On the absoluteness of orbital ω-stability

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Abstract

We show that orbital ω-stability is upwards absolute for $\aleph_0$-presented Abstract Elementary Classes satisfying amalgamation and the joint embedding property (each for countable models). We also show that amalgamation does not imply upwards absoluteness of orbital ω-stability by itself.

Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class (or *AEC*; see [1, 8] for a definition), and let (M, a, N) and (P, b, Q) be such that M, N, P and Q are structures in $\boldsymbol{K}_{\aleph_0}$ (where, for a cardinal κ, $\boldsymbol{K}_\kappa$ denotes the members of $\boldsymbol{K}$ of cardinality κ) with $M \preceq_{\boldsymbol{k}} N$, $P \preceq_{\boldsymbol{k}} Q$, $a \in N \setminus M$ and $b \in Q \setminus P$. The triples (M, a, N) and (P, b, Q) are said to be *Galois equivalent* or *orbitally equivalent* if $M = P$ and there exist $R \in \boldsymbol{K}_{\aleph_0}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi \colon N \to R$ and $\sigma \colon Q \to R$ such that π and σ are the identity on M, and $\pi(a) = \sigma(b)$. If $\boldsymbol{k}$ satisfies amalgamation (the property that if M, N and P are elements of $\boldsymbol{K}$ such that $M \preceq_{\boldsymbol{k}} N$ and $M \preceq_{\boldsymbol{k}} P$ then there exist $Q \in \boldsymbol{K}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi \colon N \to Q$ and $\sigma \colon P \to Q$ such that π and σ are the identity on M) then this relation is an equivalence relation on the class of such triples; each equivalence class is called a *Galois type* or *orbital type* (amalgamation is not necessary for orbital equivalence to be transitive; when transitivity fails one can consider the analogous notions for the transitive closure). We say that the AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is ω*-orbitally stable* if, for each $M \in \boldsymbol{K}_{\aleph_0}$, the set of equivalence classes over M as above for triples (M, a, N) as above with $N \in \boldsymbol{K}_{\aleph_0}$ is countable.

An abstract elementary class $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ over a countable vocabulary τ is called $\aleph_0$*-presentable* (among other names, including $\mathrm{PC}_{\aleph_0}$ and *analytically presented*) if the class of models $\boldsymbol{K}$ and the class of pairs corresponding to $\preceq_{\boldsymbol{k}}$ are each the set of reducts to τ of the models of an $L_{\aleph_1,\aleph_0}$-sentence in some expanded language. Equivalently, $\boldsymbol{k}$ is $\aleph_0$-presentable if it has Löwenheim-Skolem number $\aleph_0$ and the collections of subsets of ω coding (in some natural fashion) the restrictions of $\boldsymbol{K}$ and $\preceq_{\boldsymbol{k}}$ to countable structures are analytic. If $\boldsymbol{k}$ is $\aleph_0$-presentable, ω-orbital stability for $\boldsymbol{k}$ is naturally expressed as a Π^1_4 property in a countable parameter for an analytic definition for $\boldsymbol{k}$. One might hope that this property has a simpler definition, and moreover that the property is absolute between models of set theory with the same ordinals. In this note we show that ω-orbital stability is upwards absolute for $\aleph_0$-presentable abstract elementary classes $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and the joint embedding property (the property that any two elements of $\boldsymbol{K}_{\aleph_0}$ can be $\preceq_{\boldsymbol{k}}$-embedded in a common element of $\boldsymbol{K}_{\aleph_0}$, i.e., that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$

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is directed). We also present an $\aleph_0$-presented AEC for which ω-orbital stability is not upwards absolute, satisfying amalgamation but not the joint embedding property .

We note that almost ω-orbital stability, the property of not having a perfect set of representatives of distinct equivalence classes for orbital equivalence, is Π^1_2 (see [2], for instance). By Burgess's Theorem for analytic equivalence relations (see [4], Theorem 9.1.5), an $\aleph_0$-presented AEC which is almost ω-orbitally stable but not ω-orbitally stable has a countable structure with exactly $\aleph_1$ many orbital types.

1 Upward absoluteness with amalgamation and joint embedding

In this section we show that orbital ω-stability is upwards absolute for any $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and the joint embedding property. The proof below uses the notion of model-theoretic forcing from [8] and the natural generalization of the notion of orbital type to finite sequences. There may be some overlap between the material in this section and Section 4 of [9].

Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class, and let $(M, \langle a_0, \ldots, a_n\rangle, N)$ and $(P, \langle b_0, \ldots, b_q\rangle, Q)$ be such that

- $n, q \in \omega$;
- M, N, P and Q are structures in $\boldsymbol{K}_{\aleph_0}$;
- $M \preceq_{\boldsymbol{k}} N$ and $P \preceq_{\boldsymbol{k}} Q$;
- each a_i is in N and each b_i is in Q.

The triples $(M, \langle a_0, \ldots, a_n\rangle, N)$ and $(P, \langle b_0, \ldots, b_m\rangle, Q)$ are *orbitally equivalent* if $M = P$, $n = q$ and there exist $R \in \boldsymbol{K}_{\aleph_0}$ and $\preceq_{\boldsymbol{k}}$-embeddings $\pi \colon N \to R$ and $\sigma \colon Q \to R$ such that π and σ are the identity on M, and $\pi(a_i) = \sigma(b_i)$ for all $i \leq n$. As above, if $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation, then this relation is an equivalence relation on the class of such triples, and, for a fixed $M \in \boldsymbol{K}_{\aleph_0}$ the set of equivalence classes over M is the set of equivalence classes of triples with M as their first coordinate. By (a special case of) a recent result of Boney [3], if $\boldsymbol{k}$ is orbitally ω-stable (and $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation), then for each $M \in \boldsymbol{K}_{\aleph_0}$ there are just countably many equivalence classes over M in this generalized sense. Boney's arguments go through without change under the assumption that orbital equivalence (for finite tuples) is transitive, in place of amalgamation.

Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$, a subclass $\boldsymbol{K}'$ of $\boldsymbol{K}$ and an $M \in \boldsymbol{K}'$, we say that M is $\preceq_{\boldsymbol{k}}$-*universal* for $\boldsymbol{K}'$ if for each $N \in \boldsymbol{K}'$ there is a $\preceq_{\boldsymbol{k}}$-embedding of N into M; M is $\preceq_{\boldsymbol{k}}$-*maximal* for $\boldsymbol{K}'$ if there does not exist an $N \in \boldsymbol{K}$ (other than M) such that $M \preceq_{\boldsymbol{k}} N$.

For an $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ the following are easily seen to be absolute. The last of these says that there are just countably many orbital types over M.

- The statement that $\boldsymbol{K}_{\aleph_0}$ is nonempty (Σ^1_1 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).
- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).
- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies joint embedding (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).
- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, the statement that M is a $\preceq_{\boldsymbol{k}}$-universal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).
- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, the statement that M is a $\preceq_{\boldsymbol{k}}$-maximal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_1 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).

- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, and a fixed countable set of pairs (a, N) with $N \in \boldsymbol{K}_{\aleph_0}$, $M \preceq_{\boldsymbol{k}} N$ and $a \in N \setminus M$, the statement that every orbital type over M contains a member of the set (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$, M and the set) .

In light of these facts, Theorem 1.2 below shows that ω-orbital stability is upwards absolute for an $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ for which $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation and joint embedding, as it is Π^1_2 in codes for $\boldsymbol{K}$ and a $\preceq_{\boldsymbol{k}}$-universal model for $\boldsymbol{K}_{\aleph_0}$. We first show that one direction of the equivalence in Theorem 1.2 follows from weaker hypotheses. Since amalgamation is not assumed in Theorem 1.1 we use the transitive closure of the usual relation in the hypothesis regarding orbital types; inspection of the proof shows that something weaker suffices. Theorems 1.1 and 1.2 do not use the assumption of $\aleph_0$-presentability.

Theorem 1.1. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class such that*

- $\boldsymbol{K}_{\aleph_0} \neq \emptyset$;
- $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ *satisfies the joint embedding property;*
- *for some $M \in \boldsymbol{K}_{\aleph_0}$, the set of orbital types over M (for finite tuples) is countable.*

Then $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ has a universal element.

Proof. If there exists $\preceq_{\boldsymbol{k}}$-maximal element of $\boldsymbol{K}_{\aleph_0}$, then it is universal, by the joint embedding property, so assume otherwise. We use model-theoretic forcing and refer the reader to pages 162-163 of [8] for the definition of the relation $N \Vdash \phi(a_0, \ldots, a_n)$, where $N \in \boldsymbol{K}_{\aleph_0}$, $a_0, \ldots, a_n \in N$, $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ and τ is the vocabulary corresponding to $\boldsymbol{k}$. The following facts follow easily from this definition.

1. If $M, N \in \boldsymbol{K}_{\aleph_0}$, $n \in \omega$, $a_0, \ldots, a_{n-1} \in M$, $b_0, \ldots, b_{n-1} \in N$, $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ is an n-ary formula and $\pi \colon M \to N$ is an $\preceq_{\boldsymbol{k}}$-embedding such that $\pi(a_i) = b_i$ for all $i < n$, then
$$M \Vdash \phi(a_0, \ldots, a_{n-1}) \Rightarrow N \Vdash \phi(b_0, \ldots, b_{n-1}).$$
2. If $M, N, P \in \boldsymbol{K}_{\aleph_0}$, $n \in \omega$, $a_0, \ldots, a_{n-1} \in M$, $b_0, \ldots, b_{n-1} \in N$, $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ is an n-ary formula and $\pi \colon M \to P$ and $\sigma \colon N \to P$ are $\preceq_{\boldsymbol{k}}$-embeddings such that $\pi(a_i) = \sigma(b_i)$ for all $i < n$, then
$$M \Vdash \phi(a_0, \ldots, a_{n-1}) \Rightarrow \neg(N \Vdash \neg\phi(b_0, \ldots, b_{n-1})).$$
3. For every countable subset Ψ of $L_{\aleph_1,\aleph_0}(\tau)$, and every $M \in \boldsymbol{K}_{\aleph_0}$, there is an $N \in \boldsymbol{K}_{\aleph_0}$ such that $M \preceq_{\boldsymbol{k}} N$ and, for all $n \in \omega$, and $a_0, \ldots, a_{n-1} \in N$ and all n-ary formulas $\phi \in \Psi$,
$$N \Vdash \phi(a_0, \ldots, a_{n-1}) \Leftrightarrow N \models \phi(a_0, \ldots, a_{n-1}).$$
4. Since $\boldsymbol{k}$ satisfies the joint embedding property, for each sentence ϕ in $L_{\aleph_1,\aleph_0}(\tau)$ and each $M \in \boldsymbol{K}_{\aleph_0}$, $M \Vdash \phi$ or $M \Vdash \neg\phi$.

By item (3), it suffices to see that there exists a sentence $\phi \in L_{\aleph_1,\aleph_0}(\tau)$ which is the Scott sentence of a countable τ-structure, and which is forced by some (equivalently, every) element of $\boldsymbol{K}_{\aleph_0}$, as then the models of ϕ are $\preceq_{\boldsymbol{k}}$-universal for $\boldsymbol{K}_{\aleph_0}$.

To see that this does hold, we will assume some familiarity with the Scott analysis of a τ-structure (see [5, 7], for instance). This analysis, given a τ-structure M, assigns to each finite tuple $\overline{a}$ from $|M|$ and each ordinal α a formula $\phi^M_{\overline{a},\alpha}$, in such a way that (among other things)

- for all ordinals $\beta < \alpha$, and all tuples $\bar{b}$ from $|M|$, if $\phi^M_{\bar{a},\alpha} = \phi^M_{\bar{b},\alpha}$, then $\phi^M_{\bar{a},\beta} = \phi^M_{\bar{b},\beta}$;
- for any other τ-structure N and any finite tuple $\bar{b}$ from N, if $N \models \phi^M_{\bar{a},\alpha}(\bar{b})$, then $\phi^M_{\bar{a},\alpha} = \phi^N_{\bar{b},\alpha}$.

For each ordinal $\alpha < \omega_1$, we let Φ_α be the set of all formulas of the form $\phi^M_{\bar{a},\alpha}$, for some τ-structure M and some finite tuple $\bar{a}$ from M, and note that no two distinct elements of Φ_α can be satisfied by the same tuple in the same structure. By item (3) above, for each ordinal α, each $M \in \boldsymbol{K}_{\aleph_0}$ and each finite tuple $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle$ from M, there is at most one formula $\psi \in \Phi_\alpha$ such that $M \Vdash \psi(a_0, \ldots, a_{n-1})$. We call this formula $\psi_{M,\bar{a},\alpha}$ if it exists. We let Ψ_α be the set of all formulas of the form $\psi_{M,\bar{a},\alpha}$, for some $M \in \boldsymbol{K}_{\aleph_0}$. By item (2) above, the joint embedding property and our assumption on the number of orbital types for finite sequences, each set Ψ_α is countable.

Again by the assumption of orbital ω-stability for finite tuples, and the joint embedding property, there is a countable ordinal α such that for all $M, N \in \boldsymbol{K}_{\aleph_0}$, all finite tuples $\bar{a}$ from M and $\bar{b}$ from N, if $\psi_{M,\bar{a},\alpha+1}$ and $\psi_{N,\bar{b},\alpha+1}$ exist and $\psi_{M,\bar{a},\alpha} = \psi_{N,\bar{b},\alpha}$, then $\psi_{M,\bar{a},\alpha+1} = \psi_{N,\bar{b},\alpha+1}$. To see this, note that otherwise there would exist

- $M \in \boldsymbol{K}_{\aleph_0}$,
- $A \in [\omega_1]^{\aleph_1}$,
- $N_\alpha \in \boldsymbol{K}_{\aleph_0}$ ($\alpha \in A$) and
- finite tuples $\bar{a}_\alpha$, $\bar{b}_\alpha$ from N_α ($\alpha \in A$)

such that for each $\alpha \in A$, $\psi_{N_\alpha,\bar{a}_\alpha,\alpha+1}$ and $\psi_{N_\alpha,\bar{b}_\alpha,\alpha+1}$ exist and $\psi_{N_\alpha,\bar{a}_\alpha,\alpha} = \psi_{N_\alpha,\bar{b}_\alpha,\alpha}$, but

$$\psi_{N_\alpha,\bar{a}_\alpha,\alpha+1} \neq \psi_{N_\alpha,\bar{b}_\alpha,\alpha+1}.$$

The triples $(M, \bar{a}_\alpha{}^\frown\bar{b}_\alpha, N_\alpha)$ would then represent an uncountable set of distinct orbital types over M.

By item (3) above, each $M \in \boldsymbol{K}_{\aleph_0}$ has a $\preceq_{\boldsymbol{k}}$-extension $N \in \boldsymbol{K}_{\aleph_0}$ with the property that for all finite tuples $\bar{a}$ from N, $N \models \psi_{N,\bar{a},\alpha+\omega}(\bar{a})$. This implies that N has Scott rank at most α, and that N forces its own Scott sentence. $\square$

Adding amalgamation, we get an equivalence.

Theorem 1.2. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an AEC satisfying amalgamation and the joint embedding property, for which $\boldsymbol{K}_{\aleph_0}$ is nonempty. Then $\boldsymbol{K}$ is orbitally ω-stable if and only if $\boldsymbol{K}_{\aleph_0}$ has a $\preceq_{\boldsymbol{k}}$-universal member over which there are just countably many orbital types.*

Proof. The forward direction follows from Theorem 1.1. The reverse direction follows from the fact that if $\boldsymbol{k}$ satisfies amalgamation and $N \preceq_{\boldsymbol{k}} M$ are members of $\boldsymbol{K}_{\aleph_0}$, then every orbital type over N contains either an element of the form (N, a, M) for some $a \in M \setminus N$ or an element of the form (N, a, P), for some $P \in \boldsymbol{K}_{\aleph_0}$ such that $M \preceq_{\boldsymbol{k}} P$ and $a \in P \setminus M$. It follows that the cardinality of the set of orbital types over N is bounded by the cardinality of the set of orbital types over M. $\square$

2 A counterexample to both upwards and downwards absoluteness with amalgamation but not joint embedding

In the two counterexamples constructed below we let T be the theory of the structure $\langle L_{\omega_1^L}, \in\rangle$. The first counterexample uses the fact that the cardinality of the set of wellfounded models of T is the same as the cardinality of ω_1^L.

The following standard fact is not hard to verify. As usual, $\mathbb{Q}$ denotes the set of rational numbers. The last sentence of the fact can be verified by taking a generic ultrapower of $L_{\omega_1^L}$ via its version of the nonstationary ideal on ω_1.

Fact 2.1. *If M is a countable illfounded ω-model of T then the ordinals of M have ordertype $\alpha + (\mathbb{Q} \times \alpha)$ for some ordinal $\alpha \leq \omega_1^L$, where $\mathbb{Q} \times \alpha$ is given the lexicographical order. The set of ordinals $\alpha < \omega_1^L$ for which there exists a model of T whose ordinals have ordertype $\alpha + (\mathbb{Q} \times \alpha)$ is a closed unbounded subset of ω_1^L. If ω_1^L is countable, then there exists a model of T whose ordinals have ordertype $\omega_1^L + (\mathbb{Q} \times \omega_1^L)$.*

The rest of this section consists of two examples of non-absoluteness for ω-orbital stability, both of which use Fact 2.1.

In this section we present an $\aleph_0$-presentable AEC with Löwenheim-Skolem number $\aleph_0$ which satisfies amalgamation, fails the joint embedding property, and which fails to be orbitally ω-stable if and only if $\omega_1^L = \omega_1$ and $\mathbb{R} \not\subseteq L$. Let τ be the vocabulary consisting of $=$, binary symbols E and $<$, and unary symbols W_n ($n \in \omega$). Let $\boldsymbol{K}^\tau$ be the class of τ-structures M of the form

$$\langle |M|, E^M, <^M, W_n^M; n \in \omega\rangle$$

such that

- $|M|$ is a nonempty set;
- E^M is an equivalence relation on $|M|$ and $<^M$ is a subset of E^M;
- each W_n^M is either the empty set or all of $|M|$;
- for each $a \in |M|$, there exists an ω-model N of T such that
  - the ordinals of N are $[a]_{E^M}$ and $<^M \restriction [a]_{E^M}$ is the corresponding ordering,
  - $\{n \in \omega : W_n^M \neq \emptyset\}$ is not a member of N (i.e., for no $w \in N$ is it true that $N \models w \subseteq \omega$ and, for all $n \in \omega$, that $N \models$ "the n-th member of ω is in w" if and only if $W_n^M \neq \emptyset$).

Given a τ-structure M, we let r_M denote $\{n \in \omega : W_n^M \neq \emptyset\}$. Given $M, N \in \boldsymbol{K}^\tau$, we let $M \preceq_{\boldsymbol{k}^\tau} N$ if

- $|M| \subseteq |N|$;
- $E^M = E^N \cap (|M| \times |M|)$;
- $<^M = <^N \cap(|M| \times |M|)$;
- each E^N equivalence class is either contained in or disjoint from $|M|$;
- $r_M = r_N$.

This is an $\aleph_0$-presented AEC, satisfying amalgamation, but not the joint embedding property, since τ-structures M, M' for which $r_M \neq r_{M'}$ cannot be embedded into a common structure.

By Fact 2.1, if M is a τ-structure and $r_M \in L$, then there are only countably many orbital types over M, since the lengths of longest wellfounded initial segments of the orders $<^M \restriction [a]_{E^M}$ are then bounded by the least ordinal α such that $r_M \in L_\alpha$. It follows that if $\mathbb{R} \subseteq L$ then $\boldsymbol{K}^\tau$ is orbitally ω-stable. Similarly, Fact 2.1 implies that if ω_1^L is countable then $\boldsymbol{K}^\tau$ is orbitally ω-stable, since in this case there are only countably many isomorphism classes for the restriction of $<^M$ to a given E^M-class.

On the other hand, if $r \subseteq \omega$ is nonconstructible, $\omega_1^V = \omega_1^L$, and $M \in \boldsymbol{K}^\tau$ if such that $r_M = r$, then the models L_α which are countable elementary submodels of $L_{\omega_1^L}$ show that there are uncountably many orbital types over M.

Putting together these remarks, we have the following.

Theorem 2.2. *The pair $(\boldsymbol{K}^\tau, \preceq_{\boldsymbol{k}^\tau})$ forms an $\aleph_0$-presented AEC, satisfying amalgamation, and not the joint embedding property, and is orbitally ω-stable if and only if ω_1^L is countable or $\mathbb{R} \subseteq L$.*

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