at each stage. The following are the approximations we will be using:

Definition 4.12. Given an index set $\mathcal{I}$, let $\Gamma_\mathcal{I}$ denote all tuples $(\overline{G}, \mathcal{B}, \rho)$ where:

- $\overline{G} = (G, G_i, \phi : i \in \mathcal{I}) \models \Omega^p_{\mathcal{I},\{0\}}$;
- G is free abelian (this is not redundant, since $\Omega^p_{\mathcal{I},\{0\}}$ only asserts that $G \equiv_{\infty\omega} \bigoplus_\omega \mathbb{Z}$) and $\mathcal{B}$ is a basis of G;
- $\phi : G \to G$;

- $\rho : X^{\overline{G}} \to \mathrm{ON} \cup \{\infty\}$ satisfies: for all $a, b \in X^{\overline{G}}$, if $\phi(b) = a$ and $\rho(b) < \infty$ then $\rho(a) < \rho(b)$. Hence $\rho(a) \geq \mathrm{rnk}(a)$ where rnk is the rank function for $(X^{\overline{G}}, \leq^{\overline{G}})$;
- for all $a \in X$ and all nonzero $n \in \mathbb{Z}$, $\rho(a) = \rho(na)$.

When we write $\overline{G}, \overline{G}', \overline{G}^\ell$, etc., we will always have $\overline{G} = (G, G_i, \phi : i \in I)$, $\overline{G}' = (G', G'_i, \phi' : i \in I), \overline{G}^\ell = (G^\ell, G^\ell_i, \phi^\ell : i \in I)$, etc.

Definition 4.13. Suppose $\mathcal{I}, \mathcal{I}'$ are index sets with $\mathcal{I} \subsetneq \mathcal{I}'$. Suppose $(\overline{G}, \mathcal{B}, \rho) \in \Gamma_{\mathcal{I}}$ and $(\overline{G}', \mathcal{B}', \rho') \in \Gamma_{\mathcal{I}'}$. Then say that $(\overline{G}', \mathcal{B}', \rho')$ *extends* $(\overline{G}, \mathcal{B}, \rho)$ if:

- $G \subseteq G'$ and $\mathcal{B} \subseteq \mathcal{B}'$;
- for each $i \in \mathcal{I}$, $G'_i \cap G = G_i$;
- for each $i \in \mathcal{I}' \setminus \mathcal{I}$, $G'_i \cap G = 0$;
- $\phi' \restriction_G = \phi$;
- $\rho' \restriction_{X^{\overline{G}}} = \rho$.

The following lemma is immediate.

Lemma 4.14. *Suppose $\delta < \lambda^+$ is a limit ordinal, $(\mathcal{I}_\gamma : \gamma < \delta)$ is an increasing chain of index sets, and $((\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma) : \gamma < \delta)$ is a sequence satisfying each $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma) \in \Gamma_{\mathcal{I}_\gamma}$ and for $\gamma < \gamma'$, $(\overline{G}^{\gamma'}, \mathcal{B}^{\gamma'}, \rho^{\gamma'})$ extends $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma)$. Then the natural union of the chain $(\overline{G}, \mathcal{B}, \rho)$ extends each $(\overline{G}^\gamma, \mathcal{B}^\gamma, \rho^\gamma)$.*

The final set of definitions describe the embeddings we will use to arrange $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$.

Definition 4.15. If $(\overline{G}, \mathcal{B}, \rho) \in \Gamma_{\mathcal{I}}$, then say that H is a *basic subgroup* of G if H is spanned by $H \cap \mathcal{B}$.

Suppose $(\overline{G}, \mathcal{B}, \rho), (\overline{G}', \mathcal{B}', \rho') \in \Gamma_{\mathcal{I}}$. Then by a *partial* -1*-embedding* from $(\overline{G}, \mathcal{B}, \rho)$ into $(\overline{G}', \mathcal{B}', \rho')$, we mean a map f whose domain is a basic subgroup D of $\overline{G}$ satisfying the following (recall that we always write $\overline{G} = (G, G_i : i \in \mathcal{I}, \phi)$ and $\overline{G}' = (G', G'_i : i \in \mathcal{I}, \phi')$):

- D is closed under ϕ;
- $f : D \to G'$ is an injective homomorphism with $f[\mathcal{B} \cap D] \subseteq \mathcal{B}'$;
- $f[G_i] \subseteq G'_i$ for each $i \in \mathcal{I}$;
- for all $a \in D$, $\phi'(f(a)) = f(\phi(a))$.

For an ordinal $\alpha \geq 0$, say that f is a *partial α-embedding* if additionally $f[X^{\overline{G}} \cap D] \subseteq X^{\overline{G}'}$, and for all $a \in X^{\overline{G}} \cap D$, if $\rho(a) < \omega \cdot \alpha$, then $\rho(a) = \rho'(f(a))$. If $\mathrm{dom}(f) = G$ then we drop the word "partial."

Finally, we describe the construction of $\overline{G}^0_*, \overline{G}^1_*$. We will build them as a union of chains. In the outer layer, we will construct, by induction on

$n < \omega$, index sets $\mathcal{I}^n$, and, for each $\ell < 2$, $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n}) \in \Gamma_{\mathcal{I}^n}$ with a privileged element $e^n \in G^{0,n}$ for $n < \omega$, and for each $\ell < 2$ a set $\mathcal{F}^{\ell,n}$, satisfying various constraints. The goal is that $(e^n : n < \omega)$ will witness that $X^{\overline{G}^{0,n}}$ is ill-founded, and $\mathcal{F}^{\ell,n}$ will be a set of partial embeddings from $\overline{G}^{\ell,n}$ to $\overline{G}^{1-\ell,n}$, which will be used to arrange that $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$. Formally, we need the following requirements:

(1) for $n < m < \omega$, $(\overline{G}^{\ell,m}, \mathcal{B}^{\ell,m}, \rho^{\ell,m})$ extends $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$;
(2) $e_0 = 0$, and for $n > 0$, $e^n \in X^{\overline{G}^{0,n}}$; further, $\phi^{0,n+1}(e_{n+1}) = e_n$, so necessarily $\rho^{0,n}(e_n) = \infty$ for $n > 0$ ($\rho^{0,0}(e_0)$ does not make sense because $e_0 = 0 \notin X^{\overline{G}^{0,0}}$);
(3) for all $a \in X^{\overline{G}^{1,n}}$, $\rho^{1,n}(a) < \infty$;
(4) for all n, ℓ, $(\phi^{\ell,n})^n = 0$ (i.e. $\phi^{\ell,n}$ iterated n times is 0);
(5) each $\mathcal{F}^{\ell,n}$ is a set of tuples (α, D, R, f), where $-1 \leq \alpha \leq \alpha_*$, and f is a partial α-embedding from $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$ to $(\overline{G}^{1-\ell,n}, \mathcal{B}^{1-\ell,n}, \rho^{1-\ell,n})$ with domain D and range R;
(6) for each $n < m$ and each $\ell < 2$, $\mathcal{F}^{\ell,n} \subseteq \mathcal{F}^{\ell,m}$;
(7) if $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$ and $\alpha \geq 0$, then $(\alpha, R, D, f^{-1}) \in \mathcal{F}^{1-\ell,n}$ (in particular f^{-1} is a partial α-embedding);
(8) if $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$, and either $\beta < \alpha$ or $\beta = -1$, then for every $a \in G^{\ell,n+1}$, there are some $D' \supseteq D \cup \{a\}$, $R' \supseteq R$, and $f' \supseteq f$ such that $(\beta, D', R', f') \in \mathcal{F}^{\ell,n+1}$;
(9) $(\alpha_*, 0, 0, 0) \in \mathcal{F}^{0,0}$.

Having done this, let $(\overline{G}^\ell_*, \mathcal{B}^\ell_*, \rho^\ell_*)$ be the union of the chain $((\overline{G}^{\ell,m}, \mathcal{B}^{\ell,m}, \rho^{\ell,m}) : m < \omega)$, as promised by Lemma 4.14. Then $\overline{G}^0_* \not\equiv_{\infty\omega} \overline{G}^1_*$, since $(X^{\overline{G}^0_*}, \leq^{\overline{G}^0_*})$ is ill-founded (by (2)) while $(X^{\overline{G}^1_*}, \leq^{\overline{G}^1_*})$ is well-founded (by (3)). On the other hand, it is clear that for all $n < \omega$, for all $(\alpha, D, R, f) \in \mathcal{F}^{\ell,n}$ with $\alpha \geq 0$, and for all finite tuples $\overline{a} \in D$, we have $(\overline{G}^\ell_*, \overline{a}) \sim_\alpha (\overline{G}^{1-\ell}_*, f(\overline{a}))$ (by (8)). Thus $\overline{G}^0_* \sim_{\alpha_*} \overline{G}^1_*$.

So it remains to show this construction is possible. We can let $\overline{G}^{\ell,0}$ be arbitrary, subject to conditions (3) and (4) above. To extend from $\overline{G}^{\ell,n}$ to $\overline{G}^{\ell,n+1}$, we will need the following two lemmas.

Lemma 4.16. *Suppose $(\overline{G}^\ell, \mathcal{B}^\ell, \rho^\ell) \in \Gamma_{\mathcal{I}}$ for each $\ell < 2$. Suppose f is a partial -1-embedding from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^1, \mathcal{B}^1, \rho^1)$. Finally, suppose each $(\phi^i)^{n+1}$ is 0. Then we can find an index set $\mathcal{I}'$ and an extension $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ of $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ in $\Gamma_{\mathcal{I}'}$ such that $X^{\overline{G}_2} = X^{\overline{G}_1}$, f extends to a -1-embedding h from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^2, \mathcal{B}^2, \rho^2)$, and $(\phi^2)^{n+1} = 0$.*

Proof. Let D be the domain of f and let R be its range. Recall that we require D and R to be basic subgroups of G, that is, $\mathcal{B}^0 \cap D$ spans D. Let $\mathcal{I}' \supseteq \mathcal{I}$ be large enough.

Write $\mathcal{A} = \mathcal{B}^0 \setminus D$. Let $G^2 = G^1 \times \bigoplus_{\mathcal{A}} \mathbb{Z}$. Write $H = 0 \times \bigoplus_{\mathcal{A}} \mathbb{Z}$, and let $g : \mathrm{span}_{G^0}(\mathcal{A}) \cong H$ be the natural isomorphism. Let $\mathcal{B}^2$ be $\mathcal{B}^1 \cup g[\mathcal{A}]$. Define $h : G^0 \to G^2$ via $h{\restriction}_D = f$ and $h{\restriction}_{\mathrm{span}(\mathcal{A})} = g$.

Define $\phi^2 : G^2 \to G^2$ via $\phi^2{\restriction}_{G^1} = \phi^1$ and $\phi^2{\restriction}_H = h \circ \phi^0 \circ g^{-1}$. For each $i \in \mathcal{I}$, let $G^2_i = G^1_i$.

Let $(G^2_i : i \in \mathcal{I}' \setminus \mathcal{I})$ enumerate all singly generated pure subgroups of G^2 which are not contained in G^1. Note that then $X^{\overline{G}^2} = X^{\overline{G}^1}$, so we must let $\rho^2 = \rho^1$, and then clearly we are done. ■

LEMMA 4.17. *Suppose $(\overline{G}^\ell, \mathcal{B}^\ell, \rho^\ell) \in \Gamma_{\mathcal{I}}$ for each $\ell < 2$. Suppose $0 \leq \beta < \alpha$ and f is a partial α-embedding from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ such that f^{-1} is also a partial α-embedding. Finally, suppose each $(\phi^i)^{n+1}$ is* 0*. Then we can find an index set $\mathcal{I}'$ and an extension $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ of $(\overline{G}^1, \mathcal{B}^1, \rho^1)$ in $\Gamma_{\mathcal{I}'}$ such that:*

- *f extends to a β-embedding h from $(\overline{G}^0, \mathcal{B}^0, \rho^0)$ to $(\overline{G}^2, \mathcal{B}^2, \rho^2)$;*
- *h^{-1} is a partial β-embedding from $(\overline{G}^2, \mathcal{B}^2, \rho^2)$ to $(\overline{G}^0, \mathcal{B}^0, \rho^0)$;*
- *for all $a \in X^{\overline{G}^2} \setminus X^{\overline{G}^1}$, $\rho^2(a) < \omega \cdot \alpha$;*
- *$(\phi^2)^{n+1} = 0$.*

Proof. Let D be the domain of f and let R be its range. Let $\mathcal{I}' \supseteq \mathcal{I}$ be large enough.

Write $\mathcal{B}^0 = (\mathcal{B} \cap D) \cup \mathcal{A}$. Let $G^2 = G^1 \times \bigoplus_{\mathcal{A}} \mathbb{Z}$. Write $H = 0 \times \bigoplus_{\mathcal{A}} \mathbb{Z}$, and let $g : \mathrm{span}_{G^0}(\mathcal{A}) \cong H$ be the natural isomorphism. Let $\mathcal{B}^2$ be $\mathcal{B}^1 \cup g[\mathcal{A}]$. Define $h : G^0 \to G^2$ via $h{\restriction}_D = f$ and $h{\restriction}_{\mathrm{span}(\mathcal{A})} = g$.

Define $\phi^2 : G^2 \to G^2$ via $\phi^2{\restriction}_{G^1} = \phi^1$ and $\phi^2{\restriction}_H = h \circ \phi^0 \circ g^{-1}$. For each $i \in \mathcal{I}$, let $G^2_i = G^1_i$. It remains to define G^2_i for $i \in \mathcal{I}' \setminus \mathcal{I}$, and then to define ρ^2.

Let $(G^2_i : i \in \mathcal{I}' \setminus \mathcal{I})$ enumerate all singly generated pure subgroups of G^2 which are not contained in G^1 and which are not contained in $R + H$. Note that then $X^{\overline{G}^2} = X^{\overline{G}^1} \cup h[X^{\overline{G}^0}]$. We define ρ^2 as follows: Suppose $a \in X^{\overline{G}^2}$. If $a \in X^{\overline{G}^1}$ then we must let $\rho^2(a) = \rho^1(a)$. Suppose instead $a \in h[X^{\overline{G}^0}] \setminus X^{\overline{G}^1}$; write $a = h(a')$. If $\rho^0(a) < \omega \cdot \beta$ then let $\rho^2(a) = \rho^0(a)$. Otherwise, let k be largest such that there is $c' \in X^{\overline{G}^0}$ such that $(\phi^0)^k(c') = a'$, and for all $k' < k$, $(\phi^0)^{k'}(c') \in X^{\overline{G}^0}$, and finally $\rho^0(c') \geq \omega \cdot \beta$; let $\rho^2(a) = \omega \cdot \beta + k$. Note that $k \leq n$ since $(\phi_0)^{n+1} = 0$.

Now we claim this works. First of all:

CLAIM. *Suppose $a \in h[X^{\overline{G}^0}] \setminus X^{\overline{G}^1}$; write $h(a') = a$. Then $\rho^0(a') \geq \rho^2(a)$.*

Proof. This is immediate if $\rho^0(a') < \omega \cdot \beta$, so suppose instead $\rho^0(a') \geq \omega \cdot \beta$; let c', k be as in the definition of $\rho^2(a)$. Then $\rho^0(a') = \rho^0((\phi_0)^k(c')) \geq \rho^0(c') + k \geq \omega \cdot \beta + k = \rho^2(a)$. ■

We show $(\overline{G}^2, \mathcal{B}^2, \rho^2) \in \Gamma_{\mathcal{I}'}$. We must check that for all $a, b \in X^{\overline{G}^2}$ with $\phi^2(b) = a$ and $\rho^2(a) < \infty$, we have $\rho^2(b) < \rho^2(a)$. If $b \in X^{\overline{G}^1}$, then $a \in X^{\overline{G}^1}$ and this is clear.

Suppose $b \in h[X^{\overline{G}^0}] \backslash X^{\overline{G}^1}$ and $a \in X^{\overline{G}^1}$; note that $a \in f[X^{\overline{G}^0}] \subseteq h[X^{\overline{G}^0}]$; write $a = f(a')$ and $b = h(b')$.

We consider two further subcases. If $\rho^0(a') = \rho^1(a)$, then $\rho^2(a) = \rho^0(a') > \rho^0(b') \geq \rho^2(b)$, using the claim. If $\rho^0(a') \neq \rho^1(a)$, then since f, f^{-1} are both α-embeddings we must have $\rho^0(a'), \rho^1(a) \geq \omega \cdot \alpha$. Hence $\rho^2(a) = \rho^1(a) \geq \omega \cdot \alpha > \omega \cdot \beta + n \geq \rho^2(b)$.

Finally, suppose both $a, b \in h[X^{\overline{G}^0}] \backslash X^{\overline{G}^1}$. Write $a = h(a')$ and $b = h(b')$. If $\rho^0(a') < \omega \cdot \beta$ then $\rho^2(a) = \rho^0(a') > \rho^0(b') \geq \rho^2(b)$. If $\rho^0(a') \geq \omega \cdot \beta$ and $\rho^0(b') < \omega \cdot \beta$, then $\rho^2(a) \geq \omega \cdot \beta > \rho^0(b') = \rho^2(b)$. Finally, if $\rho^0(a')$ and $\rho^0(b')$ are both $\geq \omega \cdot \beta$, then let k be as in the definition of $\rho^2(b)$, i.e. $\rho^2(b) = \omega \cdot \beta + k$; clearly then $\rho^2(a) \geq \omega \cdot \beta + (k+1)$.

To finish, it is clear that for all $a' \in \overline{G}^0$, if either $\rho^0(a') < \omega \cdot \beta$ or $\rho^2(h(a')) < \omega \cdot \beta$, then $\rho^0(a') = \rho^2(h(a'))$; hence h is a β-embedding and h^{-1} is a partial β-embedding. ■

Now, suppose we are given $(\overline{G}^{\ell,n}, \mathcal{B}^{\ell,n}, \rho^{\ell,n})$, $\mathcal{F}^{\ell,n}$, and e^n satisfying (1) through (9). We explain how to get $(\overline{G}^{\ell,n+1}, \mathcal{B}^{\ell,n+1}, \rho^{\ell,n+1})$, $\mathcal{F}^{\ell,n+1}$, and e^{n+1}.

Define $G^0 = G^{0,n} \times \mathbb{Z}$ and $e^{n+1} = (0,1) \in G^0$. Let $\mathcal{I} \supseteq \mathcal{I}^n$ be sufficiently large. For each $i \in \mathcal{I}^n$ let $G^0_i = G^{0,n}_i$. Choose $(G^0_i : i \in \mathcal{I} \setminus \mathcal{I}^n)$ so as to enumerate the singly-generated pure subgroups of G^0 which are not contained in $G^{0,n}$ and which do not contain e^{n+1}. Define ϕ^0 via $\phi^0 \restriction_{G^{0,n}} = \phi^{0,n}$ and $\phi^0(e^{n+1}) = e^n$ (or if $n = 0$ then let $\phi^0(e_1) = 0$). We have defined $\overline{G}^0 \models \Omega^p_{\mathcal{I},\{0\}}$, an extension of $\overline{G}^{0,n}$. Note that $X^{\overline{G}^0} = X^{\overline{G}^{n,0}} \cup \{me^{n+1} : m \in \mathbb{Z}, m \neq 0\}$. Let $\mathcal{B}^0 = \mathcal{B}^{0,n} \cup \{e^{n+1}\}$, and define each $\rho^0(me^{n+1})$ to be ∞.

Define $G^1 = G^{1,n}$; for each $i \in \mathcal{I}^n$, let $G^1_i = G^1$, and for each $i \in \mathcal{I} \setminus \mathcal{I}^n$, let $G^1_i = 0$; let $\phi^1 = \phi^{1,n}$. Finally, let $\mathcal{F}^\ell = \mathcal{F}^{\ell,n}$ for each $\ell < 2$.

The only thing left to do is arrange for (8) to hold. For this, apply Lemmas 4.16 and 4.17 repeatedly, using Lemma 4.14 at limit stages.

This concludes the proof of Theorem 4.8, and hence of Theorem 4.7.

Acknowledgements. We would like to thank Julia Knight for pointing out a gap in a previous version. We would also like to thank Chris Laskowski and the anonymous referee, for several comments which significantly improved the exposition of the paper.

The first author was supported by the National Science Foundation under Grant No. 136974 and by European Research Council grant 338821. This is paper 1141 on Shelah's list.

The second author was partially supported by Laskowski's NSF grant DMS-1308546.

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Saharon Shelah
Institute of Mathematics
Hebrew University
Jerusalem, Israel
E-mail: shelah@math.huji.ac.il

Douglas Ulrich
Department of Mathematics
University of California, Irvine
Irvine, CA 92697, U.S.A.
E-mail: dsulrich@uci.edu

Sh:1141