PROPER TRANSLATION

HEIKE MILDENBERGER AND SAHARON SHELAH

Abstract. We continue our work on weak diamonds [11]. We show that $2^\omega = \aleph_2$ together with the weak diamond for covering by thin trees, the weak diamond for covering by meagre sets, the weak diamond for covering by null sets, and "all Aronszajn trees are special" is consistent relative to ZFC. We iterate alternately forcings specialising Aronszajn trees without adding reals (the NNR forcing from [14, Ch. V]) and $< \omega_1$-proper ${}^\omega\omega$-bounding forcings adding reals. We show that over a tower of elementary submodels there is a sort of a reduction ("proper translation") of our iteration to the c.s. iteration of simpler iterands. If we use only Sacks iterands and NNR iterands, this allows us to guess the values of Borel functions into small trees and thus derive the mentioned weak diamonds.

1. Introduction

The motive is a generalisation of diamond. We generalise the result of [11] to get the consistency of the weak diamond for the relation of covering a real by a thin tree together with $2^{\aleph_0} > \aleph_1$ and "all Aronszajn trees are special". The proof of the analogous result with CH in [11] used that no reals are added. Now the main technical work is to show that a combination of NNR forcings and adding certain innocuous reals does not destroy certain weak diamonds. (There is, of course, a match between the two "certain".) Since the NNR iterands have size $2^{\aleph_1}$, we will translate relevant parts of these iterands into the real numbers in order to guess the values of Borel functions with hereditarily countable arguments. Such a translation procedure is also established for countable support iterations of NNR iterands and other $< \omega_1$-proper ${}^\omega\omega$-bounding iterands. We call this procedure, which is the core of the current work, proper translation.

We recall the definition of a weak (or parametrised or generalised) diamond. Let A and B be sets of reals and let $E \subseteq A \times B$. Here we work only with Borel sets A and B and absolute E, so that the interpretation of the notions in various ZFC models is absolute. The set A carries the topology inherited from the reals and 2^α carries the product topology. A function $F\colon 2^{<\omega_1} \to A$ is called a Borel function if each part $F \restriction 2^\alpha$, $\alpha < \omega_1$, is a Borel function. The complexity of the set of $\aleph_1$ parts can be high.

Date: December 20, 2009, revision December 31, 2010.

2010 *Mathematics Subject Classification.* Primary 03E15, 03E17, 03E35.

Key words and phrases. Specialising Aronszajn trees, Borel computations.

The first author was supported by the Austrian "Fonds zur Förderung der wissenschaftlichen Forschung", grant no. P20229-N13 and by the Landau Center of the Minerva Foundation and by Marie Curie grant PIEF-2008-219292. The second author's research was partially supported by the United States-Israel Binational Science Foundation (Grant no. 2002323). This is the second author's publication no. 951.

Definition 1.1. *(Definition 4.4. of* [12]*) Let $\diamondsuit(A,B,E)$ be the following statement: For every Borel map $F\colon 2^{<\omega_1} \to A$ there is some $g\colon \omega_1 \to B$ such that for every $f\colon \omega_1 \to 2$ the set*

$$\{\alpha \in \omega_1 \,:\, F(f \restriction \alpha) E g(\alpha)\}$$

is stationary. Commonly, if E is not the equality $\diamondsuit(A,B,E)$ is called a weak or a parametrised or a generalised diamond.

In this paper we describe a technique that allows to translate from one proper forcing to a simpler one. We apply this technique in order to prove:

Theorem 1.2. *Let $r\colon \omega \to \omega$ such that $\lim \frac{r(n)}{2^n} = 0$. Then the conjunction of the following weak diamonds together with $2^\omega = \aleph_2$ and with "all Aronszajn trees are special" is consistent relative to ZFC:*

(a) $\diamondsuit(2^\omega, \{\lim(T) \,:\, T \subseteq 2^\omega \text{ perfect } \wedge (\forall n)|\{\eta \restriction n \,:\, \eta \in \lim(T)\}| \leq r(n)\}, \in)$,

(b) $\diamondsuit(\mathbb{R}, F_\sigma \text{ null sets}, \in)$,

(c) $\diamondsuit(\mathbb{R}, G_\delta \text{ meagre sets}, \in)$.

Remarks: a) The first weak diamond implies the other two, since there is a Borel reduction of the corresponding relations (see [12, Prop. 2.8]). So we only have to work with the relation in item (a) of the theorem, which we call "covering by thin trees".

b) We must dash some hope that our result might help to answer Juhász' question as to whether the club principle (see, e.g., [6, Second page]) implies the existence of a Souslin tree. Since our forcing has the Sacks property, in the extension $\mathrm{cof}(\mathcal{M}) = \aleph_1$. By [6, Theorem 6] the club principle does not hold in the extension.

We work with a nep (non-elementary proper) forcing as the outcome of our translation. In particular, its iterands $\mathbb{Q}_*$ will be subsets of ${}^\omega\omega$ such that $(\mathbb{Q}_*, \leq_{\mathbb{Q}_*}, \perp_{\mathbb{Q}_*})$ is Π^1_1-definable. The behaviour of the large NNR forcing (a condition has already size $\aleph_1$) over countable models above generic conditions is "faked", as in [15]. That is, we work with the original forcing $\mathbb{P}$ and another "simpler" forcing $\mathbb{P}_*$ that knows parts of $\mathbb{P}$. The relevant parts are: Let $\chi > 2^{|\mathbb{P}|}$ be a regular cardinal. Let $H(\chi)$ denote the set of sets of hereditary cardinality strictly less than χ and let $<_\chi$ be a well-order of $H(\chi)$. We use the well-order for inductive constructions. Given a countable model $M = (M, \in \cap M^2, <_\chi \cap M^2) \prec (H(\chi), \in, <_\chi)$ and a condition $p \in \mathbb{P} \cap M$, we establish a Borel function that computes pairs (q,g) with the following properties: q is a $(M,\mathbb{P})$-generic condition and $q \geq p$. The function $g\colon \mathbb{P} \cap M \to \mathbb{P}_* \cap M$ is a reduction. If $q \in G$, then the translation preserves the incompatibility of two conditions and dense subsets $I \in M$ in both directions and hence is particularly useful for evaluating $\mathbb{P}$-names in M for objects in M as $\mathbb{P}_*$-names. For iterating this translation procedure we work with towers of models as in Def. 2.1 and with $< \omega_1$-properness and with the known completeness systems for the NNR forcing. Our particular way of finding completely generic filters for the NNR iterands uses that all former iterands are ${}^\omega\omega$-bounding. After the translation procedure we guess thin trees. In order to keep the outcome of the translation in countable iteration lengths, we use that the iterands in $\mathbb{P}_*$ are non elementary proper and that hence all $\mathbb{P}_*$-names for reals can be seen in a sub-iteration of countable length, as proved in [16, Section 3].

For example we think of

$$\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq \gamma(\mathbf{k}), \beta < \gamma(\mathbf{k})\rangle$$

being a countable support iteration of proper iterands as in Definition 1.3. We do not try the more involved ones as in [14, Ch. VIII; Ch. XVIII, §23] and in [13]. The forcing notion $\mathbb{P}_*$ will be just the iteration of the oddly indexed iterands. For specialising all Aronszajn trees, we use $\gamma(\mathbf{k}) = \omega_2$. However, the translation procedure works at any iteration length.

Definition 1.3. *(a) In the following $\mathbb{P}_{2\gamma} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \,:\, \alpha \leq 2\gamma, \beta < 2\gamma\rangle$ is an alternation of forcings for the even stages $\mathbb{Q}_{2\alpha}$, $\alpha < \gamma$, and forcings of the odd stages $\mathbb{Q}_{2\alpha+1}$, $\alpha < \gamma$, of the following kind:*

Even Stages: *The iterand $\mathbb{Q}_{2\alpha}$ is an NNR forcing $Q_{\mathbf{T}}$ as in Def. 2.7 or in* [14, Ch. VI, §5] *for specialising an Aronszajn tree* $\mathbf{T}$ *(whose name is given by suitable book-keeping).*

Odd stages: *The iterand $\mathbb{Q}_{2\alpha+1}$ is a $<\omega_1$-proper ${\omega}^\omega$-bounding nep (Def. 2.4) forcing. In our special case we have that the set of conditions is a subset of the real line such that $(\mathbb{Q}, \leq_{\mathbb{Q}}, \perp_{\mathbb{Q}})$ is Π^1_1-definable.*

(b) In Section 6, for the weak diamond, we require in addition to the properties in (a) for the odd iterands $\mathbb{Q}$ that they have the Sacks property, i.e., for every $q \in \mathbb{Q}$ for every $\mathbb{Q}$-name $\underset{\sim}{f}$ for a member of ${}^\omega 2$ for every $r\colon \omega \to \omega$ with $r(n) \to \infty$ there is a $T \in \tilde{\mathbf{V}}$ and a $q' \geq q$ such that $(\forall n)(|T \cap {}^n 2| \leq r(n))$ and $q' \Vdash_{\mathbb{Q}} (\forall n)(\underset{\sim}{f} \restriction n \in T)$.

(c) For enumerating all Aronszajn trees in an iteration of length ω_2, we require in addition to item (a), that the odd iterands have the $\aleph_2$-p.i.c. (this will be explained at the end of Section 2).

For proper forcing, not adding a real is the same as not adding a new sequence of ordinals (or of members of $\mathbf{V}$), and the description "not adding new reals" is much more common. Since the evenly indexed iterands do not add reals (in an iterable manner, there is a completeness system, so that also in limit steps there are no new reals) and since the oddly indexed iterands have countable conditions and since all the iterands are proper one could hope that $\mathbb{P}_{2\gamma}$ from Def. 1.3 is equivalent to a forcing in which every condition $q \in \mathbb{Q}_{2\alpha+1}$ has a

$$\mathbb{P}_{*,2\alpha+1} = \langle \underset{\sim}{\mathbb{Q}}_{2\beta+1} \,:\, \beta < \alpha\rangle\text{-name.}$$

This will indeed be true in our special case. For the proof, we use specific features of the NNR forcing there is a countably complete completeness system that is parametrised by dominating functions. In addition we use the fact that the oddly indexed iterands (and hence, by [1, Theorem 3.5] or [14, Ch. VI]) the initial segments of the iteration are ${}^\omega\omega$-bounding.

In the core of the paper we show that the part $\mathbb{P} \cap M$ that is compatible with the computed $(M, \mathbb{P})$-generic condition is equivalent to the simpler iteration

$$\mathbb{P}_* = \langle \mathbb{P}_{*,2\alpha+1}, \underset{\sim}{\mathbb{Q}}_{2\beta+1} \,:\, \alpha \leq \gamma(\mathbf{k}), \beta < \gamma(\mathbf{k})\rangle.$$

The sequence of Borel functions $\langle \mathbf{B}_\gamma : \gamma \in \mathrm{otp}(M \cap \gamma(\mathbf{k}))\rangle$ that we are going to establish translates with the help of additional arguments $\langle \eta_\gamma : \gamma \in \mathrm{otp}(M \cap \gamma(\mathbf{k})) + 1\rangle$ the given iteration $M \cap \mathbb{P}$ to $M \cap \mathbb{P}_*$. In our application, $\mathbb{P}_*$ is just a cs iteration of Sacks forcing. In [11], where we force only with NNR forcing, the Borel functions had values just in the ground-model. This was possible since no reals were added. Now, when we add reals, the outcome of the proper translation gives us $\mathbb{P}_*$-names for the reals that are the arguments of the function F in the weak diamond.

The Borel computation of completely generic conditions is related to the apparatus of completeness systems for proper forcings not adding reals in the following way: If $\mathbb{D}(M, \mathbb{P}, p) = \{A_\eta : \eta \subset M^k\}$ and $A_\eta \subseteq \mathrm{Gen}(M, \mathbb{P}, p)$ are parts of a completeness system and at least one $A_\eta \subseteq \mathrm{Gen}^+(M, \mathbb{P}, p)$ then our parameter η is one of these very good η's and with the help of a well-ordering $<_\chi$ on $H(\chi)$ the on step Borel function $\mathbf{B}_{0,1,0}(\eta, M, \mathbb{P}, p)$ will give an upper bound of a definable member of such an $A_\eta \subseteq \mathrm{Gen}^+(M, \mathbb{P}, p)$. All the symbols about the completeness systems are explained in [14, Ch. 7] and will not be used here. Note that in this setting $\mathbb{P}$ is just one iterand. In the current work we work with Borel functions for iterations and we compute (only) generic conditions.

An analogous computation of generic conditions will also be performed for the iterated forcing. To find $\mathbb{P}_*$-names of the generics of the NNR forcing over a countable tower of models $\langle M_\alpha : \alpha \in \mathrm{otp}(\gamma(\mathbf{k}) \cap M_0) + 1\rangle$ we use a function which computes with the help of $\bar\eta \in {}^{\mathrm{otp}(M_0 \cap \gamma(\mathbf{k}))}({}^\omega\omega)$ that is $\leq^*$-increasing fast enough $(\bar M, \mathbb{P}_{\gamma(\mathbf{k})}, p)$-generic conditions and $\mathbb{P}_{*,\gamma(\mathbf{k})}$-names for these.

In order to get a weak diamond in the forcing extension we use the "invariance", that the proper translation commutes with the Mostowski collapse, so that in the end only hereditarily countable sets will be guessed, namely the collapses of $(\bar M, \mathbb{P}, p, <_\chi, \bar\beta)$. For this purpose, we use the original diamond in the ground-model. In Section 6, we apply the translation procedure to an iteration of NNR forcing and Sacks forcing, and show that the result of the proper translation and the application of any Borel function onto it is covered by a thin tree. For this, we use that in this special case the outcome of the proper translation has the Sacks property, that is, can be covered by a thin tree in the ground-model. In our imitation of Lemma 3.11 of [11], we use that there is a Laver name for these thin trees. By the results of [16, Section 3] for nep forcings, the $\mathbb{P}_*$-name of the generic given by the Borel function is equivalent to a name in the iteration of Sacks forcing of length $\mathrm{otp}(M \cap \omega_2)$ for a suitable countable model M. By the Sacks property there is a thin tree in the ground model that covers this Sacks name that depends on $\bar\eta$ and on q. With the help of Laver forcing now a bit thicker still thin tree is found that serves for $\leq^*$-cofinally many $\bar\eta$'s at once. The guessing function g_F in the weak diamond will be a suitable enumeration of the possible second thin trees and not even depend on $F \in \mathbf{V}^{\mathbb{P}_{\omega_2}}$. This leads to the weak diamonds for covering by thin trees, by meagre sets and by Lebesgue null sets. The chain condition and the reflection properties of our iterated forcing yield: if there is a function g_F witnessing the weak diamond, then this object of size $\aleph_1$ is in an initial segment of the extension after the appearance of F. In Section 6 we show that one function g

works for all old and new Borel F. Thus we get a weak diamond with switched quantifiers.

We follow the Israeli convention that the stronger forcing condition is the larger one. We assume that each poset $\mathbb{P}$ has a weakest element and denote it by $0_\mathbb{P}$. We write $p \perp q$ if p and q are incompatible, that is $\nexists r(r \geq p \wedge r \geq q)$.

Definition 1.4. *See* [9, Def. 7.1] *Let* $(\mathbb{P}, \leq_\mathbb{P})$ *and* $(\mathbb{Q}, \leq_\mathbb{Q})$ *be two notions of forcing. A function* $i \colon \mathbb{P} \to \mathbb{Q}$ *is called a* complete embedding *if it has the following properties*

(1) $p_1 \leq_\mathbb{P} p_2$ *implies* $i(p_1) \leq i(p_2)$,

(2) $p_1 \perp p_2$ *iff* $i(p_1) \perp i(p_2)$,

(3) $(\forall q \in \mathbb{Q})(\exists p \in \mathbb{P})(\forall p' \geq_\mathbb{P} p)(i(p') \not\perp q)$.

Iff there is a complete embedding then there a surjective function $\pi \colon \mathbb{Q} \to \mathbb{P}$, called a reduction or a projection, such that

(1) $q_1 \leq_\mathbb{Q} q_2$ implies $\pi(q_1) \leq \pi(q_2)$, $\pi(0_\mathbb{Q}) = 0_\mathbb{P}$,

(2) $(\forall q \in \mathbb{Q})(\forall p' \geq_\mathbb{P} \pi(q))(\exists q' \geq q)(\pi(q') = p')$.

In an iterated forcing $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma, \alpha \leq \gamma \rangle$, for $\alpha \leq \beta \leq \gamma$ there are the projections $\pi \colon \mathbb{P}_\beta \to \mathbb{P}_\alpha$, $\pi(p) = p \restriction \alpha$. These satisfy the following strengthening of property (2):

(2') $(\forall q \in \mathbb{Q})(\forall p' \geq_\mathbb{P} \pi(q))(\exists q' \geq q)(\pi(q') = p' \wedge \forall r(r \geq q \wedge \pi(r) \geq p' \to r \geq q'))$. Such a q' is denoted by $q + p'$ or $q \cup p'$.

If the identity is a complete embedding, then $\mathbb{P}$ is called a complete suborder of $\mathbb{Q}$, written $\mathbb{P} \lessdot \mathbb{Q}$. In this situation, $\mathbb{P}$-names are $\mathbb{Q}$-names at the same time. In iterations $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$, for $\alpha < \beta$, $\mathbb{P}_\alpha \lessdot \mathbb{P}_\beta$ and (2') holds. If $q_1 \perp q_2$ implies $\pi(q_1) \perp \pi(q_2)$ (this is now a "trivial" projection [1], since its inverse is a dense embedding) then $\mathbb{Q}$-names can be mapped to $\mathbb{P}$-names by just mapping the weights in the names from $\mathbb{Q}$ to their projections. We will use such a map from the set of $\mathbb{P}$-names in M for members of $\underset{\sim}{\mathbb{Q}}$ that are compatible with an $(M, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic condition to the set of $\mathbb{P}_*$-names for members of $\underset{\sim}{\mathbb{Q}}$.

The paper is organised as follows: In Section 2 we recall the NNR iterands, in Section 3 we describe the theoretical framework of proper translation. In Section 4 we show that our particular examples, that is NNR forcing alternated with some $< \omega_1$-proper ${}^\omega\omega$-bounding iterands with Π^1_1-definable $(\mathbb{Q}, \leq_\mathbb{Q}, \perp_\mathbb{Q})$, allow proper translation in the successor steps of an iteration. In Section 5 we prove that the translation procedure can be carried over limit steps in countable support iterations if the initial segments allow proper translation. In Section 6 we prove the weak diamonds for forcings whose outcome of the proper translation is a nep forcing with the Sacks property.

2. The iterands

In this section we recall some definitions and the NNR iterands.

A condition $q \in \mathbb{P}$ is $(M, \mathbb{P}, p)$-generic if $q \geq p$ and for every $\mathbb{P}$-generic filter G over $\mathbf{V}$ with $p \in G$, for every $I \in M$ that is dense in $\mathbb{P}$ (seen from M or from $\mathbf{V}$, if $M \prec (H(\chi), \in)$ for a $\chi > 2^{|\mathbb{P}|}$, this is the same) $q \Vdash I \cap \underset{\sim}{G} \neq \emptyset$. We write $\underset{\sim}{G}$ for a canonical name for the $\mathbb{P}$-generic filter. In general, q is not in M. However often it is a subset of M or an element of a small extension of M. If $\mathbb{P}$ has the c.c.c. then every condition $p \in \mathbb{P} \cap M$ is $(M, \mathbb{P})$-generic.

Remark. Every Axiom A forcing (for a definition see, e.g., [4]) is $< \omega_1$-proper, and Ishiu [7] showed the converse.

Definition 2.1. *Let $\chi > 2^{|\mathbb{P}|}$ be a regular cardinal. Let $<_\chi$ be a well-ordering of $H(\chi)$. We call $\langle M_i : i < \alpha\rangle$ a* tower of elementary submodels for $\mathbb{P}$ *and we call α the* height *of the tower, if the following holds: $\mathbb{P} \in M_0$ and for $i < \alpha$, M_i is countable and $(M, \in, <_\chi) \prec (H(\chi), \in, <_\chi)$. Here, on the M-side, we just take the inherited relations $\in$ and $<_\chi$. Let $\mathbb{P} \in M_0$ and let $\langle M_i : i \leq \alpha\rangle$ be an increasing sequence such that $\langle M_j : j \leq i\rangle \in M_{i+1}$ and for limit ordinals j, $M_j = \bigcup_{i<j} M_i$. We also work with expansions of $(H(\chi), \in, <_\chi)$. Expanded towers $(\bar{M}, \mathbb{P}, q_0, p)$ stand for towers of expansions $(M_i, \in, <_\chi, \mathbb{P}, q_0, p)$, where all the additional symbols are constants with interpretations in M_0.*

Definition 2.2. $\mathbb{P}$ *is* α-proper *if the following holds: Let $\langle M_i : i \leq \alpha\rangle$ be a tower for $\mathbb{P}$. Then for every $p \in \mathbb{P} \cap M_0$ there is a $q \geq p$ that is $(M_i, \mathbb{P}, p)$-generic for all $i \leq \alpha$. We abbreviate "$(M_i, \mathbb{P}, p)$-generic for all $i \leq \alpha$" by* $(\bar{M}, \mathbb{P}, p)$-generic. *We write "$< \gamma$-proper" for "α-proper for every $\alpha < \gamma$".*

Since the towers are continuous, for limit α, $\langle M_i : i < \alpha\rangle$-genericity is equivalent to $\langle M_i : i \leq \alpha\rangle$-genericity. For non-limit α, say $\alpha = \alpha' + 1$, any $\langle M_i : i < \alpha\rangle$ generic condition in M_α can be strengthened to an $\langle M_i : i \leq \alpha\rangle$-generic condition. However, for indecomposable α the existence of $\langle M_i : i \leq \beta\rangle$-generic conditions for any $\beta < \alpha$ does not necessarily imply that there is an $\langle M_i : i \leq \alpha\rangle$-generic condition.

Definition 2.3. *A notion of forcing $\mathbb{P}$ is called* ${}^\omega\omega$-bounding *if for all $p \in \mathbb{P}$ for all $\mathbb{P}$-names $\underset{\sim}{f}$ for functions in ${}^\omega\omega$ there are a $q \geq_{\mathbb{P}} p$ and an $g \in \mathbf{V} \cap {}^\omega\omega$ such that $q \Vdash_{\mathbb{P}} \underset{\sim}{f} \leq^* g$.*

The words "non-elementary proper" (and their acronym "nep") are used for a family of definitions [15]. For our purposes, the following instance will suffice:

Definition 2.4. *A notion of forcing is* nep *if $(\mathbb{P}, \leq_{\mathbb{P}})$ is proper and $\mathbb{P} \subseteq {}^\omega\omega$ and $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ have Π^1_1-definitions.*

Usually we write just $\mathbb{P}$ instead of $(\mathbb{P}, \leq_{\mathbb{P}})$. In our application, we use that Sacks forcing is nep and the following important property of nep forcings proved by Shelah and Spinas [16, Section 3]:

Theorem 2.5. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma\rangle$ be a cs iteration of nep iterands. Then every $\mathbb{P}$-name for a real has an equivalent $\langle \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \in a\rangle$-name for a countable $a \subseteq \gamma$ (which is the order type of a closure inside a countable elementary submodel of the name under all arguments where its conditions are not trivial).*

We will use this theorem on the $\mathbb{P}_*$-side. The iterands need not have identical definitions in general, however, in our case, each of them is just Sacks forcing. Now we recall the partial order $Q_{\mathbf{T}}$ that is also known as "the NNR forcing". These forcings for specialising an Aronszajn tree $\mathbf{T}$ without adding reals are presented in [14, Chapter V, Section 6]. We take the version from there. A slightly different version (where the side conditions are not just collections of finite subsets of the Aronszajn tree, but certain partial functions from the tree into the rational numbers) is presented in [2] and in [11, Section 2]. It is known that these forcings are $< \omega_1$-proper and are $\mathbb{D}$-complete for a simple $\aleph_1$-completeness system $\mathbb{D}$, which guarantees that their countable support iterations do not add reals, [14, Theorem V.7.1].

We will compute completely generic (Def. 2.13) conditions for the NNR iterands and interleave forcings that do add reals. After one step of adding reals, so in our setting after $\mathbb{Q}_0 * \mathbb{Q}_1 = \underset{\sim}{Q}_{\mathbf{T}} \times \mathbb{Q}_1$, we shall work with names for completely generic conditions. More general, in the forcing $\mathbb{P} * \underset{\sim}{Q}_{\mathbf{T}}$ there will be $\mathbb{P}$-names for completely $\underset{\sim}{Q}_{\mathbf{T}}$-generic conditions, for a $< \omega_1$-proper ${}^\omega\omega$-bounding $\mathbb{P}$. This is crucial for the successor step of the proper translation.

From now until the end of the current section, $\mathbb{Q}$ is the set of rational numbers. Later, it will again be a forcing. Recall, a *specialisation* of an Aronszajn tree $\mathbf{T} = (\omega_1, <_{\mathbf{T}})$ is a function $f\colon \omega_1 \to \mathbb{Q}$ such that for any $s, t \in \omega_1$, $s <_{\mathbf{T}} t \to f(s) < f(t)$. We call such a function monotone. Now we work with monotone functions f, that specialise only a part of $\mathbf{T}$, namely the union of countably many of its levels, so that the indices of the levels form a closed set C. We call such a pair (f, C) an *approximation*. For $\alpha < \omega_1$ let $T_\alpha = \{y \in T : \mathrm{otp}(\{x \in T : x <_T y\}, <_T) = \alpha\}$ denote the α-th level of $\mathbf{T}$. For $x \in T_\alpha$ and $\beta < \alpha$ we let $x\lceil\beta$ be the $y \in T_\beta$ such that $y <_{\mathbf{T}} x$. For making the notation easier, we consider only Aronszajn trees $\mathbf{T}$ whose α-th level, T_α, is $[\omega\alpha, \omega(\alpha+1))$. This is no loss of generality since specialising all these Aronszajn trees suffices. Moreover we stipulate that every node in an Aronszajn tree has infinitely many immediate successors. A specialisation of a club set of levels of an Aronszajn tree gives rise to a total specialisation.

Definition 2.6. *(See* [14, Chapter V, Section 6]*)*

(1) An approximation *is a pair (f, C) such that there are a countable ordinal α such that $C \subseteq \alpha+1$ is a closed set with $\alpha \in C$ and such that $f\colon \bigcup_{i \in C} T_i \to \mathbb{Q}$ is a partial specialisation function. The ordinal α is called* $\mathrm{last}(f)$. *We say "(f_2, C_2) extends (f_1, C_1)" and write $(f_1, C_1) \le (f_2, C_2)$ iff $f_1 \subseteq f_2$ and $C_1 \subseteq C_2$ and $(C_2 \setminus C_1) \cap (\bigcup C_1) = \emptyset$.*

(2) We say that a finite function $h\colon T_\alpha \to \mathbb{Q}$ bounds *an approximation f with* $\mathrm{last}(f) = \alpha$ *iff $\forall x \in \mathrm{dom}(h)$, $f(x) < h(x)$. More generally, if $\beta \ge \alpha =$* $\mathrm{last}(f)$, *then $h\colon T_\beta \to \mathbb{Q}$ bounds f iff $\forall x \in \mathrm{dom}(h)(f(x\lceil\alpha) < h(x))$.*

A forcing condition is an approximation together with a countable set Ψ of $\mathbf{T}$-promises. This set functions as a side-condition and ensures that the forcing and also all of its countable support iterations do not add new reals. In the current work, this property is not so decisive, since adding some mild (i.e., ${}^\omega\omega$-bounding with the Sacks property) reals would not render the procedure of proper translation impossible. However, the side-conditions are still of benefit, since they help to establish the important Lemma 2.11.

We extend the $\lceil$-notation: Let $\alpha < \gamma$. For $\bar{x} \in {}^{\omega>}T_\gamma$ we let $\bar{x}\lceil\alpha = \langle x_i\lceil\alpha : i < |\bar{x}|\rangle$.

Definition 2.7. *(See [14, Ch. V, Def. 6.2].) Γ is a* **T**-promise *iff* $\mathrm{dom}(\Gamma)$ *is club $C(\Gamma)$ in ω_1 and if there is $n \in \omega$ such that $\Gamma = \langle \Gamma(\gamma) : \gamma \in \mathrm{dom}(\Gamma)\rangle$ has the following properties:*

(a) For each $\gamma \in \mathrm{dom}(\Gamma)$, $\Gamma(\gamma)$ is a countable set of $\bar{x} \in {}^nT_\gamma$ for some $n \in \omega$.

(b) For every $\alpha < \gamma \in \mathrm{dom}(\Gamma)$, for every $\bar{x} \in \Gamma(\alpha)$ there are infinitely many $\bar{y} \in \Gamma(\gamma)$ whose ranges are pairwise disjoint such that $\bar{y}\lceil\alpha = \bar{x}$.

(c) $\Gamma(\min(C(\Gamma)) \neq \emptyset$.

Definition 2.8. *([14, Ch. V, Def. 6.4]) We say that an approximation (f, C)* fulfils *the promise Γ iff $\mathrm{last}(f) \in C(\Gamma)$, and $C \smallsetminus \min(C(\Gamma)) \subseteq C(\Gamma)$ and for every $\alpha < \beta$, $\alpha, \beta \in C(\Gamma) \cap C$ and $\bar{x} \in \Gamma(\alpha)$ for every $\varepsilon > 0$ there are infinitely many pairwise disjoint $\bar{y} \in \Gamma(\beta)$ such that $\bar{y}\lceil\alpha = \bar{x}$, $\lg(\bar{y}) = \lg(\bar{x}) = n$ and and $f(x_\ell) < f(y_\ell) < f(x_\ell) + \varepsilon$ for all $\ell < n$.*

Definition 2.9. *([14, Ch. V, Def. 6.5]) $Q_{\mathbf{T}}$ is the set of (f, C, Ψ) such that (f, C) is an approximation, and Ψ is a countable set of promises and for all $\Gamma \in \Psi$, (f, C) fulfils Γ. The partial order is defined as $(f_0, C_0, \Psi_0) \leq (f_1, C_1, \Psi_1)$ iff*

(1) f_1 extends f_0,

(2) C_1 is an end-extension of C_0 and $C_1 \setminus C_0 \subseteq \bigcap_{\Gamma\in\Psi_0} C(\Gamma)$, and

(3) $\Psi_0 \subseteq \Psi_1$.

If $p = (f, C, \Psi)$, we write $f = f^p$, $C = C^p$ and $\Psi = \Psi^p$, and we write $\mathrm{last}(p) = \mathrm{last}(f^p) = \max(C^p)$.

Now we want to extend a given condition to a stronger condition of a given height, and we want to show that the set of promises can be enlarged.

Lemma 2.10. *([14, Ch. 5, Fact 6.6], [2, Lemma 4.3], [11, Lemma 2.6], The extension lemma.) Let $\mu < \omega_1$. If $p \in Q_{\mathbf{T}}$ and if $\mathrm{last}(p) < \mu \in \bigcap_{\Gamma\in\Psi^p} \mathrm{dom}(\Gamma)$, $\bar{y} \in [T_\mu]^n$ then there is some $q \geq p$ such that $\Psi^q = \Psi^p$ and $\mathrm{last}(q) = \mu$ and $f^p(y_i\lceil\mathrm{last}(p)) < f^q(y_i) < f^p(y_i\lceil\mathrm{last}(p)) + \varepsilon$. Moreover, if $h\colon T_\mu \to \mathbb{Q}$ is finite and bounds f^p, then q can be chosen such that h bounds f^q.*

The promises enter the proof of the following important lemma.

Lemma 2.11. *([2], [14, V, Fact 6.7A], [11, Lemma 2.9]) Let* **T** *be an Aronszajn tree. Let $M \prec (H(\chi), \in)$ be a countable elementary substructure with a regular $\chi > 2^{\aleph_1}$, $Q_{\mathbf{T}} \in M$, $p \in Q_{\mathbf{T}} \cap M$, $\mu = \omega_1 \cap M$ and $h\colon T_\mu \to \mathbb{Q}$ be a finite function which bounds f^p. Let $D \in M$, $D \subseteq Q_{\mathbf{T}}$ be dense open. Then there is an $q \geq p$, $q \in D \cap M$, that h bounds q.*

Now we describe the iterations a bit more precisely than in Definition 1.3:

We assume $\mathbf{V} \models \diamondsuit_{\omega_1} + 2^{\aleph_1} = \aleph_2$ and let $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ be a countable support iteration with $\mathbb{Q}_\alpha$ an ${}^\omega\omega$-bounding $< \omega_1$-proper $\aleph_2$-p.i.c. forcing for odd α, and for even α (let us say, limit ordinals are even) $\underset{\sim}{\mathbb{Q}}_\alpha = Q_{\underset{\sim}{\mathbf{T}}_\alpha}$ being as above for some Aronszajn tree $\mathbf{T}_\alpha \in \mathbf{V}[G_\alpha]$, where the filter G_α is P_α-generic over $\mathbf{V}$, such that $\Vdash_{P_\alpha}$ "$\underset{\sim}{\mathbf{T}}_\alpha$ is an Aronszajn tree and for $\gamma < \omega_1$ its

γ-th level is $[\omega\gamma, \omega\gamma + \omega)$". The book-keeping shall be arranged so that every $\mathbb{P}_{\omega_2}$-name for an Aronszajn tree is used in some iterand.

We argue that every Aronszajn tree in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ has a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_2$. We have $|Q_{\mathbf{T}}| = \aleph_2$, so that it need not necessarily have the $\aleph_2$-chain condition. However, [14, Chapter VIII, Section 2] takes care of our iteration: By Lemma 2.11, each $Q_{\mathbf{T}}$ has the $\aleph_2$-p.i.c. (proper isomorphism condition), see [14, Chapter VIII, Def. 2.1]. The Sacks iterands with $|\mathbb{Q}_\alpha| \leq \aleph_1$ have the the $\aleph_2$-p.i.c. by [14, Lemma VIII 2.5]. Hence by [14, Chapter VIII, Lemma 2.4], $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c., if $\mathbf{V}_0$ fulfils the CH. For the readers' convenience we recall the definition of the $\aleph_2$ proper isomorphism condition:

Definition 2.12. *(See* [14, Ch. VIII, Def. 2.1]*.) Let κ be a cardinal. A notion of forcing $\mathbb{Q}$ satisfies the κ-p.i.c. if the following holds for sufficiently large χ: Suppose $i < j < \kappa$, $N_i \prec (H(\chi), \in, <_\chi)$, $\kappa \in N_j \prec (H(\chi), \in, <_\chi)$, N_j, N_i are countable, $\mathbb{Q} \in N_i \cap N_j$, $i \in N_i$, $j \in N_j$, $N_i \cap \kappa \subseteq j$, $N_i \cap i = N_j \cap j$, $p \in N_i \cap Q$, h an isomorphism from N_i to N_j, $h \restriction N_i \cap N_j$ is the identity, and $h(i) = j$.*

Then *there is a $q \geq p$ such that*

(a) $p, h(p) \leq q$, q is $(N_i, \mathbb{Q})$ and $(N_j, \mathbb{Q})$-generic, and

(b) for every q' and $r \in \mathbb{Q}$ such that $r \in N_i \cap \mathbb{Q}$ and $q \leq q'$ there is $q'' \in \mathbb{Q}$ such that $q' \leq q''$ and ($r \leq q''$ iff $h(r) \leq q''$).

Since $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c., by a lemma similar to the one of [5, 5.10], now for subsets of ω_1 instead of real numbers, every subset of ω_1 in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ for a countable support iteration $\mathbb{P}_{\omega_2}$ of proper forcings such that $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c. has a name at some stage of cofinality ω_1. So, if CH and $2^{\aleph_1} = \aleph_2$ hold in the ground-model we an carry out a desired book-keeping that enumerates all $\mathbb{P}_{\omega_2}$-names of all Aronszajn trees in the extension. Now we know that our forcing specialises all Aronszajn trees, and in the remainder of the paper we focus on its effect onto the weak diamond.

In the context of proper forcings that do not add reals we find completely $(M, \mathbb{P}, p)$-generic conditions.

Definition 2.13. *A condition q is* completely $(M, \mathbb{P}, p)$-generic *if $G = \{r \in \mathbb{P} \cap M : r \leq q\}$ is an $(M, \mathbb{P}, p)$-generic filter. G is called* bounded *and q is called a bound of G. For $p \in \mathbb{P} \cap M$, we let* $\mathrm{Gen}^+(M, \mathbb{P}, p) = \{G \subseteq M \cap \mathbb{P} : G$ *is completely $(M, \mathbb{P})$-generic and $p \in G\}$.*

Any condition stronger than a bound of G is a bound as well. However, often there are canonical upper bounds of the form $q = \bigcup_{n \in \omega} q_n$, $q_n \in M$.

3. Proper translation in countable support iterations

Since we work with iterations of lengths ω_2 and since we want to perform the translation also if it does not commute with the Mostowski collapse, we will work with ord-hc (ordinarily hereditarily countable) sets in order to keep the actual information about the ordinals. If the translation commutes with the Mostowski collapse ("is invariant", see Def. 3.7 (4)) then "ord-hc" can be replaced by the ordinary notion of "hereditarily countable" in the domains and in the ranges of the translation functions.

We recall the definition of ord-hc from [15]:

Definition 3.1. *(1)* $\mathrm{Tc}^{\mathrm{ord}}(x)$, *the hereditary closure of x relative to the ordinals, is defined by induction on $\mathrm{rk}(x) = \gamma$ as follows:*

If $\gamma = 0$ or if x is an ordinal then $\mathrm{Tc}^{\mathrm{ord}}(x) = \emptyset$. If $\gamma > 0$ and x is not an ordinal then $\mathrm{Tc}^{\mathrm{ord}}(x) = \bigcup\{\mathrm{Tc}^{\mathrm{ord}}(y) : y \in x\} \cup x$.

(2) Let κ be an uncountable regular cardinal and let $\mathbf{V}_\kappa$ be the set of sets of rank less than κ. The collection of all sets which are ordinarily hereditarily countable relatively to κ is the set

$$H_{<\aleph_1}(\kappa) = \{x \in \mathbf{V}_\kappa : \mathrm{Tc}^{\mathrm{ord}}(x) \text{ is countable and } \mathrm{Tc}^{\mathrm{ord}}(x) \cap \mathrm{On} \subseteq \kappa\}.$$

(3) We say x is an ord-hc (ordinarily hereditarily countable) set *if x is an element of $H_{<\aleph_1}(\kappa)$ for some uncountable κ.*

(4) We say x is a strict ord-hc set *if x is an element of $H_{<\aleph_1}(\kappa)$ for some uncountable κ and if x is not an ordinal.*

In our applications, κ will be the iteration length, i.e. $\aleph_2$, or it will be $(2^{\aleph_1})^+ \geq \aleph_3$ as in Theorem 3.4. We consider the ordinals as urelements i.e. $\omega \neq \{n : n \in \omega\}$. We recall the following definition also from [15, Def. 0.5].

Definition 3.2. *We define the family of* ord-hc Borel operations *to be the minimal family $\mathscr{F}$ of functions such that the following conditions are satisfied:*

(a) Each $\mathbf{B} \in \mathscr{F}$ is a function with $\leq \omega$ arguments and with each argument is designated to an ord-hc set or to an ordinal or to a truth value or to strictly ord-hc set.

(b) The arity of the value of $\mathbf{B}$ is also $\leq \omega$, and each place in the $\leq \omega$-tuple has a designation as ord-hc set or strict ord-hc set or ordinal or truth value.

(c) $\mathscr{F}$ contains the following atomic functions with the obvious interpretation:

- *(α) $\neg x$ for a truth value x,*
- *(β) $x \vee y$ for two truth values,*
- *(γ) $\bigcap_{i<\alpha} x_i$, for $\alpha \leq \omega$ and truth values x_i,*
- *(δ) the constant values "true" and "false",*
- *(ε) the following types of composition, for all $\alpha \leq \omega$ and x_n varying on truth values and for all y_n varying on hc-sets or on ordinals or on strict hc-sets:*
  - *for $n < \omega$ the composition: if x_n but not x_m for $m < n$, then y_n,*
  - *the composition: if $\neg x_n$ for every $n < \alpha$ then y_n,*
- *(ζ) $\{y_i : i < \alpha, x_i \text{ is true}\}$ for $\alpha \leq \omega$, where y_i varies on ord-hc sets or on ordinals, x_i on truth values. Note that by our convention this is always a strict hc-set, never an ordinal,*
- *(η) the truth value of "x is an ordinal" where x varies on ord-hc sets.*

(d) $\mathscr{F}$ is closed under composition (preserving the designation to strict ord-hc sets ordinals and truth values).

Definition 3.3. *Let $M \subseteq \mathbf{V}$.*

(a) We define the ord-collapse $\pi^M_{\rm ord}$ *by induction.*

$$\pi^M_{\rm ord}(x) = \begin{cases} x, & \text{if } x \in \mathrm{On}, \\ \{\pi^M_{\rm ord}(y) \,:\, y \in x \cap M\}, & \text{else.} \end{cases}$$

(b) Two structures are ord-isomorphic *if there is an isomorphism being the identity on the ordinals.*

(c) M is ord-transitive *if $\omega^M = \omega$ and $\mathrm{On}^M = M \cap \mathrm{On}$ and $x \in M \setminus \mathrm{On}$ implies $x \subseteq M$.*

The ord-hc isomorphism type of a $\tau \cup \{\in, <_\chi\}$-structure $(M, \in, <_\chi, (P)_{P\in\tau})$ is represented by its ord-collapse. The ord-collapses are ord-transitive. If M and its signature τ are countable, then the ord-hc collapse $(\pi^M_{\rm ord}(M), (\pi^M_{\rm ord}(P))_{P\in\tau}$ is in $H_{<\omega_1}(\kappa)$ for some κ.

Ord-hc Borel computations $\mathbf{B}$ have ord-hc collapses as domains. In [15] it is shown that forcing can be defined not only over transitive models but also over countable ord-transitive models. In the ord-hc Borel version of our Main Lemma 3.12, we will use ord-hc Borel computations of generics over ord-transitive models. In the application to the particular forcings of Def. 1.3, though, we will use Borel computations over coutable transitive models.

Every condition in a cs iteration of proper iterands can be realised as a subset of $H_{<\omega_1}(\mu)$ for a suitable μ. This is proved in (the proof of)

Theorem 3.4. *([14, Ch. III, Theorem 4.1 and Claim 4.1A])) Let κ be an uncountable regular cardinal. Suppose that $\langle \mathbb{P}_\alpha, \underaccent{\tilde}{\mathbb{Q}}_\beta \,:\, \alpha \le \kappa, \beta < \kappa\rangle$ is a cs iteration and*

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\underaccent{\tilde}{\mathbb{Q}}_\alpha \text{ is a proper forcing notion of size } < \kappa\text{”},$$

κ is regular and $(\forall \mu < \kappa)\mu^{\aleph_0} < \kappa$. Then $\mathbb{P}_\kappa$ (and every $\mathbb{P}_\alpha$ as well) has the κ-c.c and each $\mathbb{P}_\alpha$ for $\alpha < \kappa$ has a dense subset $\mathbb{P}'_\alpha$ of power $< \kappa$, indeed, $\mathbb{P}'_\alpha \subseteq H_{<\omega_1}(\mu)$ for some $\mu < \kappa$. Hence for $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha} 2^{\aleph_0} < \kappa$.

Theorem 3.4 is often used for forcings of size $\aleph_1$ and $\kappa = \aleph_2$. We will apply this theorem for $\mu = 2^{\aleph_1} < \kappa$ and thus get $\mathbb{P}'_{\omega_2} \subseteq H_{<\omega}(\mu)$. Indeed, in our statements about ord-hc Borel functions we will tacitly assume that $\mathbb{P}_{\omega_2}$ is $\mathbb{P}'_{\omega_2}$. However, our iteration length is only $\aleph_2$, not $\kappa > 2^{\aleph_1}$. The theorem does not give the $\aleph_2$-chain condition. The latter is derived with the help of the proper isomorphism condition (p.i.c.) [14, VIII, 2.1]. As mentioned, $Q_{\mathbf{T}}$ has the $\aleph_2$-p.i.c. and hence, under CH, the iteration $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c.

Beyond the application of Theorem 3.4 we use $Q_{\mathbf{T}} \subseteq 2^{\omega_1}$ and $Q_{\mathbf{T}} \cap N \subseteq 2^{N\cap\omega_1}$ for transitive N and we work with ordinary Mostowski collapses. Then we see that the continuation of a partial specialisation onto the level $N \cap \omega_1$ is closely connected to the existence of completely generic conditions.

Definition 3.5. *Let $\mathbf{K}$ be the class of* candidates. *$\mathbf{k} \in \mathbf{K}$ means $\mathbf{k}$ consists of the following components that fulfil the following conditions:*

(a) $\mathbb{P} = \mathbb{P}^{\mathbf{k}} = \langle \mathbb{P}^{\mathbf{k}}_\alpha, \underaccent{\tilde}{\mathbb{Q}}^{\mathbf{k}}_\beta \,:\, \alpha \le \gamma(\mathbf{k}), \beta < \gamma(\mathbf{k})\rangle$,

(b) $\mathbb{P}$ is a countable support iteration,

(c) each $\underset{\sim}{\mathbb{Q}}^{\mathbf{k}}_{\alpha}$ *is* $< \omega_1$*-proper and* ${}^\omega\omega$*-bounding,*

(d) $\mathcal{I} = \langle \mathcal{I}_\alpha \,:\, \alpha < \gamma(\mathbf{k})\rangle$*,*

(e) $\mathcal{I}_\alpha$ *is an* $\aleph_1$*-directed partial order and* $\mathcal{I}_\alpha \subseteq H_{<\omega_1}(\aleph_1)$*,*

(f) $\langle \mathbb{P}^{\mathbf{k}}_{*,\alpha}, \underset{\sim}{\mathbb{Q}}^{\mathbf{k}}_{*,\beta} \,:\, \alpha \leq \gamma(\mathbf{k}), \beta < \gamma(\mathbf{k})\rangle$ *is a cs iteration of nep (in the sense of Definition 2.4) iterands.*

Remark 3.6. Under CH, every countably directed partial order $\mathcal{I} \subseteq H_{<\omega_1}(\aleph_1)$ has a $<_{\mathcal{I}}$-cofinal subset that can be embedded into $({}^\omega\omega, \leq^*)$. So, if "sufficiently large" (in the following definitions) means $<_{\mathcal{I}}$-dominating a certain countable set, then we do not lose generality by taking $\mathcal{I}_\beta = ({}^\omega\omega, \leq^*)$ for all β as we do here. However, as in [10] one could take only almost ${}^\omega\omega$-bounding iterands $\underset{\sim}{\mathbb{Q}}_\alpha$, and then let "sufficiently large" mean "not $\leq^*$-dominated by a certain countable set". More general applications with iterands $\underset{\sim}{\mathbb{Q}}_\alpha$ preserving $<_{\mathcal{I}}$-boundedness or weak $<_{\mathcal{I}}$-boundedness in a strong iterable manner and with $\mathcal{I}$ as a domain for the Borel function is thinkable. In our case, the correspondence between the ${}^\omega\omega$-boundedness of the iterands and the existence of sufficiently large first arguments of the Borel translation function is used at a crucial point in the proof of Lemma 4.7.

We adopt the convention that p, p_i, p^+, q, q_i, q^+ and so forth are used for conditions in $\mathbb{P}_\gamma$ and in $\mathbb{P}_u = \{p \in \mathbb{P}_\gamma \,:\, \mathrm{dom}(p) \subseteq u\}$, $u \subseteq \gamma$, and conditions r, r_i, r^+ are used for conditions in $\mathbb{P}_{*,\gamma}$ and $\mathbb{P}_{*,u} = \{r \in \mathbb{P}_* \,:\, \mathrm{dom}(r) \subseteq u\}$. We also follow the convention that conditions later in the alphabet or with more or higher indices are stronger.

Before we present one of the main definitions we explain the meanings of the indices of the Borel functions $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$ computing $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}(\bar\eta, \bar M, \mathbb{P}_\gamma, q_0, p)$:

(1) $i = 0, 1$ are the two dimensions of the image of the computation: $\mathbf{B}^\alpha_{\gamma_0,\gamma,0}$ stands for the computation of a generic condition, and $\mathbf{B}^\alpha_{\gamma_0,\gamma,1}$ stands for the translator function from $\mathbb{P}_\gamma$ to $\mathbb{P}_{*,\gamma}$,

(2) $q_0 \in \mathbb{P}_{\gamma_0}$ is a given $(\bar M, \mathbb{P}_{\gamma_0})$-generic condition. It is entered as an argument in the fourth place. We do not write γ_0 as a subscript to $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$ if $\gamma_0 = 0$. As usual this will be done, when the induction is performed.

(3) $\gamma > \gamma_0$ and $p \in \mathbb{P}_\gamma$. $q_0 \in \mathbb{P}_{\gamma_0}$ and $q_0 \Vdash p \restriction \gamma_0 \in \underset{\sim}{G}_{\gamma_0}$. Then $\mathbf{B}^\alpha_{\gamma_0,\gamma,0}(\bar\eta, \bar M, \mathbb{P}, q_0, p)$ computes an $(\bar M, \mathbb{P}_\gamma, p)$-generic condition stronger than p and extending q_0.

(4) $\alpha \geq \mathrm{otp}(M_0 \cap [\gamma_0, \gamma))$. A tower $\bar M = \langle M_i \,:\, i \leq \alpha\rangle$ is used for computing $(M_0, \mathbb{P})$ generic conditions with enough completeness. The top part $\langle M_i \,:\, i \in [1, \alpha+1)\rangle$ is just a helper to compute the desired objects as values of a Borel function with the tower as an argument. Since all the forcings are $< \omega_1$-proper, we could use higher towers $\bar N \hat{} \bar M$, use only the $\bar M$-part for the computation and end up with $\bar N$-generic conditions.

(5) We write $g''I = g[I] = \{g(p) \,:\, p \in I\}$.

Definition 3.7. $\bar\eta = \langle \eta_\varepsilon \,:\, \varepsilon \in \alpha\rangle$ *is* sufficiently large *for the tower* $\bar M = \langle M_\varepsilon \,:\, \varepsilon \in \alpha + 1\rangle$ *if* $(\forall \varepsilon \in \alpha)(\forall f \in M_{\varepsilon+1})(f \leq^* \eta_\varepsilon)$.

Definition 3.8. *Let $\mathbf{k} \in \mathbf{K}$. We say that $\bar{\mathbf{B}} = \langle \mathbf{B}^\alpha_{\gamma_0,\gamma} : \gamma_0 < \gamma \le \gamma', \alpha \in \omega_1 \rangle$ is a* solution to $(\gamma', \mathbf{k})$ *if $\gamma' \le \gamma(\mathbf{k})$ and for any $\gamma \le \gamma'$, $\mathbf{B}^\alpha_{\gamma_0,\gamma}$ is an ord-hc Borel function*

$$\mathbf{B}^\alpha_{\gamma_0,\gamma} \colon \prod_{\varepsilon < \alpha} \mathcal{I}_\varepsilon \times \{(\bar{M}, \mathbb{P}_\gamma, q_0, p) : (\bar{M}, q_0, p) \text{ as below}\}$$
$$\to \mathbb{P}_\gamma \times {}^{\mathbb{P}_\gamma \cap M_0}(\mathbb{P}_{*,\gamma} \cap M)$$

with the following properties:

(a) *the second argument of $\mathbf{B}^\alpha_{\gamma_0,\gamma}$, $\bar{M} = \langle M_\varepsilon : \varepsilon \le \alpha \rangle$, is the ord-hc collapse of a tower of models for $\mathbb{P}$, expanded by constants for $\mathbb{P}$, q_0, p, we write uncollapsed structures as arguments for $\mathbf{B}$ in case it commutes:*

$$(\pi^{M_\alpha}_{\text{ord}})^{-1}(\mathbf{B}(\bar{\eta}, \pi^{M_\alpha}_{\text{ord}}(\bar{M}), \pi^{M_\alpha}_{\text{ord}}(\mathbb{P}), \pi^{M_\alpha}_{\text{ord}}(q_0), \pi^{M_\alpha}_{\text{ord}}(p))) = \mathbf{B}(\bar{\eta}, \bar{M}, \mathbb{P}, q_0, p),$$

(b) $p \in M_0 \cap \mathbb{P}_\gamma$, $\mathbb{P}_\gamma \in M_0$,

(c) *$q_0 \in \mathbb{P}_{\gamma_0}$ is an $(\bar{M}, \mathbb{P}_{\gamma_0})$-generic condition,*

(d) $q_0 \Vdash_{\gamma_0} p \restriction \gamma_0 \in \underset{\sim}{G}_{\gamma_0}$,

(e) *$\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$ is an ord-hc Borel function,*

(f) *we write $\mathbf{B}^\alpha_{\gamma_0,\gamma} = (\mathbf{B}^\alpha_{\gamma_0,\gamma,0}, \mathbf{B}^\alpha_{\gamma_0,\gamma,1})$, the values of $\mathbf{B}^\alpha_{\gamma_0,\gamma}$ are ord-hc sets of the form (q, g) such that letting $u = M_0 \cap [\gamma_0, \gamma + 1)$ we have for $\bar{\eta} = \langle \eta_\varepsilon : \varepsilon < \alpha \rangle$ sufficiently large:*
   - (α) *$\mathbf{B}^\alpha_{\gamma_0,\gamma,0}(\bar{\eta}, \bar{M}, \mathbb{P}_\gamma, q_0, p) = q \in \mathbb{P}_u$ is $\ge_{\mathbb{P}_\gamma}$ stronger than p and $(\bar{M}, \mathbb{P}_\gamma, p)$-generic, $q \restriction \gamma_0 = q_0$, $q \subseteq M_\alpha \cap \mathbb{P}_u$,*
   - (β) *$g \colon M_0 \cap \mathbb{P}_u \to M_0 \cap \mathbb{P}_{*,u}$, preserves $\le$, (note that the translator g really works well only on M_0)*
   - (γ) *$\sup g[\{p \in \mathbb{P}_u \cap M_0 : p \le_{\mathbb{P}_\gamma} q\}] \subseteq M_0 \cap \mathbb{P}_{*,u}$ exists and we denote it by $g''q$ and is $(M_0, \mathbb{P}_{*,u})$-generic,*
   - (δ) *if $q \Vdash_{\mathbb{P}_u} M_0 \models$ "$I \subseteq \mathbb{P}_\gamma$ is predense", then $g''q \Vdash_{\mathbb{P}_{*,u}}$ "$g''(I \cap M_0)$ is predense in $\mathbb{P}_{*,u} \cap M_0$."*

(g) *coherence: If $\gamma_0 < \gamma_1 < \gamma_2$ are from $M_0 \cap \gamma'$ then for all α, $\mathbf{B}^\alpha_{\gamma_0,\gamma_2}$ projects to $\mathbf{B}^\alpha_{\gamma_0,\gamma_1}$ that is for $q_0 \in \mathbb{P}_{\gamma_0}$ that is $(\bar{M}, \mathbb{P}_{\gamma_0}, p \restriction \gamma_0)$-generic, if $\mathbf{B}^\alpha_{\gamma_0,\gamma_i}(\langle \eta_\varepsilon : \varepsilon \in \alpha \rangle, M, \mathbb{P}, q_0, p \restriction \gamma_i) = (q_i, g_i)$, then $q_2 \restriction \gamma_1 = q_1$ and $g_2 \restriction M_0 \cap \mathbb{P}_{\gamma_1} = g_1$.*

Actually, in the proofs we need only the following: For any branch $b \in M_\varepsilon$ of the Aronszajn tree $\mathbf{T}$ that has a continuation on level $\text{otp}(M_\varepsilon \cap \omega_1)$ the function η_ε eventually dominates a code of the branch b. Since these branches are elements of $M_{\varepsilon+1}$, it is just easiest to require that η_ε dominates ${}^\omega\omega \cap M_{\varepsilon+1}$. We write $\eta_\varepsilon \ge^* M_{\varepsilon+1}$ for the latter.

Definition 3.9. *Assume that $\bar{\mathbf{B}}$ is a solution to $(\gamma', \mathbf{k})$. We say that $\bar{\mathbf{B}}$ is a* successful solution for $(\gamma', \mathbf{k})$ *when in addition to the previous definition*

(h) *if $\bar{\eta}$ is sufficiently large for $\bar{M} = \langle M_i : i \le \alpha \rangle$, $\alpha \ge \text{otp}[\gamma_0, \gamma) \cap M_0$, $\mathbf{B}^\alpha_{\gamma_0,\gamma}(\bar{\eta}, \bar{M}, \mathbb{P}, q_0, p) = (q, g)$, $r \not\perp g''q$, $r \in \mathbb{P}_{*,u} \cap M_0$ and for $n < \omega$,*

$I_n \subseteq$ range$(g) \subseteq M_0$, $I_n \in M_0$, I_n is predense above r in $\mathbb{P}_{*,u}$, then *there is a $q' \in \mathbb{P}_\gamma \cap M_0$, $q' \not\perp q$ with domain u such that,*

$$(\forall n < \omega) g^{-1}[I_n] \text{ is predense above } q' \text{ in } \mathbb{P}_\gamma \cap M_0.$$

(i) if $q \Vdash_{\mathbb{P}_\gamma} M_0 \models$ "$p \perp_{\mathbb{P}_\gamma} p'$, then $g''q \Vdash_{\mathbb{P}_{*,\gamma}}$ "$g(p) \perp_{\mathbb{P}_{*,\gamma}} g(p')$."

So q and $g''q$ force that g has a natural extension from $\mathbb{P}$-names in M_0 to $\mathbb{P}_*$-names in M_0 preserving predensity in both directions. In the induction, this is used for names of later conditions. Note that this backwards direction of the reduction, which says that it is trivial, works only for conditions in M_0 and it works only in the part that is forced by the generic q. Coarsely speaking, q forbids incompatibilities that are caused by the parts of the conditions that are dropped in the translation.

We can define a largeness game

$$\Game_{(\bar{M},\mathbb{P},p)} \tag{3.1}$$

played in α rounds for $\langle M_\varepsilon : \varepsilon \leq \alpha \rangle$. In round ε, the generic player plays ν_ε and the antigeneric player plays $\eta_\varepsilon \geq^* \nu_\varepsilon$. The generic player wins iff $\bar{\eta}$ is sufficiently large, that is if $\eta_\varepsilon \geq^* M_{\varepsilon+1}$. Of course, the generic player has a winning strategy. In Lemma 6.3 we will interpret such innings differently and then really use that the ν_ε's and the η_ε's can be chosen successively.

Fact 3.10. *For $\mathbf{k} \in \mathbf{K}$ and $\gamma \leq \gamma(\mathbf{k})$ let $\mathbf{k} \restriction \gamma$ be defined naturally. If $\mathbf{k} \in \mathbf{K}$ and $\gamma_0 \leq \gamma$ and $\bar{M} \prec (H(\chi), \in)$ is a sufficiently high tower and $\{\mathbf{k}, \gamma_0, \gamma\} \in M_0$, then $\bar{\mathbf{B}}$ is a solution to $(\gamma(\mathbf{k}), \mathbf{k})$ iff $\bar{\mathbf{B}}$ is a solution for $(\gamma(\mathbf{k}), \mathbf{k} \restriction \gamma)$.*

Definition 3.11. *Let $\mathbf{k} \in \mathbf{K}$.*

(1) We say that $\bar{\mathbf{B}}$ is a good for *$(\bar{\eta}, q_0, p, \bar{M}, \gamma, \mathbf{k})$ when $\bar{\mathbf{B}}$ is a successful $(\gamma, \mathbf{k})$-solution, $(\bar{M}, \mathbb{P}^{\mathbf{k}}, q_0, p)$ is a suitable second argument and $\bar{\eta}$ is large enough.*

(2) We say that $\mathbf{k}$ is good *iff $(*)_1$ implies $(*)_2$.*

Here $(*)_1$ is the following list:

(a) $\gamma_0 < \gamma_1 \leq \gamma_2 \leq \gamma(\mathbf{k})$, $\alpha_i = \mathrm{otp}([\gamma_0, \gamma_i) \cap M_0$,

(b) $\bar{M} = \langle M_\varepsilon : \varepsilon \in \alpha_2 + 1 \rangle \prec (\mathcal{H}(\chi), \in)$ is a tower of height $\alpha_2 + 1$

(c) $p \in \mathbb{P}_{\gamma_2} \cap M_0$ and $\gamma_0, \gamma_1, \gamma_2 \in M_0$,

(d) $q_0 \in \mathbb{P}_{\gamma_0}$ is $(\bar{M}, \mathbb{P}_{\gamma_0})$-generic and $q_0 \Vdash_{\gamma_0} p \restriction \gamma_0 \in \underset{\sim}{G}_{\gamma_0}$,

(e) there is an ord-hc Borel function $\bar{\mathbf{B}}_1$ that is a good for $(\bar{\eta}, \bar{M} \restriction \alpha_1, q_0, p \restriction \gamma_1, \gamma_1, \mathbf{k})$.

Here $(*)_2$ is the following statement: There are $\bar{\mathbf{B}}_2$, $\bar{\eta'}$ such that

(α) $\bar{\mathbf{B}}_2$ is a good for $(\bar{\eta'}, q_0, p, \bar{M}, \gamma_2, \mathbf{k})$,

(β) $\bar{\mathbf{B}}_2 \restriction \gamma_1 = \bar{\mathbf{B}}_1$,

(γ) $\bar{\eta'} \restriction \alpha_1 = \bar{\eta}$.

(3) We say that $\mathbf{k}$ is atomically good *if the above holds for $\gamma_2 = \gamma_1 + 1$.*

(4) Let π^M denote the Mostowski collapse of $M \prec H(\chi)$. We say that $\mathbf{k}$ is invariantly good/invariantly atomically good *if for every $(\bar{M}^i, \gamma_0^i, \gamma_1^i, (q')^i, p^i)$*

as in $(*)_1$ *of part (2) / or of part (3) for* $i = 1, 2$ *if* $\bar{N}^i$ *is the Mostowski collapse* π^{M_i} *of* $\bar{M}^i$ *(in the language* $\{\in, <_\chi\}$*) and* $\pi^{M_1}(\gamma_0^1, \gamma_1^1, q_0^1, p^1, \mathbf{k}) = \pi^{M_2}(\gamma_0^2, \gamma_1^2, q_0^2, p^2, \mathbf{k})$

(so in particular $\pi^{M_1}(\mathbb{P}^{\mathbf{k}}_{*,\bar{M}^1 \cap \gamma(\mathbf{k})}) = \pi^{M_2}(\mathbb{P}^{\mathbf{k}}_{*,\bar{M}^2 \cap \gamma(\mathbf{k})})$*) and* $\pi^{M_1}(\bar{\mathbf{B}}_1) = \pi^{M_2}(\bar{\mathbf{B}}_2)$ *is defined in a natural manner, then*

$$\pi^{M_1}(\bar{\mathbf{B}}_1) = \pi^{M_2}(\bar{\mathbf{B}}_2).$$

Lemma 3.12. *(Main Lemma)*

(1) *Assume that* $\mathbf{k}$ *satisfies*
   - *(a)* $\mathbf{k} \in \mathbf{K}$*,*
   - *(b)* $\mathbf{k}$ *is atomically good.*

   Then $\mathbf{k}$ *is good.*

(2) *If we strengthen clause (c) to* $\mathbf{k}$ *is invariantly atomically good, then* $\mathbf{k}$ *is invariantly good.*

Proof. As usual in preservation by checking like [14, Ch VI, §5]. However, we will carry out the proof of the claimed preservation in limit steps in Section 5.

First we will prove in the next section, that $\mathbf{k}$ where $\mathbb{P}^{\mathbf{k}}$ is from Definition 1.3 together with $\mathbb{P}_*$ being the iteration of the odd iterands and $\mathcal{I}_\varepsilon = ({}^\omega\omega, \leq^*)$ is invariantly atomically good. We do this first, since for this proof we introduce some notions that appear in the proof of the Main Lemma as well.

4. Proper translation for one and for two steps

We show: Let $\mathbb{P} = \mathbb{P}^{\mathbf{k}}$ be a forcing as in Definition 1.3, let $\mathbb{P}_*$ be the iteration of the odd iterands and let $\mathcal{I}_\varepsilon = ({}^\omega\omega, \leq^*)$. Then $\mathbf{k}$ is invariantly atomically good.

In the statement

$$(\forall \bar{M})(\forall p \in \mathbb{P}_\gamma \cap M_0)(\forall q_0)(q_0 \text{ is } (\bar{M}, \mathbb{P}_\gamma, p \restriction \gamma_0)\text{-generic } \to (\exists q \geq p)$$
$$(q \text{ is } (\bar{M}, \mathbb{P}, p)\text{-generic and } q \restriction \gamma_0 = q_0))$$

we want to replace the existential quantifier by a Borel function $\mathbf{B}^\alpha_{\gamma_0,\gamma,0}$ with arguments $\bar{\eta}$, $\bar{M}$, q_0 and p and finitely many relations over $\bar{M}$. In a second step, we use q's genericity over the tower of models to establish a translation function $g = \mathbf{B}^\alpha_{\gamma_0,\gamma,1}$ as in Definitions 3.8, 3.9.

Any $q' \in M_0 \cap \mathbb{P}$ that is compatible with q is mapped by the second component of $\mathbf{B}$ to some $g(q') \in \mathbb{P}_* \cap M_0$ such that dense subsets of $\mathbb{P}$ in M_0 are mapped to dense subsets of $\mathbb{P}_*$ in M_0 and vice versa and that there are generic conditions q such that g restricted to $\{p \in M_0 : p \not\perp q\}$ preserves $\leq$ and $\perp$ in both directions.

In the NNR steps $\mathbb{Q}_\gamma = Q_{\mathbf{T}}$ we know that above each $p(\gamma)$ there is some $q(\gamma)$ that is completely $(\bar{M}[G], Q_{\mathbf{T}}[G], p(\gamma)[G])$-generic. So we find a $\mathbb{P}_\gamma$-name for a filter $G(\gamma)$ such that $q(\gamma)$ bounds $G(\gamma) \cap M_0[G]$, and by induction hypothesis a $\mathbb{P}_{*,\gamma}$-name for $q(\gamma)$. Since $\mathbb{Q}_\gamma$ does not add reals, M_0 thinks the rest of the iteration looks the same in $\mathbf{V}^{\mathbb{P}_{\gamma+1}}$ and in $\mathbf{V}^{\mathbb{P}_{*,\gamma}}$. However, in the successor steps and in limit steps, this procedure is not trivial. We shall rework and adapt a part of the theory of completeness systems (see [14, Ch. V]) to the new situation in which reals are added in intermediate steps. A crucial point is that we add only reals bounded by ground model reals and that the statement "η_α dominates all the reals in $M_{\alpha+1}$" can be true with one η_α simultaneously for densely in $\mathbb{P}_\gamma \cap M_0$ many $M_0[G_\gamma]$, $\alpha = \mathrm{otp}([\gamma_0, \gamma) \cap M_0)$.

First, we recall more facts about $Q_{\mathbf{T}}$ and its completely generic conditions. The notion of forcing $Q_{\mathbf{T}}$ has size $2^{\aleph_1}$, and already the set of all approximations (f, C) to specialisations has size $\aleph_1^{\aleph_0}$. So for a countable $M \prec (H(\chi), \in, <_\chi)$, we never have $\mathbb{P} \subseteq M$. If $\mathbf{T} \in M$, we can read the definition of $\mathbb{P} = Q_{\mathbf{T}}$ in M, we call it $\mathbb{P}^M$. Since $\mathbf{T}$ is definable from $Q_{\mathbf{T}}$ ($x \not<_{\mathbf{T}} y$ iff there is an approximation with $f(x) = f(y)$), $Q_{\mathbf{T}} \in M$ implies $\mathbf{T} \in M$. So if $Q_{\mathbf{T}} \in M \prec H(\chi)$ and $\chi > 2^{2^{\aleph_1}} = 2^{|Q_{\mathbf{T}}|}$ is regular, then we get $Q_{\mathbf{T}}^M = Q_{\mathbf{T}} \cap M$ because the definition of $Q_{\mathbf{T}}$ is the same in $\mathbf{V}$ and in $H(\chi)$ and M. $Q_{\mathbf{T}} \cap M$ determines $\pi^M(Q_{\mathbf{T}})$ that is an argument in the Borel functions that are established here. In the section we work with the ordinary Mostowski collapse π^M.

The ord-hc Borel translation functions use as an argument the collapses or the ord-collapses of $(\bar{M}, \in, <_\chi, \mathbb{P}, p)$. There is a natural lifting to uncollapsed structures such that this lifting is automatically invariant in the sense of Definition 3.11 (4). In the end, we will guess all possible isomorphism types of collapsed countable structures with the help of the ordinary diamond.

Now we work on the atomic step of the proper translation for the iterands $Q_{\mathbf{T}}$. From now on we use the requirement that the α-th level of $\mathbf{T} = (\omega_1, <_{\mathbf{T}})$ is $[\omega\alpha, \omega(\alpha+1))$. Let $\chi > 2^{\aleph_2}$ be a regular cardinal. If we have a countable $M \prec (H(\chi), \in, <_\chi)$, then

$$\pi^M(\mathbf{T}) = T'_{<\mu}$$

with $N = \pi^M(M)$ and $\mu = N \cap \omega_1$ and T' is a flattened version of $\mathbf{T}$, only the levels of $M \cap \omega_1$ appear, since $\pi^M(M \cap \omega_1) = \mathrm{otp}(M \cap \omega_1) = \mu$. We take an increasing sequence $\bar{\beta} = \langle \beta_n : n \in \omega \rangle$ that is cofinal in μ. Now we take for $x_1 \subseteq M^2$ a code of the branches through $T'_{<\mu}$, for example $x_1 \colon T'_{<\mu} \to \omega$, x_1 is eventually constant on each branch. We also code in x_1 the branches through $T'_{<\mu}$ that have $<_{\mathbf{T}}$ successors in T_μ. Indeed the other branches are unimportant. If we want to find an $(M, \mathbb{P}, p)$-generic condition with last level T_μ we have to arrange that the approximations to the specialisation function do not diverge on any branch that is continued in T'_μ. The code $x = (x_1, \bar{\beta})$ are in general not in N, but they are predicates $\subseteq N^k$.

The technique of the following lemma comes from [2]. Actually a sketch of the elements of the $\aleph_1$-completeness system is also given in the end of the proof of [14, Chapter V, Theorem 6.1] on page 236. We conceive $x = (x_1, \bar{\beta})$ as one relation in M. The completeness system (that is a set, closed under countable intersections, of sets of generic filters where one set contains only completely generic filters) does not appear in our setting, since we establish functions choosing completely generic conditions over M_0 and generic conditions over a tower of models. Note that all the computation are now in the collapsed models. We use the letter M (possibly with indices) for the original elementary substructures and N for the collapsed structures.

Lemma 4.1. *Let $\psi(x,G) = \psi_0(x) \wedge \psi_1(x,G)$, with*

$$\psi_0(x) \equiv x = (x_1, \bar\beta) \wedge \bar\beta = \langle \beta_n : n \in \omega\rangle \text{ increasing}$$
$$\wedge N \cap \omega_1 = \bigcup\{\beta_n : n < \omega\}$$

and

$$\psi_1(x,G) \equiv (\forall \varepsilon > 0)(\forall t \in T'_\mu)(\exists m < \omega)(\forall n_1 < n_2 \in [m,\omega))(\forall y_1, y_2 <_{\mathbf{T}} t)$$
$$\Big((y_1 \in T_{\beta_{n_1}} \wedge y_2 \in T_{\beta_{n_2}} \wedge y_1 <_{\mathbf{T}} y_2 \to \underset{\sim}{f}[G](y_2) < \underset{\sim}{f}[G](y_1) + \frac{\varepsilon}{2^{n_2}}\Big)$$
$$\wedge \text{ “G is a filter”}$$
$$\wedge\, p \in G \wedge \forall D \in M((D \subseteq \mathbb{P} \wedge D \text{ dense in } \mathbb{P}) \to D \cap G \neq \emptyset).$$

Here M, P, x and G appear in the formulas as (names for) predicates and p is a constant. We write T'_μ instead of x (though T'_μ is not a subset of M). Let $\mu = \mathrm{otp}(M \cap \omega_1) = \sup\langle \pi^M(\beta_n) : n < \omega\rangle$ and let the $\beta_n \in M$ be increasing. If

$$(M \cup \mathcal{P}(M), \in^{M\cup\mathcal{P}(M)}, p, M, Q_{\mathbf{T}}) \models \psi_0(x),$$

then there is $G \subseteq Q_{\mathbf{T}}$, $G \in \mathrm{Gen}^+(M, Q_{\mathbf{T}}, p)$ such that

$$(M \cup \mathcal{P}(M), \in^{M\cup\mathcal{P}(M)}, p, M, Q_{\mathbf{T}}) \models \psi(x,G).$$

Proof. Let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_{\mathbf{T}}$ that are in M. Let β_n be increasing and cofinal in $\mu = \mathrm{otp}(M \cap \omega_1)$. Let $\{t_n : n \in \omega\}$ enumerate T'_μ: Now we choose by induction on $n < \omega$, p_n such that

(1) $p_0 = p$,
(2) $p_{n+1} \geq p_n \in M$,
(3) $\pi^M(\mathrm{last}(p_{n+1})) \geq \beta_{n+1}$,
(4) $p_{n+1} \in I_n$,
(5) $(\forall t \in \{t_k : k \leq n\})(\forall y <_{\mathbf{T}} t)\ \big(y \in T_{\beta_{n+1}} \to f^{p_{n+1}}(y) < f^{p_n}(y\lceil\beta_n) + \frac{1}{2^{n+1+n}}\big)$.

Then $G = \{r : (\exists n \in \omega)(r \leq p_n)\} \in \mathrm{Gen}(M, Q_{\mathbf{T}}, p)$.

Why is this choice possible? For Properties (4) and (5) we use Lemma 2.11 for h with

$$\begin{aligned} \mathrm{dom}(h) &= \{t_k\lceil\beta_{n+1} : k \leq n\}, \\ h(y) &= f^{p_n}(y\lceil\beta_n) + \frac{1}{2^{n+1+n}}, \end{aligned}$$

which is a finite function that bounds p_n and we find some p_{n+1} of length β_{n+1}.

Now we show: If $(M \cup \mathcal{P}(M), \in, p, M, Q_{\mathbf{T}}) \models \psi(x,G)$ for some x, then G has an upper bound in $Q_{\mathbf{T}}$. Again let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_{\mathbf{T}}$ that are in M. Let x be as in $\psi(x,G)$. Let $G \supseteq \{q_n : n \in \omega\}$, $q_n \in M \cap I_n$, $\mathrm{last}(q_n) = \beta_n$ such that the β_n and the q_n are increasing.

G has an upper bound q in $Q_{\mathbf{T}}$. We let $f^q \supseteq \bigcup f^{p_n}$ be a slightly larger rational variant of $\bigcup f^{p_n} \cup \{(z, \sup\{f^{p_k}(z\lceil \mathrm{last}(p_k)+1) : k \in \omega\}) : z \in T_\mu\}$. For definiteness, we stipulate that $f^q(t_n)$ is the $<_\chi$-first rational number above $\sup\{f^{p_k}(t_n\lceil \mathrm{last}(p_k)+1) : k \in \omega\} + \frac{1}{n+1}$. We let $C^q = \bigcup_{n\in\omega} C^{p_n} \cup \{\mu\}$, which is closed since for each n, $C^{p_{n+1}}$ is an end extension of C^{p_n}, $\Psi^q = \bigcup_{n\in\omega} \Psi^{p_n}$ and for $\Gamma \in \Psi^{p_n}$ we have $\mu \in \mathrm{dom}(\Gamma)$ since M and $(H(\chi), \in)$ fulfil that $\mathrm{dom}(\Gamma)$ is club in ω_1 and $N \cap \omega_1 = \mu$. Then $\mathrm{last}(q) = \mu \in \mathrm{dom}(\Gamma)$ for all $\Gamma \in \Psi^q$.

We claim that q is an upper bound of G: First we check that $q \in Q_{\mathbf{T}}$. We have that (f^q, C^q) is an approximation. Now let $H \in \Psi^q(\mu)$ be a $\mathbf{T}$-promise. For some $\mu' \geq \mu$, $k \in \omega$, $H \in \Psi^{q_k}(\mu')\lceil\mu$. Then, for any $\varepsilon > 0$ and for any $n \in \omega$ there are only finitely many $z \in T'_\mu$ such that $f^q(z) > f^{p_n}(z \restriction \beta_n) + \varepsilon$. Hence, since q_n fulfils the promise, also q fulfils the promise. $\dashv_{4.4}$

Now suppose that we do not know T'_μ and do not know the predicate x_1 and still want to have an analogue to property (5) that secures that the specialisation functions do not diverge along any branch of the Aronszajn tree that has a prolongation on level T'_μ. We give a modified definition ψ that does not refer to T'_μ but rather uses an argument $\eta \in {}^\omega\omega$ that is $\leq^*$-dominating all functions coding branches in $T'_{<\mathrm{otp}(\omega_1\cap M)}$. Now we introduce some functions $h \in {}^\omega\omega$ that code T'_μ. The additional argument $\eta \in {}^\omega\omega$ is suitable for defining a completely $(M, \mathbb{P})$-generic condition if η dominates all codes h_b of branches b of the Aronszajn tree that do have a node in T'_μ.

The transition from x_1 to η has the advantage that the η come from a countably directed system, called $\mathcal{I}_\varepsilon$ in the theoretical framework from Definition 3.5. This will be used in Lemma 4.7 to show how to work with completely generic conditions for iterands though an inital segment of the iteration adds reals.

Definition 4.2. *Let $\mathbf{T}$ be an Aronszajn tree with levels $T_\alpha = [\omega\alpha, \omega(\alpha+1))$. Let μ be a limit ordinal in ω_1. Given $\bar\beta$ converging to μ, we can write cofinally many nodes of a branch b of $T'_{<\mu}$ into a function $h_{b,\bar\beta}\colon \omega \to \omega$, such that for all n,*

$$b \cap T_{\beta_n} = \{\omega\beta_n + h_{b,\bar\beta}(n)\}$$

and we can describe each node $t = \omega\mu + k \in T_\mu$, by $h_{t,\bar\beta}\colon \omega \to \omega$, such that for all n,

$$t\lceil\beta_n = \omega\beta_n + h_{t,\bar\beta}(n).$$

If $t = \omega\beta_n + k \in T_{\beta_n}$, then we define $h_{t,\bar\beta}\colon n+1 \to \omega$, such that for all $m \leq n$,

$$t\lceil\beta_m = \omega\beta_m + h_{t,\bar\beta}(m).$$

The following lemma improves on the previous one: A completely generic condition is described by η as a parameter and any $\eta' \geq^* \eta$ can serve as a parameter as well. The description is a Borel function that commutes with the Mostowski collapse:

Lemma 4.3. *Let $M \prec (H(\chi), \in, <_\chi)$, $\mathbf{T} \in M$, and let $\pi^M\colon (M, \in) \to (N, \in)$ be the Mostowski collapse. Let $p \in Q_{\mathbf{T}} \cap M$. Let $\langle \beta_n : n < \omega\rangle$ be cofinal in*

$M \cap \omega_1$, $\beta_{n+1} > \beta_n$. We let the functions $h_{y,\bar\beta}$ be defined as in Def. 4.5. We write $\pi^M(\bar\beta)$ for $\langle \pi^M(\beta_n) : n < \omega\rangle$. Note that $h_{\pi^M(y),\pi^M(\bar\beta)} = h_{y,\bar\beta}$.

Set

$$U = (N, \in, \pi^M(<_\chi), \pi^M(\bar\beta), \pi^M(Q_{\mathbf{T}}), \pi^M(p)).$$

There is a Borel function $\mathbf{B}_{1,0} \colon \omega^\omega \times H_{<\aleph_1}(\omega_1)$, such that for every $\eta \in \omega^\omega$, if

$$(\forall y \in T'_\mu)(h_{y,\bar\beta} \leq^* \eta), \tag{4.1}$$

for

$$\hat r = \mathbf{B}_{1,0}(\eta, U)$$

the following holds: $\hat r$ is completely $(N, \pi^M(Q_{\mathbf{T}}), \pi^M(p))$-generic and and

$$\mathbf{B}_{1,0}(\eta, M, Q_{\mathbf{T}}, p) = r = ((\pi^M)^{-1})'' \hat r$$

is completely $(M, Q_{\mathbf{T}}, p)$-generic.

Proof. We verify that each step in the proof of Lemma 4.4 is Borel computable from (η, U).

We compute from η and U by induction on $n < \omega$, p'_n such that

(1) $p'_0 = \pi(p)$,

(2) $I_n \in N$ is the $\pi^M(<_\chi \cap M^2)$-least dense subset of $\pi^M(Q_{\mathbf{T}})$ such that $I_n \notin \{I_m : m < n\}$,

(3) p'_{n+1} is the $\pi^M(<_\chi \cap M^2)$-least element of N such that
   - (a) $p'_{n+1} \geq_{\pi^M(Q_{\mathbf{T}})} p'_n$,
   - (b) $\mathrm{last}(p'_{n+1}) \geq \pi^M(\beta_{n+1})$,
   - (c) $p'_{n+1} \in I_n$,
   - (d)

$$(\forall x \in \pi^M(T_{\beta_{n+1}}))$$
$$\big(h_{x,\pi^M(\bar\beta)}(n+1) \leq \eta(n+1) \to f^{p'_{n+1}}(x) < f^{p'_n}(x\lceil\pi(\beta_n)) + \frac{1}{2^{n+1+n}}\big).$$

For finding such an p'_{n+1} we use the Lemma 2.11 for the finitely many initial segments of branches $\pi^M(y \restriction (\beta_{n+1}+1))$ with $\pi^M(y(\beta_{n+1})) = y(\pi^M(\beta_{n+1})) \leq \eta(n+1)$ and with the following bound h:

$$\begin{aligned} \mathrm{dom}(h) &= \{x \in \pi(T_{\beta_{n+1}}) : h_{x,\pi(\bar\beta)}(n+1) \leq \eta(n+1)\}, \\ h(x) &= f^{p'_n}(x\lceil\pi(\beta_n)) + \frac{1}{2^{n+1+n}}. \end{aligned}$$

If Equation (4.1) holds, then η is sufficiently large to take care of all branches of $T'_{<\mu}$ that lead to points $x \in T'_\mu$. Note that if ν dominates all $h_{\bar\beta,z}$, $z \in T'_\mu$, then for every $z \in T'_\mu$ the limit $f^q(z)$ exists, because if $h_{z,\bar\beta} \leq^* \nu$, then for almost all n, $z\lceil\beta_n = \omega\beta_n + h_{z,\bar\beta}(n)$ and $h_{z,\bar\beta}(n) \leq \nu(n)$. Let $\mathbf{B}_{1,0}(\eta, U)$ be a definable (as is Lemma 4.4) upper bound of $\{p'_n : n \in \omega\}$.

Then $\mathbf{B}_{1,0}(\eta, U)$ is completely $(N, \pi^M(Q_{\mathbf{T}}), \pi^M(p))$-generic and the construction commutes with the Mostowski collapse and

$$\mathbf{B}_{1,0}(\eta, (\pi^M)^{-1}(U)) := ((\pi^M)^{-1})''\mathbf{B}_{1,0}(\eta, U)$$

is completely $(M, Q_{\mathbf{T}}, p)$-generic. ⊣

Strictly speaking we must write $U = U(M, \in, <_\chi, \mathbb{P}, p, \bar{\beta})$, since by the boundedness theorem (see, e.g., [8, Theorem 31.1]) a cofinal sequence $\bar{\beta}$ cannot be computed in a Borel manner from $(M, \in)$. The arguments $(M, \mathbb{P}, p)$ of U will change during the iteration, and one of the main tasks is to show that all the changes are Borel computable. Fortunately, since in proper forcing $\mathbb{P}$ the ordinary height of N and $N[\mathcal{G}]$ (we use the letters N and $\mathcal{G}$ for the objects after the transitive collapse) are the same for all $(M, \mathbb{P})$-generic filters G, $\pi^M(\bar{\beta})$ will not change and it does not hide features of the proof if we do not write $\bar{\beta}$ during the proof of the iteration theorem. However, $\pi^M(\bar{\beta})$ will be guessed as one component in Lemma 6.3 and will be written there. Since our notation is already heavily burdened, we write only $U(M, \mathbb{P}, p)$ and $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}(\bar{\eta}, \bar{M}, \mathbb{P}, q_0, p)$ until the end of the proof of the Main Lemma.

Lemma 4.4. *$Q_{\mathbf{T}}$ is α-proper for all $\alpha < \omega_1$, and for every α-tower $\langle M_\varepsilon : \varepsilon < \alpha\rangle$ there is a Borel function $\langle \eta_\varepsilon : \varepsilon < \alpha\rangle \mapsto \mathbf{B}^\alpha_{1,0}(\bar{\eta}, \bar{M}, \mathbb{P}, p)$ computing $(\bar{M}, \mathbb{P}, p)$ generic conditions.*

Proof. The upper bound from Lemma 4.3 gives a completely $(M, Q_{\mathbf{T}}, p)$-generic $q \geq p$. Given a tower $\bar{M}$ of countable height α, we can repeat the construction α steps, using a "diagonalised" version of Lemma 4.3 for countably many M_ε, $\varepsilon < \alpha$, and countably many enumerations of dense sets simultaneously, so that in the end we get via $\mathbf{B}^\alpha_{1,0}(\bar{\eta}, \bar{M}, \mathbb{P}, p)$ some q that is $(M_\varepsilon, Q_{\mathbf{T}})$-generic for all $\varepsilon < \alpha$. ⊣

Before we work on the translation of iterated forcings, we consider the atomic step for the oddly indexed iterands separately. There are no completely generic conditions in these steps, since they add reals. However, also we do not need to translate the conditions of these iterands to anything simpler, since each condition is already a (name for a) real. Since we assume that $\mathbb{Q}_1$ is $< \omega_1$-proper, we know that for every tower $\bar{M} = \langle M_\varepsilon : \varepsilon \leq \alpha\rangle$ and $p \in \mathbb{Q}_1 \cap M_0$ there is $q \geq_{\mathbb{Q}_1} p$ that is $(\bar{M}, \mathbb{Q}_1)$-generic. Given a well-order as one of the arguments this transition from p to q is again a Borel function, call it $(\mathbf{B}')^\alpha_{1,0}$. We demand that $q =: (\mathbf{B}')^\alpha_{1,0}$ be the $<_\chi$-least condition $\geq p$ that forces for every $I \in \bigcup_{\varepsilon\leq\alpha} M_\varepsilon$, the statement "$I \cap \underset{\sim}{G} \neq \emptyset$". The definition of $< \omega_1$-properness just says that such a condition exists. For this iteration step, we do not need an argument $\eta \in {}^\omega\omega$ however, we again use a structure prolonging the tower as an additional input to the computation. For uniformity we write (dummy) arguments η_i also in these steps.

Now we want to apply this technique of computing a completely generic condition for an iterand in a countable support iteration where $Q_{\mathbf{T}}$ is interleaved with other $< \omega_1$-proper ${}^\omega\omega$-bounding iterands that do add reals. So, along these lines, let $\mathbb{P}$ for a while stand for the initial segment of the iteration and suppose that $\mathbb{P}$ is $< \omega_1$-proper and ${}^\omega\omega$-bounding. Everything what was done in the previous three lemmas is now given $\mathbb{P}$-names and the iterated version of Lemma 4.4 will give a $\mathbb{P}$-name for some completely generic condition for $(\bar{M}[G], Q_{\mathbf{T}}[G], p[G])$ for some $\mathbb{P}$-generic filter G. We verify that this works.

Lemma 4.5. *The class* $\mathbf{k}$ *with* $\mathbb{P}^{\mathbf{k}}$ *from Definition 1.3 and* $\mathbb{P}^{\mathbf{k}}_*$ *being the cs iteration of the odd iterands is invariantly atomically good. If the odd iterands are nep, then the image* $\mathbb{P}_*$ *is nep.*

Proof. In the terms of Definition 3.11, $\gamma_1 = \gamma$ and $\gamma_2 = \gamma + 1$. The definition of q_γ is by induction and we show how to combine the atomic functions $\mathbf{B}^\alpha_{0,1,0}$ in the successor step in order to compute for every $\gamma_0 < \gamma$ and q_0 that is $(\bar{M}, \mathbb{P}_{\gamma_0}, p)$-generic and every $p \in \mathbb{P}_\gamma \cap M$ with $q_0 \Vdash p \restriction \gamma_0 \in \underset{\sim}{G}_{\gamma_0}$ an $(\bar{M}, \mathbb{P}_\gamma, p)$-generic condition $q_\gamma = \mathbf{B}_{\gamma_0,\gamma,0}(\bar{\eta}, U)$ that is longer than q_0 and at least as strong as p. Once the induction is performed, we shall set $\gamma_0 = 0$, $p_{\gamma_0} = \{0_{\mathbb{P}_0}\}$. There will be two main cases in this definition: γ successor and γ limit, and likewise there will be two cases in the proofs that $\mathbf{B}_{\gamma_0,\gamma,1}$ translates in the desired manner. So in this lemma γ is a successor. When looking at complexity, we regard q_0 as a parameter.

There are two kinds of atomic steps: first, $\mathbb{Q}_\gamma = Q_{\mathbf{T}}$ is an NNR iterand. Then we compute $(\bar{M}, Q_{\mathbf{T}}, p)$-generic conditions as worked out in the Borel function $\mathbf{B}^\alpha_{1,0} = \mathbf{B}^\alpha_{0,1,0}$ above. (The index 1 stands for the iteration length.) Then we define $g = \mathbf{B}^\alpha_{0,1,1}$, the translation function. We use one coordinate $\eta \in {}^\omega\omega$ and one helper model $M_{\alpha+1}$ at the top of the tower for this. Let $\gamma_0, \gamma \in M_0$. Let $\bar{M} = \langle M_\beta : 0 \le \beta \le \alpha + 1$ be of height $\alpha + 2$, with $\alpha = \mathrm{otp}([\gamma_0, \gamma) \cap M_0)$. We assume that q_0 is $(\bar{M}, \mathbb{P}_{\gamma_0}, p)$-generic for this tower. We first give the formulae and then we prove that they work. We let $\bar{\eta} = \langle \eta_\beta : \beta \le \alpha \rangle$. The proof is simultaneously for all $\alpha < \omega_1$. The induction trick is as follows: We are given a tower $\langle M_\varepsilon : \varepsilon \le \alpha + 1 \rangle$ as an argument for the computation. We apply the already established function for $\mathbb{P}_\gamma$ to a shifted tower $\langle M_{\varepsilon+1} : \varepsilon \le \alpha \rangle$ and thus get a sufficiently strong starting point q_γ to add a further iteration step.

First case: $\mathbb{Q}_\gamma = Q_{\mathbf{T}}$ is a NNR iterand. Let $\alpha + 1 \ge [\gamma_0, \gamma + 1) \cap M_0$. The proper translation is: By hypothesis on γ, we have $\mathbf{B}^\alpha_{\gamma_0\gamma,0}(\langle \eta_{\varepsilon+1} : \varepsilon \le \alpha \rangle, \langle M_{\beta+1} : \beta \le \alpha \rangle, \mathbb{P}_\gamma, q_0, p \restriction \gamma) =: q_\gamma$. Now q_γ is an $(M_{\alpha+1}, \mathbb{P}_\gamma, p \restriction \gamma)$-generic condition. Take $G \in M_{\alpha+1}$ that is $(M_\alpha, \mathbb{P}_\gamma, p)$-generic and compatible with q_γ.

$$\mathbf{B}^{\alpha+1}_{\gamma_0,\gamma+1,0}(\bar{\eta}, \bar{M}, \mathbb{P}_{\gamma+1}, q_0, p) = q_\gamma \hat{}\, \mathbf{B}^{\alpha+1}_{1,0}(\bar{\eta}, \bar{M}[G], \underset{\sim}{\mathbb{Q}}_\gamma[G], p(\gamma)[G]) \tag{4.2}$$

gives a generic condition, and the translating function g is

$$\begin{aligned} &\mathbf{B}^{\alpha+1}_{\gamma_0,\gamma+1,1}(\bar{\eta}, \bar{M}, \mathbb{P}, q_0, p) \\ &= \mathbf{B}^\alpha_{\gamma_0,\gamma,1}(\langle \eta_{\varepsilon+1} : \varepsilon \le \alpha \rangle, \langle M_{\varepsilon+1} : \varepsilon \le \alpha \rangle, \mathbb{P}_\gamma, q_0, p \restriction \gamma). \end{aligned} \tag{4.3}$$

So we just drop the last coordinate!

Second: $\mathbb{Q}_\gamma$ is a ${}^\omega\omega$-bounding $< \omega_1$-proper iterand with countable conditions. First compute q_γ and G as above.

$$\mathbf{B}^{\alpha+1}_{\gamma_0,\gamma+1,0}(\bar{\eta}, \bar{M}, \mathbb{P}_{\gamma+1}, q_0, p) = q_\gamma \hat{}\, \mathbf{B}'^{\alpha+1}_{1,0}(\bar{\eta}, \bar{M}[G], \underset{\sim}{\mathbb{Q}}_\gamma[G], p(\gamma)[G]) \tag{4.4}$$

gives a generic condition, and the translating function g is

(4.5)

$$\mathbf{B}^{\alpha+1}_{\gamma_0,\gamma+1,1}(\bar\eta,\bar M,\mathbb{P},q_0,p) =$$
$$\mathbf{B}^{\alpha}_{\gamma_0,\gamma,1}(\langle \eta_{\varepsilon+1} : \varepsilon\le\alpha\rangle,\langle M_{\varepsilon+1} : \varepsilon\le\alpha\rangle,\mathbb{P}_\gamma,q_0,p\restriction\gamma))\hat{}(\mathbb{P}_{*,\gamma}\text{-name of } p(\gamma)).$$

So we just let the last coordinate stand and translate its weights $p'\in\mathbb{P}_\gamma\cap M_0$ by the previous translator to $\mathbf{B}^{\alpha}_{\gamma_0,\gamma,1}(\bar\eta'\bar M',\mathbb{P}_\gamma,q_0,p')$.

We show that in the case of odd γ, that is of $\mathbb{P}_{\gamma+1}=\mathbb{P}_\gamma * \underset{\sim}{Q}_{\mathbf{T}}$, there is a $(M_\alpha[G_\gamma],\mathbb{Q}_\gamma[G_\gamma])$-generic filter $G(\gamma)$ such that $q_\gamma \Vdash$ " $q(\gamma)$ bounds $G(\gamma)$", where $\underset{\sim}{G}(\gamma)$ the canonical name for the $\underset{\sim}{\mathbb{Q}}_\gamma[G_\gamma]$-generic filter. This will guarantee the properties (f) to (j) of $\mathbf{B}_{\gamma+1}$ being a successful solution if $\mathbf{B}_\gamma$ was successful. For this aim, we performed this transition to a higher step in the tower.

The following technique is for *two step iteration in case last step is* $\mathbb{D}$*-complete* is adapted from [1, pages 58–61]. Now we work on the atomic steps for the translation function g, towards the properties of Def. 3.9. Let $\mathbb{P}$ be a poset and let $\underset{\sim}{\mathbb{Q}}=\underset{\sim}{Q}_{\mathbf{T}}\in\mathbf{V}^{\mathbb{P}}$ be a name forced by $0_{\mathbb{P}}$ to be a poset. Let χ be sufficiently large and regular (as said, $\chi=(2^{\aleph_2})^+$ is always sufficiently large) and $\langle M_i : i\le\alpha+1\rangle \prec (H(\chi),\in,<_\chi)$ be a tower of countable elementary submodel such that $\mathbb{P}$, $\underset{\sim}{Q}_{\mathbf{T}}\in M_0$. Henceforth we write just $H(\chi)$ instead of $(H(\chi),\in,<_\chi)$. We want to guarantee that

(1) the condition $(q_0,q_1)\in\mathbb{P}*\underset{\sim}{Q}_{\mathbf{T}}$ is $(M_\alpha,\mathbb{P}*\underset{\sim}{\mathbb{Q}})$-generic, and
(2) q_0 is generic over $(M_\alpha,\mathbb{P})$, and over $(M_{\alpha+1},\mathbb{P})$,
(3) and q_0 forces that q_1 is completely generic over $(M_\alpha[\underset{\sim}{G}_0],\underset{\sim}{Q}_{\mathbf{T}})$,
(4) for every $(V,\mathbb{P})$ generic filter G_0 containing q_0 there is an $(M_\alpha[G_0],\underset{\sim}{\mathbb{Q}}[G_0])$-generic filter G_1 that is bounded by q_1.

Now we write only M_0, M_1 instead of the top of a tower. Given a countable $M_0\prec H(\chi)$ such that the two step iteration $\mathbb{P}*\underset{\sim}{Q}_{\mathbf{T}}$ is in M_0, we extend every $(M_0,\mathbb{P})$-generic condition q_0 to an $(M_0,\mathbb{P}*\underset{\sim}{\mathbb{Q}})$-generic condition such that ${q'}_0\Vdash q_1$ is completely $(M_0[\underset{\sim}{G}_0],\underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}])$-generic. This is done with the help of an additional argument M_1. We strengthen q_0 to ${q'}_0$ that is also $(M_1,\mathbb{P})$-generic. In this respect and also at another point the definition depends not only on M_0 but also the countable elementary submodel $M_1\prec H(\chi)$ such that $M_0\in M_1$. We take $G_0\in M_1$ that is $(M_0,\mathbb{P})$-generic such that ${q'}_0$ is compatible with G_0. We use a tower $M_0\prec M_1\prec H(\chi)$. M_1 will help us to collect sufficiently many $(M_0,\mathbb{P})$-generic conditions thus that we can establish properties (1) to (4). So we carry on towers, since in this successor step we are using height 1 of the tower and in the limit steps we shift the genericity through the tower. We write only (M_0,M_1) but of course there could be towers of arbitrary countable height and the two would be the top of the tower.

In addition, we fix a $p\in\mathbb{P}*\underset{\sim}{Q}_{\mathbf{T}}$ which we want to extend by the seeked condition (q_0,q_1). In the following we write $\mathbf{B}^\alpha$ for $\mathbf{B}^\alpha_{0,1,0}$. We start already with an M_1-generic q_0. The following definition is used for the iterands $\mathbb{Q}_{\mathbf{T}}$.

Definition 4.6. *Let χ be sufficiently large and $M_0\prec M_1\prec(H(\chi),\in,<_\chi)$ be countable elementary submodels with $M_0\in M_1$ and $\mathbb{P}$, $\underset{\sim}{Q}_{\mathbf{T}}$, $p=(p_0,p_1)\in M_0$,*

q_0 $(M_0, \mathbb{P}, p_0)$-generic and $(M_1, \mathbb{P}, p_0)$-generic. Let $\mathbb{P}$ be a $< \omega_1$-proper ${}^\omega\omega$-bounding poset and suppose that $\underset{\sim}{Q}_{\mathbf{T}}$, $\underset{\sim}{\mathbf{B}} \in \mathbf{V}^{\mathbb{P}}$ are such that

$q_0 \Vdash_{\mathbb{P}}$ *for η dominating all branches of $T'_{<\omega_1 \cap M_0[\underset{\sim}{G}_0]}$*

$\underset{\sim}{\mathbf{B}}^\alpha_{0,1,0}(\eta, M_0[\underset{\sim}{G}_0], \underset{\sim}{Q}_{\mathbf{T}}[G_0], p_1[G_0])$

computes a completely $(M_0[\underset{\sim}{G}_0], \mathbb{Q}[\underset{\sim}{G}_0], p_1[\underset{\sim}{G}_0])$-generic condition.

We fix an $\eta \geq^* M_1$. Now choose G_0 such that q_0 compatible with G_0 and G_0 is $(M_0, \mathbb{P}, p_0)$-generic and $G_0 \in M_1$. Let $p \in \mathbb{P} * \underset{\sim}{\mathbb{Q}} \cap M_0$ be given $p = (a, \underset{\sim}{b})$ with $a \in G_0$. Then we define

$$q_1 = \mathbf{B}^{\alpha+1}_{0,1,0}(\eta, M_0[G_0], \underset{\sim}{\mathbb{Q}}[G_0], p_1[G_0]),$$

an $(M_0, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic condition containing stronger than p by the following procedure:

Let $\pi^{M_1} \colon M_1 \to N_1$ with $\pi^{M_1}(M_0) = N_0$ be the Mostowski collapse and $\mathfrak{q}_0 = (\pi^{M_1})'' q_0$. We let $G_0 \in M_1$ be an $(M_0, \mathbb{P}, p_0)$-generic filter that is compatible with q_0. Let $Q^*_0 = Q[G_0]$.

Moreover, since $\tilde{\mathbf{B}}_{0,1,i}$ is invariant, the procedure commutes with the Mostowski collapse. $\mathfrak{q}_0 = (\pi^{M_1})'' q_0$. We let $\mathcal{G}_0 = \pi^{M_1}(G_0)$ Form $N^*_0 = N_0[\mathcal{G}_0]$. Observe that $N^*_0 \in N_1$.

Thus q_1 is defined in N_1, where $b^* = \underset{\sim}{b}[G_0]$ is a condition in Q^*_0. Since M_1 is countable and since η dominates all branches of $\underset{\sim}{T}[G_0]_{<(\omega \cap M_0[G_0])}$ for all $G_0 \in M_1$ and for all evaluations of the book-keeping we have that $\mathbf{B}$ works for all possible $\mathbf{T}[G_0]$, G_0 and hence

$$q_0 \hat{}\, q_1 \text{ is } (M_0, \mathbb{P} * \mathbb{Q})\text{-generic.} \tag{4.6}$$

Moreover,

$$q_1 \text{ bounds an } (M^*_0, Q^*_0, p)\text{-generic filter and } b^* \leq q_1. \tag{4.7}$$

We define q_1 in $H(\chi)$. We cannot take the above definition verbally, because it relies on the assumption that M_0 and M_1 are elementary substructures of $H(\chi)$, something which is not expressible in $H(\chi)$. Whenever the definition above relies on some fact that happens not to hold we let G have an arbitrary value. For example if M^*_0 is not in M_1 then we let G be some arbitrary fixed M_0-generic filter. The Borel computation does invoke M_1, since we use M_1 to collect sufficiently many possible isomorphic types of $(M_0[G_0], \underset{\sim}{Q}_{\mathbf{T}}[G_0], \underset{\sim}{b}[G_0])$. Here, G_0 is a parameter and will be set $\{0_{P_0}\}$ later, so that in the end (that means in Lemma 6.3) only the possible isomorphism types of $(M_0, \in \restriction M_0, <_\chi \restriction M_0, P_\gamma, p, \bar{\beta})$ need to be guessed stationarily often alongside with names for the F and f from the statement of the weak diamond.

The following lemma shows the second part of the argument: We want to show the (q_0, q_1) given in Equation (4.7) has properties (1) to (4). This is used in $\mathbf{B}_{\gamma_0,\gamma,1}$ that replaces q_1 by a $\mathbb{P}_*$ name for q_1 and uses that q_1 determines the coordinate that is left out in the forcing $\mathbb{P}_*$. We show that the coordinates that are dropped in the translation function g have completely generic conditions and that the supremum in Definition 3.8(g) exists. Still $\mathbf{B}$ is $\mathbf{B}^\alpha_{0,1,0}$.

The point is to get a name for a completely generic condition for the $Q_{\mathbf{T}}$-iterands so that the proper translation works.

Lemma 4.7. *Compare to* [1, Lemma 5.20, the Gambit Lemma]. *Let $\mathbb{P}$ be a proper ${}^\omega\omega$-bounding poset and suppose that $\underset{\sim}{\mathbb{Q}}$, $\underset{\sim}{\mathbf{B}} \in \mathbf{V}^{\mathbb{P}}$ are such that $p = (p_0, \underset{\sim}{p_1}) \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and*

$$q_0 \Vdash_{\mathbb{P}} \underset{\sim}{\mathbf{B}} \text{ is an invariant ord-hc Borel function that } \mathbf{B}(\eta, (M_0[\underset{\sim}{G}], \underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}], p[\underset{\sim}{G}]) \text{ computes a completely } (M_0[\underset{\sim}{G}], \underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}], p[\underset{\sim}{G}])\text{-generic condition } \underset{\sim}{q_1} \geq \underset{\sim}{p_1}.$$

Let χ be sufficiently large and $M_0 \prec M_1 \prec H_\chi$ be countable elementary submodels with $M_0 \in M_1$ and $\mathbb{P}$, $\underset{\sim}{\mathbb{Q}}$, $\underset{\sim}{\mathbf{B}}, (p_0, \underset{\sim}{p_1}) \in M_0$. Suppose that $q_0 \in \mathbb{P}$ is $(M_0, \mathbb{P}, p \restriction \gamma_0)$-generic and $(M_1, \mathbb{P}, p \restriction \gamma_0)$-generic, and let $G_0 \subseteq M_0 \cap \mathbb{P}$ be an $(M_0, \mathbb{P}, p_0)$-generic filter such that q_0 is compatible with G_0. Let $(p_0, p_1) = (a, \underset{\sim}{b})$ and $a \leq q_0$. Then there is $q_1 \in \mathbf{V}^{\mathbb{P}}$ such that (q_0, q_1) is generic over $(M_0, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$ and $p_0 \leq (q_0, q_1)$ and there is G_1 that is $(M[G_0], \underset{\sim}{Q_{\mathbf{T}}}[G_0], \underset{\sim}{p_1}[G_0])$-generic such that

$$q_0 \Vdash_{\mathbb{P}} q_1 \text{ bounds the } (M_0[\underset{\sim}{G}], \underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}], p[\underset{\sim}{G}])\text{-generic filter } G_1.$$

Proof. Let G_0 be some $(M_0, \mathbb{P}, p)$-generic filter such that q_0 is compatible with G_0 and $G_0 \in M_1$. The following computation depends on G_0 but in the end it just shows that no $q_0' \geq q_0$ can force the contrary, and this is sufficient.

Let $\pi^{M_1} \colon M_1 \to N_1$, $\pi^{M_1}(M_0) = N_0$, be the transitive collapse and $\mathcal{G}_0 = (\pi^{M_1})''G_0$. We recall the computation of the invariant function $\mathbf{B}(\eta, M_0[G_0], \underset{\sim}{\mathbb{Q}}[G_0], p_1[G_0])$. Form $N_0^* = N_0[\mathcal{G}_0]$ and let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and let $\mathbf{B} = \pi^{M_1}(\underset{\sim}{\mathbf{B}})[\mathcal{G}_0]$. Then $\mathbf{B} \in N_0^*$. Thus $\mathbf{B}(\eta, N_0^*, Q_0^*, b^*)$ is defined in N_1, where $b^* = \pi^{M_1}(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in $\mathbb{Q}_0^*$. Since N_1 is countable and η dominates N_1 and hence N_0^* (independently of the choice of G_0, here we use that $\mathbb{P}$ is ${}^\omega\omega$-bounding) we have

$$\mathbf{B}(\eta, N_0^*, \mathbb{Q}_0^*, b^*) \text{ is a completely } (N_0^*, \mathbb{Q}_0^*, b^*)\text{-generic condition.} \tag{4.8}$$

(This is a crucial step.) We define q_1 as $(\pi^{M_1})^{-1}(\mathbf{B}(\eta, N_0^*, \mathbb{Q}_0^*, b^*))$.

Let $\underset{\sim}{G} \in \mathbf{V}^{\mathbb{P}}$ be the canonical name of the generic filter over $\mathbb{P}$. Then q_0 forces that π^{M_1} can be extended to a collapse $\underset{\sim}{\pi}^{M_1}$ which is onto N_0^*, that is

$$q_0 \Vdash_{\mathbb{P}} \underset{\sim}{\pi}^{M_1} \colon M_0[\underset{\sim}{G}] \to N_0^*.$$

The conclusion of our lemma follows if we show that

$$q_0 \Vdash_P q_1 \text{ bounds } \mathbf{B}_{1,0}(\eta, M_0[\underset{\sim}{G}], \underset{\sim}{\mathbb{Q}}[\underset{\sim}{G}], p). \tag{4.9}$$

So let F be $(\mathbf{V}, \mathbb{P})$-generic with $q_0 \in F$. $\underset{\sim}{\pi}[F]$ collapses $M_0[F]$ onto N_0^* and there is a function η' dominating $M_0[F]$, $\eta' \in M_1[F]$ and since $\mathbb{P}$ is ${}^\omega\omega$-bounding $\eta' \in M_1$, $F \in A'_\eta$. and hence η dominating M_1, $\eta \geq \eta'$ and so $\mathbf{B}(\eta, N_0^*, Q_0^*, b^*)$ is bounded in $\underset{\sim}{Q}[F]$ and η is independent of F. This proves equation (4.9). $\dashv_{4.7}$

End of the proof of the atomic step:

Given $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$, $\gamma_0 \leq \gamma$, $\alpha \geq \mathrm{otp}([\gamma_0, \gamma) \cap M_0)$ we define $\mathbf{B}^{\alpha+1}_{\gamma_0,\gamma+1,i}$ as in equations (4.2), (4.3), (4.4), (4.5) the list of requirements in Definitions 3.8 and 3.9

is fulfilled and we know that $\mathbf{k}$ is invariantly atomically good. The properties required for g Definition 3.9 follow in the induction step where the translation says: drop the last $Q_{\mathbf{T}}$-coordinate: Just go into the extension by $M[G_0, G_1]$ to show that in G_1 there are no incompatibilities and that if we go in the reverse direction: From any dense set of $\mathbb{P}_*$-conditions by adding the completely generic condition q as the dropped condition we get a dense set in $\mathbb{P}$ above the generic q.

$\dashv_{4.5}$

5. Proof of the Main Lemma

We show that for limit γ, we can find ord-hc Borel definitions for functions $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$ that are now desribed axiomatically as in the premise of the Main Lemma. We show that $r = \sup \mathbf{B}^\alpha_{\gamma_0,\gamma,1}[\{p \in \mathbb{P}_\gamma \cap M_0 : p \leq_{\mathbb{P}} q]$ as in item (β) of Definition 3.8(g) exists if we used a sufficiently high tower $\alpha \geq M_0 \cap [\gamma_0, \gamma)$. We inductively prove the existence of $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$. The induction is on $\alpha = \mathrm{otp}[\gamma_0, \gamma) \cap M_0$. We have a starting condition q_0 in $\mathbb{P}_{\gamma_0}$.

First we have a closer look at the Existential Completeness Lemma, since it will be invoked at several steps of our inductive computations.

Lemma 5.1. *(The Existential Completeness Lemma* [14, Lemma I 3.1]*, also called the Maximal Principle) If $q_0 \Vdash \exists x \varphi(x)$ then there is a name $\underset{\sim}{\tau}$ such that $q_0 \Vdash \varphi(\underset{\sim}{\tau})$, where $\varphi(x)$ is a formula which may mention names.* $\dashv$

Of course, the proof of this lemma uses the axiom of choice. So we will again use $<_\chi$ to make the choices definable and get an invariant ord-hc Borel function with arguments $M, \mathbb{P}, q_0, \varphi$ that computes a witness $\underset{\sim}{\tau}$: Let $M \prec H(\chi)$ and let $\pi^M(M) = N$ be the Mostowski collapse. Let $q_0, \mathbb{P}, \bar{\sigma} \in M$ and $q_0 \Vdash \exists x(\varphi(x) \wedge x \in M)$. Ord-collapse or collapse everything. Then there is Borel computation of a witness $\underset{\sim}{\tau}$: Just by induction on $<_\chi$ we choose step for the step the elements of a maximal antichain $A \subseteq D = \{q \in M : q \Vdash \not\exists x \varphi(x)$ or for some name $\underset{\sim}{\tau} \in M$, $q \Vdash \phi(\tau)\}$. Then we take for $q \in A$ a minimal witness $\underset{\sim}{\tau} \in M$ such that $q \Vdash \varphi(\underset{\sim}{\tau})\}$. Then we glue the $\{(q, \underset{\sim}{\tau}(q)) : q \in A\}$ together to one name. We will use this for ϕ that are statements about Borel functions from previous induction steps.

For carrying on $(\alpha+1)$-properness over a limit step γ the (regular) Properness Extension Lemma is used. Now we recall this lemma and verify that the proof of existence leeds to a Borel function computing a witness.

Lemma 5.2. *(The Properness Extension Lemma* [1, Lemma 2.8]*) Let $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : j < \gamma, i \leq \gamma \rangle$ be a countable support iteration of proper posets. Let λ be a sufficiently large cardinal. Let M be a countable elementary substructure of $H(\chi)$ with γ, $\mathbb{P}_\gamma \in M$. For every $\gamma_0 \in \gamma \cap M_0$ and $q_0 \in \mathbb{P}_{\gamma_0}$ that is $(M, \mathbb{P}_{\gamma_0})$-generic the following holds:*

If $\underset{\sim}{p} \in \mathbf{V}^{\mathbb{P}_{\gamma_0}}$ is such that $q_0 \Vdash_{\gamma_0} \underset{\sim}{p} \in \mathbb{P}_\gamma \cap M \wedge \underset{\sim}{p} \restriction \gamma_0 \in \underset{\sim}{G}_0$, then there is an $(M, \mathbb{P}_\gamma)$-generic condition q such that $q \restriction \gamma_0 = q_0$ and $q \Vdash_\gamma \underset{\sim}{p} \in \underset{\sim}{G}$ (where $\underset{\sim}{G}$ is *the canonical name of the generic filter over* $\mathbb{P}_\gamma$ *and the name* $\underset{\sim}{p}$ *is now viewed as a member of* $\mathbf{V}^{\mathbb{P}_\gamma}$*).*

Proof. The name $\underset{\sim}{p}_0$ is not necessarily in M but it is forced by q_0 to be a condition in $\mathbb{P}_\gamma \cap \tilde{M}$. Since we work above q_0 we can assume that $\underset{\sim}{p}_0 = \{(r, \sigma) : \sigma \in M, r \in A(\sigma) \cap M\}$ where the the $A(\sigma)$ are maximal antichains in $\mathbf{V}$. So $\underset{\sim}{p}_0$ is a predicate on M. Borel computable means Borel computable in these parameters.

Let $\{D_n : n \in \omega\}$ be an $<_\chi$-increasing enumeration of the dense subsets of $\mathbb{P}_\gamma$ that are in M.

We define by induction on $n < \omega$ a name $\underset{\sim}{p}_n \in \mathbf{V}^{\mathbb{P}_{\gamma_n}}$ and a condition $q_n \in \mathbb{P}_{\gamma_n}$ such that

1. $q_0 \in \mathbb{P}_{\gamma_0}$ is the given condition. For $n \geq 0$, $q_{n+1} \in \mathbb{P}_{\gamma_n}$ is $(M, \mathbb{P}_{\gamma_{n+1}})$-generic and $q_{n+1} \restriction \gamma_n = q_n$.
2. $\underset{\sim}{p} = \underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_{n+1}$ is a $\mathbb{P}_{\gamma_n}$-name such that
   $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p}_{n+1}$ is a condition in $\mathbb{P}_\gamma \cap M$ such that
   (a) $\underset{\sim}{p}_{n+1} \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,
   (b) $\underset{\sim}{p}_n \leq_\gamma \underset{\sim}{p}_{n+1}$,
   (c) $\underset{\sim}{p}_{n+1}$ is in D_n.

Assume that q_n and $\underset{\sim}{p}_n$ have been constructed. We define $\underset{\sim}{p}_{n+1}$ as a $\mathbb{P}_{\gamma_n}$-name by the following requirements: Imagine a generic extension $\mathbf{V}[G_n]$ made by $\mathbb{P}_{\gamma_n}$ such that $q_n \in G_n$. Then $\underset{\sim}{p}_n[G_n] \in M \cap \mathbb{P}_\gamma$ and $\underset{\sim}{p}[G_n] \restriction \gamma_n \in G_n$. Since q_n is $(M, \mathbb{P}_{\gamma_n})$-generic, $q_n \Vdash (\exists \underset{\sim}{p}_{n+1} \geq \underset{\sim}{p}_n) \wedge p_{n+1} \in D_n \wedge p_{n+1} \restriction \gamma_n \in G_n)$. Now take the existential completeness lemma and invariantly ord-hc Borel compute $\underset{\sim}{p}_{n+1}$. Now that $\underset{\sim}{p}_{n+1}$ is defined apply the inductive asumption to γ_n and to γ_{n+1} and to q_n and $\underset{\sim}{p}_n + 1 \restriction \gamma_{n+1}$. This gives $q_{n+1} = \mathbf{B}^1_{\gamma_n, \gamma_{n+1}, 0}(M, \mathbb{P}, q_n, \underset{\sim}{p}_{n+1} \restriction \gamma_{n+1})$ that satisfies the required inductive assumptions. Now we define $r = \bigcup_{n \in \omega} q_n = \mathbf{B}_{\gamma_0, \gamma, 0}(M, \mathbb{P}, q_0, p)$. ⊣

Now we use towers of models to carry on the property of $< \omega_1$-properness. We recall the α-Extension lemma:

Lemma 5.3. [1, Lemma 5.6] *Let* γ *be a countable ordinal and* $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : j < \gamma, i \leq \gamma \rangle$ *be a countable support iteration of* α*-proper posets. Let* λ *be a sufficiently large cardinal. Let* $\bar{M} = \langle M_\xi : \xi \leq \alpha \rangle$ *be an* $\alpha + 1$*-tower of countable elementary substructures of* $H(\lambda)$ *with* γ, $\mathbb{P}_\gamma$, $\alpha \in M_0$. *For every* $\gamma_0 \in \gamma \cap M_0$ *and* $q_0 \in \mathbb{P}_{\gamma_0}$ *that is* $(\bar{M}, \mathbb{P}_{\gamma_0})$*-generic the following holds:*

If $\underset{\sim}{p}_0 \in \mathbf{V}^{\mathbb{P}_\gamma}$ *is such that* $q_0 \Vdash_{\gamma_0} \underset{\sim}{p}_0 \in \mathbb{P}_\gamma \cap M_0 \wedge \underset{\sim}{p}_0 \restriction \gamma_0 \in \underset{\sim}{G}_0$, *then there is an* $(\bar{M}, \mathbb{P}_\gamma, p)$*-generic condition* q *such that* $q \restriction \gamma_0 = q_0$ *and* $q \Vdash_\gamma \underset{\sim}{p}_0 \in \underset{\sim}{G}$ *(where* $\underset{\sim}{G}$ *is the canonical name of the generic filter over* $\mathbb{P}_\gamma$ *and the name* $\underset{\sim}{p}_0$ *is now viewed as a member of* $\mathbf{V}^{\mathbb{P}_\gamma}$*).* ⊣

Now in order to establish the transition of p_0, q_0, $\bar{\eta}$ and the $\bar{M}$, $\mathbb{P}_\gamma$ to q as a Borel function (based on the hypothesis that there are already Borel functions for shorter iteration lengths) we collect some items from the proof of the α-Extension Lemma:

Now we turn to computing $\mathbf{B}_{\gamma_0,\gamma,1}$ and mapping into $\mathbb{P}_*$ while keeping dense sets. For this we need that the coordinates $p(\delta)$, $\delta < \gamma$, in $\mathbb{P}_\gamma$ that are dropped by the function $\mathbf{B}^\alpha_{\gamma_0,\gamma,1}$ in $\mathbb{P}_{*,\gamma}$ have completely generic conditions, that is conditions $q(\delta)$ determining an $(M_0[G_\delta], \mathbb{Q}_\delta)$ generic filter. The existence of completely generic conditions is equivalent to not adding reals. We show that "all the g-images of the generic conditions are bounded in $\mathbb{P}_*$" is preserved in the limit steps of the iteration and that g (or rather g'') maps dense subsets of $\mathbb{P}$ that are in M to dense subsets in M and that g^{-1} does the same. This is a combination of Abraham's proof of the Extension Lemma [1, The Extension Lemma] for $\mathbb{D}$-complete iterands with the function resulting from Lemma 4.5.

Lemma 5.4. *Let $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma'', \gamma \leq \gamma''\rangle$ be a countable support iteration of forcing posets such that each iterand $\mathbb{Q}_\alpha$ satisfies the following in $\mathbf{V}^{\mathbb{P}_\alpha}$:*

(1) $\mathbb{Q}_\gamma$ is δ-proper for every countable δ.

(2) $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$ exist for each $\gamma < \gamma''$ with the properties of Definitions 3.8 and 3.9.

Suppose that $M_0 \prec H(\chi)$ is countable, $\mathbb{P}_\gamma \in M_0$ and $p_0 \in \mathbb{P}_\gamma \cap M_0$. For any $\gamma_0 \in \gamma \cap M_0$ with $\alpha = \operatorname{otp}(M_0 \cap [\gamma_0, \gamma))$, if $\bar{M} = \langle M_\xi : \xi \leq \alpha\rangle$ is a tower of countable elementary substructures starting with the given M_0, then the following holds: Then $\mathbf{B}^\alpha_{\gamma_0,\gamma'',i}$ exists and has the properties from Defs. 3.8 and 3.9.

Let $\mathbb{P}_\gamma$ be a countable support iteration of length γ, γ a limit, with iterands $\mathbb{Q}^{\mathbf{k}}_\alpha \in \mathbf{V}^{\mathbb{P}^{\mathbf{k}}_\alpha}$ that come from a class of candidates. If $\operatorname{cf}(\gamma) > \omega$ we let $\gamma_n = \sup(\gamma \cap M_{\alpha_n})$ for $n \geq 0$. That is, each $\mathbb{P}_{\gamma_n}$, $\alpha_n = \operatorname{otp}(M_0 \cap [\gamma_0, \gamma_n)$, has its $\mathbf{B}^{\alpha_n}_{\gamma,0,\gamma_n,i}$ witnessing that $\mathbf{k}$ is invariantly good up to γ_n and for γ itself we assume that the functions below α are already established.

Now we define $\mathbf{B}^\alpha_{\gamma_0,\gamma,i}$.

Let χ be a sufficiently large regular cardinal. For $\mathbf{B}^\alpha_{\gamma_0,\gamma,0}$ we first describe a machinery for obtaining generic conditions over (transitive collapses or ord-transitive collapses) of countable submodels of $H(\chi)$. We define a function $\mathbf{B}^\alpha_{\gamma_0,\gamma,0}$ that takes five arguments, $\bar{M} \restriction [1,\alpha], \mathbb{P}_\gamma, q_0, p$ of the following types.

1. $M_0 \prec H_\chi$ is countable, $\mathbb{P}_\gamma \in M_0$, so $\gamma \in M_0$. Moreover, $p \in M_0 \cap \mathbb{P}_\gamma$.
2. $\gamma_0 \in M_0 \cap \gamma$, q_0 is an $(M_0, \mathbb{P}_{\gamma_0})$-generic condition and such that $q_0 \Vdash p \restriction \gamma_0 \in \underset{\sim}{G}_{\gamma_0}$. We assume that $q_0 \in M_1$.
3. The order type of $M_0 \cap [\gamma_0, \gamma)$ is α. $\langle \gamma_n : n \in \omega\rangle$ is a strictly increasing sequence, and $\alpha_{n+1} = \operatorname{otp}[\gamma_n, \gamma_{n+1})$, $\alpha_0 = 0$.
4. $\bar{M} = \langle M_\xi : 0 \leq \xi \leq \alpha\rangle$ is an $\alpha+1$-tower of countable elementary submodels of $H(\chi)$ and $M_0 = M$. Note that only $M_0 = M$ is the domain for the translation. The rest $\langle M_\xi : 1 \leq \xi \leq \alpha\rangle$ of the tower is used to show that the coordinates that we drop have completely generic conditions.

The value returned, $q_\gamma = \mathbf{B}^\alpha_{\gamma_0,\gamma,0}(\bar{\eta}, \langle M_i : i \leq \alpha\rangle, \mathbb{P}_\gamma, q_0, p)$ is an $(\bar{M}, \mathbb{P}_\gamma)$-generic condition that extends q_0 and is stronger than p. $\alpha = \sup\langle \alpha_n : n \in \omega\rangle$ be an increasing cofinal sequence with $\alpha_0 = 0$.

We define

$$q_\gamma = \mathbf{B}^\alpha_{\gamma_0,\gamma,0}(\langle M_i : i \leq \alpha\rangle, \langle \eta_i : i < \alpha\rangle, \mathbb{P}_\gamma, q_0, p)$$

as follows. We define by induction on $n \in \omega$ a condition $p_n \in \mathbb{P}_\gamma \cap M_0$ and an $(\langle M_i : i \leq \alpha_n\rangle, \mathbb{P}_{\gamma_n}, p_n)$-generic condition $q_n \in M_{\alpha_n+1}$ such that

1. $q_0 \in \mathbb{P}_{\gamma_0}$ is the given $(\bar{M}, \mathbb{P}_{\gamma_0})$-generic condition. For $n \geq 0$, $q_{n+1} \in \mathbb{P}_{\gamma_{n+1}}$ is $(\langle M_\xi : \alpha_{n+1} < \xi \leq \alpha\rangle, \mathbb{P}_{\gamma_{n+1}}, p_{n+1})$-generic and $q_{n+1} \restriction \gamma_n = q_n$ and it has the translation property for coordinates $\delta \in [\gamma_n, \gamma_{n+1}) \cap M_{\alpha_n}$, that is there is an $\mathbb{P}_\delta$-generic filter G_δ such that

(5.1)
$$\begin{aligned} q_n \restriction \delta \Vdash_\delta \text{“}q_n(\delta) &\text{ is completely } (M_{\alpha'}[G_\delta], \mathbb{Q}_\delta[G_\delta], p_{n+1}(\delta)[G_\delta])\text{-generic} \\ &\text{for } \alpha' = \mathrm{otp}([\gamma_n, \delta) \cap M_0) \\ &\text{and it bounds an } (M_{\alpha'}[G_\delta], \mathbb{Q}_\delta[G_\delta], p_{n+1}(\delta)[G_\delta])\text{-generic filter } G(\delta)\text{”}, \end{aligned}$$

and

$$q_{n+1} = q_n{}^\frown$$
$$\mathbf{B}^{\alpha_{n+1}-\alpha_n}_{\gamma_n,\gamma_{n+1},0}(\langle \eta_i : i \in [\alpha_n, \alpha_{n+1})\rangle, \langle M_\xi : \alpha_n < \xi \leq \alpha_{n+1}\rangle, \mathbb{P}_{\gamma_{n+1}}, q_n, \underset{\sim}{p}_{n+1} \restriction \gamma_{n+1}).$$

2. $\underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_{n+1}$ is a $\mathbb{P}_{\gamma_n}$-name such that
   $q_n \Vdash_{\gamma_n}$ “$\underset{\sim}{p}_{n+1}$ is a condition in $\mathbb{P}_\gamma$ such that
   (a) $\underset{\sim}{p}_{n+1} \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,
   (b) $\underset{\sim}{p}_n \leq_\gamma \underset{\sim}{p}_{n+1}$,
   (c) $\underset{\sim}{p}_{n+1}$ is $(\langle M_i : \alpha_{n+1} < i \leq \alpha\rangle, \mathbb{P}_\gamma, \underset{\sim}{p}_n)$ generic.

Suppose that q_n and p_n are defined. First we can find p_{n+1} by Lemma 5.3 such that $p_{n+1} \restriction \gamma_n \in G_n$. Again we invoke a computable form of the Existential Completeness Lemma. By the inductive assumption p_{n+1} exists and as above, there is a Borel manner to compute it. Now we let

$$q_{n+1} = q_n{}^\frown\mathbf{B}^{\alpha_{n+1}-\alpha_n}_{\gamma_n,\gamma_{n+1},0}(\langle \eta_i : i \in (\alpha_n, \alpha_{n+1}]\rangle, \langle M_\xi : \xi \leq \alpha_{n+1}\rangle, \mathbb{P}_{\gamma_{n+1}}, q_n, \underset{\sim}{p}_{n+1} \restriction \gamma_{n+1})$$

In the end, $q_\gamma = \bigcup\{q_n : n \in \omega\} = \mathbf{B}^\alpha_{\gamma_0,\gamma,0}(\langle \eta_i : i \in \alpha\rangle, \langle M_\xi : \xi \leq \alpha\rangle, \mathbb{P}_\gamma, q_0, p)$.

Suppose that q_n is defined and has the properties of Defs. 3.8 and 3.9. Let X be in $M_{\alpha_{n+1}+1}$ be a maximal antichain in $\mathbb{P}_{\gamma_n}$ of conditions $r \in G_n$, $G_n \in M_{\alpha_{n+1}+1}$, G_n any $\langle M_\xi : \alpha_n + 1 \leq \xi \leq \alpha_{n+1}\rangle$-generic over $\mathbb{P}_{\gamma_n}$. Observe that X is predense above q_n. For each $r_0 \in X$, define by the induction assumption $r_1 \in \mathbb{P}_{\gamma_{n+1}}$ such that r_1 with the requirements of a proper translation and $r_1 \restriction \gamma_n = r_0$. If $r_0 \in X \cap M_{\alpha_{n+1}+1}$, then r_1 is taken from $M_{\alpha_{n+1}+1}$. Now view $\{r_1 : r_0 \in X\}$ as a name $\underset{\sim}{r}$ for a condition forced by q_n to lie in $M_{\alpha_{n+1}+1}$. By induction hypothesis we can choose η_ξ, $\alpha_{n+1} < \xi \leq \alpha_{n+2}$ that $\leq^*$-dominates M_ξ and then with this argument Borel-define q_{n+1} that satisfies item 1 from the above list and such that $q_{n+1} \Vdash_{P_{\gamma_{n+1}}} \underset{\sim}{r} \in \underset{\sim}{G}_{n+1}$. So the first component of proper translation is carried on to γ_{n+1}. Now $q = \bigcup_{n\in\omega} q_n$, and Def. 3.8 is fulfilled. We define

$$\mathbf{B}^\alpha_{\gamma_0,\gamma,1}(\bar{\eta}, \bar{M}, \mathbb{P}_\gamma, q_0, p) = q.$$

Now the properties of Definition 3.9 that speak only on M_0 follow from (5.1) and the fact that $\mathbb{P}^{\mathbf{k}}_*$ is nep. $\dashv$

6. Diamonds

In this section we show that a countable support iteration of NNR iterands and Sacks iterands fulfils the weak diamonds from Theorem 1.3. The iteration length does not matter for this aim. However, since we are aiming at all Aronszajn trees are special and $2^\omega = \aleph_2$ we will use the iteration length ω_2 in the end.

Let $\mathbf{k}$ be as in Definition 1.3. Now we also use property (b) from there: That the odd stages are Π^1_1-definable and have the Sacks property. We can just use the Sacks forcing itself for these iterands. Let $\mathbb{S}$ denote the Sacks forcing, that is conditions are $p \subset 2^{<\omega}$ that are perfect trees, that is for every $s \in p$ there is some $t \supset s$, $t \in p$ such that $t\hat{}0$, $t\hat{}1 \in p$. Let $\mathbb{S}_\gamma$ denote the cs iteration of Sacks iterands of length γ.

Lemma 6.1. *Suppose that $\mathbb{P}^{\mathbf{k}}_* = \langle \mathbb{P}_{*,\alpha}, \underset{\sim}{\mathbb{Q}}_{*,\beta} : \beta < \gamma(\mathbf{k}), \alpha \le \gamma(\mathbf{k})\rangle$ is an iteration of Sacks iterands and $\mathbf{k} \in \mathbf{K}$ is invariantly good (Def. 3.11).*

(a) $\mathbf{V} \models \diamondsuit_{\omega_1}$,

(b) $\mathcal{I}^{\mathbf{k}}_\alpha = ({}^\omega\omega, \le^*)$.

Then

$$\Vdash_{\mathbb{P}_{\gamma(\mathbf{k})}} (\exists g \colon \omega_1 \to \mathcal{N}/\mathcal{M}/ \text{ thin trees})(\forall \text{ Borel } F \colon 2^{<\omega_1} \to 2^\omega)$$
$$(\forall f \colon \omega \to 2^\omega)(\text{there are stationarily many } \alpha \in \omega_1)$$
$$(F(f \restriction \alpha) \in g(\alpha)).$$

Definition 6.2. *(See* [3, Def 7.2.13]*) Let $g, h \colon \omega \to \omega$, $\lim_n h(n) = \infty$ such that $(\forall k) \lim_{n\to\infty} \frac{h(n)^k}{g(n)} = 0$. A notion of forcing $\mathbb{P}$ has the (g,h)*-bounding property *if $(\forall f \in \mathbf{V}^{\mathbb{P}} \cap \prod_{n\in\omega} g(n))(\exists S \in \mathbf{V} \cap ([\omega]^{<\omega})^\omega)(\forall n)(|S(n)| \le h(n) \wedge f(n) \in S(n))$.*

Laver forcing and Sacks forcing have the (g,h)-bounding property for any (g,h).

Lemma 6.1 will be proved by the following two lemmas. The first lemma is an extension of [11, Lemma 3.11] in which the reals are replaced by $\mathbb{S}_\gamma$-names of reals. In the applications $\alpha < \omega_1$ from the next Lemma will be the ω_2, or in general, the iteration length of the forcing under consideration, of a guessed transitive countable model M_0.

Lemma 6.3. *Suppose that*

(α) $\alpha < \omega_1$, *and*

(β) $\mathbf{B}'$ *is a Borel function from* $(\omega^\omega)^\alpha$ *to* $\mathbb{S}_\gamma$*-names of members of* 2^ω,

(α) $r \colon \omega \to \omega$ *is diverging to infinity, and* $\lim \frac{r(n)}{2^n} = 0$.

Then *we can find some* $\mathcal{S} = \mathcal{S}_{\mathbf{B}'}$ *such that*

(a) $\mathcal{S}$ *is a closed subset of* 2^ω,

(b) $(\forall n)|\{\eta \restriction n : \eta \in \mathcal{S}\}| \leq r(n)$, *so if* $\mathcal{S} = \lim(T) = \{f \in 2^\omega : \forall n f \restriction n \in T\}$, *then* $T \subseteq 2^{<\omega}$ *is a tree with* n*-th level counting less than or equal to* $r(n)$,

(c) in the following game $\partial_{(\alpha,\mathbf{B}')}$ *between two players, IN and OUT, the player IN has a winning strategy, the play lasts* α *moves and in the* ε*-th move OUT chooses* $\nu_\varepsilon \in \omega^\omega$ *and then IN chooses* $\eta_\varepsilon \geq^* \nu_\varepsilon$*. In the end IN wins iff* $\Vdash_{\mathbb{S}_\alpha} \mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \alpha\rangle) \in \mathcal{S}$.

Proof. The (g,h)-bounding property is preserved in countable support iterations [3, p. 340]. Let $g(n) = 2^n$, and $h(n) = \log(r(n))$. First we use that $\mathbb{S}_\alpha$ has the (g,h)-bounding property. For every $\bar{\rho} = \langle \rho_\eta : \eta < \alpha\rangle$ there is a slalom $S_1(\bar{\rho}) \in \mathbf{V}$, $|S_1(\bar{\rho})(n)| \leq h(n)$, $S_1(\bar{\rho})(n) \subseteq 2^n$ such that

$$(\forall n)\mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \alpha\rangle) \restriction n \in S_1(\bar{\rho})(n).$$

An analysis of the proof of this statement with forcing conditions gives that $\bar{\rho} \mapsto S_1(\bar{\rho})$ can be chosen as to be a Borel function as well, call it $\mathbf{B}''$. This is a function to the ground model. For $S_1(\bar{\rho})(n)$ there are $g'(n) = \binom{2^n}{h(n)}$ possibilities.

Assume that $\mathbb{P}^*_\alpha = \langle \mathbb{P}^*_\xi, \underset{\sim}{\mathbb{Q}}^*_\zeta : \xi \leq \alpha, \zeta < \alpha\rangle$ is a c.s. iteration of Laver forcing and assume that $p \in \mathbb{P}^*_\alpha$ and $\langle \underset{\sim}{\rho}_\xi : \xi < \alpha\rangle$ is a sequence of names for the $\mathbb{P}^*_\xi$-generics. Clearly $p \Vdash_{\mathbb{P}^*_\alpha} \mathbf{B}''(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \alpha\rangle) \in 2^\omega$.

The Laver forcing and any forcing not adding reals at all have the (g',h)-bounding property. Hence there are $p \leq p^* \in \mathbb{P}^*_\alpha$ and $\mathcal{S}$ as in (a) and (b) above such that

$$p^* \Vdash_{\mathbb{P}^*_\alpha} \mathbf{B}''(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \alpha\rangle)(n) \in \mathcal{S}(n).$$

Now $\mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \alpha\rangle) \restriction n$ has $h(n) \cdot h(n) = (\log(r(n)))^2$ possibilities for all n, and the $(\log(r(n)))^2 < r(n)$ for almost all n. Now we need to prove part (c) of the Lemma only for $\mathbf{B}''\colon (\omega^\omega)^\gamma \to V \cap \prod_{n\in\omega} g'(n)$.

Now we show that player IN can play with the strategy that imitates the Laver-generic reals over a countable elementary submodel, so that actually everything is in the ground model. $\mathbf{B}''$ is a function to the ground model and hence we now can quote [11, Lemma 3.11]. For completeness we repeat the proof.

Let $M^* \prec (H(\chi), \in)$ be countable such $\mathbf{B}''$, $\mathcal{S} \in M^*$. (So M^* is not the M from the next proof, but rather contains a non-trivial part of the power-set of that M.) Now we prove by induction on $j \leq \alpha$ for all $i < j$

$\boxtimes_{i,j}$ Assume that $\mathbb{P}^*_j \in M^*$ and $G_i \subseteq \mathbb{P}^*_i \cap M^*$ is generic over M^*, and p^* is such that $p^* \in \mathbb{P}^*_j \cap M^*$ and $p^* \restriction i \in G_i$. Then in the following game $\partial^*_{(i,j,G_i,p^*)}$ player II has a winning strategy $\sigma_{(i,j,G_i,p^*)}$. There are $j - i$ moves indexed by $\varepsilon \in [i,j)$, and in the ε-th move $(p_\varepsilon, \nu_\varepsilon, \eta_\varepsilon)$ are chosen such that player I chooses $p_\varepsilon \in \mathbb{P}_\varepsilon/G_i$, $p_\varepsilon \geq p^* \restriction \varepsilon$, and $\nu_\varepsilon \in \omega^\omega$ and player II chooses $\eta_\varepsilon \geq^* \nu_\varepsilon$.

First case: there is a $(\mathbb{P}^*_\varepsilon, M^*)$-generic $G_\varepsilon \subseteq \mathbb{P}^*_\varepsilon \cap M^*$, such that $p^*(\varepsilon) \in G_\varepsilon$ and $G_\varepsilon \supset G_i$ and $(\forall \xi \in [i,\varepsilon) \underset{\sim}{\rho}_\xi[G_\varepsilon] = \eta_\xi$ and $M^*[G_\varepsilon \cap P^*_\xi] \models p_\xi \geq p^*(\xi)$. In this case player I chooses $p_\varepsilon \in G_\varepsilon$ forcing this and so that $M^*[G_\varepsilon] \models p^*(\varepsilon) \leq_{\mathbb{P}^*_\varepsilon} p_\varepsilon$. Then player I chooses ν_ε dominating $M^*[G_\varepsilon]$ and the second player chooses $\eta_\varepsilon \geq^* \nu_\varepsilon$.

Second case: There is no such G_ε. Then player I won the play.

We prove by induction on j that player II wins the game $\Game^*_{(i,j,G_i,p^*)}$: Case 1: $j = 0$. Nothing to do. Case 2: $j = j^* + 1$. For $\varepsilon \in [i, j)$ we use the strategy for $\Game^*_{(i,j,G_i,p^*)}$, and for $\varepsilon = j$ we make the following move: We show that there is a generic G^{j^*} of ${Q^*}_{j^*}^{M^*[G_{j^*}]}$ to which $p^*(j^*)$ belongs and such that $\underset{\sim}{\rho}_{j^*}[G^{j^*}] \geq^* \nu_{j^*}$. Then the move $\underset{\sim}{\rho}_{j^*}[G^{j^*}]$ dominates $\omega^\omega \cap M^*[G_{j^*}]$ and also player I's move ν_{j^*}.

First take $q \geq p^*(j^*)$ such that q is $(M^*[G_{j^*}], {\mathbb{Q}^*}_{j^*}^{M^*[G_{j^*}]})$-generic. $q \in \mathbf{V}$ is a Laver condition. Now we take a stronger condition q' by letting $\mathrm{tr}(q) = \mathrm{tr}(q')$ and for every $s \in q'$ of length n,

$$\mathrm{suc}(q', s) = \{n \in \mathrm{suc}(q, s) \,:\, n \geq \nu_{j^*}(n)\}.$$

Now let $G^{j^*} = \{r \in M^*[G_{j^*}] \,:\, q' \geq r\}$. Since q' is a $(M^*[G_{j^*}], {\mathbb{Q}^*}_{j^*}^{M^*[G_{j^*}]})$-generic condition, G^{j^*} is a $(M^*[G_{j^*}], {\mathbb{Q}^*}_{j^*}^{M^*[G_{j^*}]})$-generic filter. The generic real is $\underset{\sim}{\rho}_{j^*}[G^{j^*}] = \bigcup\{\mathrm{tr}(p) \,:\, p \in G^{j^*}\}$. Then $q' \Vdash \underset{\sim}{\rho}_{j^*} \geq^* \nu_{j^*}$. Now player II takes $\eta_{j^*} = \underset{\sim}{\rho}_{j^*}[G^{j^*}]$. We set $G_j = G_{j^*} * G^{j^*}$. Case 3: j is a limit. Like the proof of the preservation of properness. From the proof of the preservation of properness (see, e.g., Lemma 5.1, [14, Ch. II, Theorem 3.2, Ch. II., Section 3.3, or Ch. XII, Theorem 1.8]) we get that existence of p_ε, so player I can never win the game on the ground of the second case.

The winning condition for player II is preserved in the limit steps, since it is a requirement on all formerly chosen η_ε.

Why does $\boxtimes_{i,j}$ suffice? Use $i = 0$, $j = \alpha$, $\mathbf{B}'' \in M^*$. Take $\mathbb{P}^*_\alpha \in M^*$, $p^* \in \mathbb{P}^*_\alpha \cap M^*$. Let $\sigma(0, \alpha, \{\emptyset\}, p^*)$ be a winning strategy for player II in the game $\Game^*_{(0,\alpha,\{\emptyset\},p^*)}$. During the play of $\Game_{(\alpha,\mathbf{B}'')}$ let ν_ε be chosen in stage $\varepsilon < \alpha$. The player IN simulates on the side a play of $\Game^*_{(0,\alpha,\{\emptyset\},p^*)}$: As a move of I he assumes the ν_ε chosen by OUT in the play of $\Game_{(\alpha,\mathbf{B}'')}$ and p_ε, $p_\varepsilon \restriction \delta = p_\delta$ for $\delta < \varepsilon$, the p_δ gotten from earlier simulations. Then player IN uses $\sigma(0, \alpha, \{\emptyset\}, p^*)$ for player II, applied to $(p_\varepsilon, \nu_\varepsilon)$, to compute an η_ε, which he presents in this move in $\Game_{(\alpha,\mathbf{B}')}$. So p_ε forces that there is a Laver generic $\underset{\sim}{\rho}_\varepsilon[G^\varepsilon] =: \eta_\varepsilon$ over $M^*[G_\varepsilon]$ and that $\eta_\varepsilon \geq^* \nu_\varepsilon$. The requirement $\eta_\varepsilon \geq^* \nu_\varepsilon$ is fulfilled.

Suppose that they have played. So we have $\langle \nu_\varepsilon, \eta_\varepsilon \,:\, \varepsilon < \alpha\rangle$ and there is $p = \bigcup_{\varepsilon<\alpha} p_\varepsilon \geq p^*$, and for $\varepsilon < \alpha$ there is the name for the Q^*_ε-generic real, namely $\underset{\sim}{\rho}_\varepsilon \in M^*$, such that for all $\varepsilon < \alpha$, $p \Vdash_{P^*_\alpha} \underset{\sim}{\rho}_\varepsilon = \check{\eta}_\varepsilon$. So as $p \Vdash_{\mathbb{P}^*_\alpha}$ "$\mathbf{B}''(\langle \underset{\sim}{\rho}_\varepsilon \,:\, \varepsilon < \alpha\rangle) \in \mathcal{S}$", we have $\mathbf{B}''(\langle \eta_\varepsilon \,:\, \varepsilon < \alpha\rangle) \in \mathcal{S}$. $\dashv$

Let $S \subseteq \omega_1$ be stationary and $\langle A_\delta \,:\, \delta \in S\rangle$ exemplify $\diamondsuit(S)$. For example we can take the most frequent $S = \{\alpha < \omega_1 \,:\, \alpha \text{ limit ordinal}\}$, which gives $\diamondsuit_{\omega_1}$.

Lemma 6.4. *Let $r \colon \omega \to \omega$ such that $\lim \frac{r(n)}{2^n} = 0$. Assume that $\mathbf{V} \models \diamondsuit(S)$. Then*

$$\Vdash_{\mathbb{P}_{\omega_2}} \diamondsuit(2^\omega, \{\lim(T) \,:\, T \subseteq \mathbb{R} \textit{ perfect } \wedge (\forall n)|\{\eta \restriction n \,:\, \eta \in \lim(T)\}| \leq r(n)\}, \in).$$

Proof. Let G be $\mathbb{P}_{\omega_2}$-generic over $\mathbf{V}$. We use the $\diamondsuit(S)$-sequence $\langle A_\delta : \delta \in S\rangle$ in the following manner: By easy integration and coding we have $\langle(N^\delta, \bar\beta^\delta, \underset{\sim}{f}^\delta, \underset{\sim}{F}^\delta, \underset{\sim}{C}^\delta, \mathbb{P}^\delta_{\omega_2}, p^\delta, <^\delta) : \delta \in S\rangle$ such that

(a) $\bar N^\delta$ is a transitive collapse of a tower of countable models $\bar M \prec H(\chi, \in, <_\chi)$ of height $\alpha(\delta)+1$, $\alpha(\delta) = \omega_2 \cap N^\delta_0$, $<^\delta$ is a well-ordering of $N^\delta_{\alpha(\delta)}$, U^δ codes the isomorphism type of $(\bar N^\delta, \mathbb{P}^\delta_{\omega_2}, p^\delta, \bar\beta^\delta)$. (We have a sequence $\bar\beta^\beta$ for each model in the tower, $\bar\beta^\delta$ stands for a sequene of sequences.)

(b) $N^\delta_0 \models \mathbb{P}^\delta_{\omega_2} = \langle \mathbb{P}^\delta_\alpha, \underset{\sim}{\mathbb{Q}}^\delta_\beta : \alpha \leq \omega_2^{N^\delta}, \beta < \omega_2^{N^\delta_0}\rangle$ is as in Definition 1.3.

(c) $N^\delta_0 \models (p^\delta \in \mathbb{P}^\delta_{\omega_2}$, $\underset{\sim}{f}^\delta$ is a $\mathbb{P}^\delta_{\omega_2}$-name of a member of ${}^{\omega_1}2$ $\underset{\sim}{F}^\delta \colon 2^{<\omega_1} \to 2^\omega)$.

(d) If $p \in \mathbb{P}_{\omega_2}$,

$$p \Vdash_{\mathbb{P}_{\omega_2}} \underset{\sim}{f} \in 2^{\omega_1} \wedge \underset{\sim}{F} \colon 2^{<\omega_1} \to 2^\omega \text{ is Borel}, \underset{\sim}{C} \subseteq \omega_1 \text{ is club},$$

and p, $\mathbb{P}_{\omega_2}$, $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C} \in H(\chi)$, then

$$\begin{aligned} S(p, \underset{\sim}{F}, \underset{\sim}{f}) := \{\delta \in S :\ & \text{there is a tower } \bar M \prec (H(\chi), \in, <_\chi) \\ & \text{such that } \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C}, \mathbb{P}_{\omega_2}, p \in M \\ & \text{and there is an isomorphism } h^\delta \text{ from } \bar N^\delta \text{ onto } \bar M \\ & \text{mapping } \mathbb{P}^\delta_{\omega_2} \text{ to } \mathbb{P}_{\omega_2}, \underset{\sim}{f}^\delta \text{ to } \underset{\sim}{f}, \\ & \underset{\sim}{F}^\delta \text{ to } \underset{\sim}{F}, \underset{\sim}{C}^\delta \text{ to } \underset{\sim}{C}, p^\delta \text{ to } p, <^\delta \text{ to } <_\chi\restriction M_{\alpha(\delta)}\} \end{aligned}$$

is a stationary subset of ω_1.

(e) Choose $\langle \mathbf{B}^{\alpha(\delta)}_{0,\omega_2,i} : \delta \in S\rangle$ (remember $\alpha(\delta) = \mathrm{otp}(N^\delta_0 \cap \omega_2)$) as in the proof of the Main Lemma with $U^\delta = U(\bar N^\delta, \mathbb{P}^\delta_{\omega_2}, p^\delta, \bar\beta^\delta)$.

We do not require uniformity, $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ is indeed $\langle \nu^\delta_\varepsilon, \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ since we have the dependence on the δ in the definition of $\mathbf{B}_{\alpha(\delta)}$. We assume that $N^\delta \cap \omega_1 = \delta$. Since this holds on a club set of $\delta \in \omega_1$, this is no restriction.

Now assume the $p \in G$ and $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C}$ are as in (d).
We define a function $\mathbf{B}'_{\delta,U_\delta}$ with domain $(\omega^\omega)^{\alpha(\delta)}$.

$$\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \alpha(\delta)\rangle) = \begin{cases} \mathbf{B}^{\alpha(\delta)}_{0,\omega_2,1}{}''\underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta)[\mathbf{B}^{\alpha(\delta)}_{0,\omega_2,0}(\langle \eta_\varepsilon : \varepsilon < \alpha(\delta)\rangle, U^\delta)], \\ \text{ if } \eta_\varepsilon \geq^* M_{\varepsilon+1} \text{ for } \varepsilon < \alpha(\delta); \\ \langle 0, 0, \dots, \rangle \in 2^\omega, \text{ otherwise.} \end{cases}$$

So $\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ is a Borel function. Now we choose a "very good" argument $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ that player IN plays with his strategy in $\Game_{(\alpha(\delta), \mathbf{B}'_{\delta,U_\delta})}$ from Lemma 6.3 applied to $\mathbf{B}'_{\delta,U_\delta}$ and the $(r, 2^n)$ bounding property, answering to an argument $\langle \nu^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ played by player OUT such that $\nu^\delta_\varepsilon \geq^* M_{\varepsilon+1}$.

Now we derive a guessing function g. We consider for every $\delta \in S$ a very good argument $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$. We assume that G is $\mathbb{P}_{\omega_2}$-generic over V and that $p \in G$. Then we also have by the rules of the game $\partial_{(N^\delta,\mathbb{P}^\delta,p^\delta)}$ that

$$\begin{aligned}&\mathbf{B}^{\alpha(\delta)}_{0,\omega_2,0}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle, U^\delta) \text{ is } (\bar{N}^\delta, \mathbb{P}^\delta, p^\delta)\text{-generic and}\\ &\mathbf{B}^{\alpha(\delta)}_{0,\omega_2,1}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle, U^\delta) \text{ is a translation of } M \cap \mathbb{P}_{\gamma(\delta)}\text{-names}\\ &\ \text{to } \mathbb{S}_{\alpha(\delta)}\text{-name in the domain compatible with the first.}\end{aligned}$$

Lemma 6.3 gives a closed set $\mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}}$ with small levels such that for $\delta \in S$, and we have

$$\mathbf{B}'_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle) \in \mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}}. \tag{6.1}$$

Note that $\mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}}$ does not depend on $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$. So (6.1) also holds for $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ that are the answers of player IN in the game $\partial_{(\alpha(\delta),\mathbf{B}'_{\delta,U^\delta})}$ to any good sequence $\langle \nu^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ given by the generic player that is so fast growing ν^δ_ε that $\mathbf{B}'_{\delta,U_\delta}(\langle \nu^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle)$ computes a Sacks name as in the Main Lemma. This is important, since the isomorphism h^δ does not preserve the knowledge (that is which branches are continued and what are the values of the promises in these continuations) about the level $\omega_1 \cap M_\alpha$ for the Aronszajn trees involved in $\mathbb{P}_\gamma \cap M_\alpha$.

We set

$$\mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}} =: g(\delta).$$

Both sides are conceived as Borel codes for closed sets. Since $\omega \subseteq M$ and $\omega \subseteq N^\delta$ we have that $h^\delta(\mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}}) = \mathcal{S}_{\mathbf{B}'_{\delta,U^\delta}}$. We show that g is a diamond function.

Since $\mathbb{P}_{\omega_2}$ is proper, $S(p, \underset{\sim}{f}, \underset{\sim}{F})$ is also stationary in $\mathbf{V}[G]$. Now we take a very good sequence $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle$ that is suitable so that $\mathbf{B}'_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle)$ witnesses that $\delta \in S$. So now we take the game $\partial_{(M,\mathbb{P},p)}$ for the choice of the $\langle \nu^\delta_\eta : \eta < \alpha_\delta\rangle$ and then again we take the winning strategy in the game $\partial_{(\alpha(\delta),\mathbf{B}'_{\delta,U_\delta})}$, which is unchanged by the collapse, for choosing $\langle \eta^\delta_\varepsilon : \varepsilon < \alpha_\delta\rangle$. We take q to be a bound of $\mathbf{B}_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle)$. Now we have that $q \geq p$ and

$$q \Vdash \text{“}\mathbf{B}_{\alpha(\delta),0}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle, U^\delta) \text{ is } (M,\mathbb{P})\text{-generic” and}$$

$$q \Vdash \text{“}\underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) = \mathbf{B}_{\delta,U^\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \alpha(\delta)\rangle) \text{ is a } \mathbb{S}_{\alpha(\delta)}\text{-name”.}$$

Now for $\delta \in S(p, \underset{\sim}{f}, \underset{\sim}{F})$ we have by the isomorphism property of h^δ and by (6.1),

$$q \Vdash h^{\delta\prime\prime}\underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta) = \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \ \wedge\ \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \in g(\delta) \ \wedge\ \delta \in \underset{\sim}{C}.$$

So we have that p forces that $\{\alpha \in S : F(f \restriction \delta) \in g(\delta)\}$ contains a stationary subset of $S(p, \underset{\sim}{f}, \underset{\sim}{F})$. Note that the stationary subset depends on F (and f of course), but the guessing function g does not. So actually we proved a diamond of the kind:

$\diamondsuit'(A, B, E)$ There is some $g\colon \omega_1 \to B$ such that for every Borel map $F\colon 2^{<\omega_1} \to A$ and for every $f\colon \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$ is stationary.

$\dashv$

Corollary 6.5. *If* $\mathbf{V} \models \diamondsuit \wedge 2^{\aleph_1} = \aleph_2$, *then*

$$\Vdash_{\mathbb{P}_{\omega_2}} \models \diamondsuit'(\textit{covering by thin trees}) \wedge \diamondsuit'(\mathbb{R}, \mathcal{N}, \in) \wedge \diamondsuit'(\mathbb{R}, \mathcal{M}, \in).$$

Proof. $\mathrm{Leb}(g(\delta)) = 0$ for the functions $g\colon \omega_1 \to \{\text{closed subsets of } 2^\omega\}$ from the previous lemma. Thus, for every Borel $F\colon 2^{<\omega_1} \to 2^\omega$, the function $g\colon \omega_1 \to \mathcal{N}$ is a guessing sequence showing $\Vdash_{\mathbb{P}_{\omega_2}} \diamondsuit'(\mathbb{R}, \mathcal{N}, \in)$. $\dashv_{1.2}$

References

[1] Uri Abraham. Proper forcing. In Matthew Foreman, Akihiro Kanamori, and Menachem Magidor, editors, *Handbook of Set Theory*. Kluwer, 2010.

[2] Uri Abraham and Saharon Shelah. A Δ^2_2 well-order of the reals and incompactness of $L(Q^{MM})$. *Ann. Pure Appl. Logic*, 59:1–32, 1993. math.LO/9812115.

[3] Tomek Bartoszyński and Haim Judah. *Set Theory, On the Structure of the Real Line.* A K Peters, 1995.

[4] James Baumgartner. Iterated forcing. In Adrian Mathias, editor, *Surveys in Set Theory*, volume 8 of *London Math. Soc. Lecture Notes Ser.*, pages 1–59. Cambridge University Press, 1983.

[5] Andreas Blass and Saharon Shelah. There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. *Ann. Pure Appl. Logic*, 33:213–243, 1987.

[6] Jörg Brendle Cardinal invariants of the continuum and combinatorics on uncountable cardinals. *Ann. Pure Appl. Logic*, 144:43–72, 2006.

[7] Tetsuya Ishiu. α-properness and Axiom A. *Fund. Math.*, 186(1):25–37, 2005.

[8] Alexander Kechris. *Classical Descriptive Set Theory.* Springer, 1995.

[9] Kenneth Kunen. *Set Theory, An Introduction to Independence Proofs.* North-Holland, 1980.

[10] Heike Mildenberger. Finding generic filters by playing games. *Archive Math. Logic*, 49(1):91–118, 2010.

[11] Heike Mildenberger and Saharon Shelah. Specializing Aronszajn trees and preserving some weak diamonds. *J. Applied Analysis*, 15(1):47–78, 2009.

[12] Justin Tatch Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$ principles. *Trans. Amer. Math. Soc.*, 356:2281–2306, 2004.

[13] Saharon Shelah. NNR revisited. *Preprint Sh:656.* math.LO/0003115.

[14] Saharon Shelah. *Proper and Improper Forcing, 2nd Edition.* Springer, 1998.

[15] Saharon Shelah. Properness without elementaricity. *Journal of Applied Analysis*, 10:168–289, 2004. math.LO/9712283.

[16] Saharon Shelah and Otmar Spinas. The distributivity numbers of $P(\omega)$/fin and its square. *Trans. Amer. Math. Soc.*, 352:2023–2047, 2000. math.LO/9606227.

Heike Mildenberger, Abteilung für Mathematische Logik, Mathematisches Institut, Universität Freiburg, Eckerstr. 1, 79104 Freiburg im Breisgau, Germany

Saharon Shelah, Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond Safra Campus Givat Ram, 91904 Jerusalem, Israel, and Mathematics Department, Rutgers University, New Brunswick, NJ, USA

Email address: heike.mildenberger@math.uni-freiburg.de

Email address: shelah@math.huji.ac.il