Chapter 3

Failure of 0-1 law for sparse random graph in strong logics (Sh1062)

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Dedicated to Yuri Gurevich on the Occasion of his 75th Birthday

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0 Introduction

3.0(A) The Question

Let $G_{n,p}$ be the random graph with set of nodes $[n] = \{1, \ldots, n\}$, each edge of probability $p \in [0,1]_{\mathbb{R}}$, the edges being drawn independently, see $\boxplus_2$ below. On 0-1 laws (and random graphs) see Spencer [Spe01] or Alon and Spencer [AS08], in particular on the behaviour of the random graph $G_{n,1/n^\alpha}$ for $\alpha \in (0,1)_{\mathbb{R}}$ irrational. On finite model theory see Flum and Ebbinghaus [EF06], e.g., on the logic $\mathbb{L}_{\infty,\mathbf{k}}$ (see §1) and on LFP (least fixed point[1]) logic. A characteristic example of what can be expressed by it is "in the graph G there is a path from the node x to node y"; this is close to what we use. We know that $G_{n,p}$, i.e., p constant satisfies the 0-1 law for first order logic (proved independently by Fagin [Fag76] and Glebskii et al. [GKLT69]). This holds also for many stronger logics like $\mathbb{L}_{\infty,\mathbf{k}}$ and LFP logic. If $\alpha \in (0,1)_{\mathbb{R}}$ is irrational, the 0-1 law holds for $G_{n,(1/n^\alpha)}$ and first order logic, see, e.g., [AS08].

The question we address is whether this holds also for stronger logics as above. Though our main aim is to address the problem for the case of graphs, the proof seems more transparent when we have two random graph relations (we make them directed graphs just for extra transparency). So here we shall deal with two cases A and B. In Case A, the usual graph, we have to show that there are (just first order) formulas $\varphi_\ell(x,y)$ for $\ell = 1,2$ with some special properties (actually also φ_0), see Claim 1.2. For Case B, those formulas are $R_\ell(x,y), \ell = 1,2$, the two directed graph relations. Note that (for Case B), the satisfaction of the cases of the R_ℓ are decided directly by the drawing and so are independent, whereas for Case A there are (small) dependencies for different pairs, so the probability estimates are more complicated.

In the case of constant probability $p \in (0,1)_{\mathbb{R}}$, the 0-1 law is strong: it is obtained by proving elimination of quantifier and it works also for stronger logics: $\mathbb{L}_{\infty,\mathbf{k}}$ (see §2) and so also for the LFP logic $\mathbb{L}_{\mathrm{LFP}}$. Another worthwhile case is:

$\boxplus_1$ $G_{n,1/n^\alpha}$ where $\alpha \in (0,1)_{\mathbb{R}}$; so $p_n = 1/n^\alpha$.

Again the edges are drawn independently but the probability depends on n.

The 0-1 law holds if α is irrational, but we have elimination of quantifiers only up to (Boolean combinations of) existential formulas. Do we have 0-1 law also for those stronger logics? We shall show that by proving that for some so-called scheme of interpretation $\bar\varphi$, for random enough $G_n, \bar\varphi$ interpret number theory up to m_n where m_n is not too small, e.g., $m_n \geq \log_2 \log_2(n)$.

[1] There are some variants, but those are immaterial for our perspective.

A somewhat related problem asks whether for some logic the 0-1 law holds for $G_{n,p}$ (e.g., $p = \frac{1}{2}$) but does not have the elimination of quantifiers, see [Sh:1077].

We now try to informally describe the proof, naturally concentrating on case B.

Fix reals $\alpha_1 < \alpha_2$ from $(0, \frac{1}{4})_{\mathbb{R}}$ for transparency, so $\bar{\alpha} = (\alpha_1, \alpha_2)$ letting $\alpha(\ell) = \alpha_\ell$;

$\boxplus_2$ let the random digraph $G_{n,\bar{\alpha}} = ([n], R_1, R_2) = ([n], R_1^{G_{n,\bar{\alpha}}}, R_2^{G_{n,\bar{\alpha}}})$ with R_1, R_2 irreflexive 2-place relations drawn as follows:

- (a) for each $a \neq b$, we draw a truth value for $R_2(a,b)$ with probability $\frac{1}{n^{1-\alpha_2}}$ for yes
- (b) for each $a \neq b$, we draw a truth value for $R_1(a,b)$ with probability $\frac{1}{n^{1+\alpha_1}}$ for yes
- (c) those drawings are independent.

Now for random enough digraph $G = G_n = G_{n,\bar{\alpha}} = ([n], R_1, R_2)$ and node $a \in G$; we try to define the set $S_k = S_{G,a,k}$ of nodes of G not from $\cup\{S_m : m < k\}$ by induction on k as follows:

For $k = 0$ let $S_k = \{a\}$. Assume $S_0, \dots, S_k$ has been chosen, and we shall choose S_{k+1}.

$\boxplus_3$ For $\iota = 1, 2$ we ask: is there an R_ι-edge (a,b) with $a \in S_k$ and b not from $\cup\{S_m : m \leq k\}$?

If the answer is no for both $\iota = 1, 2$, we stop and let $\text{height}(a, G) = k$. If the answer is yes for $\iota = 1$, we let S_{k+1} be the set of b such that for some a the pair (a,b) is as above for $\iota = 1$. If the answer is no for $\iota = 1$ but yes for $\iota = 2$, we define S_{k+1} similarly using $\iota = 2$.

Let the height of G be $\max\{\text{height}(a, G) : a \in G\}$.

Now we can prove that for every random enough G_n, for $a \in G_n$ or easier - for most $a \in G_n$, for every not too large k we have:

$\boxplus_4$ $S_{G_n,a,k}$ is on one hand not empty and on the other hand with $\leq n^{2\alpha_2}$ members.

This is proved by drawing the edges not all at once but in k stages. In stage $m \leq k$ we already can compute $S_{G_n,a,0}, \dots, S_{G_n,a,m}$ and we have already drawn all the R_1-edges and R_2-edges having first node in $S_{G_n,a,0} \cup \dots \cup S_{G_n,a,m-1}$; that is for every such pair (a,b) we draw the truth values of $R_1(a,b), R_2(a,b)$. For $m = 0$ this is clear. So arriving to m we can draw the edges having the first node in S_m and not dealt with earlier, and hence can compute S_{m+1}.

The point is that in the question $\boxplus_3$ above, if the answer is yes for $\iota = 1$, then the number of nodes in S_{m+1} will be small, almost surely smaller than in S_m because its expected value is $|S_m| \cdot |[n] - \bigcup_{\ell \leq m} S_\ell| \cdot \frac{1}{n^{1+\alpha_1}} \leq n^{1+2\alpha_2-(1+\alpha_1)} = n^{2\alpha_2-\alpha_1}$ and the drawings are independent so except for an event of very small probability this is what will occur. Further, if for $\iota = 1$ the answer is no but for $\iota = 2$ the answer is yes, then almost surely S_m is smaller than a number near n^{α_1} but it is known that the R_2-valency of any node of G_n is near to n^{α_2}. Of course, the "almost surely" is such that the probability that at least one undesirable event mentioned above occurs is negligible.

So the desired inequality holds.

By a similar argument, if we stop at k then there is no R_2-edge from S_k into $[n] \backslash (S_0 \cup \ldots S_k)$ so the expected value is $\geq |S_k| \cdot (n - \sum_{\ell \leq k} (S_k)) \cdot \frac{1}{n^{1-\alpha_2}}$ hence in $S_0 \cup \cdots \cup S_k$ there are many nodes, e.g., at least near $n/2$ by a crude argument. As each S_m is not too large necessarily the height of G_n is large.

The next step is to express in our logic the relation $\{(a_1, b_1, a_2, b_2)$: for some k_1, k_2 we have $b_1 \in S_{G_n, a_1, k_1}, b_2 \in S_{G_n, a_2, k_2}$ and $k_1 \leq k_2\}$.

By this we can interpret a linear order with height(G_n) members. Again using the relevant logic, this suffices to interpret number theory up to this height. Working more we can define a linear order with n elements, so we can essentially find a formula "saying" n is even (or odd).

For random graphs we have to work harder: instead of having two relations we have two formulas; one of the complications is that their satisfaction for the relevant pairs is not fully independent.

In [Sh:1096] we shall deal with the strong failure of the 0-1 for Case A, i.e., G_{n,p^α}, (e.g., can "express" n is even) and also intend to deal with the α rational case. The irrationality can be replaced by discarding few exceptions.

We thank the referee for helping to improve the presentation.

3.0(B) History

The history is nontrivial having nontrivial opaque points. I have a clear memory of the events but vague on the exact statements and more so on the proof (and a concise entry in my (private F-list, [Sh:F159])) that in January 1996, in a Conference in DIMACS, Monica McArthur gave a lecture claiming that the graph $G_{n,\alpha}$ satisfies the 0-1 law not only for first order logic (by Shelah-Spencer [ShSp:304]) but also for a stronger logic. Joel Spencer said this coud be contradicted in a way he outlined. I thought about this and saw further things, and wrote them in a letter to Monica and Joel. I understood that it was agreed that Monica would write a paper with us saying more, but eventually she left academia.

As the referee found out, MacArthur's claim in [McA97] (DIMACS) failure of the law in $\mathbb{L}^\omega_{\infty,\omega}$, but refers the proof to a paper in preparation with Spencer that never appeared. She claims also that there is 0-1 law for $\mathbb{L}^k_{\infty,\omega}$ if $k = [1/\alpha]$, referring again to the paper in preparation. The later claim is not contradicted by the results of this paper. Lynch [Lyn97] refers also to a joint paper with McArthur and Spencer that never appeared proving that for the TC (transitive closure) logic satisfies the 0-1 law.

Having sent Joel (in 2011) an earlier version of this paper, his recollection of talking to Monica was that "we hadn't really gotten a handle on the situation".

Discussing with Simi Haber (December 2011), this question arose again. Trying to recollect it was not clear to me what the logic was: inductive logic? $\mathbb{L}_{\infty,k}$? Looking at it again, I saw a proof for the logic $\mathbb{L}_{\infty,k}$. No trace of the letter or the notes mentioned above were found. The only tangible evidence is in an entry [Sh:F159] from my F-list. Joel declined a suggestion that Haber, he and I deal with it, and eventually also Haber left.

The notes on §1 are from January 2012; for §2 from Sept. 4, 2012; revised in Nov./Dec. 2014 and expanded March 2015, June 2015.

The intention was that it would appear in the Yurifest, commemorating Yuri Gurevich's 75th birthday, but it was not in a final version in time, so only a short version (with the abstract and §(0A)) appears in the Yurifest volume, [Sh:1061].

3.0(C) Preliminaries

Notation 0.1. 1) $n \in \mathbb{N}\backslash\{0\}$ will be used for "$G_n \in K_n$ random enough".
2) G, H denote graphs and M, N denote more general structures = models.
3) a, b, c, d, e denote nodes of graphs or elements of structures.
4) m, k, ℓ denote natural numbers.
5) τ denotes a vocabulary, M a model with vocabulary $\tau = \tau_M$ (see 0.1(9),(10) below).
6) $\mathscr{L}$ denotes a logic, $\mathbb{L}$ is first order logic, so $\mathbb{L}(\tau)$ is first order language (set of formulas) for the vocabulary τ. $\mathscr{L}(\tau)$ is the language for the logic $\mathscr{L}$ and the vocabulary τ.
7) $\mathbb{L}_{\text{LFP}}$ is the least fixed point logic, abbreviated LFP.
8)

- (a) Let ${}^kA$ be the set of sequences η of length k of members of A, i.e., $\eta = \langle a_0, \ldots, a_{k-1}\rangle$ where $\bigwedge\limits_{\ell<k} a_\ell \in A$, so $a_\ell = \eta(\ell)$.
- (b) For a set u, e.g., of natural number let $\bar{x}_{[u]} = \langle x_s : s \in u\rangle$.
- (c) If $\varphi(\bar{x}_m, \bar{y}) \in \mathscr{L}(\tau)$ and M is a τ-model and $\bar{b} \in {}^{\ell g(\bar{y})}M$ and $\bar{x}_m = \langle x_i : i < m\rangle$, then $\varphi(M, \bar{b}) = \{\bar{a} \in {}^mM : M \models \varphi[\bar{a}, \bar{b}]\}$.

9) Let τ_{gr} denote the vocabulary of graphs, but we may write $\mathbb{L}$(graph) or $\mathscr{L}$(graph) instead of $\mathbb{L}(\tau_{\text{gr}}), \mathscr{L}(\tau_{\text{gr}})$. So τ_{gr} consists of one two-place predicate R, (below always interpreted as a symmetric irreflexive relation).
10) Let τ_{dg} consist of two two-place predicates, below always interpreted as irreflexive relations. Let $\tau_{\mathbb{N}}$ be the vocabulary from 0.2(1).

11) We define the function $\log_*$ from $\mathbb{R}_{\geq 0}$ to $\mathbb{N}$ by:
$\log_*(x)$ is 0 if $x < 2$
$\log_*(x)$ is $\log_*(\log_2(x)) + 1$ if $x \geq 2$
12) $|u|$ is the cardinality = the number of elements of a set u.

Explanation 0.2. 1) Recall above that the vocabulary of the structure $\mathbb{N}$ is the set of symbols $\{0, 1, +, \times, <\}$ where 0,1 are individual constants (interpreted in $\mathbb{N}$ as the corresponding elements) and $+, \times$ are two-place function symbols interpreted as $+^{\mathbb{N}}, \times^{\mathbb{N}}$ the two-place functions of addition and multiplication, and $<$ is a two-place predicate (relation symbol) interpreted as $<^{\mathbb{N}}$, the usual order on $\mathbb{N}$.
2) In general

- (A) a vocabulary is a set of predicates, individual constants and function symbols each with a given arity (number of places); individual constants (like 0,1 above) are considered as 0-place function symbols
- (B) a τ-model or a τ-structure M consists of:
  - (a) its universe, $|M|$, a nonempty set of elements so $\|M\|$ is their number
  - (b) if $P \in \tau$ is an n-place predicate, P^M is a set of n-tuples of members of M
  - (c) if $F \in T$ is an n-place function symbol, then F^M is an n-place function from $|M|$ to $|M|$.

Definition 0.3. Let τ be a finite vocabulary, for simplicity with predicates only or we just consider a function as a relation; here we use $\tau_{\rm gr}, \tau_{\rm dg}$ only except when we interpret.
1) We say $\bar{\varphi}$ is in a (τ_*, τ)-scheme of interpretaion <u>when</u>: (if τ is clear from the context we may write τ_*-scheme)

- (a) $\bar{\varphi} = \langle \varphi_R(\bar{x}_{n_\tau(R)}) : R \in \tau_* \cup \{=\}\rangle$ where $n_\tau(R)$ is the arity (number of places) of R
- (b) $\varphi_R \in \mathbb{L}(\tau)$
- (c) $\varphi_=(x_0, x_1)$ is always an equivalence relation on $\{y : (\varphi(y, y)\}$; if $\varphi_=$ is $(x_0 = x_1)$, then we may omit it.

2) For a τ-model M (here a graph or diagram) and $\bar{\varphi}$ as above, let $N = N_{M,\bar{\varphi}}$ be the following structure:

- (a) $|N|$ the set of elements of N, is $\{a/\varphi_=(M) : a \in M \text{ and } M \models \varphi_=(a, a)\}$; note that $\varphi(M)$ is an equivalence relation on $\{a : M \models \varphi_=(a, a)\}$

(*b*) if $R \in \tau$ has arity m, then R^N, the interpretation of r is $\{\langle a_\ell/\varphi_=(M) : \ell < m : M \models \bigwedge_{\ell<m} \varphi_=(a_\ell, a_\ell) \wedge \varphi_R(a_0, \ldots, a_{m-1}\}$ so $a_0, \ldots, a_{m-1} \in M\}$.

Recall that here "for every random enough G_n" is a central notion.

Definition 0.4. 1) A 0-1 context consists of:

(a) a vocabulary τ, here just the one of graphs or double directed graphs, see 0.1(5),(9),(10)

(b) for each n, K_n is a set of τ-models with set of elements = nodes $[n]$, in our case graphs or double directed graphs

(c) a distribution μ_n on K_n, i.e., $\mu_n : K_n \to [0,1]_{\mathbb{R}}$ satisfying $\Sigma\{\mu_n(G) : G \in K_n\} = 1$

(d) the random structure is called $G_n = G_{\mu_n}$ and we tend to speak on G_{μ_n} rather than on the context.

2) For a given 0-1 context, let "for every random enough G_n we have $G_n \models \psi$, i.e., G satisfies ψ" and "if G_n is random enough, then ψ", etc. means that the sequence $\langle \text{Prob}(G_n \models \psi) : n \in \mathbb{N}\rangle$ converge to 1; of course, $\text{Prob}(G_n \models \psi) = \Sigma\{\mu_n(G) : G \in K_n \text{ and } G \models \psi\}$.
3) For $\bar{p} = \langle p_n = p(n) : n\rangle$ a sequence of probabilities, $G_{n,\bar{p}}$ is the case $K_n =$ graphs on $[n]$ and we draw the edges independently

(a) with probability p when $\bar{p}$ is constantly p, e.g., $\frac{1}{2}$, and

(b) with probability $p(n)$ or p_n when p is a function from $\mathbb{N}$ to $[0,1]_{\mathbb{R}}$.

Below, we add the second context because for it the proof is more transparent.

Context 0.5. 1) <u>Case A</u>:

(a) $a \in (0,1)$ is irrational

(b) $p_n = 1/n^\alpha$.

2) <u>Case B</u>:
$\bar{\alpha}^* = (\alpha_1^*, \alpha_2^*)$ where $\alpha_1^*, \alpha_2^* \in (0, 1/4)$ are irrational numbers, (natural to add linearly independent over $\mathbb{Q}$) such that $0 < \alpha_1^* < \alpha_2^* < \alpha_2^* + \alpha_2^* < 1/2$ and let $\alpha_0^* = \alpha_1^*$.

Definition 0.6. For Case A:
1) Let $K^1 := \bigcup_n K^1_n$ where we let K^1_n be the set of graphs G on $[n] = \{1, \ldots, n\}$ so $R^G \subseteq \{\{i, j\} : i \neq j \in [n]\}$.
2) For $\alpha \in (0, 1)_{\mathbb{R}}$ let $G_n = G_{n;\alpha}$ be the random graph on $[n]$ with the probability of an edge being $1/n^\alpha$ and the drawing of the edges being independent.
3) Let $\mu_n = \mu_{n;\alpha}$ be the corresponding distribution on K^1_n; so $\mu_n : K^1_n \to [0, 1]_{\mathbb{R}}$ and $1 = \Sigma\{\mu_n(M) : M \in K_n\}$, in fact, $\mu_n(G) = (1/n^\alpha)^{|R^G|} \times (1 - 1/n^\alpha)^{\binom{n}{2} - |R^G|}$.

Convention 0.7. Writing K_n means we intend K^1_n or K^2_n (see below), similarly G_n is $G_{n,\alpha}$ if Case A and $G_{n,\bar{\alpha}}$ if Case B and similarly K is K^1 or K^2.

The more transparent related case is the following:

Definition 0.8. On Case B, for $G_{n;\bar{\alpha}}$:
1) Recall $\tau_{\rm dg}$ is the vocabulary $\{R_1, R_2\}$ intended to be two directed graph relations.
2) Let $K^2 = \bigcup_n K^2_n$ where we let $K^2_n = \{G : G = ([n], R^G_1, R^G_2)$ satisfying $([n], R^G_\ell)$ is a directed graph for $\ell = 1, 2$; we may write R_ℓ instead of R^G_ℓ when G is clear from the context$\}$. We assume[2] irreflexivity, i.e., $(a, a) \notin R^G_\ell$ but allow $(a, b), (b, a) \in R^G_\ell$.
3) For reals $\alpha_1 < \alpha_2$ from $(0, \frac{1}{4})_{\mathbb{R}}$, say from 0.5(2) so $\bar{\alpha} = (\alpha_1, \alpha_2)$ let $\alpha(\ell) = \alpha_\ell$; let the random model $G_{n;\bar{\alpha}} = ([n], R_1, R_2) = ([n], R^{G_{n;\bar{\alpha}}}_1, R^{G_{n;\bar{\alpha}}}_2)$ with R_1, R_2 irreflexive relations be drawn as follows:

- (a) for each $a \neq b$, we draw a truth value for $R_2(a, b)$ with probability $\frac{1}{n^{1-\alpha_2}}$ for yes
- (b) for each $a \neq b$, we draw a truth value for $R_1(a, b)$ with probability $\frac{1}{n^{1+\alpha_1}}$ for yes
- (c) those drawings are independent.

4) We define the distribution $\mu_{n;\bar{\alpha}}$ as follows:

- (a) $\mu_n = \mu_{n;\bar{\alpha}} = \mu_{n;\alpha_1,\alpha_2}$ is the following distributions on K^2_n:
  - $\mu_n(G) = \mu^2_{n;\alpha_2}([n], R^G_1)) \cdot \mu^2_{n;-\alpha_1}([n], R^G_2))$ where
  - $\mu^2_{n;\alpha}([n], R) = (\frac{1}{n^{1-\alpha}})^{|R^G|} \cdot (1 - \frac{1}{n^{1-\alpha}})^{n(n-1)-|R^G|}$

[2]We may change the definition of K^2_n by requiring $R^G_1 \cap R^G_2 = \emptyset$, this makes little difference. We could further demand R_ℓ is asymmetric, i.e., $(a, b) \in R^G_\ell \Rightarrow (b, a) \notin R^G_0$, again this makes little difference.

(b) $G_n = G_{n;\bar{\alpha}} = G_{n;\alpha_1,\alpha_2}$ denote a random enough $G \in K^2_n$ for $\mu_{n;\bar{\alpha}}$ so n is large enough.

Observation 0.9. *For random enough (recalling 0.4(2))* $G_n = G_{n;\bar{\alpha}} = G_{n;\alpha_1,\alpha_2}$*:*

(a) *For* $a \in [n]$*, the expected value of the* R_2*-valency of* a*, that is,* $|\{b : aR_2^G b\}|$ *is* $(n-1) \cdot \frac{1}{n^{1-\alpha(2)}} \sim n^{\alpha(2)}$*;*

(b) *for every random enough* $G_{n;\bar{\alpha}}$ *for every* $a \in [n]$ *this number is close enough to* $n^{\alpha(1)}$*, e.g.,*

- $\bullet_2$ *for some* $\varepsilon \in (0, \alpha_1)_{\mathbb{R}}$*, the probability of the difference being* $\geq n^{\alpha(1)(1-\varepsilon)}$ *for at least one* $a \in [n]$*, goes to zero with* n*;*

(c) *the expected number of* R_1*-edges is* $n(n-1)/n^{+(1+\alpha_1)} \sim n^{1-\alpha_1}$ *hence the expected value of* $|\{a : aR_1 b \text{ for some } b\}|$ *is close to it;*

(d) *for every random enough* $G_{n,\bar{\alpha}}$ *the two numbers in (c) are close enough to* $n^{1-\alpha_1}$ *(similarly to (b)).*

Remark 0.10. 1) For K^2, this is a parallel of Claim 1.2 for K^1.
2) Note that the Clause (a) does not imply clause (b) in 0.9 because the a priori variance may be too large.

1 On the logic $\mathbb{L}_{\infty,\mathbf{k}}$

As the proof for $\mathbb{L}_{\infty,\mathbf{k}}$ is simpler and more transparent than for LFP, we shall explain it.

First, we try to define and then explain the logic $\mathbb{L}_{\infty,\mathbf{k}}$ for $\mathbf{k}$ a finite number.

For a vocabulary τ, we define the set $\mathbb{L}_{\infty,\mathbf{k}}(\tau)$ of formulas as the closure of the set of atomic formulas under some operation similarly to first order logic, but:

- we restrict ourselves to formulas having $< \mathbf{k}$ free variables
- we allow arbitrary conjunctions and disjunctions (that is even infinite[3] ones)
- as in first order logic we allow negation $\neg\varphi$ and existential quantifier (on one variable) $\exists x \varphi(x, \bar{y})$.

So not only does any formula in $\mathbb{L}_{\infty,\mathbf{k}}$ have $< \mathbf{k}$ free variables, but also every subformula does.

It may be helpful to recall the standard game which express equivalence. Recall (0.3(1)) that for transparency we assume the vocabulary below has only predicates and is finite.

$\boxplus$ we say $\mathscr{F}$ is an $(M_1, M_2) - \mathbb{L}_{\infty,\mathbf{k}}$-equivalence witness <u>when</u> for some vocabulary τ with predicates only

  (a) M_1, M_2 are τ-models

  (b) $\mathscr{F}$ is a nonempty set of partial isomorphisms from M_1 to M_2

  (c) if $f \in \mathscr{F}$, then $|\mathrm{dom}(f)| < \mathbf{k}$

  (d) if $f \in \mathscr{F}, A \subseteq \mathrm{dom}(f), |A| + 1 < \mathbf{k}, \iota \in \{1, 2\}$ and $a_\iota \in M_\iota$, <u>then</u> there is g such that $g \in \mathscr{F}, f{\restriction}A \subseteq g$ and $\iota = 1 \Rightarrow a_\iota \in \mathrm{dom}(g)$ and $\iota = 2 \Rightarrow a_\iota \in \mathrm{rang}(g)$.

Now

$\oplus_1$ for M_1, M_2 as in (a) of $\boxplus$ above, the following are equivalent:

  (a) M_1, M_2 are $\mathbb{L}_{\infty,\mathbf{k}}$-equivalent, i.e., for every sentence $\psi \in \mathbb{L}_{\infty,\mathbf{k}}(\tau)$, i.e., a formula with no free variables, $M_1 \models \psi \Leftrightarrow M_2 \models \psi$

[3] As we consider only finite models, countable conjunctions and injunctions are enough.

(b) there is an $(M_1, M_2) - \mathbb{L}_{\infty,\mathbf{k}}$-equivalence witness $\mathscr{F}$, i.e., as in $\boxplus$.

Also

$\oplus_2$ for $M_1, M_2, \mathscr{F}$ as in $\boxplus$ above we have

(c) if $k < \mathbf{k}, a_0, \dots, a_{k-1} \in M_1$ and $g \in \mathscr{F}$ and $\{a_\ell : \ell < k\} \subseteq \mathrm{dom}(g)$, then for every formula $\varphi(x_0, \dots, x_{k-1}) \in \mathbb{L}_{\infty,\mathbf{k}}(\tau)$ we have

$$M_1 \models \varphi[a_0, \dots, a_{k-1}] \Leftrightarrow M_2 \models \varphi[g(a_0), \dots, g(a_{k-1})].$$

* * *

Having explained the logic, how can we prove for it the failure of the 0-1 law? Consider Case B where we have two kinds of edges, R_1 and R_2. Consider η a sequence from ${}^k\{1,2\}$, see 0.1(11) and $a \neq b$. There may be $(\eta, 0, k)$-pre-paths from a to b in G, see Definition 1.6, i.e., $a = a_0, a_1, \dots, a_k = b$ such that $(a_\ell, a_{\ell+1})$ is an $R_{\eta(\ell)}$-edge for $\ell < k$.

Now depending on η there may be many such pre-paths or few. If η is constantly 2 and $k > 1/\alpha_2^*$, then there are many such pre-paths - as fixing a in $G_{n,\bar{\alpha}^*}$ the expected number of b's for which there is pre-$(\eta, 0, k)$-paths from a to b is 1 for $k = 0$, is $\approx n^{\alpha_2^*}$ for $k = 1$ is $\approx n^{2\alpha_2^*}$ for $k = 2$, etc., so for $k > 1/\alpha_2^*$ it is every $b \in G_n$; not helpful. If η is constantly 1, there are few such pre-paths and they are all short, even $\leq k$ for any random enough G_n, when $1 < \alpha_1^* k$; not helpful.

But we may choose a "Goldilock's" η, that is, such that for every initial segment of η the expected number is not too large and not too small. This means that for some a for every $k' \leq k$ for some b there is such a pre-path but not too many. We need more so that we can define by a formula from $\mathbb{L}_{\infty,\mathbf{k}}$ the set $S_{G_n,a,k'} := \{b :$ there is such pre-path from a to b of length k' but not a shorter pre-path$\}$ and it is $\neq \emptyset$; moreover, we can define the natural order on the set $\{S_{G_n,a,k} : k \leq n\}$. Fact 1.4 below indicates what kind of η's we need, and we use it proving 1.8; however, in later sections, because we have to estimate the probabilities, we shall use only a closely related definition.

Hypothesis 1.1. 1) Case A of 0.5 holds or Case B there holds.
2)

(a) for case A: $\alpha_\ell^*, \varphi_\ell(x,y), n_\ell^*$ for $\ell = 0, 1, 2$ will be as in Claim 1.2 below

(b) for case B: α_1^*, α_2^* are as in 0.5 and $\varphi_\ell(x,y) = R_\ell(x,y)$ for $\ell = 1, 2$ and let $\alpha_0^* = \alpha_1^*, \varphi_0(x,y) = \varphi_1(x,y)$

(c) let $\bar{\varphi} = \langle \varphi_\ell(x,y) : \ell = 0, 1, 2 \rangle$.

Claim 1.2. Assume we are in Case A. There are $\alpha^*_\ell, \varphi_\ell(x,y)$ and γ^*_ℓ for $\ell = 0,1,2$ such that:

(a) $0 < \alpha^*_1 < \alpha^*_0 < \alpha^*_2$ are reals $\in (0,1/4)_\mathbb{R}$ and $\gamma^*_\ell \in \mathbb{R}_{>0}$

(b) $\varphi_\ell(x,y)$ are first order formulas (in the vocabulary of graphs) even existential positive formulas such that $\varphi_\ell(x,y) \vdash x \neq y$ for random enough $G_{n;\bar\alpha}$

(c) if $G_{n;\bar\alpha}$ is random enough, <u>then</u> for every $a \in G_{n;\bar\alpha}$ the set $\varphi_2(G_{n;\alpha}, a)$ has $\approx \gamma^*_\ell n^{\alpha^*_2}$ elements, i.e., for some $\varepsilon \in (0,1)_\mathbb{R}$, if $G_{n;\bar\alpha}$ is random enough, then for every $a \in [n]$, the number of members of $\varphi_1(G_{n;\bar\alpha}, a)$ belongs to the interval $(\gamma^*_\ell n^{\alpha^*_2} - n^{\alpha^*_2(1-\varepsilon)}, \gamma^*_2 n^{\alpha^*_2} + n^{\alpha^*_2(1-\varepsilon)})$

(d) if $\ell = 0,1$ and $G_{n;\bar\alpha}$ is random enough, <u>then</u> $\{a \in [n] : \varphi_\ell(G_{n;\bar\alpha}, a) \neq \emptyset\}$ has $\approx \gamma^*_\ell n / n^{\alpha^*_\ell}$ members.

Remark 1.3. We shall use not just the statements, but also the proof of 1.2 and 1.4.

Proof. Also here we shall use freely the analysis of $G_{n,\alpha}$ for $\alpha \in (0,1)_\mathbb{R}$ irrational (see, e.g., [AS08]).

Let m^*_2, n^*_2 be such that:

(a) n^*_2 is large enough

(b) $m^*_2 \leq \binom{n^*_2}{2}$

(c) $\alpha^*_2 := (n^*_2 - 1) - \alpha m^*_2$ is positive but, e.g., $< \frac{1}{12}$.

As $\alpha \in (0,1)_\mathbb{R}$ is irrational we can find such m^*_2, n^*_2. Let H^*_2 be a random enough graph on $[n^*_2]$ with m^*_2 edges such that $(1,2) \notin R^{H^*_2}$. (Note that this "random enough" is just used for the existence proof).

We choose n^*_1, m^*_1, H^*_1 similarly except that $-\alpha^*_1 := n^*_1 - 1 - \alpha m^*_1$ is negative with value close enough, e.g., to $-\alpha^*_2/3$. Lastly, we choose n^*_0, m^*_0, H^*_0 similarly except that $-\alpha^*_0 = n^*_0 - 1 - \alpha m^*_6$ is negative and $\alpha^*_0 \in (\alpha^*_1, \alpha^*_2)$.

For $\ell = 1,2$ let $\varphi_\ell(x,y) = (\exists \ldots x_i \ldots)_{i \in [n^*_\ell]} (x = x_1 \wedge y = x_2 \wedge \bigwedge\{x_i R x_j : i,j \in [m^*_\ell]$ satisfies $i R^{H^*_\ell} j\}) \wedge \bigwedge\{x_i \neq x_j : i \neq j \in [n^*_\ell]\}$.

Now check clauses (a)-(d). Clearly α^*_1, α^*_2 satisfies clause (a) and φ_1, φ_2 are as in clause (b).

For $\ell = 1,2$, let $\gamma^*_\ell = 1$. So for any n large enough compared to n^*_1, n^*_2 and $a_1 \neq a_2 \in [n]$, the set $\mathscr{F} := \{f : f$ is a one-to-one function from $[n^*_2]$ to $[n]$ such that $f(1) = a_1, f(2) = a_2\}$ has $\prod\limits_{i < n^*_2 - 2} (n - 2 - i) \sim n^{n^*_2 - 2}$ members.

For each $f \in \mathscr{F}$ the probability of the event $\mathscr{E}_f =$ "f maps every edge of H^*_2 to an edge of $G_{n,\alpha}$" is $(\frac{1}{n^\alpha})^{m^*_2}$ so the expected value of $\{f \in \mathscr{F} : \mathscr{E}_f$ occurs$\}$

is $\approx n^{n_2^*-\alpha m_2^*-2} = \frac{1}{n^{1-\alpha_2^*}}$. Clearly as in 0.9 the expected value is as required in clause (c) and by the well known analysis of $G_{n\alpha}$ (see, e.g., [AS08]), clause (c) holds and see more in §4.

Clause (d) is proved similarly. $\square_{1.2}$

Fact 1.4. There is a sequence $\eta \in {}^{\mathbb{N}}\{1,2\}$ such that: for every $n > 0$, $\gamma_n = |(\eta{\restriction}n)^{-1}\{2\}|\alpha_2^* - |(\eta{\restriction}n)^{-1}\{1\}|\alpha_1^*$ belongs[4] to $[\alpha_2^* - \alpha_1^*, \alpha_2^* + \alpha_2^*]_{\mathbb{R}}$.

Proof. We choose $\eta(n)$ by induction on n. Let $\eta(n)$ be 2 if $\gamma_n \le \alpha_2^*$, e.g., $n = 0$ and $\eta(n)$ be 1 if $\gamma_n > \alpha_2^*$.

Easily η is as required. $\square_{1.4}$

Claim 1.5. 1) If η is as in 1.4, <u>then</u> for any m and every random enough G_n, there is an (η, m)-path in G_n, see below.
2) Moreover, also there is an $(\eta, \varepsilon\lfloor\log(n)\rfloor$-path and even an $(\eta, \lfloor n^{\varepsilon}\rfloor)$-path for appropriate $\varepsilon \in \mathbb{R}_{>0}$.

Proof. As in [AS08] on $G_{n,1/n^{\alpha}}$ and see more in §3. $\square_{1.5}$

Definition 1.6. 1) A sequence $\bar{a} = \langle a_\ell : \ell \in [m_1, m_2]\rangle$ of nodes, that is, of members of $G \in K_n$ is called a pre-(ν, m_1, m_2)-path, if $m_1 = 0$ we may omit it, <u>when</u>:

- (a) ν is a sequence of length $\ge m_2$ and $i < \ell g(\nu) \Rightarrow \nu(i) \in \{1,2\}$
- (b) if $\ell \in \{m_1, m_1+1, \ldots, m_2-1\}$, then $G \models \varphi_{\nu(\ell)}(a_\ell, a_{\ell+1})$.

2) Above we say "(ν, m_1, m_2)-path" <u>when</u> in addition:

- (c) if $m_1 \le \ell_1 < \ell_2 \le m_2$ and $\langle a'_\ell : \ell \in [m_1, \ell_2)\rangle$ is a pre-(ν, m_1, ℓ_2)-path, <u>then</u> $a'_{m_1} = a_{m_1} \wedge a'_{\ell_2} = a_{\ell_2} \Rightarrow a'_{\ell_1} = a_{\ell_1}$
- (d) if $m_1 \le \ell_1 < \ell_2 \le m_2$, <u>then</u> $a_{\ell_1} \ne a_{\ell_2}$.

3) We say "$\bar{a}$ is a (pre)-(ν, m_1, m_2)-path from a to b" <u>when</u> in addition $a_{m_1} = a \wedge a_{m_2} = b$.

Remark 1.7. 1) Note that if $\langle a_\ell : \ell \le m\rangle$ is a pre-(ν, m)-path, it is possible that $\ell_1 + 1 < \ell_2 \le m$ and $a_{\ell_1} = a_{\ell_2}$. For a (ν, m)-path this is impossible.
2) In 1.6(2)(c), really the case $\ell_2 = m_2$ suffices.
3) We use the "$\log(n)$" in case 1.5(2), but having $\log(\log(n))$ and even much less has no real effect on the proof.

Conclusion 1.8. *Let* $\mathbf{k} \ge \max\{n_0^*, n_1^*, n_2^*\}$; <u>*then*</u> G_n *fails the 0-1 law for* $\mathbb{L}_{\infty,\mathbf{k}}$.

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[4] We will also use other intervals and similar sequences.

Remark 1.9. 1) Note that if $\langle a_\ell : \ell \le m\rangle$ is a pre-(ν, m)-path, it is possible that $\ell_1 + 1 < \ell_2 \le m$ and $a_{\ell_1} = a_{\ell_2}$. For a (ν, m)-path this is impossible.
2) In 1.6(2)(c), really the case $\ell_2 = m_2$ suffices.
3) We use the "log(n)" in 1.5(2), but having $\log\log(n)$ and even much less has no real effect on the proof.

Note that we rely on 1.5(2) but we prove more in §3.

Proof. For a finite graph G and η as in 1.4 or any $\eta \in {}^{\mathbb{N}}\{1,2\}$ let $\text{length}_\eta(G)$ be the maximal m such that there is an (η, m)-path in G.

Now consider the statement

$\oplus$ there is a sentence $\psi_m = \psi_{\eta,m} \in \mathbb{L}_{\infty,\mathbf{k}}(\tau_{\text{gr}})$ such that for a finite graph $G, G \models \psi_m$ iff there is an (η, m)-path in G.

Why $\oplus$ is enough? Because then we let

$$\psi = \bigvee\{(\psi_m \wedge \neg\psi_{m+1}) : m \ge 10 \text{ and } (\log_*(m) \text{ is even})\}$$

where $\log_*(m)$ is essentially the inverse of the tower function, see 0.1(3). Note that using 1.4, 1.5(2), of course, we should be able to say much more.

Why $\oplus$ is true? First, we define the formula $\psi_{m_1,m_2}(x, y)$ for $m_1 \le m_2$ by induction on $m_2 - m_1$ as follows:

$(*)_1$ if $m_1 = m_2$ it is $x = y$

$(*)_2$ if $m_1 < m_2$ it is $(\exists x_1)[\varphi_{\eta(m_1)}(x, x_1) \wedge \psi_{m_1+1,m_2}(x_1, y)]$.

So clearly

$(*)_3$ if $G \in K_n$ and $a, b \in [n]$, then $G \models \psi_{m_1,m_2}(a, b)$ <u>iff</u> there is a pre-(η, m_1, m_2)-path from a to b.

Second, we define $\psi'_{m_2}(x, y)$ as $\psi_{0,m_2}(x, y) \wedge \bigwedge\limits_{\ell_1<\ell_2\le m_2} \neg(\exists z'_1, z''_1, z_2)[z'_1 \ne z''_2 \wedge \psi_{0,\ell_1}(x, z'_1) \wedge \psi_{0,\ell_1}(x, z''_1) \wedge \psi_{\ell_1,\ell_2}(z'_1, z_2) \wedge \psi_{\ell_1,\ell_2}(z''_1, z_2) \wedge \psi_{\ell_2,m_2}(z_2, y)]$.

This just formalizes 1.6(2)(c) so

$(*)_4$ $G \models \psi'_{m_2}(a, b)$ <u>iff</u> there is an (η, m_2)-path from a to b.

As said above this is enough. Note that complicating the sentence we may weaken the demand on G_n. $\square_{1.9}$

2 The LFP Logic

In this section we try to interpret an initial segment of number theory in a random enough $G \in K_n$, i.e., with set of nodes $[n]$. In Definition 2.2 for $G \in K_n$ and $a \in G$ we define a model $M_{G,a}$. Now in $M \in \mathbf{M}_{G,a_*}$, the equivalence classes of E^M represent natural numbers. Concentrating on Case B, starting with $\{a_*\}$ as the first equivalence class, its set of R_2-neighbors will usually be the second equivalence class. Generally, if for an equivalence class a/E^M we let the next one be the set $\mathrm{suc}(a/E^M) = \{b \in G : R_2(a', b)$ for some $a' \in a/E^M\}$, then we expect that $\mathrm{suc}(A/E^M)$ has $\approx |a/E^M| \cdot n^{\alpha_2}$ members. So if we continue in this way, shortly we get the equivalence classes cover essentially all the nodes of G. Hence we try to sometimes use the R_1-neighbors instead of the R_2-neighbors, but when? For $\mathbb{L}_{\infty,\mathbf{k}}(\tau_*)$ we can decide a priori, e.g., use η as in 1.4 and the proof of 1.8 so that the expected number will be small. But for LFP logic this is not clear, so we just say: use the R_1-neighbors if there is at least one and the R_2-neighbors otherwise, so this is close to what is done in 1.4, 1.5, 1.8 but not the same.

For case A we use φ_ℓ-neighbors instead of R_ℓ-neighbors for $\ell = 1, 2$ except that the question on existence is for φ_0-neighbors.

How do we from equivalence relations and the successor relation reconstruct the initial segment of number theory? This is exactly the power of definition by induction.

Naturally we need just

⊞ letting height(G) be the maximal number of E^M-equivalence classes for $M \in \mathbf{M}_{G,a}, a \in G$, we have:

  (∗) for every m, for every random enough $G_n, m \le \mathrm{height}(G)$; moreover, letting $f : \mathbb{N} \to \mathbb{N}$ be $f(n) = \log_*(n)$ for every random enough $G_n, f(n) \le \mathrm{height}(G_n)$.

For failure of 0-1 laws, ⊞ is enough, but we may wish to prove a stronger version, say finding a sentence ψ which for every random enough G_n, G_n satisfies ψ <u>iff</u> n is even.

We intend to return to it elsewhere; but for now note that for a set $A \subseteq G$ we can define (R is R_2 for Case B, φ_2 for Case A) $c\ell_{G_n}(A) = \{b : b \in A$ or $b \in G \backslash A$ but for no $c \in G \backslash A \backslash \{a\}$ do we have $(\forall x \in A)[R(x,c) \equiv R(x,b)]\}$. Now from a definition of a linear order on A we can derive one on $c\ell_{G_n}(A)$. We can replace R by any formula $\varphi(x,y)$. Now in our context, if we know that, with parameters, we define such A of size $\approx n^\varepsilon$ for appropriate ε, then we can define a linear order on $[n]$; why is there such A? because if $M \in \mathbf{M}_{G,a_*}$ and k is not too large, <u>then</u> there is $b \in M, \mathrm{lev}(b, M) = k$ such that there is a unique maximal $<_2^M$-path from a_* to b.

For $\mathbb{L}_{\infty,\mathbf{k}}$ this is much easier.

Context 2.1. (A) or (B):

(A) (case A of 0.5) the vocabulary τ_* is τ_{gr}, the one for the class of graphs, $\varphi_\ell(x,y), \ell = 0,1,2$ are as in 1.2 so $\in \mathbb{L}(\tau_*)$ and $\bar{\alpha}^* = (\alpha_0^*, \alpha_1^*, \alpha_2^*)$, is as there, $G = G_n = G_{n;\alpha}, K_n = K_n^1$ as in Definition 1.4,

(B) (case B of 0.5) $\tau_* = \tau_{\rm dg} = \{R_1, R_2\}, K_n = K_n^2$ and $\bar{\alpha}^* = (\alpha_1^*, \alpha_2^*)$ are as in 0.5(2) and $G_n = G_{n;\bar{\alpha}^*}$ and $\varphi_\ell(x,y) = xR_\ell y$ for $\ell = 1,2$, with G_n, K_n^2 as in Definition 0.8 and let $\alpha_0^* = \alpha_1^*, \varphi_0(x,y) = \varphi_1(x,y)$.

Definition 2.2. For $G \in K_n$ and $a_* \in G$ we define $\mathbf{M} = \mathbf{M}(G, a_*) = \mathbf{M}_{G,a_*}$ as the set of τ_1-structures of M such that (the vocabulary τ_1 is defined implicitly):

(A)
- (a) the universe of M is $P^M \subseteq [n]$
- (b) $c_*^M = a_*$, so c_* is an individual constant from τ_1
- (c) E^M is an equivalence relation on M
- (d) $<_1$ is a linear order on P^M/E^M, i.e.,
  - (α) $a_1 E^M a_2 \wedge b_1 E^M b_2 \wedge a_1 <_1^M b_1 \Rightarrow a_2 <_1^M b_2$
  - (β) for every $a, b \in P^M$ exactly one of the following holds: $a <_1^M b, b <_1^M a$ and $aE^M b$
- (e) $<_2^M$ is a partial order included in $<_1^M$
- (f) (α) a_*/E^M is a singleton and a_* is $<_2^M$-minimal, i.e., $b \in M\backslash\{a_*\} \Rightarrow a <_2^M b$
  - (β) if $a <_1^M b <_1^M c$ and $a <_2^M c$, then for some $b' \in b/E^M$ we have $a <_2^M b' <_2^M c$
  - (γ) if $a, b \in M$ and b/E^M is the immediate successor of a/E^M, then for some $a' \in a/E^M$, we have $a' <_2^M b$
- (g) if $b \in M$ is a $<_2^M$-immediate successor of $a \in M$ (i.e., $a <_2^M b$ and $\neg(\exists y)(a <_2 y <_2 b)$, equivalently, $\neg(\exists y)(a <_1 y <_1 b)$), <u>then</u> for some $\iota \in \{1,2\}$ we have $G \models \varphi_\iota[a,b]$
- (h) $P_0, P_1, P_2 = P_+, P_3 = P_\times, P_4 = P_<$ are predicates (of τ_1) with 1,1,3,3,2 places respectively such that using the definitions in clauses (B)(a),(b),(c) below, P_ℓ^M are defined in clauses (B)(d) below

(B)
- (a) for $a \in M, \text{lev}(a, M)$ is the maximal k such that there are $a_0 <_1^M a_1^M < \ldots <_1^M a_k = a$; so necessarily $a_0 = a_*$
- (b) $\text{height}(M) = \max\{\text{lev}(a,M) : a \in M\}$

(c) for $k < \text{height}(M)$ let $\iota = \iota(k, M) \in \{1, 2\}$ be such that if b, is a $<_2^M$-immediate successor of a and $k = \text{lev}(a, M)$ then $G \models \varphi_\iota[a, b]$, in the unlikely case both $\iota = 1$ and $\iota = 2$ are as required we use $\iota = 1$

(d) (α) $P_0^M = \{a_*\}$

(β) $P_1^M = \{a \in M : \text{lev}(a, M) = 1\}$

(γ) $P_2^M = \{(a, b, c) : a, b, c \in M$ and $\mathbb{N} \models$ "$\text{lev}(a, M) + \text{lev}(b, M) = \text{lev}(c, M)$"$\}$

(δ) $P_3^M = \{(a, b, c) : a, b, c \in M$ and $\mathbb{N} \models$ "$\text{lev}(a, M) \times \text{lev}(b, M) = \text{lev}(c, M)$"$\}$

(ε) $P_4^M = \{(a, b) : N \models$ "$\text{lev}(a, M) < \text{lev}(a, N)$"$\}$.

Definition 2.3. 1) Let $\iota \in \{1, 2\}$.
We say N in the ι-successor of M in $\mathbf{M}_{G,a}$ when for some k

(a) $M, N \in \mathbf{M}_{G,a}$ so $G \in K_n$ for some n

(b) $M \subseteq N$ as models so $M = N{\restriction}|M|$, recalling $|M|$ is the set of elements of M

(c) $k = \text{height}(M)$ and $\text{height}(N) = k + 1$

(d) $b \in N \backslash M$ iff $\text{lev}(b, N) = k + 1$ iff $b \in G \backslash M$ and for some[5] $a \in M$ we have $\text{lev}(a, M) = k$ and $G \models \varphi_\iota[a, b]$.

2) We may omit ι above when: $\iota = 1$ iff $(*)$ holds where:

$(*)$ there is $c \in M$ such that $\text{lev}(c, M) = k = \text{height}(G)$ and for some $b \in G \backslash M$ we have $G \models \varphi_0[b, c]$; yes not φ_1! but for case B there is no difference.

3) For a sentence ψ in the vocabulary $\tau_1 \cup \tau_*$, for $M, N \in \mathbf{M}_{G,a}$, we say N is the ψ-successor of M when for some $\iota \in \{1, 2\}$, N is the ι-successor of M and $(G, M) \models \psi \Leftrightarrow (\iota = 1)$.
4) For $G \in K_n, a_* \in G$ and $M \in \mathbf{M}_{G,a_*}$ we define $\mathbb{N}_M$ as the following structure N with the vocabulary of number theory:

(a) set of elements $\{a/E^M : a \in M\}$

(b) $0^N = a_*/E^M = P_0^N$

[5]No real harm to demand here (and in 2.2) "unique"

(c) $1^N = P_1^M$

(d) if $\mathbf{a}_\ell = a_\ell / E^M \in N, a_\ell \in M$ for $\ell = 1, 2, 3$, then

 (α) $N \models$ "$\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$" iff $(a_1, a_2, a_3) \in P_2^M$

 (β) $N \models$ "$\mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_3$" iff $(a_1, a_2, a_3) \in P_3^M$

 (γ) $N \models$ "$\mathbf{a}_1 < \mathbf{a}_2$" iff $(a_1, a_2) \in P_4^M$.

Claim 2.4. 1) If ι, G, M, a are as in 2.3(1), then there is at most one ι-successor N of M in $\mathbf{M}_{G,a}$.
1A) For some $\psi_* \in \mathbb{L}(\tau_1 \cup \tau_*)$, being a ψ_*-successor is equivalent to being a successor.
2) For a given $G \in K_n, a \in G, M \in \mathbf{M}_{G,a}$ and $\psi \in \mathbb{L}(\tau_1 \cup \tau_*)$ there is at most one ψ-successor N of M in $\mathbf{M}_{G,a}$.
3) For $G \in K_n$ and $a \in G$ there is one and only one sequence $\langle M_k : k \leq k_{G,a} \rangle$ such that:

(a) $M_k \in \mathbf{M}_{G,a}$

(b) M_0 has universe $\{a\}$

(c) M_{k+1} is the successor of M_k in $\mathbf{M}_{G,a}$, recall 2.3(2)

(d) if $k = k_{G,a}$, then there is no $N \in \mathbf{M}_{G,a}$ which is the successor of M_k in $\mathbf{M}_{G,a}$

3A) Above $\mathbb{N}_{M_k,a}$ is isomorphic to $\mathbb{N} \restriction \{0, \ldots, k\}$. Also, for every sentence ψ in $\underset{\Delta}{\mathbb{L}}$ or even in $\underset{\Delta}{\mathbb{L}}_{\text{LFP}}$ in the vocabulary of number theory there is a sentence $\varphi \in \mathbb{L}_{\text{LFP}}(\tau_*)$ such that $\mathbb{N}_{M_k,a} \models \psi \Rightarrow M_{k,a} \models \varphi$; of course, φ depends on ψ but not on G, a (and k).
4) In the LFP logic for τ_*, we can find a sequence $\bar{\varphi}$ of formulas with[6] variable $x_0, \ldots, y$ such that: for any $G \in K_n$ and $a \in G$, the sequence $\bar{\varphi}$ substituting y by a defines $N = N_{G,a}$ which is M_k for $k = k_{G,a,\psi}$ from part (3).
4A) For ψ as in 2.3(3), i.e., $\psi \in \mathbb{L}(\tau_1 \cup \tau_*)$, a sentence, the parallel of 2.4(3),(4),(5) holds for "ψ-successor", (so we should write $N_{G,a,\psi}$ instead $M_{G,a}$).
5) Letting $\text{height}(G) = \max\{\text{height}(N_{G,a}) : a \in G\}$, in LFP logic there is $\varphi_*(x)$ such that $G \models \varphi_*(a)$ iff $a \in G$ and $\text{height}(N_{G,a}) = \text{height}(G)$ iff for every $a_1 \in G, \text{height}(N_{G,a_1}) \leq \text{height}(N_{G,a})$.
6) For any sentence ψ in the vocabulary of number theory (in first order or LFP logic) there is a sentence φ in induction logic for τ_* (recalling 2.1) such that for any $G, G \models \varphi$ iff $\mathbb{N} \restriction \{0, \ldots, \text{height}(G)\} \models \psi$.

[6] The variable y stands for the parameters a; instead we may define in 2.3 one model M_k coding all $M_{a,\ell} \in \mathbf{M}_{G,a}$ for $\ell \leq k, a \in G$.

Proof. 1) Read Definition 2.3(1).
1A) Read 2.3(2).
2) Read 2.3(3) and recall part (1).
3) We choose M_k and prove its uniqueness by induction on k till we are stuck. Recalling 2.4(3)(d) we are done.
3A) Easy.
4),4A) Should be clear.
5) We can express by induction when "$\text{lev}(b_1, \mathbf{M}_{G,a_1}) \leq \text{lev}(b_2, \mathbf{M}_{G,a_2})$".
6) Should be clear but we elaborate.

Recall the formula $\varphi_*(x) \in \mathbb{L}_{\text{LFP}}(\tau_*)$ from 2.4(5). By the choice of φ_* necessarily for some $a_*, G_n \models \varphi_*[a_*]$ (as in a finite nonempty set there is a maximal member) so $\text{height}(a_*, G_n) = \text{height}(G_n)$.

Now for a given ψ, let $\varphi \in \mathbb{L}_{\text{LFP}}(\tau_*)$ say: for some (equivalently every) $a \in G_n$ such that $G_n \models \varphi_*(a)$, the model $\mathbb{N}_{G_n,a_n}$ defined in 2.3(4), which is isomorphic to $\mathbb{N}{\restriction}\{0, \ldots, \text{height}(a, N_{G_n,a_n})\}$, see 2.4(3A), satisfies ψ. $\square_{2.4}$

Conclusion 2.5. *We have "G_n fail the 0-1 law for the LFP logic; moreover; for some $\varphi \in \mathbb{L}_{\text{ind}}(\tau_*)$ we have* $\text{Prob}(G_n \models \varphi)$ *has* $\lim - \sup = 1$ *and* $\liminf = 0$*".*

Proof. Should be clear by the above, in particular 2.4(6), see 3.3, 3.4(2) for details on the probabilistic estimate needed for 1.5(2) on which we rely. But we elaborate.

Note that just the following is not sufficient:

$(*)_1$ some $\bar{\varphi}, \mathbf{m}$ satisfies

- (a) $\bar{\varphi}$ an interpretation scheme, see 0.3
- (b) $\mathbf{m}$ is a function from finite graphs to $\mathbb{N}$, depending on the isomorphism type only
- (c) for every m for every random enough $G_n, \mathbf{m}(G_n) \geq m$
- (d) for random enough $G_n, \bar{\varphi}$ defines an isomorphic copy of $\mathbb{N}{\restriction}\{0, \ldots, \mathbf{m}(G_n)\}$.

However, it is enough if we add, e.g.,

$(*)_2$ $\mathbf{m}(G_n) \geq \log_2(\log_2(n))$.

Why it is enough? Let ψ be a first order sentence in the vocabulary such that $\mathbb{N}{\restriction}\{0, \ldots, k\} \models \varphi$ iff $\log_*(k)$ belong to $\{10n + \ell : \ell = \{0, 1, 2, 3, 4\} \text{ and } n \in \mathbb{N}\}$.

Now use the interpretation $\bar{\varphi}$, i.e., we use 2.4(6) in our case.

Why $(*)_1 + (*)_2$ holds: By 3.3, 3.4(1) and 2.4(3A). $\square_{2.5}$

3 Revisiting induction

As discussed in §2, we need to prove that for random enough G_n, height(G_n) is large enough, equivalently, for some $a \in G_n$ (we shall prove that even for most), height(a, G_n) is large enough. For this a more detailed specific statement is proved - see $(*)$ in the proof of 3.3. That is, we prove that for most $a \in G_n$ (for random enough G_n): on one hand $M_{G,a,=k}$ is not too large, and, on the other hand, is not empty; and for Case A, even not too small. The computation naturally depends on what $\eta_{G,a}$ is, see 3.2(3). This is a delicate point.

For Case B, things are simpler. For each k we ask if there is an R_1-edge out of $M_{G,a,=k}$ to $G \backslash M_{G,a,k}$. If there is, clearly $M_{G,a,=k+1}$ will be quite small but not empty. If not, then necessarily $M_{G,a,=k}$ has $\leq n^{\alpha_2^* - \zeta}$ members hence the number of R_2-neighbors of members of $M_{G,a,=k}$ cannot be too large (well $< n^{\alpha_2^*} n^{\alpha_2^*}$) so we are done.

Case A seems harder, so we simplify considering only small enough k, see 3.4, hence we can consider all possible η's of length k, that is, summing the probability of the "undesirable" events on all of them; so if each has small enough probability, even the unions of all those events has small enough probability. Now we divide the η's to those which are "reasonable candidates to be $\eta_{G_n,a}$" and those which are not. For the former η's, for almost all $a \in G_n$ there is a pre-$(\eta, 0, k_*)$-path starting with a. So it is enough to prove that for almost all $a \in G_n, \eta_{G,a} {\restriction} k^*$ is one of those former η's where k_* is the relevant large enough height, e.g., $\geq \lfloor \log_2(\log_2(n)) \rfloor$. For this it is enough to prove that the other η's cannot occur and this is what we do.

In this section we fulfill promises from §2 (and §1).

Context 3.1. As in 2.1.

Below we shall use

Definition 3.2. 1) For $G \in K_n$ we define $M_k(a, G) = M_{G,a,k}$ by induction as in 2.4(3) for $\psi = \psi_*$ from 2.4(1A) and also $k = k_{G,a}$ as there and height(G) as in 2.4(6).
2) Let $M_{G,a,=k} = M_{G,a,k} \backslash \cup \{M_{G,a,m} : m < k\}$ and similarly $M_{G,a,<k}$.
3) Let $\eta = \eta_{G,a}$ be the following sequence of length $k_{G,a}$: if $\ell < k_{G,\ell}$, then $\eta(\ell) = \iota(\ell, M_{G,a}) \in \{1, 2\}$ from Definition 2.2(B)(c).

Claim 3.3. For small enough $\varepsilon \in (0,1)_{\mathbb{R}}$, for random enough G_n, for some $a \in [n], k_{G_n,a} \geq k^* = \lfloor n^\varepsilon \rfloor$ in case B and $k_{G_n,a} \geq \lfloor \log(\log(n)) \rfloor$ in case A.

Remark 3.4. It would be nice to use an $\eta \in {}^{\mathbb{N}}\{1, 2\}$ defined similarly to 1.4, say such that $\gamma_n \in [\alpha_0^*, \alpha_2^* + \alpha_2^*]$, but this is not clear. In case B, in the proof the problem is that the γ_n-s from 1.4 may be very near to α_0^*. Also the parallel problem for case A is that the answer to the question asked there is near the critical stage, so we are not almost sure about the answer.

Proof. For case A, we presently prove it, e.g., for $k^* = \lfloor \log_2(\log_2 n) \rfloor$ and for case B $k^* = [n^\epsilon]$, an overkill, but this suffices for the failure of the 0-1 law. We intend to fill the general case elsewhere. Actually for any $\varepsilon \in (0,1)_{\mathbb{R}}$ we can get $k^* = \lfloor n^{1-\varepsilon} \rfloor$.

Let $\zeta \in (0,1)_{\mathbb{R}}$ be small enough and k^* be as above.

Clearly it is enough to prove:

(∗) if $a \in [n]$ and $k < k^*$, <u>then</u> the probability that at least one of the following $(i)_{a,k}, (ii)_{a,k}, (iii)_{a,k}$ fails (assuming $\ell < k \Rightarrow (i)_{a,\ell} \wedge (ii)_{a,\ell})$ is small enough; $< \frac{1}{k \log(n)}$ suffices, being $< \frac{1}{kn^i}$ for each i for large enough n is natural)

$(i)_{a,k}$ $k \leq k_{G_n,a}$

$(ii)_{a,k}$ $M_{G_n,a=k}$ has $\leq n^{\alpha_2^* + \alpha_2^*}$ elements

$(iii)_{a,k}$ $M_{G_n,a,k}$, (noting that $(i)_{a,k}$ implies $M_{G_n,\ell,k} \neq \emptyset$) has $\geq n^{\alpha_0^* - \zeta}$ elements[7] except when $k = 0$, <u>not need</u> for Case B.

Why does (∗) hold?

<u>Case 1</u> Case B of the Context 2.1 and Definition 0.8

We are given $n \geq 1$ and $a \in [n]$; we draw the edges in k stages so by induction on k. For $k = 0$ draw the edges starting with a (of both kinds, an overkill), i.e., for $\iota \in \{1,2\}$ the truth value of $R_\iota(a,b)$ for every $b \in [n] \backslash \{a\}$, hence we can compute $M_1(a,G)$.

The induction hypothesis on stage k is that $\langle M_{G,a,i} : i \leq k \rangle$ have been computed and we have drawn the truth value of $R_\iota(c,b)$ for $b \in \cup\{M_{G,a,i} : i < k\}$ and $c \in [n] \backslash \{b\}$. If $k < k_*$ we now draw the edges $R_\iota(b,c)$ for $b \in M_{G,a,=k}$ and any $c \neq b$; actually the $c \in M_{G,i,k}$ are irrelevant and so we can compute $M_{G,a,k+1}$. Now we ask: if $(i)_{a,m} + (ii)_{a,m}$ holds for $m \leq k$ what is the probability that $(i)_{a,k+1} + (ii)_{a,k+1}$? (recalling $(iii)_{a,k+1}$ is irrelevant), i.e., is it small enough? This is easy and as required.

In details, we ask

<u>Question</u>: Are there $c \in [n] \backslash M_{G,a,k}$ and $b \in M_{G,a,=k}$ such that $(b,c) \in R_1^G$?

First note

(∗) if $|M_{G,a,=k}| \geq n^{\alpha_2^* - \zeta}$, then the probability that the answer is no is $< 1/2^n$.

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[7]We can use $\geq n^{\alpha_2^* - \alpha_0^* - \zeta}$.

[Why? We have $M_{G,a,=k} \times ([n] \backslash M_{G,a,k})$ independent drawings so their number is $\geq n^{\alpha_2^* - \zeta} n/2$, each with probability $\frac{1}{n^{1+\alpha_1^*}}$ of success and $(1 + \alpha_2^* - \zeta) - (1 + \alpha_1^*) = \alpha_2^* - \zeta - \alpha_1^* > 0$ so the probability of the no answer is $(1 - \frac{1}{n^{1+\alpha_1^*}})^{n^{(1+\alpha_2^*-\zeta)}} \sim 1/e^{(n^{\alpha_2^*-\zeta-\alpha_1^*})/2)}$; clearly more than enough.]

By $(*)$ it suffices to deal with the following two possibilities.

Possibility 1: The answer is yes.

In this case $M_{G,a,k+1}$ is well defined and $\iota(k, M_{G,a,k}) = 1, M_{G,a,=k+1} = \{c : c \in G \backslash M_{G,a,k}$ and $(b,c) \in R_1^G$ for some $b \in M_{G,a=k}\}$. Now for each $c \in [n] \backslash M_{G,a,k}$ and $b \in M_{G,a,=k}$ the probability of $(c,b) \in R_1^G$ is $\frac{1}{n^{1+\alpha_1^*}}$ hence by the independence of the drawing, recalling $|M_{G,a,=k}| \leq n^{\alpha_2^*+\alpha_2^*}$ the probability of $|M_{G,a,=k+1}| \geq n^{\alpha_2^*+\alpha_2^*}$ is negligible, e.g., $< 2^n$ so can be ignored. Also by the possibility we are in, $M_{G,a,=k+1} \neq \emptyset$.

Possibility 2: The answer is no and $|M_{G,a,=k}| \leq n^{\alpha_2^* - \zeta}$.

This is easy, too, recalling that almost surely for every $a' \in G$ the number of R_2-neighbors in the interval $[n^{\alpha_2^*} - n^{\alpha_2^*(1-\zeta)}, n^{\alpha_2^*} + n^{\alpha_2^*(1-\zeta)}]$ and so the probability that $M_{G,a,=k+1}$ is too large is negligible.

Case 2: Case A of the context 2.1

Here it helps to use "φ_0, φ_1 are distinct".

Now it suffices to prove:

$(*)_1$ for random enough G_n, for[8] $a \in M$, the following has negligible probability of failure: $(i)_{a,k}, (ii)_{a,k}, (iii)_{a,k}$.

Note that for this it seems more transparent to[9] assume $k < \log_2(\log_2(n))$ and to translate $(*)_1$ to statement on paths.

$(*)_2$ For $k \leq k_*$ let

(a) $\Omega_{=k}$ be the set of $\eta \in {}^k\{1,2\}$ such that $\alpha(\eta) := |\eta^{-1}\{2\}| \cdot \alpha_2^* - |\eta^{-1}\{1\}| \cdot \alpha_1^*$ belongs to the interval $[0, \alpha_2^* + \alpha_2^* + \zeta]$

(b) Ω_k be the set of $\eta \in {}^k\{1,2\}$ such that for every $\ell < n$ the sequence $\eta \restriction \ell = \langle \eta(0), \ldots, \eta(\ell - 1) \rangle$ belongs to $\Omega_{=\ell}$

(c) Δ_k be the set of $\eta \in {}^k\{1,2\}$ such that $\eta \in \Omega_{=k}$ and even $\eta \in \Omega_k$ but $\alpha(\eta) \leq \alpha_0^* - \zeta$

$(*)_3$ recalling that $\zeta \in (0,1)_{\mathbb{N}}$ is small enough, for any random enough G_n, for every $a \in M$ the following has probability $\leq 1/n^{\zeta}$:

[8] We allow few a's for which this fails. It suffices to have "for some a", this helps for larger k.

[9] As then, we can consider all the relevant sequences η, (and more).

- for some $k \leq \lfloor \log(\log(n)) \rfloor$ and $\eta \in {}^k\{1,2\}$ at least one of the following holds

$(a)_\eta$ $\eta \in \Omega_k$ but there is no pre-$(\eta, 0, k)$-path in G_n starting with a

$(b)_\eta$ $\eta \in \Delta_k$ but there is a pre-$(\eta, 0, k+1)$-path in G_n from a to some $b \in G_n \backslash \{a\}$ such that $G_n \models (\exists x)\varphi_0(b, x)$.

Otherwise the proof is as in the earlier case. $\square_{3.3}$

Claim 3.5. Let $G_n = G_{n;\bar{\alpha}^*}$, i.e., we are in Case B. For some $(\tau_{\mathbb{N}}, \tau_{\text{dn}})$-scheme $\bar{\varphi}$, for every random enough G_n, $\bar{\varphi}$ defines in G_n a structure isomorphic to $\mathbb{N}_{<n}$.

Proof. We make a minor change in Definition 2.3(1),(d).

Clause $(d)^+$: We require the $a \in M$ is unique.

This makes no real difference above because the probability of the occurance if even one "b with two predecessors" is small and we just need that there is one; this makes $M_{G,a,k}$ smaller but not empty.

We start as in the proof of 3.3, for $k_* = \lfloor n^\zeta \rfloor$, we use Case 1 but for stage k we draw the truth values of $\mathbb{R}_\iota(b, c)$ only when $b \in M_{G,a,=k}$ and $c \in [n]$ but $c \notin M_{G,a,0} \cup \ldots M_{G,a,k-1}$.

So there is $b \in M_{G,a,=k_*}$ and there is a unique sequence $\langle a_\ell : \ell \leq k_* \rangle, a_0 = a, a_{k_*} = b$ and $(a_\ell, a_{\ell+1}) \in R_\iota^{G_n}$ where ι is such that $M_{G,a,\ell+1}$ is the ι-successor of $M_{G,a,\ell}$ and so there are formulas $\psi_2 \in \mathbb{L}_{\text{LFP}}(\tau_{\text{dg}})$ such that not depending on the pair (a, n), $\psi_2(G, a, b) = \{(a_\ell, a_i) : \ell \leq i \leq k_*\}$.

Now

- the probability of the following event is negligible $(c < 2)$: for some $d_1 \neq d_2 \in [n] \backslash M_{n,k_*}$ for every c: if $\psi_1(c, c, a, b)$, then $(d_1, c) \in R_2^G \Leftrightarrow (d_2, c) \in R_2^G$.

Ignoring this event, the following formulas define a linear order on $[n] \backslash M_{G,a,k_*}$:

- $\psi_3(d_1, d_2, a, b)$ say: for some c we have $\psi_2(c, c, a, b) \wedge R_2(d_2, c) \wedge \neg R_2(d_1, c)$ and for any c_1 if $\psi_2(c_1, c, a, b)$, then $R_2(d_1, c') \leftrightarrow R_2(d_2, c')$.

So $\psi_3(x, y, a, b)$ defines a linear order on $[n] \backslash M_{G,a,k_*}$ which has $\geq n - \lfloor n^\zeta \rfloor$ elements. Using the same trick we get $\psi_4 \in \mathbb{L}_{\text{LFP}}(\tau_{\text{dn}})$ and $\psi_4(x, y, a, b)$ defines a linear order on $[n]$. Now the formulas $\psi_2, \ldots, \psi_4$ do not depend on n. Also for some $\psi_5 \in \mathbb{L}_{\text{LFP}}(\tau_{\text{dg}})$

- $G \models \psi_5(a, b)$ iff $\psi_4(-, -, a, b)$ defines a linear order (on G) and for some $\psi_0 \in \mathbb{L}_{\text{LFP}}(\tau_{\text{dg}})$

⊙ $G \models \psi_0[c_1, a_1, b_1, c_2, a_2, b_2]$ iff:

(a) $(a_1, b_1), (a_2, b_2) \in \psi_5(G)$

(b) $|\{c : G \models \psi_4[c, c_1, a_2, b_1)\}| = |\{c : G \models \psi_4[c, c_2, a_2, b_2]\}|$.

So the interpretation should be clear. $\square_{3.5}$

Conclusion 3.6. *[Case B] For some $\psi \in \mathbb{L}_{\rm LFP}(\tau_{\rm dg})$ for every random enough $G_n = G_{n,\bar\alpha^*}$ we have:*

- *$G_n \models \psi$ iff n is even.*

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