LARGE CONTINUUM, ORACLES
SH895

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ABSTRACT. Our main theorem is about iterated forcing for making the continuum larger than $\aleph_2$. We present a generalization of [She06] which dealt with oracles for random, (also for other cases and generalities), by replacing $\aleph_1, \aleph_2$ by λ, λ^+ (starting with $\lambda = \lambda^{<\lambda} > \aleph_1$). Well, we demand absolute c.c.c. So we get, e.g. the continuum is λ^+ but we can get cov(meagre) $= \lambda$ and we give some applications. As in "non-Cohen oracles", [She06], it is a "partial" countable support iteration but it is c.c.c.

§ 0. Introduction

Starting, e.g. with $\mathbf{V} \models$ G.C.H. and $\lambda = \lambda^{<\lambda} > \aleph_1$, we construct a forcing notion $\mathbb{P}$ of cardinality λ^+, by a partial CS iteration but the result is a c.c.c. forcing.

The general iteration theorems (treated in §1) seem generally suitable for constructing universes with $\mathrm{MA}_{<\lambda} + 2^{\aleph_0} = \lambda^+$, and taking more care, we should be able to get universes without $\mathrm{MA}_{<\lambda}$, see 0.4 below.

Our method is to imitate [She06]; concerning the differences, some are inessential: using games not using diamonds in the framework itself, (inessential means that we could have in [She06] imitate the choice here and vice versa).

An essential difference is that we deal here with large continuum - λ^+; we concentrate on the case where we shall (in $\mathbf{V}^{\mathbb{P}}$) have $\mathrm{MA}_{<\lambda}$ but e.g. non(null) $= \lambda$ and $\mathfrak{b} = \lambda^+$ (or $\mathfrak{b} = \lambda$).

It seems to us that generally:

Thesis 0.1. The iteration theorem here is enough to get results parallel to known results with $2^{\aleph_0} = \aleph_2$ replacing $\aleph_1, \aleph_2$ by λ, λ^+.

To test this thesis we have asked Bartoszyński to suggest test problems for this method and he suggests:

Problem 0.2. Prove the consistency of each of the

- (A) $\aleph_1 < \lambda < 2^{\aleph_0}$ and the λ-Borel conjecture, i.e. $A \subseteq {}^\omega 2$ is of strong measure zero iff $|A| < \lambda$
- (B) $\aleph_1 <$ non(null) $< 2^{\aleph_0}$, see 5.1

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(C) $\aleph_1 < \mathfrak{b} = \lambda < 2^{\aleph_0}$ the dual λ-Borel conjecture (i.e. $A \subseteq {}^\omega 2$ is strongly meagre iff $|A| < \lambda$)

(D) $\aleph_1 < \mathfrak{b} = \lambda < 2^{\aleph_0}+$ the dual $2^{\aleph_0}$-Borel conjecture

(E) combine (A) and (C) and/or combine (A) and (D).

Parallely Steprans suggests:

Problem 0.3. 1) Is there a set $A \subseteq {}^\omega 2$ of cardinality $\aleph_2$ of p-Hausdorff measure > 0, but for every set of size $\aleph_2$ is null (for the Lebesgue measure)?
2) The (basic product) I think $\mathfrak{b} = \mathfrak{d} \vee \mathfrak{d} = 2^{\aleph_0}$ gives an answer, what about $\mathrm{cov}(\text{meagre}) = \lambda < 2^{\aleph_0}$?

We shall deal with the iteration in §1, give an application to a problem from [She09] in §2 (and §3,§4).

Lastly, in §5 we deal with Bartoszyński's test problem (B), in fact, we get quite general such results.

It is natural to ask

Discussion 0.4. 1) In §1, we may wonder if we can give a "reasonable" sufficient condition for $\mathfrak{b} = \aleph_1$ or $\mathfrak{b} = \kappa < \lambda$? The answer is yes. It is natural to assume that we have in $\mathbf{V}$ a $<_{J^{\mathrm{bd}}_\omega}$-increasing sequence $\bar f = \langle f_\alpha : \alpha < \kappa\rangle$ of functions from ${}^\omega\omega$ with no $<^*_{J^{\mathrm{bd}}_\omega}$-upper bound and we would like to preserve this property of $\bar f$, i.e. in §1 we

(a) restrict ourselves to $\mathbf{p} \in K^1_\lambda$ such that $\Vdash_{\mathbb{P}_\mathbf{p}}$ "$\bar f$ as above".

More formally redefine K^1_λ such that

(b) replace "$\mathbb{P}$ is absolute c.c.c." by "$\mathbb{P}$ is c.c.c., preserve $\bar f$ as above and if $\mathbb{Q}$ satisfies those two conditions then also the product $\mathbb{P} \times \mathbb{Q}$ satisfies those two conditions".

This has similar closure properties, that is, the proofs do not really change.
2) More generally consider K, a property of forcing notions such that:

(a) $\mathbb{P} \in K \Rightarrow \mathbb{P}$ is c.c.c.

(b) K is closed under $\lessdot$-increasing continuous unions

(c) K is closed under composition

(d) we replace in §1 "$\mathbf{p} \in K^1_\lambda$" by "$\mathbf{p} \in K$ has cardinality $< \lambda$"

(e) we replace in §1, "$\mathbb{P}$ is absolutely c.c.c." by "$\mathbb{P} \in K$ and $\mathbb{R} \in K \Rightarrow \mathbb{P} \times \mathbb{R} \in K$".

3) What about using $\mathscr{P}(n)$-amalgamation of forcing notions? (See [She] in model theoretic version).) If we fix n this seems a natural way to get non-equality for many n-tuples of cardinal invariants; hopefully we shall return to this sometime.
4) What about forcing by the set of approximations $\mathbf{k}$? See 1.16.
5) You may wonder why here "absolute c.c.c." play a major role but is not used in [She06]. The answer is that the "absolute c.c.c." is demanded on forcing notions of cardinality $< \lambda$ which in [She06] means countable.

Definition 0.5. 1) We say a forcing notion $\mathbb{P}$ is absolutely c.c.c. when for every c.c.c. forcing notion $\mathbb{Q}$ we have $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ is c.c.c."
2) We say $\mathbb{P}_2$ is absolutely c.c.c. over $\mathbb{P}_1$ when ($\mathbb{P}_1 \lessdot \mathbb{P}_2$ and) $\mathbb{P}_2/\mathbb{P}_1$ is absolutely c.c.c.
3) Let $\mathbb{P}_1 \subseteq_{\rm ic} \mathbb{P}_2$ mean that $\mathbb{P}_1 \subseteq \mathbb{P}_2$ (as quasi orders) and if $p, q \in \mathbb{P}_1$ are incompatible in $\mathbb{P}_1$ then they are incompatible in $\mathbb{P}_2$ (the inverse follows by $\mathbb{P}_1 \subseteq \mathbb{P}_2$).

The following tries to describe the iteration theorem, this may be more useful to the reader after having a first reading of §1.

We treat λ as the vertical direction and λ^+ as the horizontal direction, the meaning will be clarified in §2; our forcing is the increasing union of $\langle \mathbb{P}^{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+ \rangle$ where $\mathbf{k}_\varepsilon \in K_2$ (so $\mathbf{k}_\varepsilon$ gives an iteration $\langle \mathbb{P}_\alpha[\mathbf{k}_\varepsilon] : \alpha < \lambda \rangle$, i.e. a $\lessdot$-increasing continuous sequence of c.c.c. forcing notions) and for each such $\mathbf{k}_\varepsilon$ each iterand $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_\varepsilon]}$ is of cardinality $< \lambda$ and for each $\varepsilon < \lambda^+$ the forcing notion $\mathbb{P}^{\mathbf{k}_\varepsilon}$ is the union of the increasing continuous sequence $\langle \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_\varepsilon]} : \alpha < \lambda \rangle$. So we can say that $\mathbb{P}^{\mathbf{k}_\varepsilon}$ is the limit of an FS iteration of length λ, each iterand of cardinality $< \lambda$ and for $\zeta \in (\varepsilon, \lambda^+), \mathbf{k}_\zeta$ gives a "fatter" iteration, which for "most" $\delta \in S(\subseteq \lambda)$, is a reasonable extension.

Question 0.6. Can we get something interesting for the continuum $> \lambda^+$ and/or get cov(meagre) $< \lambda$? This certainly involves some losses! We intend to try elsewhere.

Definition 0.7. 1) For a set x let otrcl(x), the transitive closure over the ordinals of x, be the minimal set y such that $x \subseteq y \wedge (\forall t \in y)(t \notin \text{ Ord } \rightarrow t \subseteq y)$.
2) For a set u of ordinals let $\mathscr{H}_{<\kappa}(u)$ be the set of x such that otrcl$(x) \cap$ Ord is a subset of u of cardinality $< \kappa$.

Remark 0.8. 0) We use $\mathscr{H}_{<\kappa}(u)$ (in Definition 1.3) just for bookkeeping convenience.
1) It is natural to have Ord, the class of ordinals, a class of urelements.
2) If $\omega_1 \subseteq u$ for $\mathscr{H}_{<\aleph_1}(u)$ it makes no difference, but if $\omega_1 \nsubseteq u$ and $\beta = \min(\omega_1 \backslash u)$ then β is a countable subset of u but $\notin \mathscr{H}_{<\aleph_1}(u)$. Also we use $\mathscr{H}_{<\aleph_0}(u)$ where $\omega \subseteq u$, so there are no problems.

§ 1. The iteration theorem

If we use the construction for $\lambda = \aleph_1$, the version we get is closer to, but not the same as [She06]; in this case it may be more convenient to have the forcing locally Cohen.

We now list "atomic" forcings used below coming from three sources:

(*a*) the forcing given by the winning strategies $\mathbf{s}_\delta$ (see below), i.e. the quotient $\mathbb{P}_{\mathbf{q}_\varepsilon}/\mathbb{P}_{\mathbf{p}_\varepsilon}$, see Definition 1.12

(*b*) forcing notions intended to generate $\mathrm{MA}_{<\lambda}$
[see 1.25; we are given $\mathbf{k}_1 \in K^2_f$, an approximation of size λ, see Definition 1.16, and a $\mathbb{P}_{\mathbf{k}_1}$-name $\underset{\sim}{\mathbb{Q}}$ of a c.c.c. forcing and sequence $\langle \mathscr{I}_i : i < i(*)\rangle$ of $< \lambda$ dense subsets of $\underset{\sim}{\mathbb{Q}}$. We would like to find $\mathbf{k}_2 \in K_2$ satisfying $\mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$ such that $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "there is a directed $G \subseteq \underset{\sim}{\mathbb{Q}}$ not disjoint to any $\mathscr{I}_i (i < i(*))$". We <u>do not</u> use composition, only $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]} = \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ for some $\alpha \in E_{\mathbf{k}_1} \cap E_{\mathbf{k}_2}$]]

(*c*) given $\mathbf{k}_1 \in K^2_f$, and $\underset{\sim}{\mathbb{Q}}$ which is a $\mathbb{P}_{\mathbf{k}_1}$-name of a suitable c.c.c. forcing notion of cardinality $\underset{\sim}{\lambda}$ can we find $\mathbf{k}_2$ such that $\mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$ and in $\mathbf{V}$ we have $\Vdash_{\mathbb{P}[\mathbf{k}_2]}$ "there is a subset of $\underset{\sim}{\mathbb{Q}}$ generic over $\mathbf{V}[\underset{\sim}{G}, \mathbb{P}[\mathbf{k}_2] \cap \mathbb{P}_{\mathbf{k}_1}]$".

Let us describe the roles of some of the definitions. We shall construct (in the main case) a forcing notion of cardinality λ^+ by approximations $\mathbf{k} \in K^2_f$ of size (= cardinality) λ, see Definition 1.16, which are constructed by an increasing sequence of approximations $\mathbf{p} \in K_1$ of cardinality $< \lambda$, see Definition 1.3.

Now $\mathbf{p} \in K_1$ is essentially a forcing notion of cardinality $< \lambda$, i.e. $\mathbb{P}_\mathbf{p} = (P_\mathbf{p}, \le_\mathbf{p})$, and we add the set $u = u_\mathbf{p}$ to help the bookkeeping, so (in the main case) $u_\mathbf{p} \in [\lambda^+]^{<\lambda}$. For the bookkeeping we let $P_\mathbf{p} \subseteq \mathscr{H}_{<\aleph_1}(u_\mathbf{p})$, see 0.7(2).

More specifically $\mathbf{k}$ (from Definition 1.16) is mainly a $\lessdot$-increasing continuous sequence $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha \in E_\mathbf{k}\rangle = \langle \mathbf{p}_\alpha[\mathbf{k}] : \alpha \in E_\mathbf{k}\rangle$, where $E_\mathbf{k}$ is a club of λ. Hence $\mathbf{k}$ represents the forcing notion $\mathbb{P}_\mathbf{k} = \cup\{(P_{\mathbf{p}_\alpha}, \le_{\mathbf{p}_\alpha}) : \alpha < \lambda\}$; the union of a $\lessdot$-increasing continuous sequence of forcing notions $\mathbb{P}_{\mathbf{p}_\alpha} = \mathbb{P}[\mathbf{p}_\alpha] = (P_{\mathbf{p}_\alpha}, \le_{\mathbf{p}_\alpha})$, so we can look at $\mathbb{P}_\mathbf{k}$ as a FS-iteration. But then we would like to construct say an "immediate successor" $\mathbf{k}^+$ of $\mathbf{k}$, so in particular $\mathbb{P}_\mathbf{k} \lessdot \mathbb{P}_{\mathbf{k}^+}$, e.g. taking care of (b) above so $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\mathbf{k}$-name and even a $\mathbb{P}_{\min(E_\mathbf{k})}$-name of a c.c.c. forcing notion. Toward this we choose $\mathbf{p}^{\mathbf{k}^+}_\alpha = \mathbf{p}_\alpha[\mathbf{k}^+]$ by induction on $\alpha \in E_\mathbf{k}$. So it makes sense to demand $\mathbf{p}_\alpha \le_{K_1} \mathbf{p}_\alpha[\mathbf{k}^+]$, which naturally implies that $u[\mathbf{p}_\alpha] \subseteq u[\mathbf{p}^{\mathbf{k}^+}_\alpha], \mathbb{P}_{\mathbf{p}_\alpha} \lessdot \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$. So as $\mathbf{p}_\alpha[\mathbf{k}^+]$ for $\alpha \in E_\mathbf{k}$ is $\le_{K_1}$-increasing continuous, the main case is when $\beta = \min(E_\mathbf{k} \backslash (\alpha+1)$, can we choose $\mathbf{p}_\beta[\mathbf{k}^+]$?

Let us try to draw the picture:

$$\begin{array}{ccc} \mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]} & \dashrightarrow & ? \\ \uparrow & & \uparrow \\ \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]} & \lessdot \rightarrow\longrightarrow & \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]} \end{array}$$

So we have three forcing notions, $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}, \mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}, \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$, where the second and third are $\lessdot$-extensions of the first. The main problem is the c.c.c. As in the main case we like to have $\mathrm{MA}_{<\lambda}$, there is no restriction on $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$, so it is natural to demand "$\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ is absolutely c.c.c. for $\alpha < \beta$ from $E_\mathbf{k}$" (recall $\mathbf{p}_\alpha[\mathbf{k}]$ is demanded to be $<^+_{K_1}$-increasing with α).

How do we amalgamate? There are two natural ways which say that "we leave $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ as it is".

First way: We decide that $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}^+]}$ is $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]} * \big((\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}) \times (\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]})\big)$.

[This is the "do nothing" case, the lazy man strategy, which in glorified fashion we may say: do nothing when in doubt. Note that $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ and $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ are $\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$-names of forcing notions.]

Second way: $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}$ is defined in some way, e.g. is a random real forcing in the universe $\mathbf{V}[\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}]$ and we decide that $\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}^+]}/\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}$ is defined in the same way: the random real forcing in the universe $\mathbf{V}[\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}^+]}]$; this is expressed by the strategy $\mathbf{s}_\alpha$.

[That is: retain the same definition of the forcing in the α-th place, so in some sense we again do nothing novel.]

Context 1.1. Let $\lambda = \text{cf}(\lambda) > \aleph_1$ or just[1] $\lambda = \text{cf}(\lambda) \geq \aleph_1$.

Remark 1.2. We may replace λ^+ by $\mu \geq \lambda^+$, if so we need stronger condition; mainly the C_ε-s can be of arbitrary cardinality. but not do not do it here.

Below, $\leq^+_{K_1}$ is used in defining $\mathbf{k} \in K^2_f$ as consisting also of $\leq^+_{K_1}$-increasing continuous sequence $\langle \mathbf{p}_\alpha : \alpha \in E \subseteq \lambda\rangle$ (so increasing vertically).

Definition 1.3. 1) Let K_1 be the class of $\mathbf{p}$ such that:

- (a) $\mathbf{p} = (u, P, \leq) = (u_\mathbf{p}, P_\mathbf{p}, \leq_\mathbf{p}) = (u_\mathbf{p}, \mathbb{P}_\mathbf{p})$
- (b) $\omega \subseteq u \subseteq \text{Ord}$ and $\lambda > \aleph_1 \Rightarrow \omega_1 \subseteq w$,
- (c) P is a set $\subseteq \mathscr{H}_{<\aleph_1}(u)$,
- (d) $\leq$ is a quasi-order on P,

satisfying

- (e) the pair $(P, \leq)$ which we denote also by $\mathbb{P} = \mathbb{P}_\mathbf{p}$ is a c.c.c. forcing notion.

1A) We may write $u[\mathbf{p}], P[\mathbf{p}], \mathbb{P}[\mathbf{p}]$.
2) $\leq_{K_1}$ is the following two-place relation on $K_1 : \mathbf{p} \leq_{K_1} \mathbf{q}$ iff $u_\mathbf{p} \subseteq u_\mathbf{q}$ and $\mathbb{P}_\mathbf{p} \lessdot \mathbb{P}_\mathbf{q}$ and $\mathbb{P}_\mathbf{q} \cap \mathscr{H}_{<\aleph_1}(u_\mathbf{p}) = \mathbb{P}_\mathbf{p}$; moreover, just for transparency $q \leq_{\mathbb{P}[\mathbf{q}]} p \in \mathbb{P}_\mathbf{p} \Rightarrow q \in \mathbb{P}_\mathbf{p}$.
3) $\leq^+_{K_1}$ is the following two-place relation on $K_1 : \mathbf{p} \leq^+_{K_1} \mathbf{q}$ iff $\mathbf{p} \leq_{K_1} \mathbf{q}$ and $\mathbb{P}_\mathbf{q}/\mathbb{P}_\mathbf{p}$ is absolutely c.c.c., see Definition 0.5(1).
4) K^1_λ is the family of $\mathbf{p} \in K_1$ such that $u_\mathbf{p} \subseteq \lambda^+$ and $|u_\mathbf{p}| < \lambda$.
5) We say $\mathbf{p}$ is the exact limit or the union of $\langle \mathbf{p}_\alpha : \alpha \in v\rangle, v \subseteq \text{Ord}$, in symbols $\mathbf{p} = \cup\{\mathbf{p}_\alpha : \alpha \in v\}$ when $u_\mathbf{p} = \cup\{u_{\mathbf{p}_\alpha} : \alpha \in v\}, \mathbb{P}_\mathbf{p} = \cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha \in v\}$ and $\alpha \in v \Rightarrow \mathbf{p}_\alpha \leq_{K_1} \mathbf{p}$; hence $\mathbf{p} \in K_1$.
6) We say $\mathbf{p}$ is just a limit of $\langle \mathbf{p}_\alpha : \alpha \in v\rangle$ when $u_\mathbf{p}$ is $\cup\{u_{\mathbf{p}_\alpha} : \alpha \in v\}, \mathbb{P}_\mathbf{p} \supseteq \cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha \in v\}$ and $\alpha \in v \Rightarrow \mathbf{p}_\alpha \leq_{K_1} \mathbf{p}$.
7) We say $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \alpha^*\rangle$ is $\leq_{K_1}$-increasing continuous [strictly $\leq_{K_1}$-increasing continuous] when it is $\leq_{K_1}$-increasing and for every limit $\alpha < \alpha^*, \mathbf{p}_\alpha$ is a limit of $\bar{\mathbf{p}} \restriction \alpha$ [is the exact limit of $\bar{\mathbf{p}} \restriction \alpha$], respectively.
8) In part (7) we say $(= \zeta)$-strictly $(\leq_{K_1})$-increasing continuous when it is $(\leq_{K_1}$)increasing continuous and if $\alpha = \zeta$ then $\mathbf{p}_\alpha$ is the exact limit of $\bar{\mathbf{p}} \restriction \zeta$

[1] if $\lambda = \aleph_1$, we may change the definitions of $\mathbf{k} \in K_2$, instead $\langle \mathbb{P}_\alpha[\mathbf{k}] : \alpha < \lambda\rangle$ is $\lessdot$-increasing, we carry with us large enough family of dense subsets, e.g. coming from some countable N.

Observation 1.4. 1) $\le_{K_1}$ is a partial order on K_1.
2) $\le^+_{K_1} \subseteq \le_{K_1}$ is a partial order on K_1.
3) If $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \delta \rangle$ is a $\le_{K_1}$-increasing sequence and $\cup\{\mathbb{P}_{\mathbf{p}_\alpha} : \alpha < \delta\}$ satisfies the c.c.c. and $\delta < \lambda$ then some $\mathbf{p} \in K_1$ is the union $\cup\{\mathbf{p}_\alpha : \alpha < \delta\}$ of $\bar{\mathbf{p}}$, i.e. $\cup\bar{\mathbf{p}} \in K_1$ $P_{\mathbf{p}} - \cup\{P\mathbf{p}_\alpha : \alpha < \delta\}$ and $\alpha < \delta \Rightarrow \mathbf{p}_\alpha \le_{K_1} \mathbf{p}$; this determines $\mathbf{p}$ uniquely and $\mathbf{p}$ is the exact limit of $\bar{\mathbf{p}}$.
4) If $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \delta \rangle$ is $\le_{K_1}$-increasing and $\mathrm{cf}(\delta) = \aleph_1$ implies $\{\alpha < \delta : \mathbf{p}_\alpha$ the exact limit of $\bar{\mathbf{p}} \restriction \alpha$ or just $\bigcup\limits_{\beta<\alpha} \mathbb{P}_{\mathbf{p}_\beta} \lessdot \mathbb{P}_{\mathbf{p}_\alpha}\}$ is a stationary subset of δ then $\cup\bar{\mathbf{p}} \in K_1$ is a $\le_{K_1}$-upper bound of $\bar{\mathbf{p}}$ and is the exact limit of $\bar{\mathbf{p}}$.
5) If in part (4), $\bar{\mathbf{p}}$ is also $\le^+_{K_1}$-increasing then $\alpha < \delta \Rightarrow \mathbf{p}_\alpha \le^+_{K_1} \mathbf{p}$.
6) If $\mathbf{p} \le_{K_1} \mathbf{q}$ and $\Vdash_{\mathbb{P}_{\mathbf{p}}}$"$\mathrm{MA}\aleph_1$" (less suffice) then

$$\mathbf{p} \le_{K_1^+} \mathbf{q}$$

Proof. Should be clear, e.g. in part (5) recall that c.c.c. forcing preserve stationarity of subsets of δ. $\square_{1.4}$

We now define the partial order $\le^*_{K_1}$; it will be used in describing $\mathbf{k}_1 <_{K_2} \mathbf{k}_2$, i.e. demanding $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\alpha) \le^*_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\alpha+1}, \mathbf{p}^{\mathbf{k}_2}_{\alpha+1})$ for many $\alpha < \lambda$.

Definition 1.5. 1) Let $\le^*_{K_1}$ be the following two-place relation on the family of pairs $\{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \le_{K_1} \mathbf{q}\}$. We let $(\mathbf{p}_1, \mathbf{q}_1) \le^*_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$ iff

(a) $\mathbf{p}_1 \le^+_{K_1} \mathbf{p}_2$
(b) $\mathbf{q}_1 \le^+_{K_1} \mathbf{q}_2$
(c) $\Vdash_{\mathbb{P}[\mathbf{p}_2]}$ "$\mathbb{P}_{\mathbf{q}_1}/(\underset{\sim}{G}_{\mathbb{P}[\mathbf{p}_2]} \cap \mathbb{P}_{\mathbf{p}_1}) \lessdot \mathbb{P}_{\mathbf{q}_2}/\underset{\sim}{G}_{\mathbb{P}[\mathbf{p}_2]}$"
(d) $u_{\mathbf{p}_2} \cap u_{\mathbf{q}_1} = u_{\mathbf{p}_1}$

2) Let $\le'_{K_1}$ be the following two-place relation on the family $\{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \le_{K_1} \mathbf{q}\}$ of pairs. We let $(\mathbf{p}_1, \mathbf{q}_1) \le'_{K_1} (\mathbf{p}_2, \mathbf{q}_2)$ iff clauses (a),(b),(d) from part (1) above and

(c)′ if $p_1 \in \mathbb{P}_{\mathbf{p}_1}, q_1 \in \mathbb{P}_{\mathbf{q}_1}$ and $p_1 \Vdash_{\mathbb{P}_{\mathbf{p}_1}}$ "$q_1 \in \mathbb{P}_{\mathbf{q}_1}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_1}}$" then $p_1 \Vdash_{\mathbb{P}_{\mathbf{p}_2}}$ "$q_1 \in \mathbb{P}_{\mathbf{q}_2}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_2}}$".

3) Assume $\mathbf{p}_\ell \in K_1$ for $\ell = 0, 1, 2$ and $\mathbf{p}_0 \le_{K_1} \mathbf{p}_1$ and $\mathbf{p}_0 \le_{K_1} \mathbf{p}_2$ and $u_{\mathbf{p}_1} \cap u_{\mathbf{p}_2} = u_{\mathbf{p}_0}$. We define the amalgamation $\mathbf{p} = \mathbf{p}_3 = \mathbf{p}_1 \times_{\mathbf{p}_0} \mathbf{p}_2$ or $\mathbf{p}_3 = \mathbf{p}_1 \times \mathbf{p}_2/\mathbf{p}_0$ as the triple $(u_{\mathbf{p}}, P_{\mathbf{p}} \le_{\mathbf{p}})$ as follows[2]:

(a) $u_{\mathbf{p}} = u_{\mathbf{p}_1} \cup u_{\mathbf{p}_2}$
(b) $P_{\mathbf{p}} = P_{\mathbf{p}_1} \cup P_{\mathbf{p}_2} \cup \{(p_1, p_2) : p_1 \in P_{\mathbf{p}_1} \backslash P_{\mathbf{p}_0}, p_2 \in P_{\mathbf{p}_2} \backslash P_{\mathbf{p}_0}$ and for some $p \in P_{\mathbf{p}_0}$ we have $p \Vdash_{\mathbb{P}[\mathbf{p}_0]}$ "$p_\ell \in \mathbb{P}_{\mathbf{p}_\ell}/\mathbb{P}_{\mathbf{p}_0}$" for $\ell = 1, 2\}$
(c) $\le_{\mathbf{p}}$ is defined naturally as $\le_{\mathbf{p}_1} \cup \le_{\mathbf{p}_2} \cup \{((p_1, p_2), (q_1, q_2)) : (p_1, p_2), (q_1, q_2) \in P_{\mathbf{p}}$ and $p_1 \le_{\mathbf{p}_1} q_1$ and $p_2 \le_{\mathbf{p}_2} q_2\} \cup \{(p'_\ell, (p_1, p_2)) : p'_\ell \in P_{\mathbf{p}_\ell}, (p_1, p_2) \in P_{\mathbf{p}}$ and $p'_\ell \le_{\mathbf{p}_1} p_\ell$ and $\ell \in \{1, 2\}\}$.

Remark 1.6. Why not use u instead $\mathscr{H}_{<\aleph_1}(u)$? Not a real difference but, e.g. there may not be enough elements in a union of two.

[2]If in clause (b) of 1.5(3) we would like to avoid "$p_\ell \in \mathbb{P}_{\mathbf{p}_\ell} \backslash \mathbb{P}_{\mathbf{p}_0}$" we may replace (p_1, p_2) by $(p_1, p_2, u_{\mathbf{p}_1} \cup u_{\mathbf{p}_2})$ when $\mathbf{p}_1 \ne \mathbf{p}_1 \wedge \mathbf{p}_0 \ne \mathbf{p}_2$ equivalently $\mathbf{p}_0 \ne \mathbf{p}_1 \wedge \mathbf{p}_0 \ne \mathbf{p}_2$.

Observation 1.7. 1) $\leq^*_{K_1}, \leq'_{K_1}$ are partial orders on their domains.
2) $(\mathbf{p}_1,\mathbf{q}_1) \leq^*_{K_1} (\mathbf{p}_1,\mathbf{q}_1)$ implies $(\mathbf{p}_1,\mathbf{q}_1) \leq'_{K_1} (\mathbf{p}_2,\mathbf{q}_2)$.

For the "successor case vertically and horizontally" we shall use

Claim 1.8. *Assume that $\mathbf{p}_1 \leq^+_{K_1} \mathbf{p}_2$ and $\mathbf{p}_1 \leq_{K_1} \mathbf{q}_1$ and $u_{\mathbf{p}_2} \cap u_{\mathbf{q}_1} = u_{\mathbf{p}_1}$ then $\mathbf{q}_2 \in K_1$ and $(\mathbf{p}_1,\mathbf{q}_1) \leq^*_{K_1} (\mathbf{p}_2,\mathbf{q}_2)$ when we define $\mathbf{q}_2 = \mathbf{q}_1 \times_{\mathbf{p}_1} \mathbf{p}_2$ as in 1.5(3).*

Proof. Straightforward. $\square_{1.8}$

The following claim will be applied to a pair of vertically increasing continuous sequences, one standing horizontally to the right of the other.

Claim 1.9. *Assume $\varepsilon(*) < \lambda$ is a limit ordinal and*

(a) *$\langle \mathbf{p}^\ell_\varepsilon : \varepsilon \leq \varepsilon(*)\rangle$ is $(=\varepsilon(*))$-strictly $\leq^+_{K_1}$-increasing continuous for $\ell = 1,2$*
(b) *$(\mathbf{p}^1_\varepsilon,\mathbf{p}^2_\varepsilon) \leq'_{K_1} (\mathbf{p}^1_\zeta,\mathbf{p}^2_\zeta)$ for $\varepsilon < \zeta < \varepsilon(*)$.*

Then

(α) *$\mathbf{p}^1_{\varepsilon(*)} \leq_{K_1} \mathbf{p}^2_{\varepsilon(*)}$*
(β) *for $\varepsilon < \zeta \leq \varepsilon(*)$ we have $(\mathbf{p}^1_\varepsilon,\mathbf{p}^2_\varepsilon) \leq'_{K_1} (\mathbf{p}^1_\zeta,\mathbf{p}^2_\zeta)$.*

Proof. Easy. The main point is to prove $\mathbb{P}^1_{\varepsilon(*)} \lessdot \mathbb{P}^2_{\varepsilon(*)}$, so let $q \in \mathbb{P}^2_{\varepsilon(*)}$, By clause (a) for $\ell = 2$, for $\alpha = \varepsilon(*)$ for some $\varepsilon < \varepsilon(*)$ we have $q \in \mathbb{P}^2_\varepsilon$. Recalling $\mathbb{P}^1_\varepsilon$ there is $p \in \mathbb{P}^1_\varepsilon$, and it suffice to prove that for every $p_1 \in \mathbb{P}^1_{\varepsilon(*)}$ abov e p, the conditions p_1, q are compatible in $\mathbb{P}^2)\varepsilon(*)$. Fixing p_1, let $\zeta \in (\varepsilon,\varepsilon(*)$ be such that $p_1 \in \mathbb{P}^1_\zeta$, again by clause (a) but this time for $\ell = 1$. By clause (b) and the definition of $\leq'_{K_1}$ we get that p_1, q are compatible in $\mathbb{P}^2_\zeta$ that is having a common upper bound and it serve to prove p_1, q are compatible in $\mathbb{P}^2_{\varepsilon(*)}$. This prove clause ($\alpha$), and the argument prove also clause (β). $\square_{1.9}$

For the "successor case horizontally, limit case vertically when the relevant game, i.e. the relevant winning strategy is not active" we shall use

Claim 1.10. *Assume $\varepsilon(*) < \lambda$ is a limit ordinal and*

(a) *$\langle \mathbf{p}_\varepsilon : \varepsilon \leq \varepsilon(*)\rangle$ and $\langle \mathbf{q}_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ are $(=\varepsilon(*))$-strictly $\leq^+_{K_1}$-increasing continuous*
(b) *$\mathbf{p}_\varepsilon \leq_{K_1} \mathbf{q}_\varepsilon$ for $\varepsilon < \varepsilon(*)$*
(c) *if $\varepsilon < \zeta < \varepsilon(*)$ then $(\mathbf{p}_\varepsilon,\mathbf{q}_\varepsilon) \leq'_{K_1} (\mathbf{p}_\zeta,\mathbf{q}_\zeta)$.*

Then we can choose $\mathbf{q}_{\varepsilon(*)}$ such that

(α) *$\mathbf{p}_{\varepsilon(*)} \leq_{K_1} \mathbf{q}_{\varepsilon(*)}$*
(β) *$(\mathbf{p}_\varepsilon,\mathbf{q}_\varepsilon) \leq'_{K_1} (\mathbf{p}_{\varepsilon(*)},\mathbf{q}_{\varepsilon(*)})$ for every $\varepsilon < \varepsilon(*)$*
(γ) *$\langle \mathbf{q}_\varepsilon : \varepsilon \leq \varepsilon(*)\rangle$ is $(=\varepsilon(*))$-strictly $\leq_{K^+_1}$-increasing continuous.*

Remark 1.11. We can replace $\leq'_{K_1}$ by $\leq^*_{K_1}$ in (c) and (β) of 1.10 and (b), (β) of 1.9.

Proof. But 1.9 should be clear. $\square_{1.10}$

The game defined below is the non-FS ingredient; (in the main application below, $\gamma = \lambda$), it is for the horizontal direction; it lasts $\gamma \leq \lambda$ steps but will be used in $\leq_{K^2_f}$-increasing subsequences of $\langle \mathbf{k}_i : i < \lambda^+\rangle$.

Definition 1.12. For $\delta < \lambda$ and $\gamma \le \lambda$ let $\partial_{\delta,\gamma}$ be the following game between the player INC (incomplete) and COM (complete).

A play last γ moves. In the β-th move a pair $(\mathbf{p}_\beta, \mathbf{q}_\beta)$ is chosen such that $\mathbf{p}_\beta \le^+_{K_1} \mathbf{q}_\beta$ and $\beta(1) < \beta \Rightarrow (\mathbf{p}_{\beta(1)}, \mathbf{p}_\beta) \le'_{K_1} (\mathbf{q}_{\beta(1)}, \mathbf{q}_\beta)$ and $u_{\mathbf{p}_\beta} \cap \lambda = \delta$ and $u_{\mathbf{q}_\beta} \cap \lambda = u_{\mathbf{q}_0} \cap \lambda \supseteq \delta + 1$.

In the β-th move first INC chooses $(\mathbf{p}_\beta, u_\beta)$ such that $\mathbf{p}_\beta$ satisfies the requirements and u_β satisfies the requirements on $u_{\mathbf{q}_\beta}$ (i.e. $\cup\{u_{\mathbf{q}_\alpha} : \alpha < \beta\} \cup u_{\mathbf{p}_\beta} \subseteq u_\beta \in [\lambda^+]^{<\lambda}$ and $u_\beta \cap \lambda = u_{\mathbf{q}_0} \cap \lambda$) and say $u_\beta \backslash u_{\mathbf{p}_\beta} \backslash \cup \{u_{\mathbf{q}_\gamma} : \gamma < \beta\}$ has cardinality $\ge |\delta|$ (if λ is weakly inaccessible we may be interested in asking more).
Second, COM chooses $\mathbf{q}_\beta$ as required such[3] that $u_\beta \subseteq u[\mathbf{q}_\beta]$.

A player which has no legal moves loses the play, and arriving to the γ-th move, COM wins.

Remark 1.13. 1) It is not problematic for COM to have a winning strategy. But having "interesting" winning strategies is the crux of the matter. More specifically, any application of this section is by choosing such strategies.
2) Here it is not natural to demand strict continuity for $\langle \mathbf{p}_\alpha : \sigma \le \beta \rangle$ as this fail in non-trivial cases. Still we may consider requiring more
3) Such examples are the

- (a) lazy man strategy: preserve $\mathbb{P}_{\mathbf{q}_\beta} = \mathbb{P}_{\mathbf{q}_0} \times_{\mathbb{P}_{\mathbf{p}_0}} \mathbb{P}_{\mathbf{p}_\beta}$ recalling Claim 1.8
- (b) it is never too late to become lazy, i.e. arriving to $(\mathbf{p}_{\beta(*)}, \mathbf{q}_{\beta(*)})$ the COM player may decide that $\beta \ge \beta(*) \Rightarrow \mathbb{P}_{\mathbf{q}_\beta} = \mathbb{P}_{q_{\beta(*)}} \times_{\mathbb{P}_{\mathbf{p}_{\beta(*)}}} \mathbb{P}_{\mathbf{p}_\beta}$
- (c) definable forcing strategy, i.e. preserve "$\mathbb{P}_{\mathbf{q}_\beta}/\mathbb{P}_{\mathbf{p}_\beta}$ is a definable c.c.c. forcing (in $\mathbf{V}^{\mathbb{P}[\mathbf{p}_\beta]}$)".

Definition 1.14. We say f is λ-appropriate if

- (a) $f \in {}^\lambda(\lambda + 1)$
- (b) $\alpha < \lambda \wedge f(\alpha) < \lambda \Rightarrow (\exists \beta)[f(\alpha) = \beta + 1]$
- (c) if $\varepsilon < \lambda^+, \langle u_\alpha : \alpha < \lambda \rangle$ is an increasing continuous sequence of subsets of ε of cardinality $< \lambda$ with union ε then $\{\delta < \lambda\colon \mathrm{otp}(u_\delta) < f(\delta)\}$ is a stationary subset of λ.

Convention 1.15. Below f is λ-appropriate function.

We arrive to defining the set of approximations of size λ (in the main application f_* is constantly λ); we shall later connect it to the oracle version (also see the introduction).

Definition 1.16. For f_* a λ-appropriate function let $K^2_{f_*}$ be the family of $\mathbf{k}$ such that:

- (a) $\mathbf{k} = \langle E, \bar{\mathbf{p}}, S, \bar{\mathbf{s}}, \bar{\mathbf{g}}, f \rangle$
- (b) E is a club of λ
- (c) $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha \in E \rangle$
- (d) $\mathbf{p}_\alpha \in K^1_\lambda$
- (e) $\mathbf{p}_\alpha \le_{K_1} \mathbf{p}_\beta$ for $\alpha < \beta$ from E
- (f) if $\delta \in$ acc(E) then $\mathbf{p}_\delta = \cup\{\mathbf{p}_\alpha : \alpha \in E \cap \delta\}$

[3] we could ask for equality usually

(g) $S \subseteq \lambda$ is a stationary set of limit ordinals

(h) if $\delta \in S \cap E$ (hence a limit ordinal) then $\delta + 1 \in E$

(i) $\bar{\mathbf{s}} = \langle \mathbf{s}_\delta : \delta \in E \cap S \rangle$

(j) $\mathbf{s}_\delta$ is a winning strategy for the player COM in $\Game_{\delta, f_*(\delta)}$, see 1.17(1)

(k) $\bar{\mathbf{g}} = \langle \mathbf{g}_\delta : \delta \in S \cap E \rangle$

(l)
- $\mathbf{g}_\delta$ is an initial segment of a play of $\Game_{\delta, f_*(\delta)}$ in which the COM player uses the strategy $\mathbf{s}_\delta$
- if its length is $< f_*(\delta)$ then $\mathbf{g}_\delta$ has a last move
- $(\mathbf{p}_\delta, \mathbf{p}_{\delta+1})$ is the pair chosen in the last move, call it $\text{mv}(\mathbf{g}_\delta)$
- let $S_0 = \{\delta \in S \cap E : \mathbf{g}_\delta$ has length $< f_*(\delta)\}$ and $S_1 = S \cap E \backslash S_0$

(m) if $\alpha < \beta$ are from E then $\mathbf{p}_\alpha \leq^+_{K_1} \mathbf{p}_\beta$, so in particular $\mathbb{P}_\beta / \mathbb{P}_\alpha$ is absolutely c.c.c. that is if $\mathbb{P} \lessdot \mathbb{P}'$ and $\mathbb{P}'$ is c.c.c. then $\mathbb{P}' *_{\mathbb{P}_\alpha} \mathbb{P}_\beta$ is c.c.c.; this strengthens clause (e)

(n) $f \in {}^\lambda\lambda$

(o) if $\delta \in S \cap E$ then $f(\delta) + 1$ is the length of $\mathbf{g}_\delta$

(p) for every $\delta \in E$, if $f_*(\delta) < \lambda$ then $f(\delta) \leq \text{otp}(u_{\mathbf{p}_\delta})$.

Remark 1.17. 1) Concerning clause (j), recall (using the notation of Definition 1.12) that during a play the player INC chooses $\mathbf{p}_\varepsilon$ and COM chooses $\mathbf{q}_\varepsilon, \varepsilon \leq f(\delta)$ and recalling clause (o) we see that $(\mathbf{p}_{f(\delta)}, \mathbf{q}_{f(\delta)})$ there stands for $(\mathbf{p}_\delta, \mathbf{p}_{\delta+1})$ here. You may wonder from where does the $(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon)$ for $\varepsilon < f(\delta)$ comes from; the answer is that you should think of $\mathbf{k}$ as a stage in an increasing sequence of approximations of length $f(\delta)$ and $(\mathbf{p}_\varepsilon, \mathbf{q}_\varepsilon)$ comes from the δ-place in the ε-approximation. This is cheating a bit - the sequence of approximations has length $< \lambda^+$, but as on a club of λ this reflects to length $< \lambda$, all is O.K.
2) Below we define the partial order $\leq_{K_2}$ (or $\leq_{K^2_{f_*}}$) on the set $K^2_{f_*}$, recall our goal is to choose an $\leq_{K_2}$-increasing sequence $\langle \mathbf{k}_\varepsilon : \varepsilon < \lambda^+ \rangle$ and our final forcing will be $\cup \{\mathbb{P}_{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+\}$.
3) Why clause (d) in Definition 1.18(2) below? It is used in the proof of the limit existence claim 1.24. This is because the club $E_{\mathbf{k}}$ may decrease (when increasing $\mathbf{k}$).

Note that we use $\leq^*_{K^1_f}$ "economically". We cannot in general demand (in 1.18(2) below) that for $\alpha < \beta$ from $E_{\mathbf{k}_2} \backslash \alpha(*)$ we have $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_1}_\beta) \leq^*_{K_1} (\mathbf{p}^{\mathbf{k}_2}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\beta)$ as the strategies $\mathbf{s}_\delta$ may defeat this. How will it still help? Assume $\langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is increasing, $\varepsilon(*) < \lambda$ for simplicity and $\gamma \in \cap \{E_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\} \cap \bigcap \{S_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\} \backslash \cup \{\alpha(\mathbf{k}_\varepsilon, \mathbf{k}_\zeta) : \varepsilon < \zeta < \varepsilon(*)\}$ and $\gamma_\varepsilon = \text{Min}(E_{\mathbf{k}_\varepsilon} \backslash (\gamma + 1))$ for $\varepsilon < \varepsilon(*)$. We shall have $\langle \gamma_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is increasing; there may be $\delta \in (\gamma_\varepsilon, \gamma_{\varepsilon+1})$ where $\mathbf{s}_\delta$ was active between $\mathbf{k}_\varepsilon$ and $\mathbf{k}_{\varepsilon+1}$, so it contributes to $\mathbb{P}^{\mathbf{k}_{\varepsilon+1}}_{\gamma_{\varepsilon+1}} / \mathbb{P}^{\mathbf{k}_\varepsilon}_{\gamma_\varepsilon}$.
4) If we omit the restriction $u \in [\lambda^+]^{<\lambda}$ and allow $f : \lambda \to \delta^* + 1$, replace the club E by an end segment, we can deal with sequences of length $\delta^* < \lambda^+$.

In the direct order in 1.18(3) we have $\alpha(*) = 0$. Using e.g. a stationary non-reflecting $S \subseteq S^{\delta^*}_\lambda$ we can often allow $\alpha(*) \neq 0$.
5) Is the "$\mathbf{s}_\delta$ a winning strategy" in addition for telling us what to do, crucial? The point is preservation of c.c.c. in limit of cofinality $\aleph_1$.
6) If we use $f_* \in {}^\lambda(\lambda + 1)$ constantly λ, we do not need $f_{\mathbf{k}}$ so we can omit clauses (n),(o),(p) of 1.16 and (c), and part of (e) in 1.18(2).

6A) Alternatively we can omit clause (o) in 1.16 but demand "$\prod_{\alpha<\lambda} f(\alpha)/\mathscr{D}$ is λ^+-directed", fixing a normal filter $\mathscr{D}$ on λ (and demand $S_{\mathbf{k}} \in \mathscr{D}^+$).
7) The "omitting type" argument here comes from using the strategies.
8) We may add in clause 1.16(n) that for some $\gamma < \lambda^+$ and sequence $\bar{u} = \langle u_\alpha : \alpha < \lambda\rangle$ as in 1.15(c) the set $\{\alpha < \lambda : f(\alpha) \le \text{otp}(u_\alpha)\}$ contains a club of λ.

Definition 1.18. 1) In Definition 1.16, let $E = E_{\mathbf{k}}, \bar{\mathbf{p}} = \bar{\mathbf{p}}_{\mathbf{k}}, \mathbf{p}_\alpha = \mathbf{p}^{\mathbf{k}}_\alpha = \mathbf{p}_\alpha[\mathbf{k}], \mathbb{P}_\alpha = \mathbb{P}^{\mathbf{k}}_\alpha = \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}]}, S = S_{\mathbf{k}}$ for $\ell = 0, 1$, etc. and we let $\mathbb{P}_{\mathbf{k}} = \cup\{\mathbb{P}^{\mathbf{k}}_\alpha : \alpha \in E_{\mathbf{k}}\}$ and $u_{\mathbf{k}} = u[\mathbf{k}] = \cup\{u_{\mathbf{p}^{\mathbf{k}}_\alpha} : \alpha \in E_{\mathbf{k}}\}$.
2) We define a two-place relation $\le_{K^2_f}$ on $K^2_f : \mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$ <u>iff</u> (both are from K^2_f and) for some $\alpha(*) < \lambda$ (and $\alpha(\mathbf{k}_1, \mathbf{k}_2)$ is the first such $\alpha(*) \in E_{\mathbf{k}_2}$) we have:

- (a) $E_{\mathbf{k}_2} \backslash E_{\mathbf{k}_1}$ is bounded in λ, moreover $\subseteq \alpha(*)$
- (b) for $\alpha \in E_{\mathbf{k}_2} \backslash \alpha(*)$ we have $\mathbf{p}^{\mathbf{k}_1}_\alpha \le_{K_1} \mathbf{p}^{\mathbf{k}_2}_\alpha$
- (c) if $\alpha \in E_{\mathbf{k}_2} \backslash \alpha(*)$ then $f_{\mathbf{k}_1}(\alpha) \le f_{\mathbf{k}_2}(\alpha)$
- (d) if $\gamma_0 < \gamma_1 \le \gamma_2 < \lambda, \gamma_0 \in E_{\mathbf{k}_2} \backslash (\alpha(*) \cup S_{\mathbf{k}_1}), \gamma_1 = \min(E_{\mathbf{k}_1} \backslash (\gamma_0 + 1))$ and $\gamma_2 = \min(E_{\mathbf{k}_2} \backslash (\gamma_0 + 1))$, <u>then</u> $(\mathbf{p}^{\mathbf{k}_1}_{\gamma_0}, \mathbf{p}^{\mathbf{k}_2}_{\gamma_0}) \le'_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\gamma_1}, \mathbf{p}^{\mathbf{k}_2}_{\gamma_2})$, see Definition 1.5(2) really follows from clause (h) below
- (e) if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ <u>then</u> $\delta \in S_{\mathbf{k}_2} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$; but note that if $f_{\mathbf{k}_1}(\delta) \ge f(\delta)$ we put δ into $S_{\mathbf{k}_2}$ just for notational convenience as "the game is over"
- (f) if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ <u>then</u> $\mathbf{s}^{\mathbf{k}_2}_\delta = \mathbf{s}^{\mathbf{k}_1}_\delta$ and $\mathbf{g}^{\mathbf{k}_1}_\delta$ is an initial segment of $\mathbf{g}^{\mathbf{k}_2}_\delta$
- (g) if $\mathbf{k}_1 \ne \mathbf{k}_2$ <u>then</u> $u[\mathbf{k}_1] \ne u[\mathbf{k}_2]$
- (h) if $\alpha < \beta$ are from $E_{\mathbf{k}_2} \backslash \alpha(*)$ <u>then</u> $(\mathbf{p}^{\mathbf{k}_1}_\alpha, \mathbf{p}^{\mathbf{k}_2}_\alpha) \le'_{K_1} (\mathbf{p}^{\mathbf{k}_1}_\beta, \mathbf{p}^{\mathbf{k}_2}_\beta)$, see Definition 1.5(2), i.e. if $p \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}, q \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]}$ and $p \Vdash_{\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}}$ "$q \in \mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_2]}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_\alpha[\mathbf{k}_1]}}$" <u>then</u> $p \Vdash_{\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_1]}}$ "$q \in \mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_2]}/\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_\beta[\mathbf{k}_1]}}$".

3) We define a two-place relation $\le^{\text{dir}}_{K^2_f}$ on K^2_f as follows: $\mathbf{k}_1 \le^{\text{dir}}_{K^2_f} \mathbf{k}_2$ iff

- (a) $\mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$
- (b) $E_{\mathbf{k}_2} \subseteq E_{\mathbf{k}_1}$; no real harm here if we add $\mathbf{k}_1 \ne \mathbf{k}_2 \Rightarrow E_{\mathbf{k}_2} \subseteq \text{acc}(E_{\mathbf{k}_1})$
- (c) $\alpha(\mathbf{k}_1, \mathbf{k}_2) = \text{Min}(E_{\mathbf{k}_2})$.

4) We write $K^2_\lambda, \le_{K^2_\lambda}, \le^{\text{dir}}_{K^2_\lambda}$ or just $K_2, \le_{K_2}, <^{\text{dir}}_{K_2}$ for $K^2_f, \le_{K^2_f}, \le^{\text{dir}}_{K^2_f}$ when f is constantly λ.

Remark 1.19. 1) In [She06] we may increase S as well as here but we may replace clause (e) of Definition 1.18(2) by

$(e)'$ $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ <u>iff</u> $f_{\mathbf{k}_2}(\delta) < f(\delta) \wedge \delta \in S_{\mathbf{k}_2} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$.

If we do this, is it a great loss? No! This can still be done here by choosing $\mathbf{s}_\delta$ such that as long as INC chooses u_β of certain form (e.g. $u_\beta \backslash u^{\mathbf{p}_\beta} = \{\delta\}$) the player COM chooses $\mathbf{q}_\beta = \mathbf{p}_\beta$. We can allow in Definition 1.18(2) to extend S but a priori start with $\langle S_\varepsilon : \varepsilon < \lambda^+\rangle$ such that $S_\varepsilon \subseteq \lambda$ and $S_\varepsilon \backslash S_\zeta$ is bounded in λ when $\varepsilon < \zeta < \lambda$ and demand $S_{\mathbf{k}_\varepsilon} = S_\varepsilon$.
2) We can weaken clause (e) of 1.18(2) to

$(e)''$ if $\delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_2} \backslash \alpha(*)$ and $f_{\mathbf{k}_2}(\delta) < f(\delta)$ then $\delta \in S_{\mathbf{k}_2}$.

But then we have to change accordingly, e.g. 1.18(c),(f), 1.21(c).
3) We can define $\mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$ demanding $(S_{\mathbf{k}_1}, \bar{\mathbf{s}}_{\mathbf{k}_1}) = (S_{\mathbf{k}_2}, \bar{\mathbf{s}}_{\mathbf{k}_2})$ but replace everywhere "$\delta \in S_{\mathbf{k}} \cap E_{\mathbf{k}}$" by "$\delta \in S_{\mathbf{k}} \cap E_{\mathbf{k}} \wedge f_{\mathbf{k}}(\delta) \le f(\delta)$" so omit clause (e) of 1.18.

Observation 1.20. 1) $\le_{K^2_f}$ is a partial order on K^2_f.
2) $\le^{\mathrm{dir}}_{K^2_f} \subseteq \le_{K^2_f}$ is a partial order on K^2_f.
3) If $\mathbf{k}_1 \le_{K^2_f} \mathbf{k}_2$ then $\mathbb{P}_{\mathbf{k}_1} \lessdot \mathbb{P}_{\mathbf{k}_2}$.
4) If $\langle \mathbf{k}_\varepsilon : \varepsilon < \lambda^+ \rangle$ is $<_{K^2_f}$-increasing and $\mathbb{P} = \cup\{\mathbb{P}_{\mathbf{k}_\varepsilon} : \varepsilon < \lambda^+\}$ then

- (a) $\mathbb{P}$ is a c.c.c. forcing notion of cardinality $\le \lambda^+$
- (b) $\mathbb{P}_{\mathbf{k}_\varepsilon} \lessdot \mathbb{P}$ for $\varepsilon < \lambda^+$.

Definition 1.21. 1) Assume $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\le_{K^2_f}$-increasing with $\varepsilon(*)$ a limit ordinal $< \lambda$. We say $\mathbf{k}$ is a limit of $\bar{\mathbf{k}}$ when $\mathbf{k} \in K^2_f$ and $\varepsilon < \varepsilon(*) \Rightarrow \mathbf{k}_\varepsilon \le_{K^2_f} \mathbf{k} \in K^2_f$ and for some $\alpha(*)$

- (a) $\alpha(*) = \cup\{\alpha(\mathbf{k}_\varepsilon, \mathbf{k}_\zeta) : \varepsilon < \zeta < \varepsilon(*)\}$
- (b) $E_{\mathbf{k}} \backslash \alpha(*) \subseteq \cap\{E_{\mathbf{k}_\varepsilon} \backslash \alpha(*) : \varepsilon < \varepsilon(*)\}$
- (c) $S_{\mathbf{k}} = (\cup\{S_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\}) \cap (\cap\{E_{\mathbf{k}_\varepsilon} : \varepsilon < \varepsilon(*)\}) \backslash \alpha(*)$
- (d) if $\delta \in S_{\mathbf{k}}$ then $\mathbf{g}^{\mathbf{k}_\varepsilon}_\delta$ is an initial segment of $\mathbf{g}^{\mathbf{k}}_\delta$ for every $\varepsilon < \varepsilon(*)$
- (e) $f_{\mathbf{k}}(\delta) = \cup\{f_{\mathbf{k}_\varepsilon}(\delta) : \varepsilon < \varepsilon(*)\} + 1$ for $\delta \in S_{\mathbf{k}}$.

2) Assume $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \lambda \rangle$ is $\le_{K^2_f}$-increasing continuous, see part (3) below (no viscious circle). We say $\mathbf{k}$ is a limit of $\bar{\mathbf{k}}$ when $\varepsilon < \lambda \Rightarrow \mathbf{k}_\varepsilon \le \mathbf{k} \in K^2_f$ and for some $\bar{\alpha}$

- (a) $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ is increasing continuous, $\lambda > \alpha_\varepsilon \in \cap\{E_{\mathbf{k}_\zeta} : \zeta < 1 + \varepsilon\} \backslash \cup \{\alpha(\mathbf{k}_{\zeta_1}, \mathbf{k}_{\zeta_2}) : \zeta_1 < \zeta_2 < 1 + \varepsilon\}$
- (b) $E_{\mathbf{k}} = \{\alpha_\varepsilon : \varepsilon < \lambda\} \cup \{\alpha_\varepsilon + 1 : \varepsilon < \lambda \text{ and } \varepsilon \in S\}$ and $\mathbf{p}^{\mathbf{k}}_{\alpha_\varepsilon} = p^{\mathbf{k}_\varepsilon}_{\alpha_\varepsilon}, \mathbf{p}^{\mathbf{k}}_{\alpha_\varepsilon+1} = \mathbf{p}^{\mathbf{k}_\varepsilon}_{\alpha_\varepsilon+1}$
- (c) $S_{\mathbf{k}} = \{\alpha_\varepsilon : \alpha_\varepsilon \in \mathbf{S}_{\mathbf{k}_\zeta}$ for every $\zeta < \varepsilon$ large enough$\}$
- (d) if $\delta = \alpha_\varepsilon \in S_{\mathbf{k}_\varepsilon}$ then $\mathbf{g}^{\mathbf{k}}_\delta = \mathbf{g}^{\mathbf{k}_\varepsilon}_\delta$
- (e) if $\alpha < \delta$ and $\zeta = \mathrm{Min}\{\varepsilon : \alpha \le \alpha_{\varepsilon+1}\}$ then $f_{\mathbf{k}}(\alpha) = f_{\mathbf{k}_\zeta}(\alpha)$.

3) We say that $\langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $\le_{K^2_f}$-increasing continuous when:

- (a) $\mathbf{k}_\varepsilon \le_{K^2_f} \mathbf{k}_\zeta$ for $\varepsilon < \zeta < \varepsilon(*)$
- (b) $\mathbf{k}_\varepsilon$ is a limit of $\langle \mathbf{k}_{\xi(\zeta)} : \zeta < \mathrm{cf}(\varepsilon) \rangle$ for some increasing continuous sequence $\langle \xi(\zeta) : \zeta < \mathrm{cf}(\varepsilon) \rangle$ of ordinals with limit ε, for every limit $\varepsilon < \varepsilon(*)$, by part (1) or part (2).

Definition 1.22. 1) In part (1) of 1.21, we say "a direct limit" when in addition

- (α) the sequences are $\le^{\mathrm{dir}}_{K^2_f}$-increasing
- (β) in clause (b) we have equality
- (γ) $\mathbf{p}^{\mathbf{k}}_{\min(E_{\mathbf{k}})}$ is the exact union of $\langle \mathbf{p}^{\mathbf{k}}_{\min(E_{\mathbf{k}_\varepsilon})} : \varepsilon < \varepsilon(*) \rangle$

(δ) if $\gamma \in E_{\mathbf{k}}, \xi < \varepsilon(*), \gamma \notin S^0_{\mathbf{k}_\xi}$ and $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)]\rangle$ is defined by $\gamma_\xi = \gamma, \gamma_\varepsilon = \min(E_{\mathbf{k}_\varepsilon} \backslash (\gamma+1))$ when $\xi < \varepsilon \leq \varepsilon(*)$, so $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)]\rangle$ is an $\leq$-increasing continuous sequence of ordinals, then $\mathbf{p}^{\mathbf{k}}_{\gamma_{\varepsilon(*)}}/\mathbf{p}^{\mathbf{k}}_\gamma = \cup\{\mathbf{p}^{\mathbf{k}_\varepsilon}_{\gamma_\varepsilon}/\mathbf{p}^{\mathbf{k}_\varepsilon}_\gamma : \varepsilon \in [\xi, \varepsilon(*))\}$ with the obvious meaning.

2) In part (2) of Definition 1.21 we say a "direct limit" when in addition

(α) the sequence is $\leq^{\rm dir}_{K^2_f}$

(β) α_ε is minimal under the restrictions.

3) We say that $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is $\leq^{\rm dir}_{K^2_f}$-increasing continuous or directly increasing continuous when :

(a) $\mathbf{k}_\varepsilon \leq^{\rm dir}_{K^2_f} \mathbf{k}_\zeta$ for $\varepsilon \leq \zeta < \varepsilon(*)$

(b) if $\varepsilon < \varepsilon(*)$ is a limit ordinal then for som e club C of ε $\bar{\mathbf{k}}$ is C-continuous which means that $\mathbf{k}_\varepsilon$ is a (really the) direct limit of $\bar{\mathbf{k}} \restriction C$

4) For $\varepsilon(*) < \lambda^+$ we say $\bar{C}$ is an $\varepsilon(*)$-square when

(1) $\bar{C} = \langle C_\varepsilon : \varepsilon < \varepsilon(*)\rangle$
(2) C_ε is a closed subset of ε
(3) if ε is a limit ordinal then C_ε is unboundednin ε
(4) if $\varepsilon \in C)\zeta$ then $C_\varepsilon = C_\zeta \cap \varepsilon$

5) We say that $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is $\leq^{\rm dir}_{K^2_f}$-increasing $\bar{C}$-continuous or directly increasing $\bar{C}$-continuous when

(1) $\bar{\mathbf{k}}$ is as in part (3) above
(2) $\bar{C}$ is an $\varepsilon(*)$-square
(3) for evey limit $\varepsilon < \varepsilon(*)$ $\mathbf{k}$ is C_ε-continuous

Claim 1.23. *If $\mathbf{k}_1 \leq_{K^2_f} \mathbf{k}_2$ then for some $\mathbf{k}'_2$ we have*

(a) $\mathbf{k}_1 \leq^{\rm dir}_{K^2_f} \mathbf{k}'_2$

(b) $\mathbf{k}_2 \leq_{K^2_f} \mathbf{k}'_2 \leq_{K^2_f} \mathbf{k}_2$

(c) $\mathbf{k}_2, \mathbf{k}'_2$ *are almost equal - the only differences being $E_{\mathbf{k}'_2} = E_{\mathbf{k}_2} \backslash \min(E_{\mathbf{k}'_2}), S_{\mathbf{k}'_2} \subseteq S_{\mathbf{k}_2}$, etc.*

Claim 1.24. *The limit existence claim 1) If $\varepsilon(*) < \lambda$ is a limit ordinal and $\bar{\mathbf{k}} = \langle \mathbf{k}_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is a directly increasing continuous then $\bar{\mathbf{k}}$ has a direct limit.*
2) Similarly for $\varepsilon(*) = \lambda$, i.e. if $\langle \mathbf{k}_\varepsilon : \varepsilon < \lambda\rangle$ is directly increasing continuous then there is $\mathbf{k}$ such that:

(a) $\varepsilon < \lambda \Rightarrow \mathbf{k}_\varepsilon \leq \mathbf{k}$

(b) *for each $\varepsilon < \lambda$, if $\mathbf{k}^{[\varepsilon]}$ is like $\mathbf{k}$ omitting $E_{\mathbf{k}} \cap \varepsilon$ then $\mathbf{k}_\varepsilon le_{\rm dir} \mathbf{k}^{[\varepsilon]}$.*

Proof. It is enough to prove the direct version.
1) We define $\mathbf{k} = \mathbf{k}_{\varepsilon(*)}$ as in the definition, we have no freedom left.

The main points concern the c.c.c. and the absolute c.c.c., $\leq'_{K^0_1}, \leq_{K_1}$ demands. We prove the relevant demands by induction on $\beta \in E_{\mathbf{k}_{\varepsilon(*)}}$.

Case 1: $\beta = \min(E_{\mathbf{k}_{\varepsilon(*)}})$.

First note that $\langle \mathbf{p}^\varepsilon_{\min(E_{\mathbf{k}_\varepsilon})} : \varepsilon \le \varepsilon(*)\rangle$ is increasing continuous (in K^1_λ) moreover $\langle \mathbb{P}[\mathbf{p}^{\mathbf{k}_\varepsilon}_{\min(E_{\mathbf{k}_\varepsilon})}] : \varepsilon \le \varepsilon(*)\rangle$ is increasing continuous, see clause (γ) of Definition 1.22(1). As each $\mathbb{P}[\mathbf{p}_{\min(E_{\mathbf{k}_\varepsilon})}]$ is c.c.c. if $\varepsilon < \varepsilon(*)$, we know that this holds for $\varepsilon = \varepsilon(*)$, too.

Case 2: $\beta = \delta + 1, \delta \in S^1_{\mathbf{k}} \cap E_{\mathbf{k}}$.

Since $\mathbf{s}^{\mathbf{k}}_\delta$ is a winning strategy in the game $\eth_{\delta, f_*(\delta)}$ we have $\mathbf{p}^{\mathbf{k}_{\varepsilon(*)}}_\delta \le^+_{K_1} \mathbf{p}^{\mathbf{k}_{\varepsilon(*)}}_\beta$. But what if the play is over? Recall that in Definition 1.14, $f_*(\delta) = \lambda$ or $f_*(\delta)$ is successor and $\langle f_{\mathbf{k}_\varepsilon}(\delta) : \varepsilon < \varepsilon(*)\rangle$ is (strictly) increasing, so this never happens; it may happen when we try to choose $\mathbf{k}'$ such that $\mathbf{k} <_{K^2_f} \mathbf{k}'$, see 1.25.

We also have to show: if $\alpha \in \beta \cap E_{\mathbf{k}}$ then $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\beta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$ is absolutely c.c.c. First, if $\alpha = \delta$ this holds by Definition 1.3(3) of $\le^+_{K_1}$ and the demand $\mathbf{p}_\beta \le^+_{K_1} \mathbf{q}_\beta$ in Definition 1.12 (and clause (ℓ) of Definition 1.16). Second, if $\alpha < \delta$, it is enough to show that $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\beta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\delta]$ and $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\delta]/\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$ are absolutely c.c.c., but the first holds by the previous sentence, the second by the induction hypothesis. In particular, when $\varepsilon < \varepsilon(*) \Rightarrow \mathbb{P}^{\mathbf{k}_\varepsilon}_\beta \lessdot \mathbb{P}^{\mathbf{k}}_\beta$.

Case 3: For some $\gamma, \gamma = \max(E_{\mathbf{k}} \cap \beta), \gamma \notin S^1_{\mathbf{k}}$.

As $\gamma \notin S_{\mathbf{k}}$ there is $\xi < \varepsilon(*)$ such that $\gamma \notin S^1_{\mathbf{k}_\xi}$ let $\gamma_\xi = \gamma$ and for $\varepsilon \in (\xi, \varepsilon(*)]$ we define $\gamma_\varepsilon =: \min(E_{\mathbf{k}_\varepsilon} \backslash (\beta+1))$. Now as $\bar{\mathbf{k}}$ is directly increasing continuous we have

$\circledast$ (a) $\langle \gamma_\varepsilon : \varepsilon \in [\xi, \varepsilon(*)]\rangle$ is increasing continuous
(b) $\gamma_\xi = \gamma$
(c) $\gamma_{\varepsilon(*)} = \beta$
(d) $\langle \mathbf{p}^{\mathbf{k}_\varepsilon}_{\gamma_\varepsilon} : \varepsilon \in [\xi, \varepsilon(*)]\rangle$ is increasing continuous.

So by claim 1.10 we are done, the main point is that clause (d) there holds by clause (d) of the definition of $\le_{K^2_f}$ in 1.18(2).

Case 4: $\beta = \sup(E_{\mathbf{k}} \cap \beta)$.

It follows by the induction hypothesis and 1.4(3) as $\langle \mathbf{p}^{\mathbf{k}}_\gamma : \gamma \in E_k \cap \beta\rangle$ is $\le^+_{K_1}$-increasing continuous with union $\mathbf{p}^{\mathbf{k}}_\beta$; of course we use clause (h) of Definition 1.18, so Definition 1.5(2),(5) applies.
2) Similarly. $\square_{1.24}$

The following is an atomic step toward having $\mathrm{MA}_{<\lambda}$.

Claim 1.25. *Assume*

(a) $\mathbf{k}_1 \in K^2_f$
(b) $\alpha(*) \in E_{\mathbf{k}_1}$
(c) $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}[\mathbf{p}^{\mathbf{k}_1}_{\alpha(*)}]$*-name of a c.c.c. forcing (hence* $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"*$\underset{\sim}{\mathbb{Q}}$ *is a c.c.c. forcing")*
(d) $u_* \subseteq \lambda^+$ *is disjoint to* $u[\mathbf{k}_1] = \cup\{u_{\mathbf{p}_\alpha[\mathbf{k}_1]} : \alpha \in E_{\mathbf{k}}\}$ *and of cardinality* $< \lambda$ *but* $\ge |\underset{\sim}{\mathbb{Q}}|$.

Then we can find $\mathbf{k}_2$ *such that*

(α) $\mathbf{k}_1 \le^{\mathrm{dir}}_{K^2_f} \mathbf{k}_2 \in K^2_f$

(β) $E_{\mathbf{k}_2} = E_{\mathbf{k}_1} \backslash \alpha(*)$

(γ) $u^{\mathbf{k}_2}_\alpha = u^{\mathbf{k}_1}_\alpha \cup u_*$ *for* $\alpha \in E_{\mathbf{k}_2} \cap S^1_{\mathbf{k}_1}$

(δ) $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_2]}$ *is isomorphic to* $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ *over* $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]}$

(ε) $S_{\mathbf{k}_2} = S_{\mathbf{k}_1} \backslash \alpha(*)$ *and* $\bar{\mathbf{s}}_{\mathbf{k}_2} = \bar{\mathbf{s}}_{\mathbf{k}_1} \restriction S_{\mathbf{k}_2}$

(ζ) $f_{\mathbf{k}_2} = f_{\mathbf{k}_1} + 1$

(η) *if* $\Vdash_{\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{\rho} \in {}^\omega 2$ *but* $\underset{\sim}{\rho} \notin \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]}}]$" *then* $\Vdash_{\mathbb{P}_{\mathbf{k}_2}}$ "$\underset{\sim}{\rho} \in {}^\omega 2$ *but* $\underset{\sim}{\rho} \notin \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}_1}}]$ *provided that the strategies preserve this which they do in the cases used here.*

Proof. We choose $\mathbf{p}^{\mathbf{k}_2}_\alpha$ by induction on $\alpha \in E_{\mathbf{k}_1} \backslash \alpha(*)$, keeping all relevant demands (in particular $u_{\mathbf{p}_\alpha[\mathbf{k}_2]} \cap u[\mathbf{k}_1] = u_{\mathbf{p}_\alpha[\mathbf{k}_1]}$).

Case 1: $\alpha = \alpha(*)$.

As only the isomorphism type of $\underset{\sim}{\mathbb{Q}}$ is important, without loss of generality $\Vdash_{\mathbb{P}[\mathbf{p}^{\mathbf{k}_1}_{\alpha(*)}]}$ "every member of $\underset{\sim}{\mathbb{Q}}$ belongs to u_*".

So we can interpret the set of elements of $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} * \underset{\sim}{\mathbb{Q}}$ such that it is $\subseteq \mathscr{H}_{<\aleph_1}(u_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} \cup u_*)$.

Now $\mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_1]} \lessdot \mathbb{P}_{\mathbf{p}_{\alpha(*)}[\mathbf{k}_2]}$ by the classical claims on composition of forcing notions.

Case 2: $\alpha = \delta + 1, \delta \in S_{\mathbf{k}_1} \cap E_{\mathbf{k}_1} \backslash \alpha(*)$.

The case split to two subcases.

Subcase 2A: The play $\mathbf{g}^{\mathbf{k}_1}_\delta$ is not over, i.e. $f(\delta)$ is larger than the length of the play so far.

In this case do as in case 2 in the proof of 1.24, just use $\mathbf{s}_\delta$.

Subcase 2B: The play $\mathbf{g}^{\mathbf{k}_1}_\delta$ is over.

In this case let $\mathbb{P}^{\mathbf{k}_2}_{\delta+1} = \mathbb{P}^{\mathbf{k}_1}_{\delta+1} *_{\mathbb{P}^{\mathbf{k}_1}_\delta} \mathbb{P}^{\mathbf{k}_2}_\delta$, in fact, $\mathbf{p}^{\mathbf{k}_2}_{\delta+1} = \mathbf{p}^{\mathbf{k}_1}_{\delta+1} *_{\mathbf{p}^{\mathbf{k}_1}_\delta} \mathbf{p}^{\mathbf{k}_2}_\delta$ (and choose $u_{\mathbf{p}_{\delta+1}[\mathbf{k}_2]}$ appropriately). Now possible and $(\mathbf{p}^{\mathbf{k}_1}_\delta, \mathbf{p}^{\mathbf{k}_2}_\delta) <'_{K_1} (\mathbf{p}^{\mathbf{k}_1}_{\delta+1}, \mathbf{p}^{\mathbf{k}_2}_{\delta+1})$ by 1.8.

Case 3: For some $\gamma, \gamma = \max(E_{\mathbf{k}} \cap \beta) \geq \alpha(*)$ and $\gamma \notin S_{\mathbf{k}}$.

Act as in Subcase 2B of the proof of 1.24

Case 4: $\beta = \sup(E_{\mathbf{k}} \cap \beta)$.

As in Case 4 in the proof of 1.24. $\square_{1.25}$

§ 2. $\mathfrak{p} = \mathfrak{t}$ does not decide the existence of a peculiar cut

We deal here with a problem raised in [She09], toward this we quote from there. Recall (Definition [She09, 1.10]).

Definition 2.1. Let κ_1, κ_2 be infinite regular cardinals. A (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$ is a pair $(\langle f_i : i < \kappa_1\rangle, \langle f^\alpha : \alpha < \kappa_2\rangle)$ of sequences of functions in ${}^\omega\omega$ such that:

(α) $(\forall i < j < \kappa_1)(f_j <_{J^{\rm bd}_\omega} f_i)$,
(β) $(\forall \alpha < \beta < \kappa_2)(f^\alpha <_{J^{\rm bd}_\omega} f^\beta)$,
(γ) $(\forall i < \kappa_1)(\forall \alpha < \kappa_2)(f^\alpha <_{J^{\rm bd}_\omega} f_i)$,
(δ) if $f : \omega \to \omega$ is such that $(\forall i < \kappa_1)(f \le_{J^{\rm bd}_\omega} f_i)$, then $f \le_{J^{\rm bd}_\omega} f^\alpha$ for some $\alpha < \kappa_2$,
(ε) if $f : \omega \to \omega$ is such that $(\forall \alpha < \kappa_2)(f^\alpha \le_{J^{\rm bd}_\omega} f)$, then $f_i \le_{J^{\rm bd}_\omega} f$ for some $i < \kappa_1$.

The motivation of looking at (κ_1, κ_2)-peculiar cuts is understanding the case $\mathfrak{p} > \mathfrak{t}$, (see [She09]). Also $\mathfrak{p} = \aleph_1 \Rightarrow \mathfrak{t} = \mathfrak{p}$ by the classical theorem of Rothberger and $\mathrm{MA}_{\aleph_1} + \mathfrak{p} = \aleph_2 \Rightarrow \mathfrak{t} = \aleph_2$ by [She09, 2.3].

Recall (from [She09]) that

Claim 2.2. *1) If $\mathfrak{p} < \mathfrak{t}$ <u>then</u> there is a (κ_1, κ_2)-peculiar type for some (regular) κ_1, κ_2 satisfying $\kappa_1 < \kappa_2 = \mathfrak{p}$.*
1A) If there is a (κ_1, κ_2)-peculiar cut <u>then</u> $\mathfrak{p} \le \max\{\kappa_1, \kappa_2\}$.
2) There is a (κ_1, κ_2)-peculiar cut iff there is a (κ_2, κ_1)-peculiar cut.

Proof. 1), 1A) See [She09, 1.12].
2) Trivial. $\square_{2.2}$

Observation 2.3. If $(\bar\eta^{\rm up}, \bar\eta^{\rm dn})$ is a peculiar $(\kappa_{\rm up}, \kappa_{\rm dn})$-cut and if $A \subseteq \omega$ is infinite, $\eta \in {}^\omega\omega$ <u>then</u>:

(a) $\eta <_{J^{\rm bd}_A} \eta^{\rm up}_\alpha$ for every $\alpha < \kappa_{\rm up}$ <u>iff</u> $\eta <_{J^{\rm bd}_A} \eta^{\rm dn}_\beta$ for every large enough $\beta < \kappa_{\rm dn}$

(b) $\neg(\eta^{\rm up}_\alpha <_{J^{\rm bd}_A} \eta)$ for every $\alpha < \kappa_{\rm up}$ <u>iff</u> $\neg(\eta^{\rm dn}_\beta <_{J^{\rm bd}_A} \eta)$ for every large enough $\beta < \kappa_{\rm dn}$.

Proof. <u>Clause (a)</u>: The implication $\Leftarrow$ is trivial as $\beta < \kappa_{\rm dn} \wedge \alpha < \kappa_{\rm up} \Rightarrow \eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta^{\rm up}_\alpha$. So assume the leftside.

We define $\eta' \in {}^\omega\omega$ by: $\eta'(n)$ is $\eta(n)$ if $n \in A$ and is 0 if $n \in \omega \backslash A$. Clearly $\eta' <_{J^{\rm bd}_\omega} \eta^{\rm up}_\alpha$ for every $\alpha < \kappa_{\rm up}$ hence by clause (δ) of 2.1 we have $\eta' \le_{J^{\rm bd}_\omega} \eta^{\rm dn}_\beta$ for some $\gamma < \kappa_{\rm dn}$ hence $\eta = \eta' \restriction A \le_{J^{\rm bd}_A} \eta^{\rm dn}_{\beta+1} <_{J^{\rm bd}_A} \eta^{\rm dn}_\beta$ for every $\beta \in (\gamma, \kappa_{\rm dn})$.

<u>Clause (b)</u>: Again the direction $\Leftarrow$ is obvious. For the other direction define $\eta' \in {}^\omega\omega$ by $\eta'(n)$ is $\eta(n)$ if $n \in A$ and is $\eta^{\rm up}_0(n)$ if $n \in \omega \backslash A$. So clearly $\alpha < \kappa_{\rm up} \Rightarrow \neg(\eta^{\rm up}_\alpha <_{J^{\rm bd}_\omega} \eta')$ hence $\alpha < \kappa_{\rm up} \Rightarrow \neg(\eta^{\rm up}_\alpha \le_{J^{\rm bd}_\omega} \eta)$ hence by clause (ε) of 2.1 for some $\beta < \kappa_{\rm dn}$ we have $\neg(\eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta')$. As $\eta^{\rm dn}_\beta <_{J^{\rm bd}_\omega} \eta^{\rm up}_0$, necessarily $\neg(\eta^{\rm dn}_\beta <_{J^{\rm bd}_A} \eta')$ but $\gamma \in [\beta, \kappa_{\rm dn}) \Rightarrow \eta^{\rm dn}_\beta \le_{J^{\rm bd}_A} \eta^{\rm dn}_\gamma$ hence $\gamma \in [\beta, \kappa_{\rm dn}) \Rightarrow \neg(\eta^{\rm dn}_\gamma <_{J^{\rm bd}_A} \eta') \Rightarrow \neg(\eta^{\rm dn}_\gamma <_{J^{\rm bd}_A} \eta)$, as required. $\square_{2.3}$

We need the following from [She09, 2.1]:

Claim 2.4. *Assume that $\kappa_1 \le \kappa_2$ are infinite regular cardinals, and there exists a (κ_1, κ_2)-peculiar cut in ${}^\omega\omega$.*

Then for some σ-centered forcing notion $\mathbb{Q}$ of cardinality κ_1 and a sequence $\langle \mathscr{I}_\alpha : \alpha < \kappa_2 \rangle$ of open dense subsets of $\mathbb{Q}$, there is no directed $G \subseteq \mathbb{Q}$ such that $(\forall \alpha < \kappa_2)(G \cap I_\alpha \ne \emptyset)$. Hence MA_{κ_2} fails.

Theorem 2.5. *Assume $\lambda = \mathrm{cf}(\lambda) = \lambda^{<\lambda} > \aleph_2, \lambda > \kappa = \mathrm{cf}(\kappa) \ge \aleph_1$ and $2^\lambda = \lambda^+$ and $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$.*

For some forcing $\mathbb{P}^*$ of cardinality λ^+ not adding new members to ${}^\lambda\mathbf{V}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}^*$ of a c.c.c. forcing we have $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} = \lambda^+$ and $\mathfrak{p} = \lambda$ and $\mathrm{MA}_{<\lambda}$ and there is a pair $(\bar{\eta}^{\rm up}, \bar{\eta}^{\rm dn})$ which is a peculiar (κ, λ)-cut".

Remark 2.6. 1) The proof of 2.5 is done in §4 and broken into a series of Definitions and Claims, in particular we specify some of the free choices in the general iteration theorem.
2) In 4.1(1), is $\mathrm{cf}(\delta) > \aleph_0$ necessary?
3) What if $\lambda = \aleph_2$? The problem is 3.2(2). To eliminate this we may, instead quoting 3.2(2), start by forcing $\bar{\eta} = \langle \eta_\alpha : \alpha < \omega_1 \rangle$ in $\mathbb{P}_{\mathbf{k}_0}$ and change some points.

Complementary to 2.5 is

Observation 2.7. Assume $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ and $\mu = \mathrm{cf}(\mu) = \mu^{<\lambda} > \lambda$ then for some c.c.c. forcing notion $\mathbb{P}$ of cardinality μ we have:
$\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} = \mu, \mathfrak{p} = \lambda$ and for no regular $\kappa < \lambda$ is there a peculiar (κ, λ)-cut so $\mathfrak{t} = \lambda$".

Proof. We choose $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \mu, \beta < \mu \rangle$ such that:

$\boxplus$ (*a*) $\bar{\mathbb{Q}}$ is an FS-iteration
(*b*) $\underset{\sim}{\mathbb{Q}}_\beta$ is a σ-centered forcing notion of cardinality $< \lambda$
(*c*) if $\alpha < \mu, \underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name of a σ-centered forcing notion of cardinality $< \lambda$ then for some $\beta \in [\alpha, \mu)$ we have $\underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}$
(*d*) $\mathbb{Q}_0$ is adding λ Cohens, $\langle \underset{\sim}{r}_\varepsilon : \varepsilon < \lambda \rangle$ say $\underset{\sim}{r}_\varepsilon \in {}^\omega\omega$.

Clearly in $\mathbf{V}^{\mathbb{P}_\lambda}$ we have $2^{\aleph_0} = \lambda$, also every σ-centered forcing notion of cardinality $< \mu$, is from $\mathbf{V}^{\mathbb{P}_\alpha}$ for some $\alpha < \mu$, so as μ is regular we have

$(*)$ MA for σ-centered forcing notions of cardinality $< \lambda$ and $< \mu$ dense sets.

Hence by 2.4 there is no peculiar (κ_1, κ_2)-cut when $\aleph_1 \le \kappa_1 < \kappa_2 = \lambda$ (even $\kappa_1 < \kappa_2 < \mu, \kappa_1 < \lambda < \mu$).

Lastly,

$\odot$ for $\alpha \le \mu$, in $\mathbf{V}^{\mathbb{P}_{1+\alpha}}$ for every $\eta \in {}^\omega\omega$ for every $\varepsilon < \lambda$ large enough we have $r_\varepsilon \nleq_{J^{\rm bd}_\omega} \eta$.

[Why? We prove this by induction on $\alpha \le \mu$. For $\alpha = 0$ this holds by $\boxplus(d)$. For α limit of uncountable cofinality recall $({}^\omega\omega)^{\mathbf{V}[\mathbb{P}_\alpha]} = \cup\{({}^\omega\omega)^{\mathbf{V}[\mathbb{P}_\beta]} : \beta < \alpha\}$. For α limit of cofinality $\aleph_0$ use "$\bar{\mathbb{Q}}$ is a FS-iteration". Lastly, for $\alpha = \beta + 1$ use the "of cardinality $< \lambda$" of clause (c) of $\boxplus$.] $\square_{2.7}$

§ 3. Some specific forcing

Definition 3.1. Let $\bar{\eta} =: \langle \eta_\alpha : \alpha < \alpha^* \rangle$ be a sequence of members of ${}^\omega\omega$ which is $<_{J^{\rm bd}_\omega}$-increasing or just $\le_{J^{\rm bd}_\omega}$-directed. We define the set $\mathscr{F}_{\bar{\eta}}$ and the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar{\eta}}$ and a generic real $\underset{\sim}{\nu}$ for $\mathbb{Q} = \mathbb{Q}_{\bar{\eta}}$ as follows:

(a) $\mathscr{F}_{\bar{\eta}} = \{\nu \in {}^\omega(\omega+1)$: if $\alpha < \ell g(\bar{\eta})$ then $\eta_\alpha <_{J^{\rm bd}_\omega} \nu\}$, here $\bar{\eta}$ is not[4] necessarily $<_{J^{\rm bd}_\omega}$-increasing

(b) $\mathbb{Q}$ has the set of elements consisting of all triples $p = (\rho, \alpha, g) = (\rho^p, \alpha^p, g^p)$ (and $\alpha(p) = \alpha^p$) such that
  - (α) $\rho \in {}^{\omega>}\omega$,
  - (β) $\alpha < \ell g(\bar{\eta})$,
  - (γ) $g \in \mathscr{F}_{\bar{\eta}}$, and
  - (δ) if $n \in [\ell g(\rho), \omega)$ then $\eta_\alpha(n) \le g(n)$;

(c) $\le_{\mathbb{Q}}$ is defined by: $p \le_{\mathbb{Q}} q$ iff (both are elements of $\mathbb{Q}$ and)
  - (α) $\rho^p \trianglelefteq \rho^q$,
  - (β) $\alpha^p \le \alpha^q$ and[5] $\eta_{\alpha^p} \le_{J^{\rm bd}_\omega} \eta_{\alpha^q}$
  - (γ) $g^q \le g^p$,
  - (δ) if $n \in [\ell g(\rho^q), \omega)$ then $\eta_{\alpha(p)}(n) \le \eta_{\alpha(q)}(n)$,
  - (ε) if $n \in [\ell g(\rho^p), \ell g(\rho^q))$ then $\eta_{\alpha(p)}(n) \le \rho^q(n) \le g^p(n)$.

(d) For $\mathscr{F} \subseteq \mathscr{F}_{\bar{\eta}}$ which is downward directed (by $<_{J^{\rm bd}_\omega}$) we define $\mathbb{Q}_{\bar{\eta},\mathscr{F}}$ as $\mathbb{Q}_{\bar{\eta}} \restriction \{p \in \mathbb{Q}_{\bar{\eta}} : g^p \in \mathscr{F}\}$

(e) $\underset{\sim}{\nu} = \underset{\sim}{\nu}_{\mathbb{Q}} = \underset{\sim}{\nu}_{\mathbb{Q}_{\bar{\eta}}} = \cup\{\rho^p : p \in \underset{\sim}{G}_{\mathbb{Q}_{\bar{\eta}}}\}$.

Claim 3.2. *1) If $\bar{\eta} \in {}^\gamma({}^\omega\omega)$ then $\mathscr{F}_{\bar{\eta}}$ is downward directed, in fact if $g_1, g_2 \in \mathscr{F}_{\bar{\eta}}$ then $g = \min\{g_1, g_2\} \in \mathscr{F}_{\bar{\eta}}$, i.e., $g(n) = \min\{g_1(n), g_2(n)\}$ for $n < \omega$. Also "$f \in \mathscr{F}_{\bar{\eta}}$" is absolute.*
[But possibly for every $\nu \in {}^\omega(\omega+1)$ we have: $\nu \in \mathscr{F}_{\bar{\eta}} \Leftrightarrow (\forall^* n)\nu(n) = \omega$].
2) If $\bar{\eta} \in {}^\delta({}^\omega\omega)$ is $<_{J^{\rm bd}_\omega}$-increasing and $\mathrm{cf}(\delta) > \aleph_1$ then $\mathbb{Q}_{\bar{\eta}}$ is c.c.c.
3) Moreover any set of $\aleph_1$ members of $\mathbb{Q}_{\bar{\eta}}$ is included in the union of countably many directed subsets of $\mathbb{Q}_{\bar{\eta}}$.
4) Assume $\langle \mathbb{P}_\varepsilon : \varepsilon \le \zeta \rangle$ is a $\lessdot$-increasing sequence of c.c.c. forcing notions, $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\alpha : \alpha < \delta \rangle$ is a $\mathbb{P}_0$-name of a $<_{J^{\rm bd}_\omega}$-increasing sequence of members of ${}^\omega\omega$ and $\mathrm{cf}(\delta) > \aleph_1$. For $\varepsilon \le \zeta$ let $\underset{\sim}{\mathbb{Q}}_\varepsilon$ be the $\mathbb{P}_\varepsilon$-name of the forcing notion $\mathbb{Q}_{\underset{\sim}{\bar{\eta}}}$ as defined in $\mathbf{V}^{\mathbb{P}_\varepsilon}$. Then $\Vdash_{\mathbb{P}_\zeta}$ "$\underset{\sim}{\mathbb{Q}}_\varepsilon$ is $\subseteq$-increasing and $\le_{ic}$-increasing for $\varepsilon \le \zeta$ and it is c.c.c. and $\mathrm{cf}(\zeta) > \aleph_0 \Rightarrow \underset{\sim}{\mathbb{Q}}_\zeta = \cup\{\underset{\sim}{\mathbb{Q}}_\varepsilon : \varepsilon < \zeta\}$ is c.c.c."
5) Let $\bar{\eta} \in {}^\delta({}^\omega\omega)$ be as in part (2).

(a) If $\mathscr{F} \subseteq \mathscr{F}_{\bar{\eta}}$ is downward directed (by $\le_{J^{\rm bd}_\omega}$) then $\mathbb{Q}_{\bar{\eta},\mathscr{F}}$ is absolutely c.c.c.
(b) If $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_{\bar{\eta}}$ are downward directed then $\mathbb{Q}_{\bar{\eta},\mathscr{F}_1} \subseteq_{ic} \mathbb{Q}_{\bar{\eta},\mathscr{F}_2}$.

6)

(a) $\Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$\underset{\sim}{\nu} \in {}^\omega\omega$ and $\mathbf{V}[\underset{\sim}{G}] = \mathbf{V}[\underset{\sim}{\nu}]$"
(b) $p \Vdash_{\mathbb{Q}_{\bar{\eta}}}$ "$\rho^p \triangleleft \underset{\sim}{\nu}$ and $n \in [\ell g(\rho), \omega) \Rightarrow \eta_{\alpha(p)}(n) \le \underset{\sim}{\nu}(n) \le g^p(n)$"

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[4] the central case is $\bar{\eta}$ is $\aleph_2$-directed by $<_{J^{\rm bd}_\omega}$

[5] so if $\bar{\eta}$ is $<_{J^{\rm bd}_\omega}$-increasing this can be omitted and is equivalent to $\alpha^p \le \alpha^q$

(c) $\Vdash_{\mathbb{Q}_{\bar\eta}}$ "$p \in G$ *iff* $\rho^p \triangleleft \underset{\sim}{\nu} \wedge (\forall n)(\ell g(\rho^p) \le n < \omega \Rightarrow \eta_\alpha(n) \le \underset{\sim}{\nu}(n) \le g^p(n))$"

(d) $\Vdash_{\mathbb{Q}_{\bar\eta}}$ "$\underset{\sim}{\nu} \in \mathscr{F}_{\bar\eta}$, *i.e.* $\underset{\sim}{\nu}(n) \in \mathscr{F}^{\mathbf{V}[\mathbb{Q}_{\bar\eta}]}$"

(e) $\Vdash_{\mathbb{Q}_{\bar\eta}}$ "*for every* $f \in ({}^\omega\omega)^{\mathbf{V}}$ *we have* $f \in \mathscr{F}_{\bar\eta}$ *iff* $f \in \mathscr{F}^{\mathbf{V}}_{\bar\eta}$ *iff* $\underset{\sim}{\nu} \le_{J^{bd}_\omega} f$".

Proof. 1) Trivial.
2) Assume $p_\varepsilon \in \mathbb{Q}_{\bar\eta}$ for $\varepsilon < \omega_1$. So $\{\alpha(p_\varepsilon) : \varepsilon < \omega_1\}$ is a set of $\le \aleph_1$ ordinals $< \delta$. But $\mathrm{cf}(\delta) > \aleph_1$ hence there is $\alpha(*) < \delta$ such that $\varepsilon < \omega_1 \Rightarrow \alpha(p_\varepsilon) < \alpha(*)$. For each ε let $n_\varepsilon = \mathrm{Min}\{n$: for every $k \in [n,\omega)$ we have $\eta_{\alpha(p_\varepsilon)}(k) \le \eta_{\alpha(*)}(k) \le g^{p_\varepsilon}(k)\}$. It is well defined because $\eta_{\alpha(p_\varepsilon)} <_{J^{\mathrm{bd}}_\omega} \eta_{\alpha(*)} <_{J^{\mathrm{bd}}_\omega} g^{p_\varepsilon}$ recalling $\alpha(p_\varepsilon) < \alpha(*)$ and $g^{p_\varepsilon} \in \mathscr{F}_{\bar\eta}$.

So clearly for some $\mathbf{x} = (\rho^*, n^*, \eta^*, \nu^*)$ the following set is uncountable

$$\begin{aligned}\mathscr{U} = \mathscr{U}_{\mathbf{x}} = \{\varepsilon < \omega_1 :\ & \rho^{p_\varepsilon} = \rho^* \text{ and } n_\varepsilon = n^* \text{ and } \eta_{\alpha(p_\varepsilon)} \restriction n^* = \eta^* \\ & \text{and } g^{p_\varepsilon} \restriction n^* = \nu^*\}.\end{aligned}$$

Let

$$\begin{aligned}\mathbb{Q}' = \mathbb{Q}'_{\mathbf{x}} =: \{p \in \mathbb{Q}_{\bar\eta} :\ & \ell g(\rho^p) \ge \ell g(\rho^*), \rho^p \restriction \ell g(\rho^*) = \rho^* \text{ and } \rho^p \restriction [\ell g(\rho^*), \ell g(\rho^p)) \subseteq \eta_{\alpha(*)} \\ & \text{and } \alpha(p) < \alpha(*), \text{ and } \eta_{\alpha(p)} \restriction n^* = \eta^* \text{ and } g^p \restriction n^* = \nu^* \\ & \text{and } n \in [n^*, \omega) \Rightarrow \eta_{\alpha(p)}(n) \le \eta_{\alpha(*)}(n) \le g^p(n)\}.\end{aligned}$$

Clearly

$\circledast_1$ $\{p_\varepsilon : \varepsilon \in \mathscr{U}\} \subseteq \mathbb{Q}'$

$\circledast_2$ $\mathbb{Q}' \subseteq \mathbb{Q}_{\bar\eta}$ is directed.

So we are done.
3) The proof of part (2) proves this as the set $X = \{(\rho^*, n^*, \eta^*, \nu^*) : n^* < \omega, \{\rho^*, \eta^*, \nu^*\} \subseteq {}^{\omega>}\omega\}$ is countable and $\omega_1 = \cup\{\mathscr{U}_{\mathbf{x}} : \mathbf{x} \in \mathbf{X}\}$.
4),5) First we can check clause (b) of part (5) by the definitions of $\mathbb{Q}_{\bar\eta,\mathscr{F}}, \mathbb{Q}_{\bar\eta}$. Second, concerning "$\mathbb{Q}_{\bar\eta,\mathscr{F}}$ is absolutely c.c.c." (i.e. clause (a) of part (5)) note that if $\mathbb{P}$ is c.c.c., $G \subseteq \mathbb{P}$ is generic over $\mathbf{V}$ <u>then</u> $\mathbb{Q}^{\mathbf{V}}_{\bar\eta,\mathscr{F}} = \mathbb{Q}^{\mathbf{V}[G]}_{\bar\eta,\mathscr{F}}$ and $\mathbb{Q}^{\mathbf{V}}_{\bar\eta,\mathscr{F}} \le_{\mathrm{ic}} \mathbb{Q}^{\mathbf{V}}_{\bar\eta} \le_{\mathrm{ic}} \mathbb{Q}^{\mathbf{V}[G]}_{\bar\eta}$ by clause (b) and the last one is c.c.c. (as $\mathbf{V}[G] \models$ "$\mathrm{cf}(\ell g(\bar\eta)) > \aleph_1$"). Hence $\mathbb{Q}^{\mathbf{V}}_{\bar\eta,\mathscr{F}}$ is c.c.c. even in $\mathbf{V}[G]$ as required. Turning to part (4), letting $\mathscr{F}_\varepsilon = (\mathscr{F}_{\bar\eta})^{\mathbf{V}[\mathbb{P}_\varepsilon]}$, clearly $\Vdash_{\mathbb{P}_{\varepsilon_2}}$ "$\underset{\sim}{\mathbb{Q}}_{\varepsilon_1} = \underset{\sim}{\mathbb{Q}}_{\bar\eta,\mathscr{F}_{\varepsilon_1}}$" for $\varepsilon_1 < \varepsilon_2 < \zeta$. Now about the c.c.c., as $\mathbb{P}_\varepsilon$ is c.c.c., it preserves "$\mathrm{cf}(\delta) > \aleph_1$", so the proof of part (1) works.
6) Easy, too. $\square_{3.2}$

Definition 3.3. Assume $\bar A = \langle A_\alpha : \alpha < \alpha^* \rangle$ is a $\subseteq^*$-decreasing sequence of members of $[\omega]^{\aleph_0}$. We define the forcing notion $\mathbb{Q}_{\bar A}$ and the generic real $\underset{\sim}{w}$ by:

(A) $p \in \mathbb{Q}_{\bar A}$ <u>iff</u>
  (a) $p = (w, n, A_\alpha) = (w_p, n_p, A_{\alpha(p)})$,
  (b) $w \subseteq \omega$ is finite,
  (c) $\alpha < \alpha^*$ and $n < \omega$,

(B) $p \le_{\mathbb{Q}_{\bar A}} q$ <u>iff</u>
  (a) $w_p \subseteq w_q \subseteq w_p \cup (A_{\alpha(p)} \backslash n_p)$
  (b) $n_p \le n_q$
  (c) $A_{\alpha(p)} \backslash n_p \supseteq A_{\alpha(q)} \backslash n_q$

(C) $\underset{\sim}{w} = \cup\{w_p : p \in \underset{\sim}{G}_{\mathbb{Q}_{\bar A}}\}$.

Claim 3.4. *Let $\bar{A}$ be as in Definition 3.3.*
1) $\mathbb{Q}_{\bar{A}}$ is a c.c.c. and even a σ-centered forcing notion.
2) $\Vdash_{\mathbb{Q}_{\bar{A}}}$ "$\underset{\sim}{w} \in [\omega]^{\aleph_0}$ is $\subseteq^* A_\alpha$ for each $\alpha < \alpha^*$" and $\mathbf{V}[\underset{\sim}{G}] = \mathbf{V}[\underset{\sim}{w}]$.
3) Moreover, for every $p \in \mathbb{Q}_{\bar{A}}$ we have $\Vdash$ "$p \in \underset{\sim}{G}$ iff $w_p \subseteq \underset{\sim}{w} \subseteq (A_{\alpha(p)} \backslash n_p) \cup w_p$".

Proof. Easy. $\square_{3.4}$

Claim 3.5. *Assume $\bar{\eta} \in {}^\delta({}^\omega\omega)$ is $\leq_{J^{bd}_\omega}$-increasing.*
1) If $\mathscr{F} \subseteq \mathscr{F}_{\bar{\eta}}$ is downward cofinal in $(\mathscr{F}_{\bar{\eta}}, <_{J^{bd}_\omega})$, i.e. $(\forall \nu \in \mathscr{F}_{\bar{\eta}})(\exists \rho \in \mathscr{F})(\rho <_{J^{bd}_\omega} \nu)$ and $\mathscr{U} \subseteq \delta$ is unbounded then $\mathbb{Q}_{\bar{\eta} \restriction \mathscr{U}, \mathscr{F}} = \{p \in \mathbb{Q}_{\bar{\eta}} : \alpha^p \in \mathscr{U}$ and $g^p \in \mathscr{F}\}$ is (not only $\subseteq \mathbb{Q}_{\bar{\eta}}$ but also is) a dense subset of $\mathbb{Q}_{\bar{\eta}}$.
2) If $\mathrm{cf}(\delta) > \aleph_0$ and $\mathbb{R}$ is Cohen forcing then $\Vdash_{\mathbb{R}}$ "$\mathbb{Q}^{\mathbf{V}}_{\bar{\eta}}$ is dense in $\mathbb{Q}^{\mathbf{V}[\underset{\sim}{G}]}_{\bar{\eta}}$".

Remark 3.6. 1) We can replace "$\eta_\alpha \leq_{J^{\rm bd}_\omega} \rho$" by "$\rho$ belongs to the F_σ-set $\mathbf{B}_\alpha$", where $\mathbf{B}_\alpha$ denotes a Borel set from the ground model, i.e. its definition.
2) Used in 4.4.

Proof. 1) Check.
2) See next claim. $\square_{3.5}$

Claim 3.7. *Let $\bar{\eta} = \langle \eta_\gamma : \gamma < \delta \rangle$ is $\leq_{J^{bd}_\omega}$-increasing in ${}^\omega\omega$.*
1) If $\mathbb{P}$ is a forcing notion of cardinality $< \mathrm{cf}(\delta)$ <u>then</u> $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}^{\mathbf{V}}_{\bar{\eta}}$ is dense in $\mathbb{Q}^{\mathbf{V}[\underset{\sim}{G}_{\bar{\eta}}]}$".
2) A sufficient condition for the conclusion of part (1) is:

$\odot^{\mathrm{cf}(\delta)}_{\mathbb{P}}$ *for every $X \in [\mathbb{P}]^{\mathrm{cf}(\delta)}$ there is $Y \in [\mathbb{P}]^{<\mathrm{cf}(\delta)}$ such that $(\forall p \in X)(\exists q \in Y)(p \leq q)$.*

2A) We can weaken the condition to: if $X \in [\mathbb{P}]^{\mathrm{cf}(\delta)}$ then for some $q \in \mathbb{P}$, $\mathrm{cf}(\delta) \leq |\{p \in X : p \leq_{\mathbb{P}} q\}|$.
3) If $\langle A_\alpha : \alpha < \delta^* \rangle$ is $\subseteq^*$-decreasing sequence of infinite subsets of ω and $\mathrm{cf}(\delta^*) \neq \mathrm{cf}(\delta)$ <u>then</u> $\odot^{\mathrm{cf}(\delta)}_{\mathbb{Q}_{\bar{A}}}$ holds.

Proof. 1) By part (2).
2) Let $\mathscr{U} \subseteq \delta$ be unbounded of order type $\mathrm{cf}(\delta)$. Assume $p \in \mathbb{P}$ and $\underset{\sim}{\nu}$ satisfies $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu} \in \mathscr{F}^{\mathbf{V}[\underset{\sim}{G}]}_{\bar{\eta}}$". So for every $\gamma \in \mathscr{U}$ we have $p \Vdash_{\mathbb{P}}$ "$\eta_\gamma <_{J^{\rm bd}_\omega} \underset{\sim}{\nu} \in {}^\omega\omega$", hence there is a pair (p_γ, n_γ) such that:

$(*)$ (a) $p \leq_{\mathbb{P}} p_\gamma$
(b) $n_\gamma < \omega$
(c) $p_\varepsilon \Vdash_{\mathbb{P}}$ "$(\forall n)(n_\gamma \leq n < \omega \Rightarrow \eta_\varepsilon(n) < \underset{\sim}{\nu}(n))$.

We apply the assumption to the set $X = \{p_\varepsilon : \gamma \in \mathscr{U}\}$ and get $Y \in [\mathbb{P}]^{<\mathrm{cf}(\delta)}$ as there. So for every $\gamma \in \mathscr{U}$ there is q_γ such that $p_\gamma \leq_{\mathbb{P}} q_\gamma \in Y$. As $|Y \times \omega| = |Y| + \aleph_0 < \mathrm{cf}(\delta) = |\mathscr{U}|$ there is a pair $(q_*, n_*) \in Y \times \omega$ such that $\mathscr{U}' \subseteq \delta$ is unbounded where $\mathscr{U}' := \{\gamma \in \mathscr{U} : q_\gamma = q_*$ and $n_\gamma = n_*\}$. Lastly, define $\nu_* \in {}^\omega(\omega+1)$ by $\nu_*(n)$ is 0 if $n < n_*$ is $\cup\{\eta_\alpha(n) + 1 : \alpha \in \mathscr{U}'\}$ when $n \geq n_*$.
Clearly

$\circledast$ (a) $\nu_* \in {}^\omega(\omega+1)$
(b) $\gamma \in \mathscr{U}' \Rightarrow \eta_\alpha \restriction [n_*, \omega) < \nu_* \restriction [n_*, \omega)$
(c) if $\gamma < \delta$ then $\eta_\alpha <_{J^{\rm bd}_\omega} \nu_*$

(d) $\nu_* \in \mathscr{F}^{\mathbf{V}}_{\bar\eta}$

(e) $p \le q_*$

(f) $q_* \Vdash_{\mathbb{P}}$ "$\nu_* \le \underset{\sim}{\nu}$".

So we are done.
2A) Similarly.
3) If $\mathrm{cf}(\delta^*) < \mathrm{cf}(\delta)$ let $\mathscr{U} \subseteq \delta^*$ be unbounded of order type $\mathrm{cf}(\delta^*)$ and $\mathbb{Q}'_{\bar A} = \{p \in \mathbb{Q}_{\bar A} : \alpha^p \in \mathscr{U}\}$, it is dense in $\mathbb{Q}_{\bar A}$ and has cardinality $\le \aleph_0 + \mathrm{cf}(\delta^*) < \mathrm{cf}(\delta)$, so we are done.

If $\mathrm{cf}(\delta^*) > \mathrm{cf}(\delta)$ and $X \in [\mathbb{P}]^{\mathrm{cf}(\delta)}$, let $\alpha(*) = \sup\{\alpha^p : p \in X\}$ and $Y = \{p \in \mathbb{Q}_{\bar A} : \alpha^p = \alpha(*)\}$.
The rest should be clear. $\square_{3.7}$

§ 4. Proof of Theorem 2.5

Choice 4.1. 1) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) > \aleph_0\}$ stationary.
2) $\bar\eta$ is as in 4.2 below, so possibly using a preliminary forcing of cardinality $\aleph_2$ we have such $\bar\eta$.

Definition/Claim 4.2. 1) Assume $\kappa = \mathrm{cf}(\kappa) \in [\aleph_2, \lambda)$ and $\bar\eta = \langle \eta_\alpha : \alpha < \kappa \rangle$ is an $<_{J^{\rm bd}_\omega}$-increasing sequence in ${}^\omega\omega$ and $\delta \in \lambda \backslash \omega_1$ a limit ordinal and $\gamma \le \lambda$. Then the following $\mathbf{s} = \mathbf{s}_{\delta,\gamma}$ is a winning strategy of COM in the game $\partial_{\delta,<\gamma}$: COM just preserves:

$\otimes$ (a) if for every $\zeta < \varepsilon$ we have $(\alpha) + (\beta)$ then we have $(*)$ where
- (α) $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$ where $\underset{\sim}{\mathbb{Q}}_{\bar\eta}$ is from 3.1 and in $\mathbf{V}^{\mathbb{P}[\mathbf{p}_\zeta]}$, i.e. is a $\mathbb{P}_{\mathbf{p}_\zeta}$-name
- (β) $\mathbb{P}_{\mathbf{p}_\zeta} * \underset{\sim}{\mathbb{Q}}_{\bar\eta} \lessdot \mathbb{P}_{\mathbf{p}_\varepsilon} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$
- $(*)$ $\mathbb{P}_{\mathbf{q}_\varepsilon} = \mathbb{P}_{\mathbf{p}_\varepsilon} * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$, so we have to interpret $\mathbb{P}_{\mathbf{q}_\varepsilon}$ such that its set of elements is $\subseteq \mathscr{H}_{<\aleph_1}(u^{\mathbf{q}_\varepsilon})$ which is easy, i.e. it is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: $p \in \mathbb{P}_{\mathbf{p}_\varepsilon}$ and $\underset{\sim}{r}$ is a canonical $\mathbb{P}_{\mathbf{p}_\varepsilon}$-name of a member of $\underset{\sim}{\mathbb{Q}}_{\bar\eta}$ (i.e. use $\aleph_0$ maximal antichains, etc.)$\}$

(b) if in (a) clause (α) holds but (β) fail then
- (α) the set of elements of $\mathbb{P}_{\mathbf{q}_\varepsilon}$ is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: for some $\zeta < \varepsilon$ and $(p', \underset{\sim}{r}) \in \mathbb{P}_{\mathbf{q}_\zeta}$ we have $\mathbb{P}_{\mathbf{p}_\varepsilon} \models$ "$p' \le p$"$\}$
- (β) the order is defined naturally

(c) if in (a), clause (α) fail, let ζ be minimal such that it fails, and then
- (α) the set of elements of $\mathbb{P}_{\mathbf{q}_\varepsilon}$ is $\mathbb{P}_{\mathbf{p}_\varepsilon} \cup \{(p, \underset{\sim}{r})$: for some $\xi < \zeta$ and p' we have $(p', \underset{\sim}{r}) \in \mathbb{P}_{\mathbf{q}_\zeta}$ and $\mathbb{P}_{\mathbf{p}_\varepsilon} \models$ "$p' \le p$"$\}$
- (β) the order is natural.

Remark 4.3. In 4.2 we can combine clauses (b) and (c).

Proof. By 3.2 this is easy, see in particular 3.2(4). $\square_{4.2}$

Technically it is more convenient to use the (essentially equivalent) variant.

Definition/Claim 4.4. 1) We replace $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \mathbb{Q}_{\bar\eta}$ by $\mathbb{P}_{\mathbf{q}_\zeta} = \mathbb{P}_{\mathbf{p}_\zeta} * \mathbb{Q}_{\bar\eta, \mathscr{F}_\zeta}$ where

$$\mathscr{F}_\zeta = \{\nu : \text{ for some } \varepsilon \le \zeta, \nu \in \mathscr{F}_{\bar\eta}^{\mathbf{V}[\mathbb{P}[\mathbf{p}_\varepsilon]]} \text{ but for no } \xi < \varepsilon \text{ and } \nu_1 \in \mathscr{F}_{\bar\eta}^{\mathbf{V}[\mathbb{P}[\mathbf{p}_\xi]]} \text{ do we have } \nu_1 \le_{J^{\rm bd}_\omega} \nu\}.$$

2) No change by 3.5(1).

Remark 4.5. In 4.2 we can use $\underset{\sim}{\bar\eta} = \langle \eta_\alpha : \alpha < \kappa \rangle$ say a $\mathbb{P}_{\mathbf{k}_0}$-name, but then for the game $\partial_{\delta, f(\delta)}$ we better assume $\delta \in E_{\mathbf{k}_0}$ and $\underset{\sim}{\bar\eta}$ is a $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\delta]$-name.

Definition/Claim 4.6. 1) Let $\mathbf{k}_* \in K^2_\lambda$ and $\underset{\sim}{\nu}_\alpha$ $(\alpha < \lambda)$ be chosen as follows:

(a) $E_{\mathbf{k}_*} = \lambda$ and $u[\mathbf{p}^{\mathbf{k}_*}_\alpha] = \omega_1 + \alpha$ hence $u[\mathbf{k}_*] = \lambda$
(b) $\mathbb{P}^{\mathbf{k}_*}_\alpha$ is $\lessdot$-increasing continuous
(c) $\mathbb{P}^{\mathbf{k}_*}_{\alpha+1} = \mathbb{P}^{\mathbf{k}_*}_\alpha * \underset{\sim}{\mathbb{Q}}_{\bar\eta}$ and $\underset{\sim}{\nu}_\delta$ is the generic (for this copy) of $\underset{\sim}{\mathbb{Q}}_{\bar\eta}$ where $\bar\eta$ is from 4.2
(d) $S_{\mathbf{k}_*} = S$ (a stationary subset of λ), $\delta \in S \Rightarrow \mathrm{cf}(\delta) > \aleph_0$

(e) for each $\delta \in S_{\mathbf{k}_*}, \mathbf{s}^{\mathbf{k}_*}_\delta = \mathbf{s}_{\delta,\lambda}$ is from 4.2 or better 4.4

(f) $\mathbf{g}^{\mathbf{k}_*}_\delta$ is $\langle(\mathbf{p}^{\mathbf{k}_*}_\delta, \mathbf{p}^{\mathbf{k}_*}_{\delta+1})\rangle$, $\mathrm{mv}(\mathbf{g}^{\mathbf{k}_*}_\delta) = 0$, only one move was done.

2) If $\mathbf{k}_* \le_{K_2} \mathbf{k}$ then $\Vdash_{\mathbb{P}_\mathbf{k}}$ "the pair $(\langle \underset{\sim}{\nu}_\alpha : \alpha < \lambda\rangle, \langle \eta_i : i < \kappa\rangle)$ is a (λ,κ)-peculiar cut".

Proof. Clear (by 4.2). $\square_{4.6}$

Definition 4.7. Let $\mathbb{P}^*$ be the following forcing notion:

(A) the members are $\mathbf{k}$ such that
  (a) $\mathbf{k}_* \le_{K_2} \mathbf{k} \in K^2_\lambda$
  (b) $u[\mathbf{k}] = \cup\{u[\mathbf{p}^\mathbf{k}_\alpha] : \alpha \in E_\mathbf{k}\}$ is an ordinal $< \lambda^+$ (but of course $\ge \lambda$)
  (c) $S_\mathbf{k} = S_{\mathbf{k}_*}$ and $\mathbf{s}^\mathbf{k}_\delta = \mathbf{s}^{\mathbf{k}_*}_\delta$ for $\delta \in S_\mathbf{k}$

(B) the order: $\le_{K^2_\lambda}$.

Definition 4.8. We define the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ as

$$\cup\{\mathbb{P}^\mathbf{k}_\lambda : \mathbf{k} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}^*}\} = \cup\{\mathbb{P}_\mathbf{p}[\mathbf{p}^\mathbf{k}_\alpha] : \alpha \in E_\mathbf{k} \text{ and } \mathbf{k} \in \underset{\sim}{G}_{\mathbb{P}^*}\}.$$

Claim 4.9. *1) $\mathbb{P}^*$ has cardinality λ^+.*
2) $\mathbb{P}^*$ is strategically $(\lambda+1)$-complete hence add no new member to ${}^\lambda\mathbf{V}$.
3) $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ is c.c.c. of cardinality $\le \lambda^+$".
4) $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ is a forcing notion of cardinality λ^+ neither collapsing any cardinal nor changing cofinalities.
5) If $\mathbf{k} \in \mathbb{P}^*$ <u>then</u> $\mathbf{k} \Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_\mathbf{k} \lessdot \underset{\sim}{\mathbb{Q}}^*$" hence $\Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}_*} \lessdot \underset{\sim}{\mathbb{Q}}^*$".

Proof. 1) Trivial.
2) By claim 1.24.
3) $\underset{\sim}{G}_{\mathbb{P}^*}$ is $(< \lambda^+)$-directed.
4),5) Should be clear. $\square_{4.9}$

Claim 4.10. *If $\mathbf{k} \in \mathbb{P}^*$ and $G \subseteq \mathbb{P}_\mathbf{k}$ is generic over $\mathbf{V}$ <u>then</u>*

(a) $\langle \underset{\sim}{\nu}_\alpha[G \cap \mathbb{P}_{\mathbf{k}_*}] : \alpha < \lambda\rangle$ is $<_{J^{bd}_\omega}$-decreasing and $i < \kappa \Rightarrow \eta_i <_{J^{bd}_\omega} \underset{\sim}{\nu}_\alpha[G \cap \mathbb{P}_{\mathbf{k}_*}]$, (this concerns $\mathbb{P}_{\mathbf{k}_*}$ only)

(b) if $\rho \in ({}^\omega\omega)^{\mathbf{V}[G]}$ and $i < \kappa \Rightarrow \eta_i <_{J^{bd}_\omega} \rho$ <u>then</u> for every $\alpha < \lambda$ large enough we have $\underset{\sim}{\nu}_\alpha[G] <_{J^{bd}_\omega} \rho$

(c) if $\rho \in ({}^\omega\omega)^{\mathbf{V}[G]}$ and $i < \kappa \Rightarrow \eta_i \nleq_{J^{bd}_\omega} \rho$ <u>then</u> for every $\alpha < \lambda$ large enough we have $\underset{\sim}{\nu}_\alpha[G] \nleq_{J^{bd}_\omega} \rho$.

Proof. Should be clear. $\square_{4.10}$

Claim 4.11. *1) If $\mathbf{k} \in \mathbb{P}^*$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_\mathbf{k}$-name of a c.c.c. forcing of cardinality $< \lambda$ and $\alpha \in E_\mathbf{k}$ and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}[\mathbf{p}^\mathbf{k}_\alpha]$-name <u>then</u> for some $\mathbf{k}_1$ we have:*

(a) $\mathbf{k} \le_{K_1} \mathbf{k}_1 \in \mathbb{P}^*$

(b) $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "there is a subset of $\underset{\sim}{\mathbb{Q}}$ generic over $\mathbf{V}[G_{\mathbb{P}_{\mathbf{k}_1}} \cap \mathbb{P}[\mathbf{p}^\mathbf{k}_\alpha]]$".

2) In (1) if $\Vdash_{\mathbb{P}[\mathbf{p}^\mathbf{k}_\alpha] * \underset{\sim}{\mathbb{Q}}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_\mathbf{k}}]$" <u>then</u> $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ "there is $\rho \in {}^\omega 2$ not in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_\mathbf{k}}]$".

Proof. 1) By 1.25.
2) By part (1) and clause (η) of 1.25. $\square_{4.11}$

Proof. Proof of Theorem 2.5 We force by $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ where $\mathbb{P}^*$ is defined in 4.7 and the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ is defined in 4.8. By Claim 4.9(4) we know that no cardinal is collapsed and no cofinality is changed. We know that $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} \le \lambda^+$" because $|\mathbb{P}^*| = \lambda^+$ and $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ has cardinality $\le \lambda^+$", so $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ has cardinality λ^+, see 4.9(3),(4).

Also $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}}$ "$2^{\aleph_0} \ge \lambda^+$" as by 4.9(2) it suffices to prove: for every $\mathbf{k}_1 \in \mathbb{P}^*$ there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \le_{K_2} \mathbf{k}_2$ and forcing by $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ adds a real, which holds by 4.11(2).

Lastly, we have to prove that $(\langle \eta_i : i < \kappa\rangle, \langle \underset{\sim}{\nu}_\alpha : \alpha < \lambda\rangle)$ is a peculiar cut. In Definition 2.1 clauses $(\alpha), (\beta), (\gamma)$ holds by the choice of $\mathbf{k}_*$. As for clauses $(\delta), (\varepsilon)$ to check this it suffices to prove that for every $f \in {}^\omega\omega$ they hold, so it is suffice to check it in any sub-universe to which $(\bar\eta, \bar\nu), f$ belong. Hence by 4.9(1) it suffices to check it in $\mathbf{V}^{\mathbb{P}_{\mathbf{k}}}$ for any $\mathbf{k} \in \mathbb{P}^*$. But this holds by 4.6(2). $\square_{2.5}$

§ 5. Quite general applications

Theorem 5.1. *Assume* $\lambda = \mathrm{cf}(\lambda) = \lambda^{<\lambda} > \aleph_2$ *and* $2^\lambda = \lambda^+$ *and* $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$. *Then for some forcing* $\mathbb{P}^*$ *of cardinality* λ^+ *not adding new members to* ${}^\lambda\mathbf{V}$ *and* $\mathbb{P}^*$*-name* $\underset{\sim}{\mathbb{Q}}^*$ *of a c.c.c. forcing it is forced, i.e.* $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ *that* $2^{\aleph_0} = \lambda^+$ *and*

- (a) $\mathfrak{p} = \lambda$ *and* $\mathrm{MA}_{<\lambda}$
- (b) *for every regular* $\kappa \in (\aleph_1, \lambda)$ *there is a* (κ, λ)*-peculiar cut* $(\langle \eta^\kappa_i : i < \kappa\rangle, \langle \nu^\kappa_\alpha : \alpha < \lambda\rangle)$ *hence* $\mathfrak{p} = \mathfrak{t} = \lambda$
- (c) *if* $\mathbb{Q}$ *is a (definition of a) Suslin c.c.c. forcing notion defined by* $\bar{\varphi}$ *possibly with a real parameter from* $\mathbf{V}$, *then we can find a sequence* $\langle \nu_{\mathbb{Q},\underset{\sim}{\eta},\alpha} : \alpha < \lambda\rangle$ *which is positive for* $(\mathbb{Q}, \underset{\sim}{\eta})$*, see* [She04]*, e.g.* non(null) $= \lambda$
- (d) *in particular* $\mathfrak{b} = \mathfrak{d} = \lambda$.

Remark 5.2. 0) In clause (c) we can let $\mathbb{Q}$ be a c.c.c nep forcing (see [She04]), with $\mathfrak{B}, \mathfrak{C}$ of cardinality $\le \lambda$ and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real (i.e. member of ${}^\omega 2$).
1) Concerning 5.1 as remarked earlier in 1.19(1), if we like to deal with Suslin forcing defined with a real parameter from $\mathbf{V}^{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^+}$ and similarly for $\mathfrak{B}, \mathfrak{C}$ we in a sense have to change/create new strategies. We could start with $\langle S_\alpha : \alpha < \lambda^+\rangle$ such that $S_\alpha \subseteq \lambda, \alpha < \beta \Rightarrow |S_\alpha \backslash S_\beta| < \lambda$ and $S_{\alpha+1} \backslash S_\alpha$ is a stationary subset of λ. But we can code this in the strategies, do nothing till you know the definition of the forcing.
2) We may like to strengthen 5.1 by demanding

- (c) for some $\mathbb{Q}$ as in clause (c) of 5.1, $\mathrm{MA}_{\mathbb{Q}}$ holds or even for a dense set of $\mathbf{k}_1 \in \mathbb{P}^*$, see below, there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \le_{K_2} \mathbf{k}_2$ and $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ is $\mathbb{Q}^{\mathbf{V}[\mathbb{P}_{\mathbf{k}_1}]}$.

For this we have to restrict the family of $\mathbb{Q}$'s in clause (c) such that those two families are orthogonal, i.e. commute. Note, however, that for Suslin c.c.c forcing this is rare, see [She04].
3) This solves the second Bartoszynski test problem, i.e. (B) of Problem 0.2.
4) So $(\bar{\varphi}, \mathbb{Q}, \nu, \underset{\sim}{\eta})$ in clause (c) of 5.1 satisfies

- (a) $\nu \in {}^\omega 2$
- (b) $\bar{\varphi} = (\varphi_0, \varphi_1, \varphi_2), \Sigma_1$ formulas with the real parameter ν
- (c) $\mathbb{Q}$ is the forcing notion defined by:
  - set of elements $\{\rho \in {}^\omega 2 : \varphi_0[\rho]\}$
  - quasi order $\le_{\mathbb{Q}} = \{(\rho_1, \rho_2) : \rho_1, \rho_2 \in {}^\omega 2$ and $\varphi_1(\rho_1, \rho_2)\}$
  - incompatibility in $\mathbb{Q}$ is defined by φ_3
- (d) $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a real, i.e. $\langle p_{n,k} : k \le \omega\rangle$ a (absolute) maximal antichain of $\mathbb{Q}, \mathbf{t}_k = \langle \mathbf{t}_{n,k} : k < \omega\rangle, \mathbf{t}_{k,n}$ a truth value.

Proof. The proof is like the proof of 2.5 so essentially broken to a series of definitions and Claims. $\square$

Claim 5.3. Claim/Choice:
Without loss of generality there is a sequence $\langle S_\alpha : \alpha < \lambda^+\rangle$ *such that:*

- (a) $S_\alpha \subseteq S^\lambda_{\aleph_0}$ *is stationary*

(b) *if* $\alpha < \beta$ *then* $S_\alpha \backslash S_\beta$ *is bounded (in* λ*)*

(c) $\diamondsuit_{S_{\alpha+1} \backslash S_\alpha}$ *and* $\diamondsuit_{S^\lambda_{\aleph_0} \backslash \cup \{S_\alpha : \alpha < \lambda^+\}}$.

Proof. E.g. by a preliminary forcing. $\square$

Definition 5.4. Let $\mathbb{P}^*$ be the following forcing notion:

(A) The members are $\mathbf{k}$ such that
  (a) $\mathbf{k} \in K^2_\lambda$
  (b) $u[\mathbf{k}] = \cup \{u[\mathbf{p}^{\mathbf{k}}_\alpha] : \alpha \in E_{\mathbf{k}}\}$ is an ordinal $< \lambda^+$ (but of course $\geq \lambda$)
  (c) $S_{\mathbf{k}} \in \{S_\alpha : \alpha < \lambda^+\}$.

(B) The order: $\leq_{K^2_\lambda}$.

Definition 5.5. We define the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}^*$ as

$$\cup \{\mathbb{P}^{\mathbf{k}}_\lambda : \mathbf{k} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}^*}\} = \cup \{\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha] : \alpha \in E_{\mathbf{k}} \text{ and } \mathbf{k} \in \underset{\sim}{G}_{\mathbb{P}^*}\}.$$

Claim 5.6. *As in 4.9:*
1) $\mathbb{P}^*$ *has cardinality* λ^+.
2) $\mathbb{P}^*$ *is strategically* $(\lambda+1)$*-complete hence add no new member to* ${}^\lambda\mathbf{V}$.
3) $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ *is c.c.c. of cardinality* $\leq \lambda^+$".
4) $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ *is a forcing notion of cardinality* λ^+ *neither collapsing any cardinal nor changing cofinalities.*
5) If $\mathbf{k} \in \mathbb{P}^*$ *then* $\mathbf{k} \Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}} \lessdot \underset{\sim}{\mathbb{Q}}^*$" *hence* $\Vdash_{\mathbb{P}^*}$ "$\mathbb{P}_{\mathbf{k}_*} \lessdot \underset{\sim}{\mathbb{Q}}^*$".

Proof. 1) Trivial.
2) By claim 1.24.
3) $\underset{\sim}{G}_{\mathbb{P}^*}$ is $(< \lambda^+)$-directed.
4),5) Should be clear. $\square_{4.9}$

Claim 5.7. *Assume*

(A) (a) $\mathbf{k} \in \mathbb{P}^*$
  (b) $S_{\mathbf{k}} = S_\alpha, \alpha < \lambda^+$
  (c) $\underset{\sim}{\nu}$ *is a* $\mathbb{P}^{\mathbf{k}}_\varepsilon$*-name of a member of* ${}^\omega 2, \varepsilon < \kappa$
  (d) $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{k}_1}$*-name of a c.c.c. Suslin forcing and* $\underset{\sim}{\eta}$ *a* $\underset{\sim}{\mathbb{Q}}$*-name both definable from* $\underset{\sim}{\nu}$.

Then there is $\mathbf{k}_2$ *such that*

(B) (a) $\mathbf{k}_1 \leq \mathbf{k}_2$
  (b) $S_{\mathbf{k}_2} = S_{\alpha+1}$
  (c) *if* $\varepsilon \in S_{\alpha+1} \backslash S_\alpha$ *then* $\mathbb{P}^{\mathbf{k}_2}_{\varepsilon+1} = \mathbb{P}^{\mathbf{k}_2}_\varepsilon * \underset{\sim}{\mathbb{Q}}$ *and* $\underset{\sim}{\eta}_\varepsilon$ *is the copy of* $\underset{\sim}{\eta}$
  (d) *if* $\varepsilon \in S_{\alpha+1} \backslash S_\varepsilon$ *then the strategy* $\mathbf{st}_\varepsilon$ *is as in 4.2, using* $\underset{\sim}{\mathbb{Q}}$ *instead of* $\underset{\sim}{\mathbb{Q}}_{\bar{\eta}}$.

Proof. Straightforward. $\square_{4.10}$

Claim 5.8. *Like 4.11:*
1) If $\mathbf{k} \in \mathbb{P}^*$ *and* $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_{\mathbf{k}}$*-name of a c.c.c. forcing of cardinality* $< \lambda$ *and* $\alpha \in E_{\mathbf{k}}$ *and* $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]$*-name then for some* $\mathbf{k}_1$ *we have:*

(a) $\mathbf{k} \le_{K_2} \mathbf{k}_1 \in \mathbb{P}^*$

(b) $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"there is a subset of* $\underset{\sim}{\mathbb{Q}}$ *generic over* $\mathbf{V}[G_{\mathbb{P}_{\mathbf{k}_1}} \cap \mathbb{P}[\mathbf{p}^{\mathbf{k}}_\alpha]]$.

2) In (1) if $\Vdash_{\mathbb{P}_{\mathbf{k}} * \underset{\sim}{\mathbb{Q}}}$ *"there is* $\rho \in {}^\omega 2$ *not in* $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$*" then* $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"there is* $\rho \in {}^\omega 2$ *not in* $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}_{\mathbf{k}}}]$*".*

Proof. 1) By 1.25.
2) By part (1) and clause (η) of 1.25. $\square_{4.11}$

Claim 5.9. *Assume* $\kappa \in [\aleph_2, \lambda)$ *is regular,* $\mathbf{k} \in \mathbb{P}^*$ *and* $S_{\mathbf{k}} = S_\alpha$ *and* $\Vdash_{\mathbb{P}_{\mathbf{k}}}$ *"*$\langle \underset{\sim}{\eta}_\varepsilon : \varepsilon < \kappa\rangle$ *is increasing. Then we can find* $\mathbf{k}_1$ *such that* $\mathbf{k} \le \mathbf{k}_1 \in \mathbb{P}^*$ *and* $\gamma(*) < \lambda, \mathbb{P}_{\mathbf{k}_1}$*-name* $= \langle \underset{\sim}{\nu}_i : i \in S_{\alpha+1} \backslash S_\alpha \backslash \gamma(*)\rangle$ *such that*

$(*)_1$ $\Vdash_{\mathbb{P}_{\mathbf{k}_1}}$ *"*$(\langle \underset{\sim}{\eta}_\varepsilon : \varepsilon < \kappa\rangle, \langle \nu_i : i \in S_{\alpha+1} \backslash S_\alpha \backslash \gamma(*)\rangle)$ *is a* (κ, λ)*-peculiar uct*

$(*)_2$ *moreover if* $\mathbf{k}_1 \le \mathbf{k}_2 \in \mathbb{P}^*$ *this still holds.*

Proof. As in the proof of 2.5. $\square$

Proof. Proof of Theorem 5.1

We force by $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ where $\mathbb{P}^*$ is defined in 5.4 and the $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}$ is defined in 5.5. By Claim 5.6(4) we know that no cardinal is collapsed and no cofinality is changed. We know that $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*}$ "$2^{\aleph_0} \le \lambda^+$" because $|\mathbb{P}^*| = \lambda^+$ and $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ has cardinality $\le \lambda^+$", so $\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*$ has cardinality λ^+, see 5.6(3),(4).

Also $\Vdash_{\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}}$ "$2^{\aleph_0} \ge \lambda^+$" as by 4.9(2) it suffices to prove: for every $\mathbf{k}_1 \in \mathbb{P}^*$ there is $\mathbf{k}_2 \in \mathbb{P}^*$ such that $\mathbf{k}_1 \le_{K_2} \mathbf{k}_2$ and forcing by $\mathbb{P}_{\mathbf{k}_2}/\mathbb{P}_{\mathbf{k}_1}$ add a real, which holds by 5.8(2). Similarly $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}^*}$ "$\mathrm{MA}_{<\lambda}$ for $< \lambda$ dense subsets" by 5.8(1) hence $\mathfrak{p} \ge \lambda$ follows; as $\mathfrak{p} \le \lambda$ by clause (b) we have proved clause (a) of 5.1.

Clause (b) of 5.1 is proved as in the proof of 2.5, that is by 5.9.

As for clause (c) we are given $\mathbf{k}_0$ and $\mathbb{Q}, \underset{\sim}{\nu}, \underset{\sim}{\eta}$ such that $\underset{\sim}{\nu}$ is a $(\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*)$-name of a real and $\underset{\sim}{\mathbb{Q}}$ is a Suslin c.c.c. forcing definable (say by $\bar\varphi_0$) from the real $\underset{\sim}{\nu}$ and η a $(\mathbb{P}^* * \underset{\sim}{\mathbb{Q}}^*)$-name of $\underset{\sim}{\mathbb{Q}}$-name for $\mathbb{Q}$ of a real defined by $\aleph_0$ maximal antichain of $\underset{\sim}{\mathbb{Q}}$, absolutely of course.

As $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}}^*$ satisfies the c.c.c.", for some $\mathbf{k}_1 \in \mathbb{P}^*$ above $\mathbf{k}_0$ and $\mathbb{P}_{\mathbf{k}_1}$-name $\underset{\sim}{\nu}'$ of a member of ${}^{\lambda \ge}2$ and $\underset{\sim}{\eta}'$ is a $\mathbb{P}_{\mathbf{k}_1}$-name in $\mathbb{Q}_{\bar\varphi, \nu'}$ we have $\mathbf{k}_1 \Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\nu} = \underset{\sim}{\nu}' \wedge \underset{\sim}{\eta} = \underset{\sim}{\eta}'$".

As $\mathbb{P}_{\mathbf{k}_1}$ satisfies the c.c.c. for some $\varepsilon < \lambda, (\mathbf{k}_1, \varepsilon, \underset{\sim}{\nu}', \mathbb{Q}_{\underset{\sim}{\nu}'}, \underset{\sim}{\eta}')$ satisfies the assumptions on $(\mathbf{k}, \varepsilon, \underset{\sim}{\nu}', \underset{\sim}{e}ta')$ is as in 5.7 so there is $\mathbf{k}_2$ and $\langle \underset{\sim}{\eta}_\alpha : \alpha \in S_{\alpha+1} \backslash S_\alpha\rangle$ as there. So $\mathbf{k}_0 \le \mathbf{k}_1 \le \mathbf{k}_2$ and

$(*)$ if $\mathbf{k}_2 \le \mathbf{k}_3$ then for a club of $\zeta < \lambda, \underset{\sim}{\nu}'$ is a $\mathbb{P}^{\mathbf{k}_3}_\zeta$-name and $\underset{\sim}{\eta}_\zeta$ is $(\underset{\sim}{\mathbb{Q}}_{\bar\varphi, barv'}, \underset{\sim}{\eta})$-generic over $\mathbf{V}^{\mathbb{P}_\zeta[\mathbf{k}_3]}$.

This is clearly enough, so clause (a) of 5.1 holds. For clause (d) of 5.1, first Random real forcing is a Suslin c.c.c. forcing so non(null) $\le \lambda$ follows from clause (c) and non(nul) $\ge \lambda$ follows from clause (a).

Lastly, $\mathfrak{b} \ge \lambda$ by $\mathrm{MA}_{<\lambda}$ and we know $\mathfrak{d} \ge \mathfrak{b}$. As dominating real forcing = Hechler forcing is a c.c.c. Suslin forcing so by clause (c) we have $\mathfrak{d} \le \lambda$, together $\mathfrak{d} = \mathfrak{b} = \lambda$, i.e. clause (d) holds. $\square_{5.1}$

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