An Extension of M. C. R. Butler's Theorem on Endomorphism Rings

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Abstract We will prove the following theorem: Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.

Keywords Endomorphism rings • Butler's theorem

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1 Introduction

The main purpose of this paper is to honor the memory of Rüdiger Göbel, a dear friend and colleague, who passed away much too early. He made significant contributions on *realizing rings as endomorphism rings of abelian groups and modules* in many different settings. Most of this work can be found in the excellent monographs [4] and [5]. When Rüdiger came to Essen University, he started a successful research seminar. Among the first batch of papers studied was A. L. S. Corner's celebrated paper [2], where he proved that each countable torsion-free reduced ring R is the endomorphism ring of a countable torsion-free reduced abelian

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group G. If the additive group of such a ring R has finite rank n, then the group G can be constructed such that G has rank $\leq 2n$. Corner also provided examples of rings R such that the corresponding group G must have rank equal to $2n$. On the other hand, Zassenhaus [9] proved that for every ring R with identity and free additive group of finite rank, there is some abelian group M such that $R \subseteq M \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R = \mathrm{End}_{\mathbb{Z}}(M)$, i.e., R and M have the same rank.

Soon after [9] was written, Butler [1] generalized Zassenhaus's result replacing "free abelian of finite rank" by "locally free abelian of finite rank". Reid and Vinsonhaler [7] extended this result by replacing the ring of integers by certain Dedekind domains. More recently, Zassenhaus's result was generalized in [3] to rings with free additive groups of countable rank, whose elements are all algebraic over $\mathbb{Q}$. We will combine the results in [3] and [7] to obtain:

Theorem 1.1 *Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.*

After reading Corner's paper [2], it became a goal of Rüdiger's to remove the cardinality barrier in this result. Eventually, this was accomplished by utilizing powerful combinatorial tools such as the diamond principle and Shelah's Black Box.

2 The Results

Notation 2.1 *Let D denote a countable Dedekind domain of characteristic zero and with infinitely many prime ideals. Let F be its field of fractions. It follows that for any prime ideal P of D, the localization D_P of D at P is a PID with unique maximal ideal pD_P for some $p \in P$. Let $\widehat{D}_P$ denote the P-adic closure of D_P. Let $f(x) \in F[x]$. Then $f(x) \in D_P[x]$ with the leading coefficient a unit in D_P for all but finitely many prime ideals P of D. Define $N_P(f)$ to be the number of roots of $f(x)$ in $\widehat{D}_P$. We call D an* ***admissible domain*** *if for all $f(x) \in F[x]$ the set of prime ideals P of D with $N_P(f) \geq 1$ is infinite. If E is some D-module, then we call E* ***torsion-free*** *if $se = 0$ for $s \in D$ and $e \in E$ implies $s = 0$ or $e = 0$. Moreover, E is called* ***locally free****, if the localization $E_P = E \otimes_D D_P$ is a free D_P-module for all prime ideals P of D. If R is some ring and $a \in R$, we define the map $a\cdot$ from R to R to be the left multiplication by the element a, i.e., $(a\cdot)(x) = ax$ for all $x \in R$.*

Our main result will be the following:

Theorem 2.1 *Let D be an admissible domain and E a countable, torsion-free and locally free D-algebra such that each $a \in E$ is algebraic over F. Then there exists a locally free left E-submodule M of $FE = E \otimes_D F$ such that $E \subseteq M$ and $\mathrm{End}_D(M) = E\cdot$, the ring of left multiplications by elements of E.*

2.1 The Proof of Theorem 1.1

Before we turn to the proof of Theorem 2.1, note that Theorem 1.1 will be an immediate consequence provided:

Proposition 2.2 *Let D be the ring of algebraic integers of some finite Galois field extension of $\mathbb{Q}$. Then D is an admissible Dedekind domain.*

We need to show for all $f(x) \in F[x]$ the existence of infinitely many prime ideals P of D with $N_P(f) \geq 1$. We will line up some results from algebraic number theory to obtain this proposition. Note that for any $f(x) \in F[x]$ there exists some $d \in D$ with $df(x) \in F[x]$. Thus, we may restrict to polynomials $f(x) \in D[x]$. Furthermore, any polynomial is a product of irreducible ones and we may restrict to irreducible $f(x) \in D[x]$.

We recall the following, well-known version of Hensel's Lemma [6, Proposition 2, p. 43]:

Lemma 2.3 *Let $1 \in S$ be a commutative ring and $\mathfrak{m}$ an ideal of S such that S is complete in the $\mathfrak{m}$-adic topology. Let $f(x) \in S[x]$ and $a \in S$ be such that $f(a) \in f'(a)^2\mathfrak{m}$. Then there exists some $b \in S$ such that $f(b) = 0$ and $b - a \in f'(a)^2\mathfrak{m}$.*

Applying this to our situation:

Remark 2.4 Let P be a prime ideal of D and let $f(x) \in D[x]$ be irreducible of degree n over F. Then $f(x)$ has only simple roots and thus has non-zero discriminant $\Delta(f)$. Let P be a prime ideal of D such that $\Delta(f) \notin P$. Then $f(x) \bmod P$ has no multiple roots. Assume that $a \in \widehat{D}_P$ is such that $f(a) \in p\widehat{D}_P$. Then $f'(a) \notin p\widehat{D}_P$ and we may apply Lemma 2.3 to obtain $b \in \widehat{D}_P$ with $f(b) = 0$ and $b - a \in p\widehat{D}_P$. Thus, for irreducible $f(x) \in D[x]$, $f(a) \in p\widehat{D}_P$ implies $N_P(f) \geq 1$.

By the above it is sufficient to show that for any irreducible $f(x) \in D[x]$, there are infinitely many prime ideals P of D such that $f(x) \bmod P$ has a root in D/P. Hensel's Lemma will then provide a root of $f(x)$ in $\widehat{D}_P$.

First we recall some well-known definitions that are in [6] and many other sources.

Let k be an algebraic number field and K a Galois extension of k with Galois group G. Let $\mathscr{O}_k$ ($\mathscr{O}_K$) denote the ring of algebraic integers in k (K). Let $\mathfrak{p}$ be a prime (ideal) of $\mathscr{O}_k$ and $\mathfrak{P}$ a prime of $\mathscr{O}_K$ lying over $\mathfrak{p}$. Then $\mathscr{O}_K/\mathfrak{P}$ is a finite extension of the finite field $\mathscr{O}_k/\mathfrak{p}$ and thus a finite field of order $n_{\mathfrak{P}}$ with cyclic Galois group $G = \langle\overline{\sigma}\rangle$ over $\mathscr{O}_k/\mathfrak{p}$ where $\overline{\sigma}(x) = x^{n_{\mathfrak{P}}} \bmod \mathfrak{P}$. Let $G_{\mathfrak{P}} = \{g \in G : g\mathfrak{P} = \mathfrak{P}\}$ denote the *decomposition group* of $\mathfrak{P}$ and $T_{\mathfrak{P}} = \{g \in G : \overline{g} = \mathrm{id}_{\mathscr{O}_K/\mathfrak{P}}\}$ the *inertia group* of $\mathfrak{P}$. Then there exists some coset $\sigma T_{\mathfrak{P}} \in G_{\mathfrak{P}}/T_{\mathfrak{P}}$ which induces $\overline{\sigma}$. Any element of that coset is called a *Frobenius automorphism* which we denote by $\sigma(\mathfrak{P}, K/k)$. Now we need a celebrated theorem due to Chebotarev [6, Theorem 10, page 169]:

Theorem 2.5 (Chebotarev) *Let K be a Galois extension of k with Galois group G. Let $\emptyset \neq C \subseteq G$ be some set invariant under conjugations with $|C| = c$ and $[K : k] = n$. Let*

$$M = \{\text{primes } \mathfrak{p} \text{ of } k \mid \mathfrak{p} \text{ is unramified in } K \text{ and there is some prime } \mathfrak{P} \text{ of } K \text{ lying over } \mathfrak{p} \text{ such that } \sigma(\mathfrak{P}, K/k) \in C\}.$$

Then the set M has a density and this density is $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$ for all $C \subsetneq G$.

The definition of density in this context can be found in [6, page 167]. All we need to know is that only infinite sets have a positive density.

Now we need a result from [8]. We maintain our current notations.

Theorem 2.6 ([8, Theorem 1]) *Let $f(x) \in \mathcal{O}_k[x]$ have degree $n \geq 2$ and be irreducible over k. Let $N_{\mathfrak{p}}(f)$ be the number of roots of $f(x)(\text{mod } \mathcal{O}_k/\mathfrak{p})$ in $\mathcal{O}_k/\mathfrak{p}$. Let*

$$P_0(f) = \{\mathfrak{p} \text{ prime in } \mathcal{O}_k \mid N_{\mathfrak{p}}(f) = 0\}.$$

Then $P_0(f)$ has density $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$.

This shows that the set of all primes $\mathfrak{p}$ **not** in $P_0(f)$ has positive density and thus is infinite, completing the proof of Proposition 2.2.

Here is an outline of Serre's argument [8, page 432]: First, disregard all (finitely many) primes $\mathfrak{p}$ of $\mathcal{O}_k$ that are ramified or contain non-zero coefficients of $f(x)$. Let K be the splitting field of $f(x)$ over k with Galois group G and $\sigma = \sigma(\mathfrak{P}, K/k)$. Moreover, let X be the set of the n distinct roots of $f(x)$ in K. It turns out that $N_{\mathfrak{p}}(f)$ is the number of fixed points of $\sigma \restriction X$. Now put

$$G_0 = \{g \in G \mid g \restriction X \text{ has no fixed point}\}$$

and note that G_0 is invariant under conjugation, with $G_0 \subsetneq G$ since $\text{id}_K \notin G_0$. Now apply Theorem 2.5 with $C = G_0$.

2.2 *The Proof of Theorem 2.1*

We start with an easy observation.

Proposition 2.7 *Let $1 \in S$ be a commutative ring, A some S-algebra, and $\tau \in A$. Let $f(x) = \sum_{i=0}^{m} f_i x^i \in S[x]$, the polynomial ring over S. Then*

$$f(x) = f(\tau) + (x - \tau)(f_m \tau^{m-1} + g(\tau, x))$$

where $g(\tau, x) \in \text{span}_{\mathbb{Z}[x, f_0, \ldots, f_m]}\{\tau^j : 0 \leq j \leq m - 2\}$.

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Proof We evaluate

$$f(x) = f((x-\tau)+\tau) = \sum_{i=0}^{m} f_i[(x-\tau)+\tau]^i = \sum_{i=0}^{m} f_i\left[\sum_{j=0}^{i}\binom{i}{j}(x-\tau)^j\tau^{i-j}\right]$$

$$= \sum_{i=0}^{m} f_i\left[\tau^i + \sum_{j=1}^{i}\binom{i}{j}(x-\tau)^j\tau^{i-j}\right]$$

$$= f(\tau) + \sum_{i=0}^{m} f_i(x-\tau)\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}$$

$$= f(\tau) + (x-\tau)\left[\sum_{i=0}^{m} f_i\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}\right].$$

The highest power of τ that might occur in $\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}$ is τ^{i-1}. Note that

$$\sum_{j=1}^{m}\binom{m}{j}(x-\tau)^{j-1}\tau^{m-j} = \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k(-\tau)^{j-1-k}\right]\tau^{m-j}$$

$$= \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k\tau^{m-1-k}(-1)^{j-1-k}\right].$$

Thus τ^{m-1} only occurs for $k=0$ and with coefficient $\sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$. Recall that $\sum_{j=0}^{m}\binom{m}{j}(-1)^j = 0$ and thus $1 = \binom{m}{0} = -\sum_{j=1}^{m}\binom{m}{j}(-1)^j = \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$.

This shows that $f(x) = f(\tau) + (x-\tau)\left[f_m\tau^{m-1} + g(\tau,x)\right]$ where $g(\tau,x) \in \mathrm{span}_{\mathbb{Z}[x,f_0,\dots,f_m]}\{\tau^j : 0 \le j \le m-2\}$. □

Corollary 2.8 *Same notation as in the proposition. Let S be an integral domain with Q its field of fractions and $c \in S$ such that $f(c) \neq 0 = f(\tau)$. Then*

$$(c-\tau)^{-1} = \frac{1}{f(c)}(f_m\tau^{m-1} + g(\tau,x)) \in QA.$$

We also want to list:

Proposition 2.9 *Let F be a field and V some vector space over F. If $\tau \in \mathrm{End}_F(V)$ is algebraic over F, then τ has only finitely many eigenvalues.*

Proof There exists some monic polynomial $f(x) \in F[x]$ such that $f(\tau) = 0$. Let $0 \neq v \in V$ be an eigenvector of τ with eigenvalue λ. Then $I = \{g(x) \in F[x] : g(\tau \restriction_{vF}) = 0\} = (x-\lambda)F[x]$ is an ideal of $F[x]$ and $f(x) \in I$. This shows that λ is a root of $f(x)$, of which there are only finitely many. □

Lemma 2.10 *Let $\tau \in \mathrm{End}_D(E^+)$ such that τ is algebraic over F. Let $0 \neq e \in E$ and Π a finite number of prime ideals of D. Then there exists a prime ideal $P \notin \Pi$ of D and $c \in D$ such that $c - \tau$ is an automorphism of FE^+ and $e \notin E_P(c - \tau)$. Moreover, $E_P(c - \tau)^{-1} \subseteq p^{-k}E_P$ for some natural number k where $PD_P = pD_P$.*

Proof Let $g(x) = \sum_{i=0}^{n} g_i x^i \in F[x]$ be the minimal polynomial of τ over F with $g_n = 1$. Let $V = eF[\tau]$, a finite dimensional τ-invariant F-subspace of FE. Put $\theta = \tau \restriction_V$, the restriction of τ to V, and $f(x) = \sum_{i=0}^{m} f_i x^i \in F[x]$ the monic minimal polynomial of θ. Then $f(x)$ is a divisor of $g(x)$ and the set of all prime ideals Q of D for which $h(x) \notin D_Q[x]$ for any monic divisor $h(x)$ of $g(x)$ is finite. We may enlarge Π to contain the finitely many exceptions. By Proposition 2.7, we have, for any $s \in D$, that $g(s) = (s - \tau)(\tau^{n-1} + \sum_{i=0}^{n-2} s_i \tau^i)$ where $s_i \in D_Q$ for all prime ideals $Q \notin \Pi$. We infer that $s - \tau$ is an automorphism of FE^+ whenever $g(s) \neq 0$. In this case, we have that $E_Q(s - \tau)^{-1} \subseteq \frac{1}{g(s)} E_Q$. A similar statement holds for $s - \theta$.

Since D is admissible, there is an infinite set of prime ideals Q of D such that $f(x)$ has a root γ in the Q-adic completion of the discrete valuation domain D_Q. We choose such a prime ideal $P \notin \Pi$. Let $P = D \cap pD_P$ for some $p \in P$.

Let $V = eF[\tau] = e \cdot \mathrm{span}_F\{1, \tau, \tau^2, \ldots, \tau^{m-1}\}$ be the τ-invariant subspace of FE generated by e. Note that $\{e, e\theta, e\theta^2, \ldots, e\theta^{m-1}\}$ is a basis of V over F.

Let $V_P = V \cap E_P$, which is a free D_P-module of rank m. Let $W_P = e \cdot \mathrm{span}_{D_P}\{1, \theta, \theta^2, \ldots, \theta^{m-1}\}$, a free D_P-module of rank m. Since D_P is a PID, the Stacked Basis Theorem for finite rank free modules holds and we infer that $p^h V_P \subseteq W_P$ for some natural number h.

Let $\gamma_0 \in D$ be such that $\gamma \equiv \gamma_0 \bmod p^{h+1} D_P$. Then $f(\gamma_0 + p^{h+j}) \equiv 0 \bmod p^{h+1} D_P$ for all natural numbers $j \geq 1$. We infer the existence of some $c \in D$ such that

(1) $g(c) \neq 0$
(2) $f(c) \equiv 0 \bmod p^{h+1} D_P$.

Note that this implies $f(c) \neq 0$ and $g(c) \equiv 0 \bmod p^{h+1} D_P$ as well.

It follows from the above that $c - \tau \in \mathrm{End}_F(FE^+)$ is bijective with $E_P(c - \tau)^{-1} \subseteq \frac{1}{g(c)} E_P$. Moreover, $c - \theta \in \mathrm{End}_F(V)$ is bijective as well.

Assume that $e(c - \theta)^{-1} \in E_P$.

Since $e(c - \theta)^{-1} \in eF[\theta] = V$ as well, we infer that $e(c - \theta)^{-1} \in V_P$ and thus $p^h e(c - \theta)^{-1} = \frac{p^h}{f(c)} \left[e\theta^{m-1} + e\psi \right] \in W_P$ for some $\psi \in \mathrm{span}_{D_P}\{1, \theta, \theta^2, \ldots, \theta^{m-2}\}$. This is a contradiction since $\frac{1}{p} e\theta^{m-1} \notin W_P$. □

Corollary 2.11 *Let Π be a finite set of prime ideals of D and $0 \neq \psi \in \mathrm{End}_F(FE^+)$ such that $1\psi = 0$. Let $t \in E$ be such that $0 \neq t\psi$. Then there is a prime ideal $P \notin \Pi$ of D and a free D_P-submodule M_P of FE^+ such that*

(1) $E_P \subseteq M_P$,
(2) $M_P\psi \nsubseteq M_P$ and
(3) For each $x \in FE$ we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.

Note that (2) holds for any $\varphi \in \mathrm{End}_F(FE^+)$ in place of ψ such that $1\varphi = 0$ and $t\psi = t\varphi$.

Proof Let $0 \neq e = t\psi$. We may assume that $e \in E$. Define $\tau \in \mathrm{End}_F(FE^+)$ by $\tau(x) = xt$ for all $x \in FE$. Then $\tau E \subseteq E$ since E is a ring. Since t is algebraic over F, so is τ and we can apply Lemma 2.10 and find a prime ideal $P \notin \Pi$ and $c \in D$ such that $\sigma = c - \tau \in \mathrm{End}_F(FE^+)$ is bijective, $e \notin E_P\sigma$ and $E_P\sigma \subseteq E_P$. Moreover, $E_P\sigma^{-1} \subseteq p^{-k}E_P$ for some natural number k. We infer $p^kE_P \subseteq E_P\sigma \subseteq E_P$.

Let $M_P = p^{-k}E_P\sigma$. Since σ is injective, M_P is a free D_P-module.

Then $E \subseteq E_P \subseteq p^{-k}E_P\sigma = M_P$ and (1) holds. Moreover, $E_P \cdot M_P \subseteq M_P$ since the multiplication in FE is associative.

Since $1\psi = 0$, we have $-1\sigma\psi = -(c1 - \tau)\psi = t\psi = e$ and $p^{-k}e \in M_P\psi$ but $p^{-k}e \notin p^{-k}E_P\sigma = M_P$. This shows that $M_P\psi \nsubseteq M_P$ and we have (2).

Let $x \in FE$. Then $x(p^{-k}E_P\sigma) = p^{-k}(xE_P)\sigma$ is contained in $p^{-k}E_P\sigma$ if and only if $xE_P \subseteq E_P$ by the injectivity of σ. Since $1 \in E_P$, this holds if and only if $x \in E_P$, and (3) follows. □

Let $\mathrm{End}^0(FE^+) = \{\varphi \in \mathrm{End}_F(FE^+) : 1\varphi = 0\}$ be the set of all linear transformations of FE^+ that map the identity element of E to zero. Then $\mathrm{End}_F(FE^+) = \mathrm{End}^0(FE^+) \oplus ((FE^+)\cdot)$. There exists a countable subset $1 \notin B$ of E such that $FE = \mathrm{span}_F(B \cup \{1\})$. Note that if $0 \neq \varphi \in \mathrm{End}^0(FE^+)$, then there exists some $b \in B$ such that $b\varphi \neq 0$. Moreover, $b\varphi$ is an element of the *countable* (E is countable, cf. Notation 2.1 and Theorem 2.1) set FE. This shows that there exists a countable list $\{\varphi_n : n \in \mathbb{N}\}$ of elements of $\mathrm{End}^0(FE^+)$ such that for all $\tau \in \mathrm{End}^0(FE^+)$ there exists some $n \in \mathbb{N}$ and $b \in B$ such that $\tau(b) = \varphi_n(b) \neq 0$. We apply Corollary 2.11 repeatedly to find a sequence of distinct prime ideals P_n of D and free D_{P_n}-modules M_{P_n} with properties

(1_n) $E_{P_n} \subseteq M_{P_n}$
(2_n) $M_{P_n}\varphi_n \nsubseteq M_{P_n}$ and
(3_n) If $x \in FE$, then $xM_{P_n} \subseteq M_{P_n}$ if and only if $x \in E_{P_n}$.

If Q is a prime ideal not in the list $\{P_n : n \in \mathbb{N}\}$, we put $M_Q = E_Q$. Then we have

(1) $E_P \subseteq M_P$ for all prime ideals P of D and also
(3) For each $x \in FE$, we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.

Now let $M = \bigcap_P M_P$, where the intersection runs over all prime ideals P of D. Then $E \subseteq M$ by (1), and M is locally free since all M_P are free D_P-modules. Recall that $\mathrm{End}_D(M) = \bigcap_P \mathrm{End}_{D_P}(M_P)$. By (3) we get that

$$((FE)\cdot) \cap \mathrm{End}_D(M) = (E\cdot).$$

Let $0 \neq \psi \in \mathrm{End}^0(FE^+)$. Then there exists some $n \in \mathbb{N}$ such that, for some $b \in B$, we have $b\psi = b\varphi_n \neq 0$. By ($2_n$), we have that $M_{P_n}\psi \nsubseteq M_{P_n}$ which shows that $\mathrm{End}^0(FE^+) \cap \mathrm{End}_D(M) = \{0\}$. Let $\varphi \in \mathrm{End}_D(M)$. Then $\varphi = \psi + (x\cdot)$ for some $x \in FE$ and $\psi \in \mathrm{End}^0(FE^+)$. Pick $0 \neq s \in D$ with $sx \in E$. Then $s\varphi = s\psi + s(x\cdot) = s\psi + (sx\cdot)$, where $sx \in E$, and we infer

$$s\varphi - (sx\cdot) = s\psi \in \mathrm{End}^0(FE^+) \cap \mathrm{End}_D(M) = \{0\}.$$

Thus $\psi = 0$ and $\varphi = x\cdot$ for some $x \in E$ by condition (3). We conclude that $\mathrm{End}_D(M) = E\cdot$, as promised. □

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