Sh:1132

Math. Log. Quart. **65,** No. 4, 511–516 (2019) / **DOI** 10.1002/malq.201900045

Ultrafilter extensions do not preserve elementary equivalence

Denis I. Saveliev[1*] and **Saharon Shelah**[2,3]

[1] Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetny Lane, 19, build. 1, Moscow 127051, Russia
[2] Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Givat Ram, Jerusalem 9190401, Israel
[3] Department of Mathematics, Rutgers, The State University of New Jersey, Hill Center—Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, United States of America

Received 16 June 2019, revised 14 October 2019, accepted 25 October 2019
Published online 16 December 2019

We show that there are models $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1$ elementarily embeds into $\mathcal{M}_2$ but their ultrafilter extensions $\boldsymbol{\beta}(\mathcal{M}_1)$ and $\boldsymbol{\beta}(\mathcal{M}_2)$ are not elementarily equivalent.

1 Introduction

The ultrafilter extension of a first-order model is a model in the same vocabulary, the universe of which consists of all ultrafilters on the universe of the original model, and which extends the latter in a canonical way. This construction was introduced in [4]. The article [5] is an expanded version of [4]; it contains a list of problems, one of which is solved here.

The main precursor of the general construction was the ultrafilter extension of semigroups, called often the Čech-Stone compactification of semigroups. This particular case was discovered in 1970s and became since then an important tool for getting various Ramsey-theoretic results in combinatorics, algebra, and dynamics; the textbook [3] is a comprehensive treatise of this area. For theory of ultrafilters and for model theory we refer the reader to the standard textbooks [2] and [1], respectively.

Recall the construction of ultrafilter extensions and related basic facts.

Definition 1.1 For a set M, an ultrafilter D on M, and a formula $\varphi(x, \ldots)$ with parameters $x, \ldots$, we let

$$\forall^D x(\varphi(x, \ldots)) \text{ if and only if } \{a \in M : \varphi(a, \ldots)\} \in D.$$

It is easy to see that the ultrafilter quantifier is self-dual: it coincides with $\exists^D x$, defined as $\neg\forall^D x \neg$, since D is ultra. Note also that if D is the principal ultrafilter given by some $a \in M$, then $\forall^D x(\varphi(x, \ldots))$ is reduced to $\varphi(a, \ldots)$, and that if, e.g., D_1, D_2 are two ultrafilters on M then $\forall^{D_1} x_1 \forall^{D_2} x_2(\varphi(x_1, x_2, \ldots))$ means $\{a_1 \in M : \{a_2 \in M : \varphi(a_1, a_2, \ldots)\} \in D_2\} \in D_1$.

Definition 1.2 Let $\mathcal{M}$ be a model in a vocabulary τ with the universe M. Define the model $\boldsymbol{\beta}(\mathcal{M})$ and the function j_M as follows:

(a) the universe of $\boldsymbol{\beta}(\mathcal{M})$ is $\boldsymbol{\beta}(M)$, the set of ultrafilters on M,
(b) $j_M : M \to \boldsymbol{\beta}(M)$ is such that for all $a \in M$, $j_M(a)$ is the principal ultrafilter on M given by a, i.e., $j_M(a) = \{A \subseteq M : a \in A\}$,
(c) if $P \in \tau$ is an n-ary predicate symbol (other than the equality symbol), let

$$P^{\boldsymbol{\beta}(\mathcal{M})} = \{(D_1, \ldots, D_n) : \forall^{D_1} x_1 \ldots \forall^{D_n} x_n (P^{\mathcal{M}}(x_1, \ldots, x_n))\},$$

(d) if $F \in \tau$ is an n-ary function symbol, let $F^{\boldsymbol{\beta}(\mathcal{M})}(D_1, \ldots, D_n) = D$ if and only if

$$\forall A \subseteq M (A \in D \Leftrightarrow \forall^{D_1} x_1 \ldots \forall^{D_n} x_n (F^{\mathcal{M}}(x_1, \ldots, x_n) \in A)).$$

* Corresponding author; e-mail: d.i.saveliev@iitp.ru

Sh:1132

The model $\boldsymbol{\beta}(\mathcal{M})$ is the *ultrafilter extension* of the model $\mathcal{M}$, and j_M is the *natural embedding* of $\mathcal{M}$ into $\boldsymbol{\beta}(\mathcal{M})$. The use of words "extension" and "embedding" is easily justified:

Proposition 1.3 *If $\mathcal{M}$ is a model in a vocabulary τ, then $\boldsymbol{\beta}(\mathcal{M})$ is also a model in τ and j_M isomorphically embeds $\mathcal{M}$ into $\boldsymbol{\beta}(\mathcal{M})$.*

Proof. Cf. [4, 5]. □

The following result, called the First Extension Theorem in [5], shows that the ultrafilter extension lifts certain relationships between models.

Theorem 1.4 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models in the same vocabulary with the universes M_1 and M_2, respectively, and let h be a mapping of M_1 into M_2 and $\widetilde{h}$ its (unique) continuous extension of $\boldsymbol{\beta}(M_1)$ into $\boldsymbol{\beta}(M_2)$:*

$$\begin{array}{ccc} \boldsymbol{\beta}(\mathcal{M}_1) & \overset{\widetilde{h}}{\dashrightarrow} & \boldsymbol{\beta}(\mathcal{M}_2) \\ {\scriptstyle j_{M_1}}\uparrow & & {\scriptstyle j_{M_2}}\uparrow \\ \mathcal{M}_1 & \xrightarrow{h} & \mathcal{M}_2 \end{array}$$

If h is a homomorphism (epimorphism, isomorphic embedding) of $\mathcal{M}_1$ into $\mathcal{M}_2$, then $\widetilde{h}$ is a homomorphism (epimorphism, isomorphic embedding) of $\boldsymbol{\beta}(\mathcal{M}_1)$ into $\boldsymbol{\beta}(\mathcal{M}_2)$.

Proof. Cf. [4, 5]. □

Actually, Theorem 1.4 is a special case of a stronger result, called the Second Extension Theorem in [5]. Here we omit its precise formulation, which involves topological concepts, and note only that it generalizes the standard topological fact stating that the Čech-Stone compactification is the largest one, to the case when the underlying discrete space M carries an arbitrary first-order structure. This confirms that the construction of ultrafilter extensions given in Definition 1.2 is canonical in a certain sense.

Theorem 1.4 holds also for certain other relationships between models (e.g., for so-called homotopies and isotopies, cf. [4, 5]). A natural task is a characterization of such relationships. In particular, one can ask whether elementary embeddings or elementary equivalence lift under ultrafilter extensions. This task was posed in [5] (cf. Problem 5.1 there and comments before it).

In this note, we answer this particular question in the negative. In fact, we establish a slightly stronger result:

Theorem 1.5 (Main Theorem) *There exist models $\mathcal{M}_1$ and $\mathcal{M}_2$ in the same vocabulary such that $\mathcal{M}_1$ elementarily embeds into $\mathcal{M}_2$ but their ultrafilter extensions $\boldsymbol{\beta}(\mathcal{M}_1)$ and $\boldsymbol{\beta}(\mathcal{M}_2)$ are not elementarily equivalent:*

$$\begin{array}{ccc} \boldsymbol{\beta}(\mathcal{M}_1) & \overset{\not\equiv}{\dashrightarrow} & \boldsymbol{\beta}(\mathcal{M}_2) \\ {\scriptstyle j_{M_1}}\uparrow & & {\scriptstyle j_{M_2}}\uparrow \\ \mathcal{M}_1 & \xrightarrow{\prec} & \mathcal{M}_2 \end{array}$$

Of course, it follows that neither elementary embeddings nor elementary equivalence are preserved under ultrafilter extensions. The construction of such models $\mathcal{M}_1$ and $\mathcal{M}_2$ will be provided in the next section.

We conclude this section with the following natural questions on possible general results in this direction.

Problem 1.6 Characterize (or at least, provide interesting necessary or sufficient conditions on) theories T such that $\mathcal{M}_1 \equiv \mathcal{M}_2$ implies $\boldsymbol{\beta}(\mathcal{M}_1) \equiv \boldsymbol{\beta}(\mathcal{M}_2)$ for all $\mathcal{M}_1, \mathcal{M}_2 \vDash T$.

Problem 1.7 Characterize those theories for which the implication from Problem 1.6 holds with elementary embeddings instead of elementary equivalence.

2 Proof of the Main Theorem

First we define a vocabulary τ and construct two specific models $\mathcal{M}_1$ and $\mathcal{M}_2$ in τ. Then we shall show that these models are as required. Let τ be the vocabulary consisting of two unary predicate symbols P_1 and P_2, two binary predicate symbols R_1 and R_2, and one binary function symbol F.

Definition 2.1 Let $\mathcal{M}_1$ be a model in τ having the universe M_1 and defined as follows:

(a) $M_1 = \mathbb{N} \sqcup \wp(\mathbb{N})$, the disjoint sum of $\mathbb{N}$ and $\wp(\mathbb{N})$ (which we shall identify with their disjoint copies),
(b) $P_1^{\mathcal{M}_1} = \mathbb{N}$,
(c) $P_2^{\mathcal{M}_1} = \wp(\mathbb{N})$,
(d) $R_1^{\mathcal{M}_1} = \{(n, a) : n \in \mathbb{N} \wedge a \in \wp(\mathbb{N}) \wedge n \in a\}$, i.e., the intersection of the membership relation with $\mathbb{N} \times \wp(\mathbb{N})$,
(e) $R_2^{\mathcal{M}_1}$ is a relation such that
  (α) $R_2^{\mathcal{M}_1} \cap (\mathbb{N} \times \mathbb{N})$ is the usual order on $\mathbb{N}$,
  (β) $R_2^{\mathcal{M}_1} \cap (\wp(\mathbb{N}) \times \wp(\mathbb{N}))$ is a linear order on $\wp(\mathbb{N})$ with no endpoints,
  (γ) if $a \in \mathbb{N} \Leftrightarrow b \notin \mathbb{N}$ then $R_2^{\mathcal{M}_1}(a, b)$ is defined arbitrarily (really this case will not be used), and
(f) $F^{\mathcal{M}_1}$ is an unordered pairing function mapping $\mathbb{N}$ into $\mathbb{N}$ and $\wp(\mathbb{N})$ into $\wp(\mathbb{N})$, i.e., satisfying the following conditions:
  (α) if either $a_1, b_1, a_2, b_2 \in \mathbb{N}$ or $a_1, b_1, a_2, b_2 \in \wp(\mathbb{N})$, then

$$F^{\mathcal{M}_1}(a_1, b_1) = F^{\mathcal{M}_1}(a_2, b_2) \;\Leftrightarrow\; \{a_1, b_1\} = \{a_2, b_2\},$$

  (β) if $a, b \in \mathbb{N}$ then $F^{\mathcal{M}_1}(a, b) \in \mathbb{N}$,
  (γ) if $a, b \in \wp(\mathbb{N})$ then $F^{\mathcal{M}_1}(a, b) \in \wp(\mathbb{N})$,
  (δ) if $a \in \mathbb{N} \Leftrightarrow b \notin \mathbb{N}$ then $F^{\mathcal{M}_1}(a, b)$ is defined arbitrarily (really this case will not be used).

Proposition 2.2 *Assume $\lambda \geq 2^{\aleph_0}$. Then there exists a model $\mathcal{M}_2$ in τ such that $\mathcal{M}_1 \prec \mathcal{M}_2$ and $|P_1^{\mathcal{M}_2}| = |P_2^{\mathcal{M}_2}| = \lambda$.*

P r o o f. Let $\mathcal{M}_3$ be λ-saturated and $\mathcal{M}_1 \prec \mathcal{M}_3$. By the λ-saturatedness, for each $i \in \{1, 2\}$ we have $|P_i^{\mathcal{M}_3}| \geq \lambda$, so we can pick $A_i \subseteq P_i^{\mathcal{M}_3}$ with $|A_i| = \lambda$. By the downward Löwenheim-Skolem Theorem, there exists a model $\mathcal{M}_2$ with the universe M_2 such that $\mathcal{M}_2 \prec \mathcal{M}_3$, $M_1 \cup A_1 \cup A_2 \subseteq M_2$, and $|M_2| = \lambda$, whence it follows that $\mathcal{M}_2$ is a required model.

Alternatively, we can use a version of the upward Löwenheim-Skolem Theorem by picking two sets of constants, C_1 and C_2, with $|C_1| = |C_2| = \lambda$ and adding to the elementary diagram of $\mathcal{M}_1$ the formulas $P_i(c_i)$ for all $c_i \in C_i$, $i \in \{1, 2\}$. The obtained theory is consistent (by compactness), so extract its submodel of cardinality λ (by the downward Löwenheim-Skolem Theorem) and reduce it to the required model $\mathcal{M}_2$ in the original vocabulary τ. □

Clearly, this observation is of a general character; a similar argument allows to obtain, for every model, its elementary extension in which all predicate symbols are interpreted by relations of the same cardinality.

To simplify reading, we introduce the following shorthand notation for the ultrafilter extensions of the models $\mathcal{M}_1$ and $\mathcal{M}_2$. For $\ell \in \{1, 2\}$, let $\mathcal{N}_\ell = \boldsymbol{\beta}(\mathcal{M}_\ell)$, $N_\ell = \boldsymbol{\beta}(M_\ell)$, $j_\ell = j_{M_\ell}$. It is easy to observe that $P_1^{\mathcal{N}_\ell}$ consists of all ultrafilters D on M_ℓ such that $P_1^{\mathcal{M}_\ell} \in D$ (so for $\ell = 1$ this means $\mathbb{N} \in D$), and $P_1^{\mathcal{N}_\ell} \setminus \{j_\ell(n) : n \in P_1^{\mathcal{M}_\ell}\}$ consists of all such non-principal ultrafilters and that $P_2^{\mathcal{N}_\ell}$ consists of all ultrafilters D on M_ℓ such that $P_2^{\mathcal{M}_\ell} \in D$ (so for $\ell = 1$ this means $\wp(\mathbb{N}) \in D$), and $P_2^{\mathcal{N}_\ell} \setminus \{j_\ell(A) : A \in P_2^{\mathcal{M}_\ell}\}$ consists of all such non-principal ultrafilters.

Now we are going to construct a specific sentence ψ which will be satisfied in $\mathcal{N}_1$ but not in $\mathcal{N}_2$. First we define two auxiliary formulas φ_1 and φ_2: For $i \in \{1, 2\}$, let $\varphi_i(x)$ be the formula $P_i(x) \wedge \forall y\, (P_i(y) \rightarrow F(x, y) = F(y, x))$. Thus $\varphi_i(x)$ means that x is in the center in a sense. Actually, only φ_2 will be used to construct ψ.

Proposition 2.3 *Assume $i, \ell \in \{1, 2\}$. For every $D \in N_\ell$, $\mathcal{N}_\ell \vDash \varphi_i(D)$ if and only if $D \in \{j_\ell(a) : a \in P_i^{\mathcal{M}_\ell}\}$.*

P r o o f. This follows from the four lemmas below.

Lemma 2.4 *If $D \notin P_i^{\mathcal{N}_\ell}$ then $\mathcal{N}_\ell \vDash \neg\, \varphi_i(D)$.*

Proof. By the first conjunct in φ_i. □

Lemma 2.5 *If $D_1 \in P_i^{\mathcal{N}_\ell}$ and $D_2 = j_\ell(a)$ for some $a \in P_i^{\mathcal{M}_\ell}$, then $\mathcal{N}_\ell \vDash F(D_1, D_2) = F(D_2, D_1)$.*

Proof. We must check that $F^{\mathcal{N}_\ell}(D_1, D_2) = F^{\mathcal{N}_\ell}(D_2, D_1)$. It suffices to show that, for any $A \subseteq P_i^{\mathcal{M}_\ell}$, $A \in F^{\mathcal{N}_\ell}(D_1, D_2)$ if and only if $A \in F^{\mathcal{N}_\ell}(D_2, D_1)$. By Definition 1.2, we have $A \in F^{\mathcal{N}_\ell}(D_1, D_2)$ if and only if $\forall^{D_1} x_1 \forall^{D_2} x_2 (F^{\mathcal{M}_\ell}(x_1, x_2) \in A)$. But $D_2 = j_\ell(a)$ for an $a \in P_i^{\mathcal{M}_\ell}$, i.e., D_2 is a principal ultrafilter given by a. Hence $\forall^{D_2} x_2$ is reduced by replacing the bounded occurrence of the variable x_2 with a (as we have noted after Definition 1.1), whence we have that $A \in F^{\mathcal{N}_\ell}(D_1, D_2)$ if and only if $\forall^{D_1} x_1 (F^{\mathcal{M}_\ell}(x_1, a) \in A)$. Similarly, we get that $A \in F^{\mathcal{N}_\ell}(D_2, D_1)$ if and only if $\forall^{D_1} x_1 (F^{\mathcal{M}_\ell}(a, x_1) \in A)$. Since $a \in P_i^{\mathcal{M}_\ell}$, we have $F^{\mathcal{M}_\ell}(a, b) = F^{\mathcal{M}_\ell}(b, a)$ for all $b \in P_i^{\mathcal{M}_\ell}$ by Definition 2.1(f)(α). And as $P_i^{\mathcal{M}_\ell} \in D_1$, the required equivalence follows. □

Lemma 2.6 *If $D_1 \in P_1^{\mathcal{N}_1} \setminus \{j_1(n) : n \in P_1^{\mathcal{M}_1}\}$, then there exists $D_2 \in P_1^{\mathcal{N}_1}$ such that $F^{\mathcal{N}_1}(D_1, D_2) \neq F^{\mathcal{N}_1}(D_2, D_1)$.*

Proof. Actually, we shall prove a slightly stronger assertion: if $D_1, D_2 \in P_1^{\mathcal{N}_1} \setminus \{j_1(n) : n \in P_1^{\mathcal{M}_1}\}$ are such that $D_1 \neq D_2$, then $F^{\mathcal{N}_1}(D_1, D_2) \neq F^{\mathcal{N}_1}(D_2, D_1)$. So assume that D_1, D_2 are distinct non-principal ultrafilters on M_1 such that $\mathbb{N} \in D_1 \cap D_2$. By $D_1 \neq D_2$, there is $A_1 \in \wp(\mathbb{N})$ such that $A_1 \in D_1$ and $A_2 = \mathbb{N} \setminus A_1 \in D_2$. Let

$$B_1 = \{F^{\mathcal{M}_1}(n_1, n_2) : n_1 \in A_1 \wedge n_2 \in A_2 \wedge (n_1, n_2) \in R_2^{\mathcal{M}_1}\} \text{ and}$$

$$B_2 = \{F^{\mathcal{M}_1}(n_1, n_2) : n_1 \in A_1 \wedge n_2 \in A_2 \wedge (n_2, n_1) \in R_2^{\mathcal{M}_1}\}.$$

Recall that $R_2^{\mathcal{M}_1} \cap (\mathbb{N} \times \mathbb{N})$ is the usual order $<$ on $\mathbb{N}$, so the last conjuncts in the definition of B_1 and B_2 mean just $n_1 < n_2$ and $n_2 < n_1$, respectively. Now our stronger assertion clearly follows from claims (a) to (c) below:

(a) $B_1 \cap B_2 = \varnothing$,
(b) $B_1 \in F^{\mathcal{N}_1}(D_1, D_2)$,
(c) $B_2 \in F^{\mathcal{N}_1}(D_2, D_1)$.

It remains to verify these claims. For (a), note that if there is some $c \in B_1 \cap B_2$, then

(α) since $c \in B_1$, we can find $n_1 < n_2$ such that $F^{\mathcal{M}_1}(n_1, n_2) = c$, $n_1 \in A_1$, $n_2 \in A_2$, and
(β) since $c \in B_2$, we can find $m_2 < m_1$ such that $F^{\mathcal{M}_1}(m_1, m_2) = c$, $m_1 \in A_1$, $m_2 \in A_2$.

So, since by Definition 2.1(f)(α), $F^{\mathcal{M}_1}$ is an unordered pairing function, we conclude $\{n_1, n_2\} = \{m_1, m_2\}$. However, then $n_1 < n_2$ and $m_2 < m_1$ imply $n_1 = m_2$ and $n_2 = m_1$, which contradicts to $n_1 \in A_1$, $m_2 \in A_2$.

For (b), note that $\{n_2 \in A_2 : n_2 > n_1\} \in D_2$ because of $A_2 \in D_2$ and D_2 is non-principal. It follows $\forall^{D_2} n_2 (F(n_1, n_2) \in B_1)$. But $A_1 \in D_1$, so we get $\forall^{D_1} n_1 \forall^{D_2} n_2 (F(n_1, n_2) \in B_1)$. By Definition 1.2(d), this gives claim (b).

For (c), argue similarly. □

The fourth lemma (and its proof) generalizes the previous one.

Lemma 2.7 *If $i, \ell \in \{1, 2\}$ and $D_1 \in P_i^{\mathcal{N}_\ell} \setminus \{j_\ell(a) : a \in P_i^{\mathcal{M}_\ell}\}$, then there exists $D_2 \in P_i^{\mathcal{N}_\ell}$ such that $F^{\mathcal{N}_\ell}(D_1, D_2) \neq F^{\mathcal{N}_\ell}(D_2, D_1)$.*

Proof. Let D_1 be a non-principal ultrafilter on $P_i^{\mathcal{M}_\ell}$. It follows from Definition 2.1(e) and $\mathcal{M}_1 \prec \mathcal{M}_2$ that $R_2^{\mathcal{M}_\ell}$ is a linear order on $P_i^{\mathcal{M}_\ell}$. One of the two following possibilities occurs:

Case 1: there is an initial segment I of the linearly ordered set $(P_i^{\mathcal{M}_\ell}, R_2^{\mathcal{M}_\ell})$ such that $I \in D_1$ but if $I_1 \subset I$ is another initial segment of the set then $I_1 \notin D_1$ (this I necessarily has no last element).

Case 2: there is a final segment J of the linearly ordered set $(P_i^{\mathcal{M}_\ell}, R_2^{\mathcal{M}_\ell})$ such that $J \in D_1$ but if $J_1 \subset J$ is another final segment of the set then $J_1 \notin D_1$ (this J necessarily has no first element).

To see this, we observe the following general facts: If $(X, <)$ is a linearly ordered set, for any ultrafilter D on X define the initial segment I_D and the final segment J_D of $(X, <)$ as follows:

$$I_D = \bigcap \{I \in D : I \text{ is an initial segment of } (X, <)\} \text{ and}$$

Sh:1132

$$J_D = \bigcap\{J \in D : J \text{ is a final segment of } (X, <)\}.$$

It is easy to see, that if D is principal, then $I_D \cap J_D = \{x\}$ for $\{x\} \in D$; and if D is non-principal then (I_D, J_D) is a cut and either I_D or J_D, but not both, is in D. Furthermore, if I_D is in D, then so are all final segments of I_D, $S \cap I_D$ is cofinal in I_D for all $S \in D$, and I_D does not have a greatest element whenever D is non-principal; and symmetrically for J_D in D. (More details related to ultrafilter extensions of linearly ordered sets can be found in [6].)

In our situation, D_1 is non-principal, so we have either $I_{D_1} \in D_1$, in which case we get *Case 1* with $I = I_{D_1}$, or $J_{D_1} \in D_1$, in which case we get *Case 2* with $J = J_{D_1}$.

In *Case 1*, choose an ultrafilter D_2 on $P_i^{\mathcal{M}_\ell}$ such that $I \in D_2$, if $I_1 \subset I$ is an initial segment of $(P_i^{\mathcal{M}_\ell}, R_2^{\mathcal{M}_\ell})$ then $I_1 \notin D_2$, $D_2 \neq D_1$. Now we can repeat the proof of Lemma 2.6 mutatis mutandis, i.e., we can find $A_1 \in D_1 \backslash D_2$ such that $A_1 \subseteq I$ and $A_2 = I \backslash A_1 \in D_2$, then define

$$B_1 = \{F^{\mathcal{M}_\ell}(a_1, a_2) : a_1 \in A_1 \wedge a_2 \in A_2 \wedge (a_1, a_2) \in R_2^{\mathcal{M}_\ell}\} \text{ and}$$

$$B_2 = \{F^{\mathcal{M}_\ell}(a_1, a_2) : a_1 \in A_1 \wedge a_2 \in A_2 \wedge (a_2, a_1) \in R_2^{\mathcal{M}_\ell}\},$$

etc.

In *Case 2*, the proof is symmetric: we only replace I with J, initial segments with final ones, and $x R_2^{\mathcal{M}_\ell} y$ with $y R_2^{\mathcal{M}_\ell} x$. □

These four lemmas complete the proof of Proposition 2.3. □

Now everything is ready in order to provide a sentence ψ having the required property.

Definition 2.8 Let ψ be the following sentence in τ:

$$\forall x_1 \forall x_2((P_1(x_1) \wedge P_1(x_2) \wedge x_1 \neq x_2) \to \exists y(\varphi_2(y) \wedge R_1(x_1, y) \wedge \neg R_1(x_2, y))).$$

Proposition 2.9 *Let $\ell \in \{1, 2\}$. Then $\mathcal{N}_\ell \vDash \psi$ if and only if $\ell = 1$.*

Proof. 1. First we show that $\mathcal{N}_1 \vDash \psi$. Let D_1, D_2 satisfy the antecedent of ψ, i.e., $D_1, D_2 \in P_1^{\mathcal{N}_1}$ and $D_1 \neq D_2$. We should find $b \in N_1$ such that $\mathcal{N}_1 \vDash \varphi_2(b) \wedge R_1(D_1, b) \wedge \neg R_1(D_2, b)$. Since D_1, D_2 are distinct ultrafilters on M_1 such that $P_1^{\mathcal{M}_1} \in D_1 \cap D_2$, we can choose $A_1 \subseteq P_1^{\mathcal{M}_1}$ such that $A_1 \in D_1$ and $A_1 \notin D_2$. Then $A_1 \in P_2^{\mathcal{M}_1}$ clearly follows from Definition 2.1(b),(c). So $b = j_1(A_1) \in P_2^{\mathcal{N}_1}$, and hence, by the "if" part of Proposition 2.3, $\mathcal{N}_1 \vDash \varphi_2(b)$.

It remains to show the conjunction of $(D_1, b) \in R_1^{\mathcal{N}_1}$ and $(D_2, b) \notin R_1^{\mathcal{N}_1}$. To this end, note that for any ultrafilter D concentrated on $P_1^{\mathcal{M}_1}$ and any $A \in P_2^{\mathcal{M}_1}$, by Definition 1.2(c), the formula $(D, j_1(A)) \in R_1^{\mathcal{N}_1}$ means $\forall^D n \forall^{j(A)} B((n, B) \in R_1^{\mathcal{M}_1})$. Recalling that $R_1^{\mathcal{M}_1}$ is the membership relation (Definition 2.1(d)) and reducing $\forall^{j(A)} B$, we see that the latter formula is equivalent to $\forall^D n(n \in A)$, and so, to $A \in D$. Since we have $A_1 \in D_1$ and $A_1 \notin D_2$, this gives the required conjunction.

2. Now we show that $\mathcal{N}_2 \vDash \neg\psi$. Define a function G from $P_1^{\mathcal{N}_2}$ into $\wp(P_2^{\mathcal{M}_2})$ by $G(D) := \{b \in P_2^{\mathcal{M}_2} : \{a \in P_1^{\mathcal{M}_2} : (a, b) \in R_1^{\mathcal{M}_2}\} \in D\}$. Recall that $|P_1^{\mathcal{M}_2}| = |P_1^{\mathcal{M}_2}| = \lambda$ (Proposition 2.2). Therefore, $|\operatorname{dom}(G)| = |\beta(|P_1^{\mathcal{M}_2}|) = |\beta(\lambda)| = 2^{2^\lambda} > 2^\lambda$, while $|\operatorname{ran}(G)| \leq |\wp(P_2^{\mathcal{M}_2})| = |\wp(\lambda)| = 2^\lambda$, whence we conclude that G is not one-to-one. Take $S \in \wp(P_2^{\mathcal{M}_2})$ such that $|G^{-1}(S)| > 1$, pick $D_1, D_2 \in G^{-1}(S)$ such that $D_1 \neq D_2$, and show that D_1, D_2 witness the failure of the sentence ψ. Note that $\mathcal{N}_2$ satisfies the antecedent of ψ, i.e., $\mathcal{N}_2 \vDash P_1(D_1) \wedge P_1(D_2) \wedge D_1 \neq D_2$, by the condition $D_1, D_2 \in G^{-1}(S) \subseteq P_1^{\mathcal{N}_2}$. So to finish, it suffices to show $\mathcal{N}_2 \vDash \neg\exists y(\varphi_2(y) \wedge R_1(D_1, y) \wedge \neg R_1(D_2, y))$. Toward a contradiction, assume that there is $b \in N_2$ such that $\mathcal{N}_2 \vDash \varphi_2(b) \wedge R_1(D_1, b) \wedge \neg R_1(D_2, b)$. But since $\mathcal{N}_2 \vDash \varphi_2(b)$, by the "only if" part of Proposition 2.3, we see that $b = j_2(A)$ for some $A \in P_2^{\mathcal{M}_2}$. So we obtain $R_1^{\mathcal{N}_2}(D_1, j_2(A))$ and $\neg R_1^{\mathcal{N}_2}(D_2, j_2(A))$.

By Definition 1.2(c), $R_1^{\mathcal{N}_2}(D_1, j_2(A))$ means $\forall^{D_1} a \forall^{j_2(A)} b((a, b) \in R_1^{\mathcal{M}_2})$, whence reducing $\forall^{j_2(A)} b$ we get $\forall^{D_1} a((a, A) \in R_1^{\mathcal{M}_2})$, i.e., $\{a \in P_1^{\mathcal{M}_2} : (a, A) \in R_1^{\mathcal{M}_2}\} \in D_1$. Similarly, $R_1^{\mathcal{N}_2}(D_2, j_2(A))$ is equivalent to $\{a \in P_1^{\mathcal{M}_2} : (a, A) \in R_1^{\mathcal{M}_2}\} \in D_2$, and hence, $\neg R_1^{\mathcal{N}_2}(D_2, j_2(A))$ is equivalent to $\{a \in P_1^{\mathcal{M}_2} : (a, A) \in R_1^{\mathcal{M}_2}\} \notin D_2$. Therefore, $A \in G(D_1)$ and $A \notin G(D_2)$, which, however, contradicts to the choice of D_1, D_2. This completes the proof. □

So we have constructed two models $\mathcal{M}_1, \mathcal{M}_2$ in τ with $\mathcal{M}_1 \prec \mathcal{M}_2$ and a τ-sentence ψ such that $\mathcal{N}_1 = \boldsymbol{\beta}(\mathcal{M}_1) \vDash \psi$ and $\mathcal{N}_2 = \boldsymbol{\beta}(\mathcal{M}_2) \vDash \neg\, \psi$, thus witnessing $\boldsymbol{\beta}(\mathcal{M}_1) \not\equiv \boldsymbol{\beta}(\mathcal{M}_2)$. This proves the Main Theorem (Theorem 1.5).

Acknowledgements We are grateful to the reviewer for reading the paper carefully and making helpful remarks. This research was partially supported by European Research Council grant 338821. The first author was also supported by Russian Foundation for Basic Research grant 17-01-00705.

References

[1] C. C. Chang and H. J. Keisler, Model Theory, 3rd edition, Studies in Logic and the Foundations of Mathematics Vol. 73 (North-Holland, 1990).
[2] W. W. Comfort and S. Negrepontis, The Theory of Ultrafilters, Grundlehren der mathematischen Wissenschaften Vol. 211 (Springer, 1974).
[3] N. Hindman and D. Strauss, Algebra in the Stone-Čech Compactification, Theory and Applications, 2nd ed., De Gruyter Textbook (de Gruyter, 2012).
[4] D. I. Saveliev, Ultrafilter Extensions of Models, Lecture Notes in Computer Science Vol. 6521 (Springer, 2011), pp. 162–177.
[5] D. I. Saveliev, On ultrafilter extensions of models, in: The Infinity Project, A 2009–2011 Research Programme, edited by S.-D. Friedman, M. Koerwien, and M. Müller (Centre de Recerca Matemàtica, 2012), pp. 599–616.
[6] D. I. Saveliev, Ultrafilter extensions of linearly ordered sets, Order **32**(1), 29–41 (2015).

