STRUGGLING WITH THE SIZE OF INFINITY

SUPPLEMENTARY MATERIAL TO **THE PAUL BERNAYS LECTURES 2020**

SAHARON SHELAH

Abstract. This paper contains supplementary material to the three Paul Bernays lectures 2020, videos of which can be found here:

https://video.ethz.ch/speakers/bernays/2020.html

The Paul Bernays lectures 2020 were three lectures held on August 31st and September 1st 2020. Due to the COVID-19 pandemic, the lectures were not held at ETH Zürich in front of a live audience, but where given in form of a "Zoom Webinar". The recordings of these lectures can be found at the ETH video repository. This document contains the following:

I thank the ETH Zürich for the great honour of inviting me to give the Paul Bernays lectures, and the audience for coming to hear, and I thank all who help in the rehearsals. Naturally the choice of topics reflects my personal opinion (or prejudices, if you are not so kind).

Date: 2020-10-01.
Number E90 in Shelah's list of publications.

Titles and abstracts

Lecture 1 (Aug 31 2020, 5pm)
Cardinal arithmetic: Cantor's paradise.

We explain Hilbert's first problem.
Specifically it asks the value of the continuum- Is the number of real numbers equal to $\aleph_1$ (the first infinite cardinal above $\aleph_0$, which is the number of natural numbers).
Recall that Cantor (1870s) introduce infinite numbers- just equivalence classes of sets under "there is a bijection" The problem really means "what are the laws of cardinal arithmetic= the arithmetic of infinite numbers". We review the history, (including Gödel in the 1930s and Cohen in the 1960s), mention other approaches, explain what is undecidable and mainly some positive answers we now have. This will be mainly on cofinality arithmetic, the so called pcf theory; but will also mention cardinal invariants of the continuum.

Lecture 2 (Sep 1st, 2.15pm)
How large is the continuum?

After the works of Gödel and Cohen told us that we cannot decide what is the value of the continuum, that is what $\aleph$ is the number of real numbers; still this does not stop people from having opinions and argument. One may like to adopt extra axioms which will decide it (usually as $\aleph_1$ or $\aleph_2$), and argue that they should and eventually will be adopted. We feel that assuming the continuum is small make us have equalities which are incidental. So if we can define 10 natural cardinals which are uncountable but at most the continuum, and if the continuum is smaller than $\aleph_{10}$, at least two of them will be equal, without any inherent reasons. Such numbers are called cardinal invariants of the continuum, and they naturally arise from various perspectives. We like to show they are independent, That is, there are no non-trivial restrictions on their order. More specifically we shall try to explain Cichon's diagram and what we cannot tell about it. References [Sh:1044], [Sh:1122], [Sh:1004].

Lecture 3 (Sep 1st, 4.30pm)
Cardinal invariants of the continuum: are they all independent?

Experience has shown that in almost all cases; if you define a bunch of Cardinal invariants of the continuum, then modulo some easy inequalities, by forcing (the method introduced by Cohen), there are no more restrictions. Well, those independence results have been mostly for the case the continuum being at most $\aleph_2$, but his seem to be just our lack of ability, as the problems are harder.
But this opinion ignores the positive side of having forcing, being able to prove independence results: clearing away the rubble of independence results, the cases where we fail may indicate there are theorems there. We shall on the one hand deal with cases where this succeed and on the other hand with cofinality arithmetic, what was not covered in the first lecture.

Additional topics not mentioned in the abstracts.

A posteriory, it turned out that the lectures also dealt with *weakening the axiom of choice*, and with *pcf-theory*, i.e., the laws of cofinality arithmetic.

Further reading

Popular science media. An exposition for the general public on infinite cardinals, cardinal invariant of the continuum and in particular $\mathfrak{p} = \mathfrak{t}$, appeared is Quanta Magazine (Kevin Hartnett. *Mathematicians Measure Infinities and Find They're Equal.* Sep. 12, 2017), reposted in Scientific American and translated into German in Spektrum der Wissenschaft (*Von Unendlichkeit zu Unendlichkeit*). Related are articles in the Christian Science Monitor (by the editorial board, *The awards and rewards of grasping infinity*, Sep. 19, 2017) and the London Times (Tom Whipple. *The riddle of infinity? Here's an answer you can count on.* Sep. 15, 2017).

Another exposition, in German, focusing on Cichoń's Maximum, appeard in Spektrum der Wissenschaft (Manon Bischoff, *Ordnung in den Unendlichkeiten*, 14. Aug. 2019).

For mathematicians. An exposition for mathematicians on $\mathfrak{p} = \mathfrak{t}$, by Casey and Malliaris, can be found arXiv:1709.02408; and on cardinal arithmetic in [Sh:E25] and earlier [Sh:400a] (see parts of firsts and last lecture).

The Slides used in the Talks

Sh:F1171 2020-08-30_3 (Saharon's variant)

STRUGGLING WITH THE SIZE OF INFINITY

SAHARON SHELAH

Abstract. First Lecture: Cardinal arithmetic: Cantor paradise

Abstract: We explain Hilbert first problem- Specifically it asks the value of the continuum- Is the number of real number equal to $\aleph_1$ - the first infinite cardinal Above $\aleph_0$ =the number of natural numbers. Recall that Cantor introduce infinite numbers- just equivalence classes of sets under "there is a bijection" The problem really means "what are the laws of cardinal arithmetic= the arithmetic of infinite numbers". We review the history, (including Godel and Cohen) mention other approaches, explain what is undecidable and mainly some positive answers we now have. This will be mainly on cofinality arithmetic, the so called pcf theory; but will also mention cardinal invariants of the continuum.

Second Lecture Title: How large is the continuum?

Abstract: After the works of Godel and Cohen told us that we cannot decide what is the value of the continuum, that is what $\aleph$ is the number of real numbers; still this does not stop people from having opinions and argument. One may like to adopt extra axioms which will decide it (usually as $\aleph_1$ or $\aleph_2$), and argue that they should and eventually will be adopted. We feel that assuming the continuum is small make us have equalities which are incidental. So if we can define 10 natural cardinals which are uncountable but at most the continuum, and the continuum is smaller than $\aleph_{10}$, at least two of them will be equal, without any inherent reasons. Such numbers are called cardinal invariants of the continuum, and they naturally arise from various perspectives. We like to show they are independent, That is, there are no non-trivial restrictions on their order. More specifically we shall try to explain Cichon diagram and what we cannot tell about it. References [FisGolKelShe:1044], [GolKelShe:1122], [She:1004].

Third Lecture Title: Cardinal invariants of the continuum: are they all independent?

Abstract: Experience has shown that in almost all cases; if you define A bunch of Cardinal invariants of the continuum, then modulo some easy inequalities, by forcing (the method introduced by Cohen), there are no more restrictions. Well, those independence results have been mostly for the case the continuum Being at most $\aleph_2$, but his seem to be just our lack of ability, as the problems are harder. But this opinion ignores the positive side of having forcing, being able to prove independence results: clearing away the rubble of independence results, the cases where we fail may indicate there are theorems there. We shall on the one hand deal with cases where this succeed and on the other hand with cofinality arithmetic, what was not covered in the first lecture.

Date: 2020-08-30.3.

2010 *Mathematics Subject Classification.* FILL.

Key words and phrases. FILL.

Paper number F1171; Bernays lectures; thecre-lacement References like [Sh:950, Th0.2=Ly5] mean that the internal label of Th0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number F1970a in Saharon Shelah's list.

1

We thank the ETH for giving me the great honour of inviting me to give the Bernays lectures, and the audience for coming to hear, and thank all who help in the rehearsals,... Naturally the choice reflect my personal opinion (or prejudices if you are not so kind).

Lecture I:Cardinal arithmetic: Cantor paradise

§ 1.

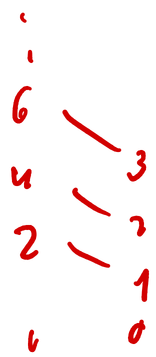
The arithmetic of infinite numbers was discovered by Cantor.

But actually it is very old. Primitives people do not know large numbers like 564 or even 56, but they use one-to-one matching in barter.

Galileo has noticed that: The number of even natural numbers is the same as the number of natural numbers,..

much more can be said on this but another times'

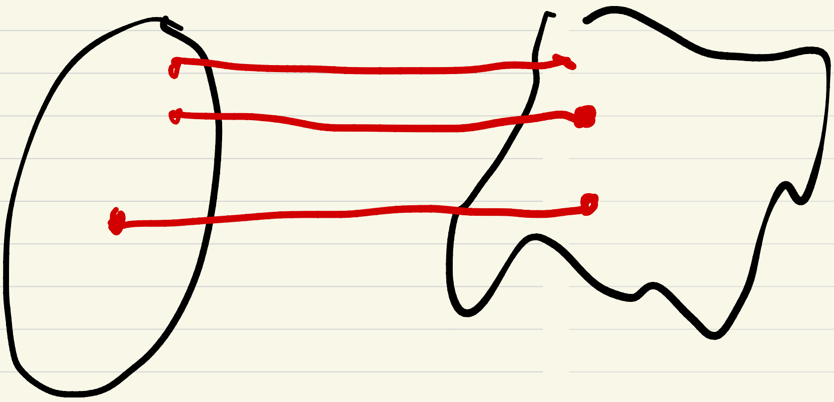
I suspect the attributions to another time is the most quoted one



1.2

Hume's principle or HP—the terms were coined by George Boolos—says that the number of Fs is equal to the number of Gs if and only if there is a one-to-one correspondence (a bijection) between the Fs and the Gs. HP can be stated formally in systems of second-order logic. Hume's principle is named for the Scottish philosopher David Hume.

HP plays a central role in Gottlob Frege's philosophy of mathematics. Frege shows that HP and suitable definitions of arithmetical notions entail all axioms of what we now call second-order arithmetic. This result is known as Frege's theorem, which is the foundation for a philosophy of mathematics known as neo-logicism.



1.3

Cantor consider also infinite numbers = the number of elements= the cardinality of (infinite) sets. Now the arithmetical operation are naturally defined:

(A) sum $\lambda + \mu$ correspond to (disjoint) union
(B) similarly to the sum of many which lead us to
(C) product $\lambda \times \mu$ correspond to (Cartesian) products
(D) similarly to product of many, which lead us to
(E) power κ^μ correspond to the operation A^B, the set of functions from B to A

This lead to

Theorem 1.1. *All the arithmetical laws about equalities holds also in this context like* $\lambda \times (\mu \times \kappa) = (\lambda \times \mu)\kappa$

However, not so for the inequalities

Theorem 1.2. *(1)* $\lambda + 1 = \lambda$ *for infinite* λ
(2) $\lambda + \mu = \max\{\lambda, \mu\}$ *for infinite* λ, μ
(3) $\lambda \times \mu = \max\{\lambda, \mu\}$ *for infinite* λ, μ

For example, the number of points in the plan is equal to the number of point in the line

Thesis 1.3. This contradict Aristotle dictum (the whole is bigger the the part. But actually this is a good description of the way we handle numbers: for the national deficient to the number of electron in the universe.

Sh:F1171 2020-08-30_3 (Saharon's variant)

1.4

Let us phrase this again and add a little

1.1. **Cardinal Arithmetic.** Recall (Cantor), that we call two sets A, B equivalent (or equinumerous) if there is a one-to-one mapping from A onto B; the number of elements of A is the equivalence class of A denoted by $|A|$, we call it also *the power* or *the cardinality* of A. Having defined infinite numbers, we can naturally ask ourselves what is the natural meaning of the arithmetical operations and the order. There can be little doubt concerning the order:

- $|A| \leq |B|$ iff A is equivalent to some subset of B.

We know that

- any two infinite cardinals are comparable so it is really a linear order
- any cardinal λ has a successor λ^+, which means that
- $\lambda < \mu \Leftrightarrow \lambda^+ \leq \mu$.

Well, but what about the arithmetical operations? There are natural definitions for the basic operations:

- addition (such that $|A \cup B| = |A| + |B|$ when A, B are disjoint),
- multiplication (such that $|A \times B| = |A| \times |B|$),
- exponentiation (such that $|A|^{|B|} = |{}^B A|$, where ${}^B A = \{f : f$ a function from B to $A\}$).

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1.5

So maybe all infinite numbers are equal?
Not so because

Theorem 1.4. *1) $2^\lambda > \lambda$, equivalently the number of subsets of X is strictly bigger then the number of members of X.*
2) Every infinite number λ has a successor λ^+,

SO we can define $\aleph_0$ = the number of natural number, the first infinite cardinal, $\aleph_1$ it's successor $\aleph_{n+1} - (\aleph_n)^+$, then $\aleph_\omega = \Sigma_n \aleph_n$, etc

Now as addition and multiplication of two infinite numbers is trivial, maybe also the rules of exponentations are easy; so modulo the above

Conjecture 1.5 (Cantor, Hausdorff)**.** The GCH = generalized continuum hypothesis]: The two operation increasing an infinite number 2^λ and λ^+ are the same?
We can rephrase this as $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ where $\aleph_\alpha$ is the α-s cardinal

This is (essentially)

Conjecture 1.6. HILBERT first problem: $2^{\aleph_0} = \aleph_1$? in other words; is every uncountable set of reals has the same cardinality as the set of reals?

As in other problems, Hilbert phrase the simplest case. Viewing the above, it is clearly the [redacted] meaning is
FIND ALL THE RULES of CARDINAL ARITHMETIC
because if the conjecture is true, then we know them. ~~I think the meaning~~

Is such reformulation legitimate? As an argument, I can cite, from the book [Br] on Hilbert's problems, Lorentz's article on the thirteenth problem. The problem was

(∗) Prove that the equation of the seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable with the help of any continuous functions of only two variables.

for reals

Nobody take 7 (and the triple (a,b,c)) seriously, Hilbert just state the simplest open case

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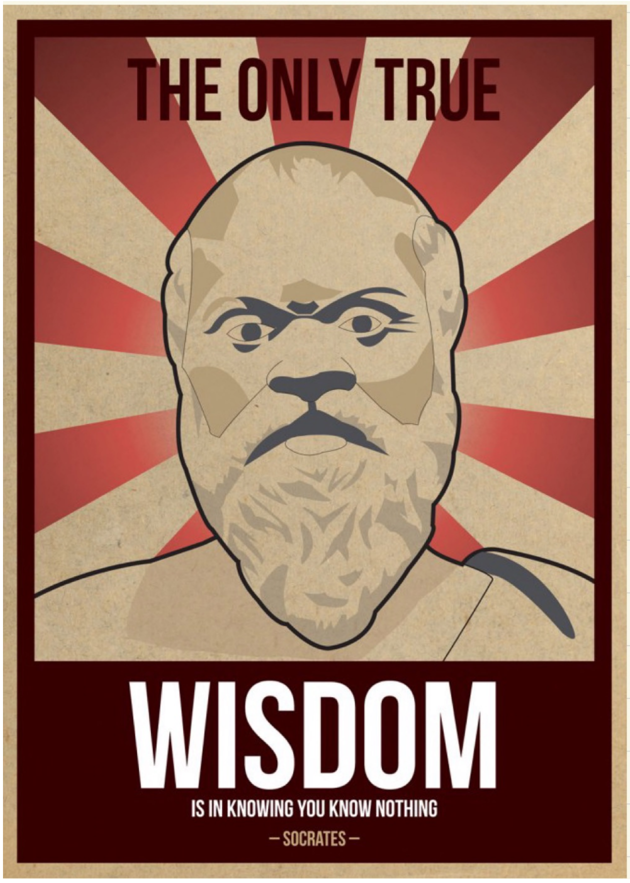
2.1

§ 2. Do we know that we do not know?

SOCRATES



2.2

A well known semi-joke is:
If something seem impossible, do not give up.
First prove that it is impossible.
Then give up
But mathematician take this seriously:
They waste centuries trying to prove the fifth postulate (on parallel) can be proved; then they prove it cannot be proved.
¡Mathematicains take this seriously because

(a) graveyards are full in the tombs of irreplaceable heroes. .
(b) Mathematics is full of statements known to be false till proved true

2.3

Liar paradox

The paradox's name translates as pseudómenos lógos (ψευδόμενος λόγος) in Ancient Greek. One version of the liar paradox is attributed to the Greek philosopher Eubulides of Miletus who lived in the 4th century BC. Eubulides reportedly asked, "A man says that he is lying. Is what he says true or false?"[2]

Godel 1. incompleteness.
no reasonable axiom system is enough

This proof has and will continue to have many profound and important descendants, relatives and applications, such as equi-consistency results, the unsolvability of the halting problem (there is no algorithm to decide whether a computer program will terminate or not), the negative solution of Hilbert’s 10th problem (there is no algorithm to decide whether a polynomial with integer coefficients has an integer root), cuts in models of PA, the Paris-Harrington theorem, reverse mathematics and Boolean relation theory

This may look a sophist proof; becase though we cannot prove in Peano arithmetic its consistency, we all know it is consitent.

Now Godel also prove

Theorem 2.1. *Maybe GCH holds*

How did he proved it? in short, by being a miser; he

A. he put in only the ordinals (representatives of order types of $\mathbb{P}$ which are linear orders which are well ordered; that is any non-empty set has a first elemtns; so we can carry inductions.

B. closing under: adding only sets which are easily definable subsets of what we have so far

This start socalled "inner model theory, Jensen continue; but not for here and now.

This is fine moreover great BUT as above it give only

ASYMETRIC INDEPENDENCE

maybe holds, but tell nothing on failures

Sh:F1171: 2020-08-30_4 (11 of 24)

2.5

The dark ages of. set theory — before Cohen
It was known much is. no decidable but no concrete way to show it for the problems that interest them

Usually the dark ages of set theory are used in derogative way but actually they were mostly very successful-proving what they could prove

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$\aleph_{n+1} = (\aleph_n)^+$

Next Cohen prove

Theorem 2.2. *We do not know whether the continuum is* $\aleph_1$ *or* $\aleph_2$ *or whatever.* †

Subsequently, Solovay (on the $\aleph_n$-s) and Easton (generally)

Theorem 2.3. *On the the function* $\lambda \mapsto 2^\lambda$ *there are only classical limitations for* $\lambda = \aleph_0$ *and* λ *successor. But* <u>not</u> *for so called singular cardinals like* $\aleph_\omega = \Sigma_n \aleph_n$

How those proofs were done: instead tightening the belt as Godel does, we expand the universe of sets; we use a partial order $\mathbb{P}$ and add a new directed subset, not in the same universe, which meet every dense "old" subset.
This is called forcing, much work was done on this.

But, using so called large cardinals, Magidor proved that GCH may hold up to $\aleph_\omega$ but not at $\aleph_\omega$. $= \Sigma_n \aleph_n$ first singular

[Mag77a] Menachem Magidor, *On the singular cardinals problem i*, Israel J. Math. **28** (1977), 1–31.
[Mag77b] ——, *On the singular cardinals problem ii*, Annals Math. **106** (1977), 517–547.

encouraging

† Cohen not only make Godel result symmetric, the method gave both possibilities

Silver proved that there are some restriction - letting $\aleph_{\omega_1}$ be the first cardinal below which there are $\aleph_1 cardinals, if$ GCH hold before it, is hold for $\lambda = \aleph_{\omega_1}$

[Sil74] Jack Silver, *On the singular cardinal problem*, Proceedings of the International congress of Mathematicians (Vancouver), vol. I, 1974, pp. 265–268.

[GalHaj75] Fred Galvin and Andras Hajnal, *Inequalities for cardinal powers*, Annals Math. **101** (1975), 491–498.

[DevJen75] Keith J. Devlin and Ronald B. Jensen, *Marginalia to a theorem of silver*, Proceedings of the Logic Colloquium Kiel 1974 (Berlin) (G. H. Müller, A. Oberschelp, and K. Potthoff eds.), Lecture Notes in Mathematicas, vol. 499, Springer, 1975, pp. 115–142.

See Handbook of Set Theory in particular on forcing see Gitik [Git10] there
There is much to be said on those; but not for here and now

3.1

§ 3. Cardinal invariants

What are cardinal invariants of the continuum?

We can measure the continuum by the number of reals, the Cantor definition. BUT we can measure it in some other ways- so we have definition of a cardinal Let me give some examples

2.2
3.2

For functions $f, g \in {}^\omega\omega$, we write $f \leq^* g$ to mean $\forall^\infty x\,(f(x) \leq g(x))$.

2.1 Definition A family $\mathcal{D} \subseteq {}^\omega\omega$ is *dominating* if for each $f \in {}^\omega\omega$ there is $g \in \mathcal{D}$ with $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of any dominating family, $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ dominating}\}$.

2.2 Definition A family $\mathcal{B} \subseteq {}^\omega\omega$ is *unbounded* if there is no single $f \in {}^\omega\omega$ such that $g \leq^* f$ for all $g \in \mathcal{B}$. The *bounding number* $\mathfrak{b}$ (sometimes called the *unbounding number*) is the smallest cardinality of any unbounded family.

for any ideal have four numbers

2.7 Definition Let $\mathcal{I}$ be a proper ideal of subsets of a set X, containing all singletons from X.

- The *additivity* of $\mathcal{I}$, $\mathbf{add}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union not in $\mathcal{I}$.
- The *covering number* of $\mathcal{I}$, $\mathbf{cov}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union X.
- The *uniformity* of $\mathcal{I}$, $\mathbf{non}(\mathcal{I})$, is the smallest cardinality of any subset of X not in $\mathcal{I}$.
- The *cofinality* of $\mathcal{I}$, $\mathbf{cof}(\mathcal{I})$ is the smallest cardinality of any subset $\mathcal{B}$ of $\mathcal{I}$ such that every element of $\mathcal{I}$ is a subset of an element of $\mathcal{B}$. Such a $\mathcal{B}$ is called a *basis* for $\mathcal{I}$.

major cases : the null ideal

the meagre ideal

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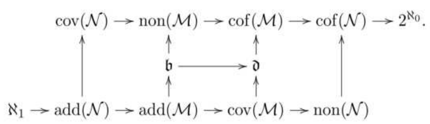
3.3

Cichon diagram [[page 1.4]]



An arrow from $\mathfrak{x}$ to $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\mathrm{cof}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$ and $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$. A (by now) classical series of theorems [Bar84], [BJS93], [CKP85], [JS90], [Kam89], [Mil81], [Mil84], [RS83] and [RS85] proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent with ZFC, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, Ch. 7].)

Too many:

It is known from ancient times that
(A) There are only ten numberd in Cichon diagram; because
(B) $\mathrm{add}(meagre) = \min\{\mathfrak{b}, \mathrm{cov}(meagre)\}$
(C) $\mathrm{cf}(meagre) = \max\{, \mathrm{non}(meagre), \mathfrak{d}\}$

For long this have looked an impossible dream;

an unreasonable aim, but at last:

Theorem 3.1. *After some forcing, there are really ten numbers in Cichon diagram*

CAN WE GET REALY TEN?

For long this have looked an impossible dream;

an unreasonable aim, but at last:

Theorem 3.1. *After some forcing, there are really ten numbers in Cichon diagram*

Better $2^{\aleph_0} > \aleph_{10}$ others

none =

4.1

§ 4. Not all is independent

Thesis 4.1. (A) When nothing works fixing the machine, try to read the manual.

(B) When all attempts to prove independence, try prove a theorem in ZFC

Thesis 4.2. The rubble thesis: after forcing tell us what cannot be proved, we can concentrate on the good cases no longer camouflages by the cacophony of independent cases

Definition 1.1. (See, e.g., [8].) *We define several properties which may hold of a family $D \subseteq [\mathbb{N}]^{\aleph_0}$, i.e., a family of infinite sets of natural numbers. Let $A \subseteq^* B$ mean that $\{x : x \in A,\ x \notin B\}$ is finite.*

- *D has a* pseudo-intersection *if there is an infinite $A \subseteq \mathbb{N}$ such that for all $B \in D$, $A \subseteq^* B$.*
- *D has the s.f.i.p. (strong finite intersection property) if every nonempty finite subfamily has infinite intersection.*
- *D is called a tower if it is well ordered by $\supseteq^*$ and has no infinite pseudo-intersection.*

Then,

$$\mathfrak{p} = \min\{|\mathcal{F}| \ : \ \mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0} \text{ has the s.f.i.p. but has no infinite pseudo-intersection}\},$$
$$\mathfrak{t} = \min\{|\mathcal{T}| \ : \ \mathcal{T} \subseteq [\mathbb{N}]^{\aleph_0} \text{ is a tower}\}.$$

Clearly, both $\mathfrak{p}$ and $\mathfrak{t}$ are at least $\aleph_0$ and no more than $2^{\aleph_0}$. It is easy to see that $\mathfrak{p} \leq \mathfrak{t}$, since a tower has the s.f.i.p. By a 1934 theorem of Hausdorff $\aleph_1 \leq \mathfrak{p}$ [13]. In 1948 [37], Rothberger proved (in our terminology) that $\mathfrak{p} = \aleph_1$ implies $\mathfrak{p} = \mathfrak{t}$, which begs the question of whether $\mathfrak{p} = \mathfrak{t}$.

Problem 1. *Is $\mathfrak{p} = \mathfrak{t}$?*

4 4.2

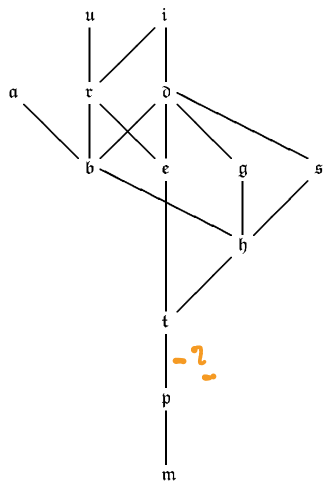


Figure 1: Hasse Diagram of Combinatorial Characteristics

Th $\mathfrak{p} = \mathfrak{t}$

Sh:998 Malliaris, M., & Shelah, S. (2016).
Cofinality spectrum theorems in model theory, set theory, and general topology.
J. Amer. Math. Soc., 29(1), 237–297.
arXiv: 1208.5424 DOI: 10.1090/jams830
MR: 3402699

Sh:F1171 2020-08-30_3 (Saharon's variant)

§ 5. Cardinal arithmetic

Thesis 5.1.

(a) The impression was that cardinal exponentiation essentially has no non-classical restrictions, well except some anomalities

(b) this is wrong; actually there are two phenomena which should be separated

(c) phenomena 1 is the function $\lambda \mapsto 2^\lambda$ for λ successor or $\aleph_0$ or so called regular cardinals; for this there are no classical restriction;

(d) $(\lambda, \kappa) \mapsto \lambda^\kappa$ for $\lambda \leq 2^\kappa$; here there are serious restriction; the concentration on $\lambda \mapsto 2^\lambda$ obscure this. WE may concentrate on $\lambda^{\aleph_0}$ equivalently on $\Pi_n \lambda_n$

, ⇒ pp pcf

The simplest case is $\Pi_n \aleph_n$; as $\lambda_n^{\aleph_n}$ easy

5.2

So the first non trivial case is the (infinite) product of those numbers: $\prod_n \aleph_n$, or equivalently,

(1) what is $\aleph_\omega^{\aleph_0}$?

where $\aleph_\omega$ is the sum of the $\aleph_n$'s.

If the continuum (i.e. $2^{\aleph_0}$) is above all the $\aleph_n$, then this product is equal to the continuum; so assume that the continuum is one of them.

Let $\aleph_{\omega_n}$ be the first cardinal below which there are $\aleph_n$ infinite cardinals.

Theorem 2. $\prod_n \aleph_n < \aleph_{\omega_4}$ *when the product is not* $2^{\aleph_0}$. Cantor would

You may think this is a typographical error (in fact almost all who saw it for the first time were convinced this is a typographical error) and we still do not know:

Dream/Question: Why the hell is it four? Can we replace it by one? Is 4 an artifact of the proof or the best possible bound?

I think the four looks strange because we are looking at the problems from a not so good perspective.

5.3

Thesis 1.3. *(1) We should not concentrate on the function $\lambda \mapsto 2^\lambda$ as was traditional but rather on $(\lambda, \kappa) \mapsto \lambda^\kappa$ or even on products of relatively few (but infinitely many) cardinals* even $\lambda^{\aleph_0}$

(2) Replacing λ^κ by $\mathrm{cf}([\lambda]^\kappa)$ we get a much more "robust" theory; there are more answers and less independence results, "we cannot answer".

(This lead to pdf theory, we shall say more in the third lecture)

$\mathrm{cf}([\aleph_1]^{\aleph_0}) = \aleph_1$

$\lambda^\kappa = \mathrm{cf}([\lambda]^\kappa) + 2^\kappa$

5.4

rephrase the GCH as

(3) for every regular $\kappa < \lambda$ we have $\lambda^\kappa = \lambda$.

Ahah, now that we have two parameters we can look again at "for most pairs of cardinals (3) holds." However, this is a bad division, because, say, a failure for $\kappa = \aleph_1$ implies a failure for $\kappa = \aleph_0$.
To rectify this we suggest another division, we define "λ to the revised power of κ", for κ regular $< \lambda$ as

$$\lambda^{[\kappa]} = \mathrm{Min}\Big\{|\mathscr{P}| : \mathscr{P} \text{ a family of subsets of } \lambda \text{ each of cardinality } \kappa \text{ such that any subset of } \lambda \text{ of cardinality } \kappa \text{ is contained in the union of } < \kappa \text{ members of } \mathscr{P}\Big\}.$$

This answers the criticism above and is a better slicing because:

(A) for every $\lambda > \kappa$ we have: $\lambda^\kappa = \lambda$ iff $2^\kappa \leq \lambda$ and for every regular $\theta \leq \kappa$, $\lambda^{[\theta]} = \lambda$.

$\lambda^{\aleph_1} = \lambda \Rightarrow \lambda^{\aleph_0} = \lambda$

So we can say for almost

5.5

(B) By Gitik, Shelah [GiSh 344], the values of, e.g., $\lambda^{[\aleph_0]}, \dots, \lambda^{[\aleph_n]}$ are essentially independent.

Weak

Now we rephrase the generalized continuum hypothesis as:

(4) for most pairs (λ, κ), $\lambda^{[\kappa]} = \lambda$

Is such reformulation legitimate? As an argument, I can cite, from the book [Br] on Hilbert's problems, Lorentz's article on the thirteenth problem. The problem was

(∗) Prove that the equation of the seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable with the help of any continuous functions of only two variables.

Nobody take 7 (and the triple (a,b,c)) seriously, Hilbert just state the simplest open case

GCH $\aleph_n = \beth_n$

$(\lambda^{\kappa})^{\kappa} \overset{?}{=} \lambda^{\kappa \times \kappa} \overset{?}{=} \lambda^{\kappa}$

5.6

$\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$, $\beth_\omega = \Sigma_n \beth_n$

Back to firmer grounds, how will we interpret "for most"? The simplest ways are to say "for each λ for most κ" or "for each κ for most λ". The second interpretation holds in a non-interesting way: for each κ for many λ's, $\lambda^\kappa = \lambda$ hence $\lambda^{[\kappa]} = \lambda$ (e.g. μ^κ when $\mu \geq 2$). So the best we can hope for is: for every λ for most small κ's (remember we have restricted ourselves to regular κ quite smaller than λ). To fix the difference we restrict ourselves to $\lambda > \beth_\omega > \kappa$. Now what is a reasonable interpretation of "for most $\kappa < \beth_\omega$"? The reader may well stop and reflect. As "all" is forbidden (by [GiSh 344] even finitely many exceptions are possible), the simplest offer I think is "for all but boundedly many".
So the best we can hope for is ($\beth_\omega$ is for definiteness):

ZFC

Th. if $\lambda > \beth_\omega$, for every large enough regular $\kappa < \beth_\omega$, $\lambda^{[\kappa]} = \lambda$ (and similarly replacing $\beth_\omega$ by any strong limit cardinal).

Conclusion for every $\lambda \geq \beth_\omega$ for some n and[2] $\mathscr{P} \subseteq [\lambda]^{<\beth_\omega}$ of cardinality λ, every $a \in [\lambda]^{<\beth_\omega}$ is the union of $< \beth_n$ members of $\mathscr{P}$.

exactly!

Sh:460 Shelah, S. (2000). The generalized continuum hypothesis revisited. Israel J. Math., 116, 285–321. arXiv: math/9809200 DOI: 10.1007/BF02773223 MR: 1759410

829 Shelah, S. (2006). More on the revised GCH and the black box. Ann. Pure Appl. Logic, 140(1-3), 133–160

Sh:F1171 2020-08-31_3 (Saharon's variant)

Summary

(A) We happily live in Cantor's paradise, and the arithmetic of infinite numbers= Cardinal arithmetic is central

(B) Cardinal invariants are essential in understanding sets of reals and all related sets of the same cardinality as the continuum

(C) Forcing is great telling us on the one hand what we cannot prove and directing us to what maybe we should try to prove

(D) Great problems do not necessarily have one interpretation: at least by our interpretation we have have presented here a positive solution to Hilbert first problem

II

2

Abstract.

First Lecture: **Cardinal arithmetic: Cantor's paradise**

Abstract: We explain Hilbert's first problem.
Specifically it asks the value of the continuum- Is the number of real numbers equal to $\aleph_1$ (the first infinite cardinal above $\aleph_0$, which is the number of natural numbers).
Recall that Cantor (1870s) introduce infinite numbers- just equivalence classes of sets under "there is a bijection" The problem really means "what are the laws of cardinal arithmetic= the arithmetic of infinite numbers". We review the history, (including Gödel in the 1930s and and Cohen in the 1960s), mention other approaches, explain what is undecidable and mainly some positive answers we now have. This will be mainly on cofinality arithmetic, the so called pcf theory; but will also mention cardinal invariants of the continuum.

Second Lecture Title: **How large is the continuum?**

Abstract: After the works of Gödel and Cohen told us that we cannot decide what is the value of the continuum, that is what $\aleph$ is the number of real numbers; still this does not stop people from having opinions and argument. One may like to adopt extra axioms which will decide it (usually as $\aleph_1$ or $\aleph_2$), and argue that they should and eventually will be adopted. We feel that assuming the continuum is small make us have equalities which are incidental. So if we can define 10 natural cardinals which are uncountable but at most the continuum, and the continuum is smaller than $\aleph_{10}$, at least two of them will be equal, without any inherent reasons. Such numbers are called cardinal invariants of the continuum, and they naturally arise from various perspectives. We like to show they are independent, That is, there are no non-trivial restrictions on their order. More specifically we shall try to explain Cichon's diagram and what we cannot tell about it. References [FGKS17], [GKS19], [She17].

Third Lecture Title: **Cardinal invariants of the continuum: are they all independent?**

Abstract: Experience has shown that in almost all cases; if you define a bunch of Cardinal invariants of the continuum, then modulo some easy inequalities, by forcing (the method introduced by Cohen), there are no more restrictions. Well, those independence results have been mostly for the case the continuum being at most $\aleph_2$, but his seem to be just our lack of ability, as the problems are harder.
But this opinion ignores the positive side of having forcing, being able to prove independence results: clearing away the rubble of independence results, the cases where we fail may indicate there are theorems there. We shall on the one hand deal with cases where this succeed and on the other hand with cofinality arithmetic, what was not covered in the first lecture.

Lecture II: Many cardinal invariants or CICHON meets PYTHAGORAS

§ 7. How large is the continuum?

Abstract: After the works of Gödel and Cohen told us that we cannot decide what is the value of the continuum, that is what $\aleph$ is the number of real numbers; still this does not stop people from having opinions and argument. One may like to adopt extra axioms which will decide it (usually as $\aleph_1$ or $\aleph_2$), and argue that they should and eventually will be adopted.

We feel that assuming the continuum is small make us have equalities which are incidental. So if we can define 10 natural cardinals which are uncountable but at most the continuum, and the continuum is smaller than $\aleph_{10}$, at least two of them will be equal, without any inherent reasons. Such numbers are called cardinal invariants of the continuum, and they naturally arise from various perspectives.

We like to show they are independent, That is, there are no non-trivial restrictions on their order. More specifically we shall try to explain Cichon's diagram and what we cannot tell about it.

Why Pythagoras? As he and his school have strong belief in numerology, in particular the number ten.

So the main theorem here

Theorem 7.1. *In some forcing extension, there really are 10 cardinals in Cichon's diagram* here is a reference about Pythagoras and the number 10, which I found at https://www.britannica.com/topic/number-symbolism/7

10 was the Pythagorean symbol of perfection or completeness

The number 1 symbolized unity and the origin of all things, since
all other numbers can be created from 1 by adding enough copies of it.
The number 2 was symbolic of the female principle, 3 of the male.
The number 4 represented justice.
The most perfect number was 10, because 10=1+2+3+4.

The number 10 is also related to space.

A single point corresponds to 1, a line to 2 (determined by 2

points), a triangle to 3, and space to 4.

Thus 10 also symbolized all possible spaces

What are cardinal invariants of the continuum?

We can measure the continuum by the number of reals, the Cantor definition. BUT we can measure it in some other ways- so we have definition of a cardinal.

Let me give some examples:

2.2

3.2

For functions $f, g \in {}^{\omega}\omega$, we write $f \leq^* g$ to mean $\forall^\infty x\,(f(x) \leq g(x))$.

2.1 Definition A family $\mathcal{D} \subseteq {}^{\omega}\omega$ is *dominating* if for each $f \in {}^{\omega}\omega$ there is $g \in \mathcal{D}$ with $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of any dominating family, $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ dominating}\}$.

2.2 Definition A family $\mathcal{B} \subseteq {}^{\omega}\omega$ is *unbounded* if there is no single $f \in {}^{\omega}\omega$ such that $g \leq^* f$ for all $g \in \mathcal{B}$. The *bounding number* $\mathfrak{b}$ (sometimes called the *unbounding number*) is the smallest cardinality of any unbounded family.

for any ideal have four numbers

2.7 Definition Let $\mathcal{I}$ be a proper ideal of subsets of a set X, containing all singletons from X.

- The *additivity* of $\mathcal{I}$, $\mathbf{add}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union not in $\mathcal{I}$.
- The *covering number* of $\mathcal{I}$, $\mathbf{cov}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union X.
- The *uniformity* of $\mathcal{I}$, $\mathbf{non}(\mathcal{I})$, is the smallest cardinality of any subset of X not in $\mathcal{I}$.
- The *cofinality* of $\mathcal{I}$, $\mathbf{cof}(\mathcal{I})$ is the smallest cardinality of any subset $\mathcal{B}$ of $\mathcal{I}$ such that every element of $\mathcal{I}$ is a subset of an element of $\mathcal{B}$. Such a $\mathcal{B}$ is called a *basis* for $\mathcal{I}$.

major cases : the null ideal

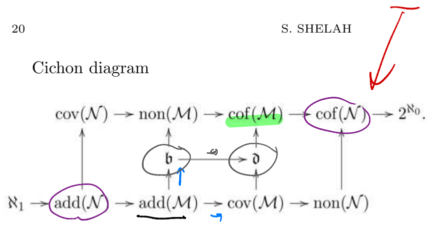
the meagre ideal

Sh:F1171 2020-08-30_2 (Saharon's variant)

6.4

~~II 1.4~~

Cichon diagram



An arrow from $\mathfrak{x}$ to $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \le \mathfrak{y}$. Moreover, $\mathrm{cof}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$ and $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$. A (by now) classical series of theorems [Bar84], [BJS93], [CKP85], [JS90], [Kam89], [Mil81], [Mil84], [RS83] and [RS85] proves these (in)equalities in ZFC and shows that they are the only ones provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's Diagram are consistent with ZFC, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, Ch. 7].)

It is known from ancient times that

(A) There at most only ten numbers in Cichon's diagram; because

(B) $\mathrm{add}(meagre) = \min\{\mathfrak{b}, \mathrm{cov}(meagre)\}$

(C) $\mathrm{cf}(meagre) = \max\{\mathrm{non}(meagre), \mathfrak{d}\}$

Theorem 3.1. *After some forcing, there are really ten numbers in Cichon's diagram*

There has been much discussion on what is the continuum; of course we cannot decide in ZFC, but this will never stop people from arguing, because:

Thesis 7.2. It is interesting to argue only about matters which cannot be decided because then you can never be proved you are wrong

Thesis 7.3 (Woodin)**.** : Extra axioms of set theory which though not obvious even to phrase will be accepted because they are good = give the right picture

He has some candidates which give $\aleph_1$ or $\aleph_2$.

Those include and go much further than DC (the axiom of determinacy) as it give "all reasonably definable sets of reals are nice (the relevant game is determined)"

IMPORTANT ... YES

TRUE ?

I believe Caesar had said on new words apply here:

Thesis 7.4. 1) The axioms of set theory are good if you do notice they are new like the axiom of choice; see [She03],
2) The axiom of determinacy is very important , interesting etc but NOT true; it should be investigated as well as others contradictory to it so let us called them semi-axioms
3) For different directions, different semi axioms are natural

Moreover

I read that some nasty professor have said that some researchers in the social sciences, ask 100 people 100 questions and try hundred correlations, and AHA some are of them past the test of being significant-probability of error being less than 5%

Similarly, if the continuum is $\aleph_1$ or just $\aleph_8$ looking at ten invariants some are equal!! BUT they are not naturally so

Thesis 7.5. The continuum large is better as it avoids unnecessary, non-natural equalities, so Cichon's diagram has ten numbers OR we are missing a real relation which was camouflaged by the restriction of $2^{\aleph_0} \leq \aleph_2$.

However note that this thesis does not tell us what the order of the cardinals in the diagram should be, maybe we should look for semi-axioms answering this.

Sh:F1171 2020-08-30_3 (Saharon's variant)

6.6

There are many cardinal invariants arising from different directions- algebra, general topology, measure theory and set theory. The following diagram of twenty will give a modest indication of its complexity

BLASS diagrame

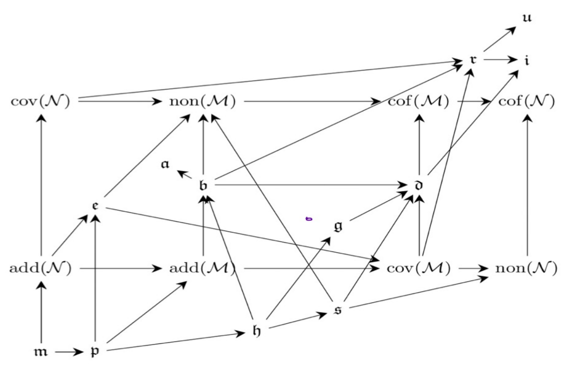


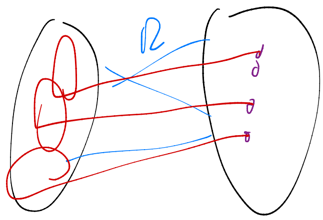
FIGURE 1. The ZFC implications

Many cardinal invariant, though definitely not all fall under the following (see Blass article, on Tukey duality)

Definition 7.6. 1) For a relation R we define an invariant $\mathrm{inv}(R)$ as the minimal cardinality of a subset Y of $\mathrm{Rang}(R)$ which covers, that is for every $x \in \mathrm{dom}(R)$ there is $y \in Y$ satisfying xRy
2) The relation dual to R, called $\mathrm{dual}(R)$ is defined by:
$\mathrm{dual}(R)(x, y)$ iff $\neg R(y, x)$
3) we shall say that the cardinal invariant $\mathrm{inv}(\mathrm{dual}(R))$ is the dual of the cardinal invanriant $\mathrm{inv}(R)$

We shall use this thus cutting the number by half

Thesis 7.7. The cardinals in Cichon's diagram come in pairs- one and its dual



Exact full citation is like our health: boring moral duty except when it concern you , when you are ill (or afraid as when the plague is around), we shall give the relevant most recent ones.

[1066] Goldstern, M., Mejia A.D., & Shelah S. (2016). The left side of Cichon's diagram Proc. Amer. Math. Soc., 144(9), 4025-4042. arXiv: 1504.04192 DOI: 10.1090/proc/13161 MR: 3513558

[1122] Goldstern, M., Kellner, J., & Shelah, S. (2019) Cichon's maximum, Ann. of Math. (2), 190(1), 113-143. arXive: 1708.03691

[1131] Kellner, J., Shelah, S. & Tanasie, A. (2019). Another or ordering of the ten cardinal characteristics in Cichon's diagram Comment. Math. Univ. Carolin., 60(1), 61-95. arXive: 1712.00778

[1166] Goldstern, M.,Kellner, J., Mejia A.D., & Shelah S. Controlling cardinal cahracteristics without adding reals, Preprint. arXive: 2006.09826

[1177] Goldstern, M.,Kellner, J., Mejia A.D., & Shelah S.
Cichon's maximum without large cardinals Journal of the European Mathematical Society (JEMS). arXive 1906.06608

[1199] Goldstern, M.,Kellner, J., Mejia A.D., & Shelah S. Preservation of splitting families and cardinal characteristics of the continuum, Preprint, arXive: 2007.13500

§ 8. One ingredient of the proof: cofinality

Definition 8.1. (1) The cofinality $\mathrm{cf}(\mathbb{P})$ of a partial order $\mathbb{P}$ is the minimal cardinality of a subset Y such that for every $x \in \mathbb{P}$ there is $y \in Y$ satisfying $x \leq_{\mathbb{P}} y$; So this is exactly $\mathrm{inv}(\leq_{\mathbb{P}},$

(2) Recall an ordinal is the order type of a well ordering; identified with the set of smaller ordinals, so is a well ordered set

(3) A cardinal = infinite number is identified or represented, with the first ordinal of this cardinality, We let $\aleph_\alpha$ be the α-th infinite cardinal

(4) A cardinal is regular if it is equal to its cofinality, RECALL that $\aleph_0$ and all successor cardinals ($\aleph_{\alpha+1}$ are regular, and $\aleph_\omega$ is the first singular = non-regular cardinal

But this is a very important case

Thesis 8.2. (1) Cofinalities are much easier to understand then cardinalities;

(2) So even if you are intersted in cardinalities, many times it is better to analyse the related cofinalities

Let me give examples

$$\mathrm{cf}(\aleph_n) = \aleph_n$$

$$\mathrm{cf}(\aleph_\omega) = \mathrm{cf}(\sum_n \aleph_n) = \aleph_0$$

Example 8.3. In model theory:

1) (Löwenheim-Skolem) For given cardinals $\lambda_1, \lambda_2, \kappa_1, \kappa_2$ we can ask:

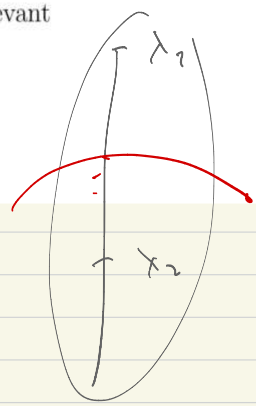
(A) for model M , if P_1^M, P_2^M has cardinality λ_1, λ_2 respectively then M has a submodel N such that P_1^N, P_2^N has cardinality κ_1, κ_2 respectively . The answer is "rarely", (cases of Chang conjecture)

(B) similarly for cofinalities: the answer is usually yes, if the cardinals are regular then usually the asnwer is yes; e.g if $\lambda_1 > \lambda_2 > \kappa_1 + \kappa_2$

2) This is true also when we take ultra-powers (or relatives like Boolean ultra-powers, see below)

3) [Compactness] If we expand first order logic by adding the quantifier "a definable set is uncountable" we get compactness only for countable set of sentences; but if we add the quantifier "the cofinality of a definable linear order is uncountable" we get a fully compact logic

Here only the first example is relevant



Example 8.4. In set theory - cardinality arithmetic via cofinality arithmetic = pcf theory

Thesis 8.5. 1) A forcing notion $\mathbb{P}$ is intended for describing an extension of the universe of set; that is we add a directed subset $\mathbf{G}$ which is genric, random; satisfying it is not disjiont to any dense set (hence cannot be in our present universe) the point is that properties of the new universe $\mathbf{V}[\mathbf{G}]$ gotten for the present universe $\mathbf{v}$ by adding $\mathbf{G}$
2) This can be translated to Boolean algebras $\mathbb{B} = \mathbb{B}_{\mathbb{P}}$ and it was hoped that the rich knowledge of algebras will help in forcing. But so far it help only in the other direction
3) However sometimes it is more transparent to consider a forcing notion $\mathbb{P}$ as just a model rather then using it as intended, that forcing with it. That is we forget looking at the relations between the two universes, we look at $\mathbb{P}$ and use submodels, ultra-powers of it etc.

For example we can start with $\mathbb{P}$, let D be an ultrafilter over a set I and consider the forcing notion $\mathbb{P}^I/D$ But, we usually like to preserve cardinals in the extension, so a mjajor case is asssumin the ccc (countable chain condition) That is among any uncountably many comditions= member of $\mathbb{P}$ some pairs are compatible (= have a common upper bound)

But ultra-products while preserving many properties (all first order one, Loś theorem) do not preserve this.

(Look at forcing notion as a model)

Claim 8.6. *1) If D is $\aleph_1$-complete, the ccc is preserved*
2) With care this enable us to manipulate cofinalities and cardinalities

Recall that the ccc mean that for any uncountable set of members, there are two compatible ones;

Recall that this is the simplest property ensuring no cardinal is collapsed no cofinality is changes

But here the so called large cardinals appear. If we use the Downward LS then we do not need them.

uf is the dual maximal ideal of the BA P(I) .a =

More formally

Claim 8.7. *If (A) then (B) where:*

(A) *(a)* $\mathfrak{B} = (\mathscr{H}(\chi), \in), \lambda > \mu \geq \theta \geq \sigma, \lambda$ *regular for transparency (usually* $\mu = \theta = \sigma^\sigma$ *large such that* D *exist)*

(b) $\mathbb{B} = \mathbb{B}_{\lambda,\mu,\theta}$ *the completion of the Boolean Algebra generated by* λ *maximal antichain each of size* μ *such that intersection of* $< \theta$ *generators (no two from the same maximal antichain) is positive*

(c) D *is a* σ*-complete ultrafilter on* $\mathbb{B}$

(d) $\mathfrak{A} = \mathfrak{B}^{\mathbb{B}}/D, \mathbf{j} : \mathfrak{B} \to \mathfrak{A}$ *the canonical embedding, may identiy* $\mathfrak{A}$ *with its Mostowski Collapse*

(B) *if* $\mathfrak{B} \models$ *"I is a directed partial order"* $J \subseteq \mathscr{T}$ *is cofinal" then*

(a) *if* J *has cardinality* $< \sigma$ *then* $\mathrm{cf}(j(I)) = \mathrm{cf}(I) = \mathrm{cf}(J)$ *(so* $\mathrm{cf}(\mathbf{j}(I))$ *is the cofinality of the partial order* $\mathbf{j}(I)^{\mathfrak{A}}$

(b) *if* J *is* μ^+*-directed then* $\mathrm{cf}(\mathbf{j}(I)^{\mathfrak{A}}) = \mathrm{cf}(I)$

(c) *if* $J \approx ([\mu]^{<\kappa}, \subseteq), \kappa < \sigma, \kappa < \mu, \mathrm{cf}(\mathbf{j}(I)^{\mathbb{B}}) = \mu^{\mathbb{B}}/D$

(d) *if* $J \approx (\kappa, <)$ *then* $\mathrm{cf}(\mathbf{j}(I)^{\mathfrak{B}})$ *is* $\mathrm{cf}((\kappa, <)^{\mathbb{B}}/D)$ *and:*

(α) *if* $\kappa = \mathrm{cf}(\kappa) < \sigma$ *then* $\mathrm{cf}((\kappa, <)^{\mathbb{B}}/D)) = \kappa$

(β) *if* $\kappa = \mathrm{cf}(\kappa) > \mu$ *then* $\mathrm{cf}((\kappa, <)^{\mathbb{B}}/D)) = \kappa$

(γ) *if* $\kappa = \mathrm{cf}(\kappa) \in [\sigma, \lambda]$ *then for suitable* D *we get* $\mathrm{cf}((\kappa, <)^{\mathbb{B}}/D)) = \lambda$.

Proof. Should be clear. $\square_{8.7}$

We suggest you NOT to try to understand it this, rather look at the cases we shall apply it

Remark 8.8. This is used in the works on ≥ 10 cardinal, I_ℓ a partial order, $J_\ell \subseteq I_k$ cofinal invariants. There we stand with $J_\ell \cong ([\lambda_\ell]^{<\kappa_\ell}; \subseteq)$ with $\kappa_0 = \mathrm{cf}(\kappa_0) < \lambda_\ell$ with $\kappa_0 < \kappa_1 < \kappa_\ell < \ldots < \lambda_\infty$, in the beginning $\bigwedge_\ell \lambda_\ell = \lambda_\infty$ and changing by a finite sequence of Boolean-ultra-powers.

The point is:

(∗) given $\langle(\lambda_\ell, \kappa_\ell) : \ell < n\rangle, k < n$, if $\mathbb{B} = \mathbb{B}_{\lambda,\sigma,\sigma,\sigma}, \kappa_k < \sigma < \kappa_{k+1}, D$ a σ-complete ultrafilter on $\mathbb{B}$ then:
- (a) if $\ell > k$ then $I^{\mathbb{B}}_\ell/D$ has a cofinal subset $\cong ([\lambda_\ell]^{<\kappa_\ell}, \subseteq)$
- (b) if $\ell < k$ then $I^{\mathbb{B}}/D$ has a cofinal subset $\cong ([\lambda'_\ell]^{<\kappa_\ell}, \subseteq), \lambda'_\ell = |\lambda^{\mathbb{B}}_\ell/D|$.

NOTE the following cardinal inequality $n + 1 < 2n$. The idea is that we shall find a forcing dealing with $n = 1$ cardinal invanriants and upgrading it by Boolean ultra-powers to one giving $2n$

Question 8.9. Why dealing with the partial order $[\lambda]^{<\kappa} = \{u : u \subseteq \lambda, u$ has cardinality $< \kappa\}$ is relevant?

Thesis 8.10. If you want to shoot your arrow exactly to the desired spot, you should aim elsewhere

Eg if you like to be second best...

Consider non$(null)$, cf$(null)$, and assume that $\{B_\alpha : \alpha < \lambda\}$ list the Borel null sets (=of Lebsgue measure zero) naturally partially ordered and for each $u \subseteq \lambda$ of cardinality $< \kappa$ we have $C_u = B_{u(\alpha)}$ cover

Assume more over that $X \subseteq \lambda$ has cardinality λ and no subset of $B_\alpha : \sigma \in X$ has an upper bound .

This give a translation:

add$(null)$ = add$([\lambda]^{<\kappa}$ and cf$(null) = cf([\lambda]^{<\kappa})$

So assume we like to have $\kappa_1 < \kappa_2 < \dots < \kappa_n < \lambda < \chi_n < \chi_{n-1} \dots$ chi_0

Now we use n times Boolean ultrapowers, preserving the κ_ι-s but pushing λ to the χ_ι

That is by downward induction of $m \leq n$ in stage m we have

$\iota \leq m \Rightarrow (\lambda, \kappa_\iota) \mapsto (\chi_m, \kappa_\iota)$

$\iota > m \Rightarrow (\lambda, \kappa_\iota) \mapsto (\chi_\iota, \kappa_\iota)$

§ 9. Forcing and averaging

Above we have translate a forcing notion dealing with $n+1$ cardinal invanriants to one dealing with $2n$, but we have to start with $\mathbb{P}$ as described there We use Finite Support Iteration forcing.

For each cardinal invariant we are dealing with, we attach a forcing notion pedantically (but important here) a definition φ of one.

Naturally the forcing notions $\mathbb{Q}_\iota$ are related to the cardinal invariant and the natural relation R we are interested in. Here specifically we choose:

Amoeba: this is the natural forcing adding a measure zero Borel set of reals including all the old one ($\iota = 1$)

Hechler: adding a dominating function from ω to ω, ($\iota = 2$)

random real forcing: ; it add a real running out of all old Borel sets of measure zero, ($\iota = 3$)

Eventually different: a member of $\Pi_N n!$ which is eventually different from any old such function (in some cases we need a more complicated forcing), ($\iota = 4$)

Cohen forcing: which is the natural way to add a real running away from all old meagre sets, ($\iota = 0$)

For every coordinate $\alpha < \lg(\mathbf{q})$ we have a memory $u_\alpha \subseteq \alpha$ and generic η , A major necessary point is that the memory is not transitive.

Why? we shall have some (definitions) of forcing notions φ_ι for $\iota < n$ (presently $\iota = 4$) and a relation R_ι defining the relevant invaniant and we promise that the set $W_\iota = \{\alpha < \lg(\mathbf{q}) : \mathbb{Q}_\alpha$ is φ_ι as interpreted in $\mathbf{V}[\mathbb{P}_{u_\alpha}]$ This is normally a partition of $\lg(\mathbf{q})$ and in hour scheme tthis is a cofinal sub-family of of $[\lg(\mathbf{q})]^{<\kappa(\iota)}$

SO every set $\{\eta_\alpha : \alpha \in W_\iota\}$ gives us a R_ι-cofinal subset (or R_ι-cover) of $\mathrm{rang}(R_\iota)$.

For example, for $\iota = 1$ the forcing notion is amoeba and the the relation is inclusion (of Borel null sets, actually F_σ-sets (= countable union of closed sets).

For $\iota = 2$ we have Hechler forcing adding a function from ω to ω domination all old such functions (so for $\alpha \in W_\iota$ it dominate (= bigger for all but finitely many places) all such function from $\mathbf{V}[\langle \eta_\gamma : \gamma \in u_\alpha \rangle]$.

Dealing with more invariants we have to add more φ_ι-s and W_ι; some does not fit this frame so well that is definable by a Borel two place relation R

For example We may add $W_{0.1}$ using it to increase the MA number $\mathfrak{m}$. but not for now.

Now usually $\lg(\mathbf{q})$ is a regular cardinal; which is above all the κ_ι; now the forcing described above does not give the desired result After considerable work it gives reasonable values to the cardinals in the left side of the diagram; like additivity of the relevant ideals; but it cofinality (natural way for add$(null)$, $\mathfrak{b}$ and cov$(null)$, the non$(meagre)$ was treated separately. But what about their dual?

Thesis 9.1. If you like to arrive to X, never try to go there, try to arrive to Y, another one (NOT to any other, just for suitable one)

The manipulation of the cofinalities, by Boolean-ultra-powers or suitable elementary sub-models do this

Additional point, have now to explain the role of, are the special cardinal κ and the ultra-filters.

(a) for some $\alpha \leq \lg(\mathbf{q})$, the forcing notion $\mathbb{Q}_\alpha$ has cardinality $< \kappa$ equivalently $\alpha \in W_\iota, \kappa_\iota < \kappa$

(b) for the other coordinates α we have the memory $u_\alpha \subseteq \alpha$ and an ultrafilter (or ultra-filters) $\underset{\sim}{D}_\alpha$ a $\mathbb{P}_\alpha$-name such that $\underset{\sim}{D}_\alpha$ respect u_α and the forcing is closed under averages by this ultrafilter.

Why the need? usually we use FS iteration with full memory; this has various good properties; but here we cannot use it moreover the memory is not transitive Now in this case various properties like not adding random reals are lost because we have a product; the ultrafilter help us to transfer information to regain some lost properties The point is that if $\gamma < \beta < \alpha < \lg(\mathbf{q})$ and $\gamma \in u_\beta, \beta \in u_\alpha$ but $\beta \notin u_\alpha$ then $\underset{\sim}{\eta}_\alpha$ know, the forcing introducing it depend on $\underset{\sim}{\eta}_\gamma$, but the connection is obscure. we need "($\underset{\sim}{\eta}_\alpha$ (that is the forcing $\underset{\sim}{\mathbb{Q}}_\alpha$) should know enough about $\underset{\sim}{\eta}_\gamma$", we should not let it have the full information BUT "the right amount secrets should be whispered" The condition about ultra-filters do it.

§ 10. Summary

(1) Cardinal invariants of the continuum are important and an exciting direction
(2) Assuming the continuum (that is $2^{\aleph_0}$) is small makes us have many "incidental" equalities
(3) So it is worthwhile to find ways to force the continuum to be large controlling as many cardinal invariants as we can, in particular making them not equal.
(4) This will enable us to see through the cacophony of having so many cardinal invariant which are very independent and single out candidates for true theorems, provable relationships
(5) The method of Boolean ultra-powers does not give us directly relevant forcing notions, it enable us strengthen worthwhile forcing notions; it helps the rich to be super-rich, doubling your gains
(6) Specifically, Cichon's diagram has no redundancy- you may have fully ten different numbers there.

Lecture III Are they all independent

A theme of this lecture is

Thesis 10.1. (A) When nothing works fixing the machine, try to read the manual.

(B) When all attempts to prove independence, try prove a theorem in ZFC

§ 11. Between general topology and model theory

The works on $\mathfrak{p}, \mathfrak{t}$ and the works on Keisler order are with Maryanthe Malliaris

Totally unrelated is a problem in model theory ‘‘what are the maximal theories in Keisler order $\triangleleft$ and the related $\triangleleft^*$;

Definition 11.1. For a countable complete first order theories T_1, T_2 let $T_1 \triangleleft T_2$ mean that:

if K_ℓ is a model of T_ℓ for $\ell = 1, 2$ and s so called regular ultrafilter D on a cardinal λ;
if $(M_2)^\lambda/D$ is λ^+-saturated then also $(M_1)^\lambda/D$

Advances on this were made using classification theory, but not for now and here

The relevant property is

Definition 11.2. A first order complete T has the SOP_2 when for some (first order) formula $\varphi(x, y)$ (but we can use finite tuples instead x, y) there are a model M of T and $a_\eta \in M$ for η a finite sequence of zeros and ones such that : the model M satisfies

(A) if η is an initial segment of ν then M satisfies $\varphi(a_\eta, a_\nu)$
(B) $\varphi(x, a_\eta), \varphi(x, a_\nu)$ are contradictory when η, ν are incomparable

This was totally unrelated to the $\mathfrak{p}, \mathfrak{t}$ Problem

Definition 7.1. (1) The cofinality $\mathrm{cf}(\mathbb{P})$ of a partial order $\mathbb{P}$ is the minimal cardinality of a subset Y such that for every $x \in \mathbb{P}$ there is $y \in Y$ satisfying $x \leq_{\mathbb{P}} y$; So this is exactly $\mathrm{inv}(\leq_{\mathbb{P}}$,

(2) Recall an ordinal is the order type of a well ordering; identified with the set of smaller ordinals, so is a well ordered set

(3) A cardinal = infinite number is identified or represented, with the first ordinal of this cardinality, We let $\aleph_\alpha$ be the α-th infinite cardinal

(4) A cardinal is regular if it is equal to its cofinality, RECALL that $\aleph_0$ and all successor cardinals ($\aleph_{\alpha+1}$ are regular, and $\aleph_\omega$ is the first singular = non-regular cardinal)

$\aleph_n \nearrow \aleph_\omega$

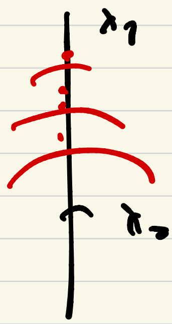
Thesis 7.2. (1) Cofinalities are much easier to understand then cardinalities;

(2) So even if you are intersted in cardinalities, many times it is better to analyse the related cofinalities

Let me give examples

Example 7.3. In model theory:
1) (Lowenheim-Skolem) For given cardinls $\lambda_1, \lambda_2, \kappa_1, \kappa_2$ we can ask:
A. for model M , if P_1^M, P_2^M has cardinality λ_1, λ_2 respectively then M has a submodel N such that P_1^N, P_2^N has cardinality κ_1, κ_2 respectively . The answer is "rarely", (cases of Chang conjecture)
B. similarly for cofinalities: the answer is usually yes, if the cardinals are regular then usually the asnwer is yes; e.g if $\lambda_1 > \lambda_2 > \kappa_1 + \kappa_2$
2) This is true also when we take ultra-powers (or relatives like Boolean ultra-powers, see below)
3) [Compactness] If we expand first order logic by adding the quantifier "a definable set is uncountable" we get compactness only for countable set of sentences; but if we add the quantifier "the cofinality of a definable linear order is uncountable" we get a fully compact logic

Here only the first example is relevant



||| 1.3

Definition 1.1. (See, e.g., [8].) *We define several properties which may hold of a family $D \subseteq [\mathbb{N}]^{\aleph_0}$, i.e., a family of infinite sets of natural numbers. Let $A \subseteq^* B$ mean that $\{x : x \in A,\ x \notin B\}$ is finite.*

- *D has a* pseudo-intersection *if there is an infinite $A \subseteq \mathbb{N}$ such that for all $B \in D$, $A \subseteq^* B$.*
- *D has the s.f.i.p. (strong finite intersection property) if every nonempty finite subfamily has infinite intersection.*
- *D is called a tower if it is well ordered by $\supseteq^*$ and has no infinite pseudo-intersection.*

Then,

$$\mathfrak{p} = \min\{|\mathcal{F}| \ : \ \mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0} \text{ has the s.f.i.p. but has no infinite pseudo-intersection}\},$$

$$\mathfrak{t} = \min\{|\mathcal{T}| \ : \ \mathcal{T} \subseteq [\mathbb{N}]^{\aleph_0} \text{ is a tower}\}.$$

Clearly, both $\mathfrak{p}$ and $\mathfrak{t}$ are at least $\aleph_0$ and no more than $2^{\aleph_0}$. It is easy to see that $\mathfrak{p} \leq \mathfrak{t}$, since a tower has the s.f.i.p. By a 1934 theorem of Hausdorff $\aleph_1 \leq \mathfrak{p}$ [13]. In 1948 [37], Rothberger proved (in our terminology) that $\mathfrak{p} = \aleph_1$ implies $\mathfrak{p} = \mathfrak{t}$, which begs the question of whether $\mathfrak{p} = \mathfrak{t}$.

Problem 1. *Is $\mathfrak{p} = \mathfrak{t}$?*

appear.

[1] 1.4

Cantor proved in 1874 that the continuum is uncountable, i.e., $\aleph_0 < 2^{\aleph_0}$ [6]. The study of cardinal invariants or characteristics of the continuum illuminates this gap by studying connections between cardinals measuring the continuum which arise from different perspectives: combinatorics, algebra, topology, and measure theory. Although there are many cardinal invariants and many open questions about them (see, e.g., the surveys of van Douwen [8], Vaughan [49], and Blass [5]), the problem of whether $\mathfrak{p} = \mathfrak{t}$ is the oldest and so holds a place of honor. (Moreover, usually if such an equality was not obviously true it was consistently false, by forcing.)

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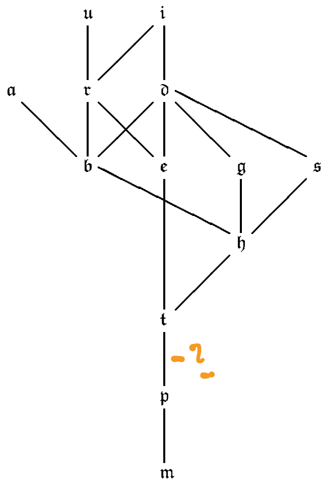

Diagram of Combinatorial Characteristics

Theorem 11.3. *(1)* $\mathfrak{p} = \mathfrak{t}$

(2) A first order countable complete T, if T is SOP_2 then it is $\triangleleft$-maximal

(3) For $\triangleleft^*$ we get "iff"; (well using a case of GCH)

Cofinality spectrum

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Thesis 12.3.
(a) The impression was that cardinal exponentiation essentially has no non-classical restrictions, well except some anomalies
(b) this is wrong; actually there are two phenomena which should be separated
(c) phenomenon 1: the function $\lambda \mapsto 2^\lambda$ for λ successor or $\aleph_0$ or so called regular cardinals; for this there are no classical restriction;
(d) phenomena 2A: the function $(\lambda, \kappa) \mapsto \lambda^\kappa$ for $\lambda \geq 2^\kappa$; here there are serious restriction; the concentration on $\lambda \mapsto 2^\lambda$ obscures this.

WE may concentrate on $\lambda^{\aleph_0}$ equivalently on $\prod_n \lambda_n$

(e) phenomena 2B: better is the function $(\lambda, \kappa) \mapsto cf([\lambda]^\kappa)$ for $\lambda > \kappa$, regular for transparency, see below
(f) phenomena 2C: best is the function $(\lambda, \kappa) \mapsto cf(\lambda^{[\kappa]})$ for $\lambda > \kappa$, regular for transparency; see below
(g) so best is to have: for all regular $\lambda > \kappa$ we have $\lambda^{[\kappa]} = \lambda$; but not possible , so
(h) the best we can hope for is "for most (λ, κ) we have $\lambda^{[\kappa]} = \lambda$;

Recall

Claim 12.4. *(1) For $\lambda > \kappa$ we have $\lambda^\kappa = cf([\lambda]^\kappa) + 2^\kappa$*
(2) For $\lambda > \kappa$, regular for transparency, we have $\lambda^{[\kappa]} = \lambda$ if and only if $(\forall \theta \leq \kappa)[\lambda^{[\theta]} = \lambda]$

Definition 12.5. (1) $cf[\lambda]^\kappa$ mean the cofinality of $([\lambda]^\kappa, \subseteq)$
(2) for $\lambda < \theta$ is the minimal cardinality of a subset $\mathscr{P}$ of $[\lambda]^\theta$ such that every $u \in [\lambda]^\theta$ is included in the union of $< \theta$ members of $\mathscr{P}$

The simplest case is $\prod_n \aleph_n$;

$\lambda^{\aleph_0}$

$\prod \aleph_n < \aleph_{\omega_4}$

or $= 2^{\aleph_0}$

why the hell 4?

Notation 12.6. Let $\mathfrak{a}$ denote a set of successor or just regular cardinals whichare biggen than the cardinality of $\mathfrak{a}$,

We use also $\mathbf{a}, \mathfrak{g}, \mathbf{b}, \mathfrak{c}, \mathbf{c}, \mathfrak{d}, \mathbf{d}$

Definition 12.7. (1) A cardinal λ is a possible cofinality of $\mathbf{a}$ if there is an ultrafilter $\mathscr{D}$ on $\mathbf{a}$ such that the linear order $\Pi\mathbf{a}/\mathscr{D}$ is λ; see below

(2) Let $\mathrm{pcf}(\mathbf{a})$ be the set of possible cofinalities of $\mathfrak{a}$.

(3) We say $\mathscr{D}$ is an ultrafilter of the set I when it is the compliment on a maximal ideal on the Boolean algebra $\mathscr{P}(I)$

Alternatively is is a family of subset of I closed under increasing, under intersection of 2 , such that the empty set does not belongs to it but for every $X \subseteq I$ exactly one of $X, I \setminus X$ belongs to D

(4) The order on $\Pi\mathfrak{a}/\mathscr{D}$ is:

for $f, g \in \Pi\mathfrak{a}$, that is functions with domain $\mathfrak{a}$ such that $\theta \in \mathfrak{a} \Rightarrow f(\theta), g(\theta) \in \theta$ we have

$f \leq_{\mathscr{D}} g$ if and only if the set $\{\theta \in \mathfrak{a} : f(\theta) \leq g(\theta)$ belong to $\mathscr{D}$

You can think of $\mathfrak{a} = \{\theta_n; n \in \omega\}$ and so essentially we are looking at $\Pi_n \theta_n$

Lastly,

Theorem 12.8 (The continuity theorem). *If μ is singular cardinal, so its cofinality is $< \mu$, and its confinality is uncountable, eg $\aleph_{\omega_1}$, then for som eclosed unbounded set C of cardinal $< \mu$ we have $\mu^+ = \max \mathrm{pcf}\{\lambda^+ : \lambda \in C\}$*

Since there are $2^{2^{|I|}}$ ultrafilters on I, pcf $\mathbf{a}$ could be quite large a priori; this would restrict heavily its applications to cardinal arithmetic. Fortunately there are various uniformities present that lead to a useful structure theory for pcf.

THE PCF. THEOREM

1. $\mathrm{pcf}\,\mathbf{a}$ *contains at most* $2^{|\mathbf{a}|}$ *cardinals;*

2. $\mathrm{pcf}\,\mathbf{a}$ *has a largest element* $\max\mathrm{pcf}\,\mathbf{a}$;

3. $\mathrm{cof}\prod\mathbf{a} = \max\mathrm{pcf}\,\mathbf{a}$

4. *For each* $\lambda \in \mathrm{pcf}\,\mathbf{a}$ *there is a subset* $\mathbf{b}_\lambda$ *of* $\mathbf{a}$ *such that*
   a. $\lambda = \max\mathrm{pcf}\,\mathbf{b}_\lambda$, *and*
   b. $\lambda \notin \mathrm{pcf}(\mathbf{a} - \mathbf{b}_\lambda)$;

$\mathcal{G}_0 = \{x_n : n\}.$

$\min \mathrm{pf}(\mathcal{G}_0) = x_0/n$

$b_{x_0/n} = \{x_0/n\}$

but:

5. *If* $\mathcal{J}_\lambda$ *is the ideal on* I *generated by the sets* $\mathbf{b}_\mu$ *for* $\mu < \lambda$, *then for each* $\lambda \in \mathrm{pcf}\,\mathbf{a}$ *there are functions* f_i^λ $(i < \lambda)$ *such that*
   a. *for* $i < j$ *we have* $f_i^\lambda < f_j^\lambda \bmod \mathcal{J}^\lambda$;
   b. *for any* $f \in \prod \mathbf{a}$ *and* $\lambda \in \mathrm{pcf}\,\mathbf{a}$ *there is some* $i < \lambda$ *such that* $f < f_i^\lambda \bmod \langle \mathcal{J}_\lambda, (\mathbf{a} - \mathbf{b}_\lambda) \rangle$.

both dominating
& increasing.

For $\operatorname{cof}\lambda \le \kappa < \lambda$ we define the *pseudopower* $\mathrm{pp}_\kappa(\lambda)$ as follows.

5.1. Definition. 1. $\mathrm{pp}_\kappa(\lambda)$ is the supremum of the cofinalities of the ultraproducts $\prod \mathbf{a}/\mathscr{F}$ associated with a set of at most κ regular cardinals below λ and an ultrafilter $\mathscr{F}$ on $\mathbf{a}$ containing no bounded set bounded below λ.

2. $\mathrm{pp}(\lambda)$ is $\mathrm{pp}_{\operatorname{cof}\lambda}(\lambda)$.

Let $\mathrm{PP}_\kappa(\lambda)$ be the set of cofinalities whose supremum was taken to get $\mathrm{pp}_\kappa(\lambda)$. This turns out to have the simplest possible structure.

5.3. Convexity Theorem. *If $\kappa \in [\operatorname{cof}\lambda, \lambda)$ then $\mathrm{PP}_\kappa(\lambda)$ is an interval in the set of regular cardinals with minimum element λ^+.* //

( ) Hence $PP_\kappa(\lambda) \subseteq \lambda_{(2_2)}^*$

4.4. Localization Theorem. *Let $\mathbf{a}$ be a set of κ distinct regular cardinals with $\lambda > \kappa$ for all $\lambda \in \mathbf{a}$; and suppose $\mathbf{b} \subseteq \operatorname{pcf}\mathbf{a}$ with $\lambda > |\mathbf{b}|$ for all λ in $\mathbf{b}$. If $\mu \in \operatorname{pcf}\mathbf{b}$ then $\mu \in \operatorname{pcf}\mathbf{a}$, and for some $\mathbf{c} \subseteq \mathbf{b}$ of cardinality at most κ we have $\lambda \in \operatorname{pcf}\mathbf{c}$.*

§ 13. WHY THE HELL $\aleph_{\omega_4}$?

Now we try to explain how do we get the 4 above, recalling the better than $\aleph_{\omega_1}$ we cannot prove (well, except it you will prove that some well established large cardinals do not exist).

FIRST ROUND:

First step, we should replace $\Pi_n \aleph_n$ by the cofinality of $[\aleph_\omega]^{\aleph_0}$ because $\Pi_n \aleph_n = {\aleph_\omega}^{\aleph_0} = \text{cf}([\aleph_\omega]^{\aleph_0}) + 2^{\aleph_0}$.

This just say that we are proving a stronger theorem.

Second step it to prove that $\text{cf}([\aleph_\omega]^{\aleph_0})$ is equal to $\sup\{\text{cf}(\Pi_n/\mathscr{D}; u \subseteq \omega = \mathbb{N}, \mathscr{D}$ an ultrafilter on $u\}$, we know by th pcf theorem that this is eqaul to $\text{cf}\Pi_n \aleph_n$

Third so we may try to investigate the set $PP(\aleph_\omega)$ which is the set $\{\text{cf}(\Pi_{n \in u}; u \subseteq \omega = \mathbb{N}\}$ APPLY the pcf LAWS

Fourth step, this set (discarding $\aleph_0$) is an initial segment of the set of successor ordinals.

Fifth step, on this set there is a natural structure coming from $\kappa \in \text{pcf}(\mathfrak{a})$ for $\mathfrak{a} \subseteq PP(\aleph_\omega)$

SECOND ROUND

We shall investigate it using so called guessing of clubs and the pcf laws, till we get a contradiction of $PP(\aleph_\omega)$ is too large

It is enough to consider the set $\mathbf{a}_* = \{\aleph_{\alpha+1} : \alpha < \omega_4\}$, endow it with the topology= the closure operation $\text{pcf}(\mathbf{a})$ for $\mathfrak{a} \subseteq \mathfrak{a}_*$.

Why does it help?

THIRD ROUND

We have to recall the rules of pcf, which tell us

(A) (Continuity) if δ is ω_1 or just any limit ordinal of uncountable cofinality (here $< \aleph_4$) then for some increasing continuous sequence $\langle \alpha_i : i < \mathrm{cf}(\delta) \rangle$ with limit δ we have: $\aleph_{\delta+1}$ is equal to

$$\max \mathrm{pcf}(\{\alpha_{i+1} : i < \mathrm{cf}(\delta)\}$$

(B) (monotonicity) if $\mathbf{a}, \mathbf{b} \subseteq \mathbf{a}_*$ and $\mathbf{b} \subseteq \mathrm{pcf}(\mathbf{a})$ then $\mathrm{pcf}(\mathbf{b}) \subseteq \mathrm{pcf}(\mathfrak{a})$

(C) (local character) if $\mathbf{a}, \mathbf{b} \subseteq \mathbf{a}_*$ and $\lambda \in \mathrm{pcf}(\mathbf{b}), \mathbf{b} \subseteq \mathrm{pcf}(\mathbf{a})$ then for some $\mathbf{b}^* \subseteq \mathbf{b}$ of cardinality at most that of $\mathbf{a}$ we have $\lambda \in \mathrm{pcf}(\mathbf{b}^*)$

(D) (being a closure operation) of course $\mathbf{a} \subseteq \mathrm{pcf}(\mathbf{a})$

FOURTH ROUND We can now forget pcf and cardinal arithmetic.

We have a well ordered set of cardinality $\aleph_4$, even order type ω_4 which satisfies the laws above; the rest of the proof use only them. So if you ask about "why the hell is it four? why not three? even better two? or, best one? The answer is in this part, which does not work for too small number.

We need the following.

Theorem 13.1 (club guessing). *Assume $\lambda = \aleph_1$ or just a regular uncountable cardinal, (we need it for $\lambda \le \aleph_3$. Then there is a sequence $\bar{\mathscr{P}}$ such that*

(a) $\bar{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda^+ \rangle$
(b) $\mathscr{P}_\alpha$ is a family of closed subsets of the ordinal α
(c) if $\alpha \in u \in \mathscr{P}_\beta$ then $u \cap \alpha \in \alpha$
(d) if E is a closed unbounded subset of λ^+ and θ is a cardinal $< \lambda$ then there is a limit ordinal $\delta \in E$ of cofinality $\mathrm{cf}(\theta)$ such that: some unbounded (necessarily closed) subset u of δ from $\mathscr{P}_\delta$ is included in E

How does this compare with Jensen's principle?
It is much weaked but proved in ZFC.

§ 14. The resurrection of the dead

Thesis 14.1. (A) We all know that the axiom of choice is true.

I agree

(B) We all know the the research into versions of the axiom of choice is a dead end area.

I had agree BUT have changed my mind

More accurately I first start to prove and then change my mind

(C) A central reason for deserting the non-choice set theory was because we could not prove much

(C) Advancing in a direction which is very active area is praise worthy, but resurrecting a direction everyone consider dead is too

Definition 14.2. (1) let Ax^4_α for an ordinal α, usually a cardinal mean that the set $[\lambda]^{\aleph_0}$ can be well ordered

(2) Let Ax_4 be the statement that for every ordinal α we have Ax^4_α

(3) Let DC, the axiom of dependent choice mean that if $\mathscr{T}$ is a tree consisting of finite sequences ordered by being initial segment, closed under initial segment, is there is no maximal member then there is an ω branch, that is a no ω- sequence every initial segment of which belong to $\mathscr{T}$.

Univ Earlier forcing

(AC = $\bigwedge_\lambda$ P(λ) well ordered)

Th (ZF) Łoś conjecture = Morley theorem is true

ZF+ model theory not empty

Discussion 14.3. Not that with restricted choice; some obvious properties of cardinals λ become problematic

(*) a successor cardinal λ^+ may be singular , e.g of countable cofinality

Theorem 14.4. *Assume* Ax_4.

(1) There is a class of successor cardinal which are regular
(2) moreover, "usually" a successor of singular cardinal is regular [4]

Theorem 14.5. *Assume* Ax_4

(1) *For every cardinals* $\lambda > \kappa$ *the set* ${}^{\kappa}\lambda = \{: \eta$ *a function from* λ *to* $\lambda\}$ *is covered by to essentaiily* $\mathscr{P}(\mathscr{P}(\kappa)))$ *sets which are well ordered*
(2) *The pcf theorem still true with very modest adaptations*

[835] [1005]

Conclusion 0.3. *[DC] Assume $[\lambda]^{\aleph_0}$ is well ordered for every λ.*
1) If 2^{2^κ} is well ordered then for every $\lambda, [\lambda]^\kappa$ is well ordered.
2) For any set Y, there is a derived set Y_* so called $\mathrm{Fil}^4_{\aleph_1}(Y)$ of power near $\mathscr{P}(\mathscr{P}(Y))$ such that $\Vdash_{\mathrm{Levy}(\aleph_0,Y)}$ "for every $\lambda, {}^Y\lambda$ is well ordered".

Thesis 0.4. 1) If $\mathbf{V} \models$ "ZF + DC" and "every $[\lambda]^{\aleph_0}$ is well orderable" then $\mathbf{V}$ looks like the result of starting with a model of ZFC and using $\aleph_1$-complete forcing notions like Easton forcing, Levy collapses, and more generally, iterating of κ-complete forcing for $\kappa > \aleph_0$.
2) This approach is dual to investigating $\mathbf{L}[\mathbb{R}]$ - here we assume ω-sequences are understood (or weaker versions) and we try to understand $\mathbf{V}$ (over this), there over the reals everything is understood.

Also though our original motivation was to look at consequences of Ax_4, this was shadowed here by the try to use weaker relatives; see more in [She:1005].

Explanation 0.5. How do we analyze $[\mu]^\kappa$ or equivalently ${}^\kappa\mu$ here? We use $\aleph_1$-complete filters on κ and a well ordering of $[\alpha]^{\aleph_0}$ for appropriate α or less. We will consider $f : \kappa \to \mu$; now for every $\aleph_1$-complete filter D on κ, the ordinal $\mathrm{rk}_D(f)$ gives us some information on α, but if $A, \kappa\backslash A \in D^+$ and $f{\restriction}A = 0_A$, then $\alpha = 0$ but we have no information on $f{\restriction}(\kappa\backslash A)$, then $\alpha = 0$ but we have no information on $f{\restriction}(\kappa\backslash A)$. Trying to correct this we consider the ideal $J[f, D] = \{A \subseteq \kappa : A = \emptyset$ mod D or $A \in D^+$ but $\mathrm{rk}_{D+(A)}(f) > \alpha\}$, this is an $\aleph_1$-complete ideal and so we may consider the pair $\bar{D} = (D_1, D_2) = (D, \mathrm{dual}(J[f, D]))$. Now α and the pair $\bar{D}$ gives more information on f; they determine f modulo D_2. This is not enough so we use an algebra $\mathscr{B}$ on μ with no infinite decreasing sequence of sub-algebras built using the assumption "$[\mu]^{\aleph_0}$ is well ordered". So there is $Z \in D_2$ such that $A = cl_{\mathscr{B}}(\mathrm{Rang}(f{\restriction}Z))$ is $\subseteq$-minimal.

Now the triple (D_1, D_2, Z) and the ordinal α almost determines f, we need one more piece of information with domain $\kappa : h(i) = \mathrm{otp}(\alpha \cap Z)$, hence an ordinal $< \mathrm{hrtg}(\mathrm{Rang}(f))$. So we need a bound on it which depends on the choice of $\mathscr{B}$, usually it is $\mathrm{hrtg}([\kappa]^{\aleph_0})$, natural by the construction of $\mathscr{B}$.

So $f{\restriction}Z$ is uniquely determined by the ordinal $\mathrm{rk}_D(f)$ and the quadruple (D_1, D_2, Z, h), which belongs to a set defined from κ, independently of μ.

Lastly, considering all such filters D (recalling we are assuming DC) we can find countably many quadruple (D_1^n, D_2^n, Z^n, h^n) which together are enough as $\bigcup\limits_n Z^n = \kappa$.

Claim [redacted]. [DC] *Assume that μ is a singular cardinal of cofinality $\kappa > \aleph_0$ (no* GCH *needed), the parameter $X \subseteq \mu$ codes in particular the tree $\mathscr{T} = {}^{\kappa>}\lambda$ and the set $\mathscr{P}(\mathscr{P}(\kappa))$ and $F : {}^{\omega}\mu \to \mu$ which satisfies "(μ, F) has no infinite decreasing ω-chain of subalgebras"; in particular, from X a well orderings of $[\lambda]^{<\kappa} \cup \mathscr{P}(\mathscr{P}(\kappa))$ are definable. Then (with this parameter) we can define a well ordering of the set of κ-branches of the tree $({}^{\kappa>}\lambda, \triangleleft)$.*

§ 15. Summary

(A1) For long set theorist did not $\aleph_\omega^{\aleph_0}$. look at for good reason

(A2) Even after the results of Easton; everybody know that $2^{\aleph_\omega}$ is just a problem of finding more complicated forcing

(A3) Reasonable but false

(B) For long general topologists and set theorist, including myself, know that surely possibly $\mathfrak{p} < \mathfrak{t}$ it just require more complicated forcing; as generally making the continuum at least $\aleph_3$ is harder;

(B2) Very reasonable but false

Claim 15.1. *WE ARE ALL VERY WISE: PARTICULARLY A POSTERITY*

Answer: GCH was extensively used say in the sixties

BUT because they believe, but they could not

prove otherwise; but now this is not the case

Questions and Answers

Question and Answers for Lecture 1.

Yuval Paz 06:01 PM Is there a good motivation for this specific definition of $[\lambda]^\kappa$?

Goldstern 06:02 PM
Saharon briefly sketched his motivation: you do not want to have a dependence on values $[\lambda]^{\kappa'}$ for smaller $\kappa' < \kappa$, so you declare that unions of size < lambda can be obtained for "free" (so you succeed in making them independent)

Expanded: Recall that Easton's result tells us that on 2^λ for λ successor (or $\aleph_0$ or just regular) we cannot prove anything more than the classical results. This leads us to consider the cofinality of $[\lambda]^\kappa$ for regular $\lambda > \kappa$, recalling that $\lambda^\kappa = \mathrm{cf}([\lambda]^\kappa) + 2^\kappa$ and on 2^κ we can say nothing. So the best we could have aspired to is

$(*)_1$ for all regular $\lambda > \kappa$ we have $\mathrm{cf}([\lambda]^\kappa) = \lambda$; called the strong hypothesis.

But finer independence results exclude this. Now it is natural to to ask at least that "for most such pairs (λ, κ)", but the monotonicity of this function in κ makes this unreasonable. So we shall restate $(*)_1$ in a way that monotonocity disappear. Toward this we define revised power

$(*)_2$ for regular $\lambda > \kappa$ we define $\lambda^{[\kappa]}$, the revised power of λ by κ as the minimal cardinality of a subset $\mathscr{P}$ of $[\lambda]^\kappa$ such that
- every subset of λ of cardinality κ is included in the union of $< \kappa$ members of κ

Indeed, speaking on $\lambda^{[\kappa]} = \lambda$ makes us naturally to bisect $(*)_1$ because
$(*)_3$ for every regular $\lambda > \kappa$ we have $\mathrm{cf}([\lambda]^\kappa) = \lambda$ if and only if for every regular $\theta \leq \kappa$ we have $\lambda^{[\theta]} = \lambda$.

So the best we can hope for is
$(*)_4$ for "most" regular $\lambda > \kappa$ we have $\lambda^{[\kappa]} = \lambda$
More in the third lecture.

We can interpret the question differently: is this reformulating legitimate? The natural criterion is considering other Hilbert problem, we have given above only a partial quotation so let us expand in the next answer.

Vadim Kulikov 06:12 PM
By "positive solution", do you mean CH or not-CH (i.e. there exists a set of reals...)?
This question has been answered live

Expanded: By positive answer (of Hilbert first problem) we do not mean CH or a value for $2^{\aleph_0}$. though we think it should be quite large. We mean that first taking considering the independence result of Easton and then more, the best we could have hoped for is $(*)_4$ stated in the previous answer. In short, a weak version of GCH, which is best when we take into account what is impossible to prove by the independence results.
The specific choice of "most" seem very reasonable; though we can hope for more (e.g. essentially replacing $\beth_\omega$ by $\aleph_\omega$).

Is such reformulation legitimate? As an argument, we can cite, from the book [Br] on Hilbert's problems, Lorentz's article on the thirteenth problem. The problem was

(A) Prove that the equation of the seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable with the help of any continuous functions of only two variables.

Lorentz does not even discuss the change from 7 to n and he shortly discard the polynomial and changed the question to (see

(B) Prove that there are continuous functions of three variables not represented by continuous functions of two variables.

Then, he discusses Kolmogorov's solution and improvements (giving negative solution). He opens the second section with ([Br, p.421,16-22]): "that having disproved the conjecture is not solving it, we should reformulate the problem in the light of the counterexamples and prove it, which in his case: (due to Vituvskin) the fundamental theorem of the Differential Calculus: there are r-times continuously differential functions of n variables not represented by superpositions of r times continuously times differential functions of less than n variables".

Concerning the fifth problem, Gleason (who makes a major contribution to its solution) says (in [AAC90]): "Of course, many mathematicians are not aware that the problem as stated by Hilbert is not the problem that has been ultimately called the Fifth Problem. It was shown very, very early that what he was asking people to consider was actually false. He asked to show that the action of a locally-euclidean group on a manifold was always analytic, and that's false. It's only the group itself that's analytic, the action on a manifold need not be. So you had to change things considerably before you could make the statement he was concerned with true. That's sort of interesting, I think. It's also part of the way a mathematical theory develops. People have ideas about what ought to be so and they propose this as a good question to work on, and then it turns out that part of it isn't so."

In our case, I feel that while the discovery of **L** (the constructible universe) by Gödel and the discovery of forcing by Cohen are fundamental discoveries in set theory, things which are and will continue to be in its center, forming a basis for flourishing research, and they provide for the first Hilbert problem a negative solution which justifies our reinterpretation of it. Of course, it is very reasonable to include independence results in a reinterpretation.

Jouko A Väänänen 06:14 PM
In your result about $2^{\aleph_\omega}$, why $\aleph_{\omega_4}$ Where does the "4" come from? Why not "3"?
This question has been answered live

Expanded: Maybe the 4 rather then 3 (or even 1) is an artifact of human failure rather that of nature. Still, I have looked at it several times and I know that many have immediately react "this is a misprint, you cannot really seriously mean 4", and people have read and represent the proof; so surely many tries (but mathematicians normally do not record their failures, probably as it is like dog biting a man- being so common no point to record it).

Also we cannot go below $\aleph_{\omega 1}$, that is as long as some suitable so called large cardinals are not proven inconsistent, which seem very unlikely.

So it seem we are stuck with 4 for the time being.

Anyhow, we can ask why the present proof give 4?

In short, moving our focus to pcf/cofinality arithmetic give us a topology on the set of regular cardinals which satisfies various laws. Some say that when we have too many laws society would collapse, anyhow in our case it lead to contradiction.

So the 4 look bizarre considering cardinal arithmetic, but probably would not look so in the context of the pcf laws. The long answer is to listen to the (second part of the) third lecture.

Neil Barton 06:15 PM

Thanks for the talk! Here, and a little in 'Logical Dreams' you seemed to suggest that ZFC was somehow inevitable, because of its ease of use. I wondered the extent to which you thought this was really inevitable. One can imagine, for instance, that mathematicians became very committed to every set being countable (perhaps the continuum being a proper class), and then thinking that ZFC studies *small* countable models (or inner models missing out subsets), with ZFC being the study of these worlds but the 'real' world containing only countable sets. (There are other thought experiments, e.g. we might deny Replacement.) So, to what extent is ZFC 'inevitable'?

This question has been answered live

Expanded:

There are several answers, not necessarily compatible ones (as in classical excuses: I did not borrow it, I have return it whole and I have borrow it broken).

First, suppose we encounter some nice aliens and find out they have adopted such set theory, not so unreasonable considering the Egyptian system of writing fractions and calculating with them; make their life hard but do not lead to a different mathematics. When we define the constructible universe **L** of Gödel, there is no difference. Also we can can define/cod sets of reals or equivalence classes of various cardinalities Of course there will be some differences, actually differences in stress: the cardinal ${\aleph_\omega}^{\mathbf{L}}$ will be a (mild) large cardinal; Borel determinacy will be proved only assuming suitable large cardinals (in inner models). But hose do not seem to me essential.

Second I think ZFC is really the natural choice. Would you think twice about the direct product of two ring $R_1 \times R_2$, the group of homomorphisms from one Abelian group to another, $Hom(G, H)$, use the sub-group of G generated by a set? Essentially you are accepting ZFC.

Andrés Villaveces 06:16 PM

Can you *place* (or recall) the moment (or the mathematical situation or construction) that led you originally to see that cofinality of $[\lambda]^\kappa$ was *more robust* than the usual power? What made you conjecture (and then prove) it?

This question has been answered live

Expanded: History is not so logical and ordered. The years 1974-6 were very exciting for cardinal arithemetic (and set theory); I have felt I have come late to the party. I still wrote [Sh:68] which lead me to the question

($*$) does $\aleph_{\omega+1} \in \text{pcf}\{\aleph_n : n < \omega\}$

It was a sufficient condition for $\aleph_{\omega+1}$ having a Jonsson algebra.

I have thought about it; thought it natural, it bug me and think it is a good problem but did not really think it is so central, did not particularly work on it.

Later I work on [Sh:111] which deal with bounds on μ for cardinals like "$\aleph_\delta = \beta_\delta$ the ω_1-fix point, ill luck - editor waiting for a referee report which was not actually promised and typing problems- delay it for many years

in 1980 visiting Harvey Friedman with leo Harrington and Hugh Woodin; hearing on relevant advances of the later I have work on a different direction on so called strong covering lemma and on bounding $2^{\aleph_\omega}$ really $(\aleph_\omega)^{\aleph_0}$ when $\aleph_\omega = \beth_\omega$ or just $\aleph_\omega > 2^{\aleph_0}$. For this develop pcf for set of κ cardinals which are bigger than 2^κ . Using $2^{\aleph_0}$ rather then $\beth_2$ (or 2^κ instead 2^{2^κ}) take considerable work. I was very happy about it, the bound naturally was $\aleph_\delta, \delta = (2^{\aleph_0})^+$. My impression is that people thinks well of the result but look at pcf as a technical point. I disagree, but not strongly enough to continue to work on it; and/or do not see what is the next step.

Still was very interested, thought it fundamental but did not continue rather try better cardinal bound- covering more cardinals; continuing [Sh:111] in [Sh:256], [Sh:333], the cardinals were small- before the first inaccessible or having not too many inaccesible cardinals below it. Mentioning the problem $\aleph_{\omega+1} \in \text{pcf}\{\aleph_n : n < \omega\}$ I was not convincing.

All those use filters which were $\aleph_1$ complete so deal with cardinals of uncountable cofinlaity, less related was [Sh:233])

Next comes working on some problems of Monk on Boolean algebras and their cardinal invariants; see the first half of [Sh:345]. Thinking how to solve those, It occur to me that maybe we can analyze pcf of sets of regular cardinals which are just above

this excite me beyond it use for those problems become I become convince that pcf is central and fruitful, and work on it in the later eighties

Coming to MSRI in fall 1989, I have met Leo and tell him the exciting news. He was very nice but when interrogated he answer "Hilbert first, cardinal arithmetic are central in set theory? great problem but this lemon was squeezed dry; the cardinals dealt with ([Sh:256] are not so exciting". Going back to the apartment I have worked for few days on [Sh:400], aided by the recent work on [Sh:309], [Sh:331].

Later dealing (in [Sh:454]) with a problem in (very) general topology (of Kishor Kale communicated to me by Wilfrid Hodges) I realize that to solve it in full generality (not only the original problem) it will help to have : if $\mu = \beth_\delta$ is strong limit singular, then for cardinals in the intervals $(\mu, 2^\mu)$ something like the RGCH will help, this was done in [Sh:454a] and lead to [Sh:460]; I still like the general topology problem- but probably most will look at it as a case of the birth of a pearl.

<u>Hanjo</u> 06:17 PM

Both in the abstract and in the lecture Prof. Shelah maintained that the problem of the size of the continuum is equivalent to: ‘‘What are the laws of cardinal arithmetic, i.e. the arithmetic of infinite numbers". This sounds plausible to me as a layperson, because the step from $\aleph_0$ to $\aleph_1$, in Cantor's work sets up the very idea of an arithmetic of infinite numbers so to speak. But can one rule out that there other laws, and if so how?

This question has been answered live

Expanded: Pedantically I would say that Hilbert first problem means findsing the laws of cardinal arithemtic. Concerning the existence of further laws of cardinal arithemtic, absolutely I do not think it is the end each generation put its layer, more than enough left.

Jouko A Väänänen 06:18 PM

In mathematics sometimes assuming CH simplifies a proof. What about revised GCH? Can it be used to simplify proofs?

This question has been answered live

Expanded: It is a reasonable approach; but I am not so excited by simplifying proofs. I had and still am convinced that it should help significantly in combinatorial set theory and it applications; I have expected more , a partial explanation may be that I have not systematically try to find ones, just use it when it naturally arise Yes there are applications.. Hope for more... But have not dedicate myself to it.

Question and Answers for Lecture 2. Vadim Kulikov 02:35 PM

If I remember correctly, Sy Friedman had a programme to show that continuum is large. I don't know if he succeeded (Martin?) and if his programme is related in any way to the present argumentation of Shelah?

Martin Goldstern 02:40 PM

I thought he was mainly interested in the ‘‘global" structure of the universe, involving inner models and class forcing

Expanded: Sorry, I do not know

Neil Barton

Martin Goldstern 03:05 PM

In his question, Neil Barton has pointed out the ‘‘Strong inner model hypothesis" and its implication for the size of the continuum.

Jouko Väänänen 02:36 PM

Suppose we find new cardinal invariants in the future. Should we then think it pushes continuum up? Is there some reason to think there is an upper bound for the number of cardinal invariants, which are mutually consistent? On the other hand the continuum has some cardinality. So it cannot be pushed up without end.

Martin Goldstern 02:43 PM

In an old paper with Shelah ([Sh:448]), and a later paper by Kellner and Shelah ([Sh:872], [Sh:961]), there is an uncountable (later: perfect) set of very simple cardinal characteristics (all defined by closed relations), which can all be forced to be different (in the same model).

Ralf-Dieter Schindler 02:46 PM

Even worse, aren't there $2^{2^{\aleph_0}}$ cardinal invariants of the continuum? :)

Martin Goldstern 02:52 PM

Nice point. But if you allow all possible relations, then trivially all cardinals below c are the values of some cardinal invariant. So I think it makes sense to look at definable or even projective relations.

See also Blass' survey [Bls10] ("Simple cardinal characteristics of the continuum", 1991).

Expanded: Unlike Pythagoras I see no inherent significance in the number 10, surely - there are many more cardinal invaraints; and anyhow I think the continuum is above $\aleph_\omega$. Martin ask is the continuum a fix point of the alephs; this is reasonable but less persuasive; in fact weakly inaccessible and every real valued measurable were suggested. Naturally just large enough is very reasonable the others are reasonable.

Ali Sadegh Daghighi 02:50 PM

The idea of continuum being large is in contrast with Woodin's Ultimate L program in which continuum takes its minimum possible value, $\aleph_1$. The way that Ultimate L is presented suggests that Woodin believes that $\aleph_1$ is the "true" value of continuum. What is your idea about the general direction of Woodin's program and its possible mathematical and philosphical implications, Saharon?

This question has been answered live

Expanded: I believe Woodin's approach is very interesting, important and probably is the most interesting one for some direction. As is the axiom of determinacy for the family of projective sets. But from other perspective (closer to my heart), other axioms are more interesting.

In particular, an reasonable axiom which implies that many natural cardinal invariants are distinct will may be very intersting.

Neil Barton 03:01 PM

Re: Vadim, the Strong Inner Model Hypothesis implies that the continuum is larger than $\aleph_\alpha$ for any α that is countable in L . We don't know, however, if it's consistent (relative to large cardinals). In any case (my question): If an axiom implied that these cardinal invariants were all simultaneously separable, would it constitute evidence in favour of the axiom?

Martin Goldstern 03:07 PM

Thank you, Neil, for answering Vadim's question. In your question, I assume you mean ‘‘different", not just ‘‘separable (by some forcing notion)", right?

Expanded: Certainly yes , as I have said in the lecture. So e.g. an axiom implying there are ten cardinal invariants in Cichon diagram is very interesting; (but it has to look like an axiom, not just a consistency results a ‘‘semi axiom" in the terminology of [Sh:E23]

David Jose Fernandez Breton 03:10 PM

A question maybe for the end: there is an emphasis that assuming the Continuum too small will "accidentally" force some cardinal characteristics to be equal. But something we learn from Ramsey theory is that sometimes when a structure is too large one winds up with some large interesting substructures that are also there "by accident" (think the three people at a party of at least six). Could it be that there is some tension between these two desideratum (to avoid accidental equalities among cardinal characteristics and at the same time to avoid certain accidental structure coming from Ramsey theory) forcing us to consider a continuum that is not too small, but also not too large?

This question has been answered live

Expanded: I love Ramsey theory, and certainly it is interesting to have such results. In the direction you mention- maybe there are but I do not know. However in second thought there are Ramsey like results assuming the continuum is not too small, and there are consistency results with not CH

See Sierpinski, on the existence of an independent set of size n iff the continuum is at least $\aleph_n$ and see more [Sh:49] which get a a two-cardinal theorem by proving a combinatorial theorem when the continuum is at least $\aleph_\omega$, there are also consistency results of such theorems.

Andrés Villaveces 03:13 PM

Seeing this description of the iteration of the different forcings, there seems to be a kind of mystery: why doesn't one kind of forcing really destroy what has been achieved by the other kinds? (Does this reveal some kind of ‘‘global" structural properties of the null ideal vs the meager ideal?)

This question has been answered live

Expanded: This is the main point of of the recent works on Cichon diagram. It was known since ancient times (that is during the last millennium) that we can handle two of those cardinal invariants; increase one preserving the other. The whole point here is to simultaneously increase each cardinal invariant, no ‘‘harming" those which should be smaller that it

Jouko Väänänen 03:18 PM

Again: Suppose we find new cardinal invariants in the future. Should we then think it pushes continuum up? Is there some reason to think there is an upper bound for the number of cardinal invariants, which are mutually consistent? On the other hand the continuum has some cardinality. So it cannot be pushed up without end.

This question has been answered live

Expanded: I doubt; but in second thoughts, maybe there is a definition of super-nice cardinal invariants for which there are only finitely many solutions. This will be very interesting, but I have doubts, and have come up with no candidate

Vadim Kulikov 03:26 PM

I would be still interested in Shelah's opinion on the Strong Inner model hyp

This question has been answered live

Expanded: I have heard but did not digest it; would be glad to look at it more seriously

Lorenz Halbeisen You have given the ten cardinals in Cichon's diagram ten different values in a linear ordering. Is the linear ordering unique or are there different linear orderings.

Expanded: Yes, some of the works cited give different linear orderings. The number of linear orderings possible by our knowledge is very finite still I doubt I would like to look at all of them.

The main question in this regard is making non(meagre) $<$ cov(meagre) because of FS iterations of of uncountable length this necessarily fail. This involve problems on iterated forcing, a direction I am fond of.

Question and Answers for Lecture 3. Andrés Villaveces 05:11 PM

Can you say a bit more on how the Keisler order, SOP_2 and $\mathfrak{p} = \mathfrak{t}$ are related? (A priory they would seem to be speaking of very different things...)

This question has been answered live

Expanded: Yes, everyone knew that there is no connection; in fact such a connection was not a possibility you will even consider; including me. More on the connection, in short; for Keisler's order we take an ultra-power $N = M^\lambda/D$ of a model M coding enough set theory, (usually by a regular ultrafilter D on λ). Now inside N we investigate pseudo finite sets, linear order and trees. In this context we can define $\mathfrak{p}(M), \mathfrak{t}(M)$ and will prove they are equal, this suffice. In the proof we mainly ask what about possible cuts of such a linear order, what is their pairs of cofinalities (the upper and lower). If one cofinality is small, does it determine the other? if they are both small is the cut symmetric? (that is has the same lower and upper cofinality)

For pseudo finite trees we ask if increasing sequences have upper bounds. All are related to the saturation of ultra-powers of relevant models. For $\mathfrak{p} = \mathfrak{t}$ we first force by infinite sets of natural number modulo finite. This forcing produce an ultrafilter on the old power set of ω but it add no new sequence of length $< \mathfrak{t}$ (and even so called $\mathfrak{h}$). Now we can use it to take ultra-power N of $M = (\mathscr{H}(\chi) \in)$ for χ larger enough then we have a parallel situation.

We may use this opportunity to say more on the proof. For models M, N as above, we define $\mathfrak{p}(M)$ as the saturation of N and $\mathfrak{t}$ will be the least length of an outside increasing sequence of non-standard members of "the tree of increasing sequences of natural numbers $< \mathbf{n}$, a non-standard natural numbers". This can be proved equal to the saturation of $N \restriction \tau_T$ for T a so called SOP_2-theory interpreted in N (say with parameters; the point is the SOP_2 is related to trees). For the set theoretic problem, as the name indicate it is equal to $\mathfrak{t}$.

In more details, $\mathfrak{p}$ is the minimal cardinality of a family of non-empty N-definable subsets of a pseudo finite sets in N, which is closed under intersection of two, but have empty intersection. Now $\mathfrak{p}(N)$, in the model theoretic case is goodness of D, equivalently the saturation of N. The $\mathfrak{t}(M)$ is defined similarly but using a family of definable sets which is linearly ordered (or well ordered). For the set-theoretic question we use $\mathfrak{p}$ and $\mathfrak{t}$. So we "just" need to prove that $\mathfrak{p}(N) = \mathfrak{t}(N)$.

Toward this let $cf - spec(N)$ the the set of pairs (θ, κ) of regular cardinals such that for some cut of a pseudo finite natural number, θ is the cofinality of the lower part of the cut and κ is the cofinality of the upper part of the cut inverted, and we investigate this spectrum. One preliminary step in is that $\mathfrak{p}(N) = \min\{\theta + \kappa : (\theta, \kappa) \in cf - spec(N))\}$ and $\mathfrak{t}(N) = \min\{\theta : (\theta, \theta) \in cf - spec(N)\}$. So to prove $\mathfrak{p}(N) = \mathfrak{t}(N)$ we need to show that is $(\theta, \kappa) \in cf - spec(N)$ has minimal $\sigma = \theta + \kappa$ then $\mathfrak{t} = \sigma$.

From this, in fact the proof was done in three stages, some months separating each. First the case $\theta = \aleph_0$. Second the case $\theta^+ < \mathfrak{p}(N)$ The last was when $\theta^+ = \kappa < \mathfrak{p}(N)$.

Neil Barton 05:29 PM

I wonder: Given these weakenings of AC, what are Saharon's views on the prospects of axiom systems with cardinals that imply $\neg AC$ (e.g. ZF + "There exists a super-Reinhardt cardinal")?

This question has been answered live

Expanded: I have not seriously looked into it. It is interesting, and may have great, interesting implication, but I do not know.

Vadim Kulikov 05:30 PM
So the take away is: CH is false and GCH is true?
This question has been answered live

Expanded: Yes, this is certainly a short and witty way to express the results.

Menachem Kojman 05:30 PM
What will change if you require that all subsets of size $\lambda > \aleph_0$ are well ordered?
This question has been answered live

Expanded: May some day it will be prove that if we have a well ordering of $[\alpha]^{\aleph_1}$ this will have exciting results but so far it does not help.

On the other hand I have not dedicate myself to this direction.

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