

## SMALL-LARGE SUBGROUPS OF THE REALS

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ABSTRACT. We are interested in subgroups of the reals that are small in one and large in another sense. We prove that, in ZFC, there exists a non-meager Lebesgue null subgroup of  $\mathbb{R}$ , while it is consistent that there is no non-null meager subgroup of  $\mathbb{R}$ . This answers a question from Filipczak, Roslanowski and Shelah [5].

### 1. INTRODUCTION

Subgroups of the reals which are small in one and large in another sense were crucial in Filipczak, Roslanowski and Shelah [5]. If there is a non-meager Lebesgue null subgroup of  $(\mathbb{R}, +)$ , then there is no translation invariant Borel hull operation on the  $\sigma$ -ideal  $\mathcal{N}$  of Lebesgue null sets. That is, there is no mapping  $\psi$  from  $\mathcal{N}$  to Borel sets such that for each null set  $A \subseteq \mathbb{R}$ :

- $A \subseteq \psi(A)$  and  $\psi(A)$  is null, and
- $\psi(A + t) = \psi(A) + t$  for every  $t \in \mathbb{R}$ .

Parallel claims hold true if “Lebesgue null” is interchanged with “meager” and/or  $(\mathbb{R}, +)$  is replaced with  $({}^\omega 2, +_2)$ .

If  $\mathcal{M}$  is the  $\sigma$ -ideal of meager subsets of  $\mathbb{R}$  (and  $\mathcal{N}$  is the null ideal on  $\mathbb{R}$ ) and  $\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{N}, \mathcal{M}\}$ , then various set theoretic assumptions imply the existence of a subgroup of  $\mathbb{R}$  which belongs to  $\mathcal{I}$  but not to  $\mathcal{J}$ . But in [5, Problem 4.1] we asked if the existence of such subgroups can be shown in ZFC. This question is interesting *per se*, regardless of its connections to translation invariant Borel hulls.

The present paper presents two theorems. First, in Theorem 2.3 we give ZFC examples of null non-meager subgroups of  $({}^\omega 2, +_2)$  and  $(\mathbb{R}, +)$ , respectively. Next in Theorem 4.1 we show that it is consistent with ZFC that every meager subgroup of  $({}^\omega 2, +_2)$  and/or  $(\mathbb{R}, +)$  has Lebesgue measure zero. This answers

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*Date:* May 23, 2016.

*2010 Mathematics Subject Classification.* Primary 03E35; Secondary: 28A05, 54A05.

*Key words and phrases.* Null ideal, meager ideal, additive subgroups of the reals, Borel hulls, forcing.

Both authors acknowledge support from the United States-Israel Binational Science Foundation (Grant no. 2010405). Publication 1081 of the second author.

[5, Problem 4.1]. Also, our results give another example of a strange asymmetry between measure and category.

**Notation** Our notation is rather standard and compatible with that of classical textbooks (like Jech [6] or Bartoszyński and Judah [1]). However, in forcing we keep the older convention that *a stronger condition is the larger one*.

- (1) The Cantor space  ${}^\omega 2$  of all infinite sequences with values 0 and 1 is equipped with the natural product topology, the product measure  $\lambda$  and the group operation of coordinate-wise addition  $+_2$  modulo 2.
- (2) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet  $\alpha, \beta, \gamma, \delta$ . Finite ordinals (non-negative integers) will be denoted by letters  $i, j, k, \ell, m, n$  while integers will be called  $L, M$ .
- (3) Most of our intervals will be intervals of non-negative integers, so  $[m, n) = \{k \in \omega : m \leq k < n\}$  etc. They will be denoted by letter  $J$  (with possible indices). However, we will also use the notation  $[0, 1)$  to denote the unit interval of reals.
- (4) The Greek letter  $\kappa$  will stand for an uncountable cardinal such that  $\kappa^{\aleph_0} = \kappa \geq \aleph_2$ .
- (5) For a forcing notion  $\mathbb{P}$ , all  $\mathbb{P}$ -names for objects in the extension via  $\mathbb{P}$  will be denoted with a tilde below (e.g.,  $\tilde{\tau}, \tilde{X}$ ), and  $\dot{G}_{\mathbb{P}}$  will stand for the canonical  $\mathbb{P}$ -name for the generic filter in  $\mathbb{P}$ .
- (6) We fix a well ordering  $\prec^*$  of all hereditarily finite sets.
- (7) The set of all partial finite functions with domains included in  $\omega$  and with values in 2 is denoted  ${}^\omega 2$ .

## 2. NULL NON-MEAGER

Here we will give a ZFC construction of a non-meager Lebesgue null subgroup of the reals. The main construction is done in  ${}^\omega 2$  and then we transfer it to  $\mathbb{R}$  using the standard binary expansion  $\mathbf{E}$ .

**Definition 2.1.** Let  $D_0^\infty = \{x \in {}^\omega 2 : (\exists^\infty i < \omega)(x(i) = 0)\}$  and for  $x \in D_0^\infty$  let  $\mathbf{E}(x) = \sum_{i=0}^{\infty} x(i)2^{-(i+1)}$ .

**Proposition 2.2.** (1) *The function  $\mathbf{E} : D_0^\infty \rightarrow [0, 1)$  is a continuous bijection, it preserves both the measure and the category.*

- (2) *Assume that*
  - (a)  $x, y, z \in D_0^\infty$ ,  $\mathbf{E}(z) = \mathbf{E}(x) + \mathbf{E}(y)$  modulo 1, and
  - (b)  $n < m < \omega$  and both  $x \upharpoonright [n, m]$  and  $y \upharpoonright [n, m]$  are constant.*Then  $z \upharpoonright [n, m - 1]$  is constant.*
- (3) *Assume that*
  - (a)  $x, y \in D_0^\infty$ ,  $0 < \mathbf{E}(x)$  and  $\mathbf{E}(y) = 1 - \mathbf{E}(x)$ ,
  - (b)  $n < m < \omega$  and  $x \upharpoonright [n, m]$  is constant.

Then  $y \upharpoonright [n, m - 1]$  is constant.

*Proof.* (1) Well known, cf. Bukovský [4, §2.4].

(2,3) Straightforward (just consider the possible constant values and analyze how the addition is performed).  $\square$

**Theorem 2.3.** (1) *There exists a null non-meager subgroup of  $(\omega 2, +_2)$ .*

(2) *There exists a null non-meager subgroup of  $(\mathbb{R}, +)$ .*

*Proof.* (1) For  $k \in \omega$  let  $n_k = \frac{1}{2}k(k+1)$  and let  $D$  be a non-principal ultrafilter on  $\omega$ . Define

$$H_D = \left\{ x \in \omega 2 : (\exists m < \omega) (\exists j < 2) (\{k > m : x \upharpoonright [n_k, n_{k+1} - m] \equiv j\} \in D) \right\}.$$

(i)  $H_D$  is a subgroup of  $(\omega 2, +_2)$ .

Why? Suppose that  $x_0, x_1 \in H_D$  and let  $m_\ell < \omega$  and  $j_\ell < 2$  be such that

$$A_\ell \stackrel{\text{def}}{=} \{k > m_\ell : x_\ell \upharpoonright [n_k, n_{k+1} - m_\ell] \equiv j_\ell\} \in D.$$

Let  $m = \max(m_0, m_1)$  and  $j = j_0 \dot{-} j_1$ . Then  $A_0 \cap A_1 \in D$  and for each  $k \in A_0 \cap A_1$  we have  $(x_0 \dot{-} x_1) \upharpoonright [n_k, n_{k+1} - m] \equiv j$ . Hence  $x_0 \dot{-} x_1 \in H_D$ .

(ii)  $H_D \in \mathcal{N}$ .

Why? For each  $m < k < \omega$  and  $j < 2$  we have

$$\lambda(\{x \in \omega 2 : x \upharpoonright [n_k, n_{k+1} - m] \equiv j\}) = 2^{m-(k+1)}$$

and therefore for each  $m < \omega$  and  $j < 2$

$$\lambda(\{x \in \omega 2 : (\exists^\infty k)(x \upharpoonright [n_k, n_{k+1} - m] \equiv j)\}) = 0.$$

Now note that  $H_D \subseteq \bigcup_{m < \omega} \bigcup_{j < 2} \{x \in \omega 2 : (\exists^\infty k)(x \upharpoonright [n_k, n_{k+1} - m] \equiv j)\}$ .

(iii)  $H_D \notin \mathcal{M}$ .

Why? Suppose that  $W$  is a dense  $\Pi_2^0$  subset of  $\omega 2$ . Then we may choose an increasing sequence  $\langle k_i : i \in \omega \rangle$  and a function  $f \in \omega 2$  such that

$$\left\{ x \in \omega 2 : (\exists^\infty i)(x \upharpoonright [n_{k_i}, n_{k_{i+1}}] = f \upharpoonright [n_{k_i}, n_{k_{i+1}}]) \right\} \subseteq W.$$

Let  $A = \bigcup \{[k_{2i}, k_{2i+1}] : i \in \omega\}$  and  $B = \bigcup \{[k_{2i+1}, k_{2i+2}] : i \in \omega\}$ . Then either  $A \in D$  or  $B \in D$ . Let  $x_A, x_B \in \omega 2$  be such that, for each  $i \in \omega$ ,

$$\begin{aligned} x_A \upharpoonright [n_{k_{2i}}, n_{k_{2i+1}}] &\equiv 0, & x_A \upharpoonright [n_{k_{2i+1}}, n_{k_{2i+2}}] &= f \upharpoonright [n_{k_{2i+1}}, n_{k_{2i+2}}] && \text{and} \\ x_B \upharpoonright [n_{k_{2i+1}}, n_{k_{2i+2}}] &\equiv 0, & x_B \upharpoonright [n_{k_{2i}}, n_{k_{2i+1}}] &= f \upharpoonright [n_{k_{2i}}, n_{k_{2i+1}}]. \end{aligned}$$

Then  $x_A, x_B \in W$  and either  $x_A \in H_D$  or  $x_B \in H_D$ . Consequently,  $W \cap H_D \neq \emptyset$ .

(2) Consider  $H_D^* = \mathbf{E}[H_D \cap D_0^\infty] + \mathbb{Z}$ . It follows from 2.2(1) that  $H_D^*$  is a Lebesgue null meager subset of  $\mathbb{R}$ . We will show that it is a subgroup of  $(\mathbb{R}, +)$ .

Suppose that  $x_0, x_1 \in H_D \cap D_0^\infty$  and  $L_0, L_1 \in \mathbb{Z}$  and we will argue that  $(\mathbf{E}(x_0) + L_0) + (\mathbf{E}(x_1) + L_1) \in H_D^*$ . Let  $m_\ell < \omega$  be such that

$$A_\ell \stackrel{\text{def}}{=} \{k > m_\ell : x_\ell \upharpoonright [n_k, n_{k+1} - m_\ell] \text{ is constant}\} \in D$$

and let  $m = \max(m_0, m_1) + 1$ . Choose  $y \in D_0^\infty$  and  $M \in \{0, 1\}$  such that  $\mathbf{E}(x_0) + \mathbf{E}(x_1) = \mathbf{E}(y) + M$ . It follows from 2.2(2) that for every  $k \in A_0 \cap A_1$ ,  $k > m$ , we have that  $y \upharpoonright [n_k, n_{k+1} - m]$  is constant and since  $A_0 \cap A_1 \in D$  we conclude  $y \in H_D$ . Consequently,  $(\mathbf{E}(x_0) + L_0) + (\mathbf{E}(x_1) + L_1) = \mathbf{E}(y) + (M + L_0 + L_1) \in H_D^*$ .

Now assume that  $x \in H_D \cap D_0^\infty$ ,  $L \in \mathbb{Z}$  and we will argue that  $-(\mathbf{E}(x) + L) \in H_D^*$ . If  $\mathbf{E}(x) = 0$  then the assertion is clear, so assume also  $\mathbf{E}(x) > 0$ . Let  $m < \omega$  be such that

$$A \stackrel{\text{def}}{=} \{k > m : x \upharpoonright [n_k, n_{k+1} - m] \text{ is constant}\} \in D.$$

Choose  $y \in D_0^\infty$  such that  $1 - \mathbf{E}(x) = \mathbf{E}(y)$ . It follows from 2.2(3) that for every  $k \in A$ ,  $k > m + 1$ , we have that  $y \upharpoonright [n_k, n_{k+1} - (m + 1)]$  is constant. Consequently,  $y \in H_D$  and  $-(\mathbf{E}(x) + L) = \mathbf{E}(y) - 1 - L \in H_D^*$ .  $\square$

*Remark 2.4.* A somewhat simpler non-meager null subgroup of  $({}^\omega 2, +_2)$  is

$$H_D^- = \left\{x \in {}^\omega 2 : \{k \in \omega : x \upharpoonright [n_k, n_{k+1}] \equiv 0\} \in D\right\}.$$

The group  $H_D$ , however, was necessary for our construction of  $H_D^* < \mathbb{R}$ .

**Corollary 2.5.** *There exists no translation invariant Borel hull for the null ideal on  ${}^\omega 2$  and/or on  $\mathbb{R}$ .*

### 3. SOME TECHNICALITIES

Here we prepare the ground for our consistency results.

**3.1. Moving from  $\mathbb{R}$  to  ${}^\omega 2$ .** First, let us remind connections between the addition in  $\mathbb{R}$  and that of  ${}^\omega 2$  (via the binary expansion  $\mathbf{E}$ , see 2.1).

**Definition 3.1.** Let  $J = [m, n)$  be a non-empty interval of integers and  $c \in \{0, 1\}$ . For sequences  $\rho, \sigma \in {}^J 2$  we define  $\rho \otimes_c \sigma$  as the unique  $\eta \in {}^J 2$  such that

$$\left( \sum_{i=m}^{n-1} \rho(i) 2^{-(i+1)} + \sum_{i=m}^{n-1} \sigma(i) 2^{-(i+1)} + c \cdot 2^{-n} \right) - \sum_{i=m}^{n-1} \eta(i) 2^{-(i+1)} \in \{0, 2^{-m}\}.$$

For notational convenience we also set  $\rho \otimes_2 \sigma = \rho +_2 \sigma$  (coordinate-wise addition modulo 2).

The operation  $\otimes_c$  is defined on the set  ${}^J 2$ , so it does depend on  $J$ . We may, however, abuse notation and use that same symbol  $\otimes_c$  for various  $J$ .

**Observation 3.2.** *Let  $m, \ell, n$  be integers such that  $m < \ell < n$  and let  $J = [m, n)$ .*

- (1) For each  $c \in \{0, 2\}$ ,  $({}^J 2, \otimes_c)$  is an Abelian group.
- (2) If  $\rho, \sigma \in {}^J 2$  and  $\rho(\ell) = \sigma(\ell)$ , then  $(\rho \otimes_0 \sigma) \upharpoonright [m, \ell] = (\rho \otimes_1 \sigma) \upharpoonright [m, \ell]$ .
- (3) If  $\rho, \sigma \in {}^J 2$  and  $(\rho \otimes_0 \sigma)(\ell) = 0$ , then  $(\rho \otimes_0 \sigma) \upharpoonright [m, \ell] = (\rho \otimes_1 \sigma) \upharpoonright [m, \ell]$ .
- (4) Suppose that  $r, s \in [0, 1)$ ,  $\rho, \sigma, \eta \in D_0^\infty$ ,  $\mathbf{E}(\rho) = r$ ,  $\mathbf{E}(\sigma) = s$  and  $\mathbf{E}(\eta) = r + s$  modulo 1. Then
  - if  $\sum_{i \geq n} ((\rho(i) + \sigma(i))/2^{i+1}) \geq 2^{-n}$ , then  $\eta \upharpoonright J = (\rho \upharpoonright J) \otimes_1 (\sigma \upharpoonright J)$ ;
  - if  $\sum_{i \geq n} ((\rho(i) + \sigma(i))/2^{i+1}) < 2^{-n}$ , then  $\eta \upharpoonright J = (\rho \upharpoonright J) \otimes_0 (\sigma \upharpoonright J)$ .

**3.2. The combinatorial heart of our forcing arguments.** For this subsection we fix a strictly increasing sequence  $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$ .

**Definition 3.3.** We define  $\bar{m}[\bar{n}] = \langle m_i : i < \omega \rangle$ ,  $\bar{N}[\bar{n}] = \langle N(i) : i < \omega \rangle$ ,  $\bar{J}[\bar{n}] = \langle J_i : i < \omega \rangle$ ,  $\bar{H}[\bar{n}] = \langle H_i : i < \omega \rangle$ ,  $\bar{\pi}[\bar{n}] = \langle \pi_i : i < \omega \rangle$  and  $\mathbf{F}[\bar{n}]$  as follows.

We set  $m_0 = 0$  and then inductively for  $i < \omega$  we let

$$(*)_1 \quad m_{i+1} = 2^{n_{m_i} + 1081}.$$

Next, for  $i < \omega$ ,

$$(*)_2 \quad N(i) = n_{m_i}, \quad J_i = [N(2^i), N(2^{i+1})], \quad \text{and}$$

$$(*)_3 \quad H_i = \{a \subseteq {}^{J_i} 2 : (1 - 2^{-N(2^i)}) \cdot 2^{|J_i|} \leq |a|\}.$$

We also set  $\pi_i : |H_i| \rightarrow H_i$  to be the  $\prec^*$ -first bijection from  $|H_i|$  onto  $H_i$ .

Finally, for  $\eta \in \prod_{m < \omega} (m + 1)$  we let

$$(*)_4 \quad \mathbf{F}_0[\bar{n}](\eta) = \{x \in {}^\omega 2 : (\forall i < \omega) (x \upharpoonright J_i \in \pi_i(\eta(|H_i| - 1)))\} \quad \text{and}$$

$$\mathbf{F}[\bar{n}](\eta) = \{x \in {}^\omega 2 : (\forall^\infty i < \omega) (x \upharpoonright J_i \in \pi_i(\eta(|H_i| - 1)))\}.$$

**Lemma 3.4.** For every  $\eta \in \prod_{m < \omega} (m + 1)$ ,  $\mathbf{F}_0[\bar{n}](\eta) \subseteq {}^\omega 2$  is a closed set of positive Lebesgue measure, and  $\mathbf{F}[\bar{n}](\eta)$  is a  $\Sigma_2^0$  set of Lebesgue measure 1.

*Proof.* Note that  $J_i \cap J_j = \emptyset$  and  $|H_i| < |H_j|$  for  $i < j$ , and  $\sum_{i=0}^{\infty} 2^{-N(2^i)} < 1$ .  $\square$

**Lemma 3.5.** Let  $i < \omega$ ,  $c \in \{0, 2\}$  and let  $\eta \in {}^{J_i} 2$ . Suppose that for each  $\ell < 2^i$  and  $x < 2$  we are given a function  $\mathcal{Z}_\ell^x : H_i \rightarrow {}^{J_i} 2$  such that  $\mathcal{Z}_\ell^x(a) \in a$  for each  $a \in H_i$ . Then there are  $a^0, a^1 \in H_i$  such that for every  $\ell < 2^i$  there is  $k \in [m_{2^i + \ell}, m_{2^i + \ell + 1})$  satisfying

$$(\mathcal{Z}_\ell^0(a^0) \upharpoonright [n_k, n_{k+1}]) \otimes_c^k (\mathcal{Z}_\ell^1(a^1) \upharpoonright [n_k, n_{k+1}]) = \eta \upharpoonright [n_k, n_{k+1}),$$

where  $\otimes_c^k$  denotes the operation  $\otimes_c$  on  $[n_k, n_{k+1}) 2$ .

*Proof.* We start the proof with the following Claim.

**Claim 3.5.1.** If  $\mathcal{A} \subseteq H_i$ ,  $|\mathcal{A}| \leq 2^{|J_i| - N(2^i) - i}$  and  $x < 2$ , then there is  $b \in H_i$  such that  $\mathcal{Z}_\ell^x(b) \notin \{\mathcal{Z}_\ell^x(a) : a \in \mathcal{A}\}$  for each  $\ell < 2^i$ .

*Proof of the Claim.* Note that  $|\{\mathcal{Z}_\ell^x(a) : \ell < 2^i \text{ \& } a \in \mathcal{A}\}| \leq 2^i \cdot 2^{|J_i| - N(2^i) - i} = 2^{|J_i| - N(2^i)}$ , so letting  $b = {}^{J_i}2 \setminus \{\mathcal{Z}_\ell^x(a) : \ell < 2^i \text{ \& } a \in \mathcal{A}\}$  we have  $b \in H_i$ . Since  $\mathcal{Z}_\ell^x(b) \in b$  we see that  $b$  is as required in the claim.  $\square$

It follows from Claim 3.5.1 that we may pick sequences  $\langle a_j^0 : j < j^* \rangle \subseteq H_i$  and  $\langle a_j^1 : j < j^* \rangle \subseteq H_i$  with  $\mathcal{Z}_\ell^x(a_{j_1}^x) \neq \mathcal{Z}_\ell^x(a_{j_2}^x)$  for  $j_1 < j_2 < j^*$ ,  $\ell < 2^i$ ,  $x < 2$  and such that  $j^* > 2^{|J_i| - N(2^i) - i}$ . Now, by induction on  $\ell < 2^i$ , we choose sets  $X_\ell, Y_\ell \subseteq j^*$  and integers  $k_\ell \in [m_{2^i+\ell}, m_{2^i+\ell+1})$  such that the following demands are satisfied.

- (i)  $X_{\ell+1} \subseteq X_\ell \subseteq j^*$ ,  $Y_{\ell+1} \subseteq Y_\ell \subseteq j^*$ ,
- (ii) if  $j_0 \in X_\ell$  and  $j_1 \in Y_\ell$  then

$$(\mathcal{Z}_\ell^0(a_{j_0}^0) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) \otimes_c^{k_\ell} (\mathcal{Z}_\ell^1(a_{j_1}^1) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) = \eta \upharpoonright [n_{k_\ell}, n_{k_\ell+1}),$$

- (iii)  $\min(|X_\ell|, |Y_\ell|) \geq j^* \cdot 2^{N(2^i) - N(2^i+\ell+1) - \ell - 1}$ .

We stipulate  $X_{-1} = Y_{-1} = j^*$  and we assume that  $X_{\ell-1}, Y_{\ell-1}$  have been already determined (and  $\min(|X_{\ell-1}|, |Y_{\ell-1}|) \geq j^* \cdot 2^{N(2^i) - N(2^i+\ell) - \ell}$  if  $\ell > 0$ ). Let

$$\begin{aligned} X^* &= \{j \in X_{\ell-1} : |X_{\ell-1}| \cdot 2^{N(2^i+\ell) - N(2^i+\ell+1) - 1} \leq \\ &\quad |\{j' \in X_{\ell-1} : \mathcal{Z}_\ell^0(a_{j'}^0) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}_\ell^0(a_j^0) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1))\}|\}, \\ Y^* &= \{j \in Y_{\ell-1} : |Y_{\ell-1}| \cdot 2^{N(2^i+\ell) - N(2^i+\ell+1) - 1} \leq \\ &\quad |\{j' \in Y_{\ell-1} : \mathcal{Z}_\ell^1(a_{j'}^1) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}_\ell^1(a_j^1) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1))\}|\}. \end{aligned}$$

**Claim 3.5.2.**  $|X^*| \geq \frac{1}{2}|X_{\ell-1}|$  and  $|Y^*| \geq \frac{1}{2}|Y_{\ell-1}|$ .

*Proof of the Claim.* Assume towards contradiction that  $|X^*| < \frac{1}{2}|X_{\ell-1}|$ . Then for some  $\nu_0 \in [N(2^i+\ell), N(2^i+\ell+1))2$  we have

$$|\{j \in X_{\ell-1} \setminus X^* : \nu_0 \subseteq \mathcal{Z}_\ell^0(a_j^0)\}| \geq |X_{\ell-1} \setminus X^*| \cdot 2^{N(2^i+\ell) - N(2^i+\ell+1)} > \frac{1}{2}|X_{\ell-1}| \cdot 2^{N(2^i+\ell) - N(2^i+\ell+1)}.$$

Let  $j \in X_{\ell-1} \setminus X^*$  be such that  $\nu_0 \subseteq \mathcal{Z}_\ell^0(a_j^0)$ . Then  $j \in X^*$ , a contradiction.

Similarly for  $Y^*$ .  $\square$

**Claim 3.5.3.** For some  $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$  we have that both  $|\{\mathcal{Z}_\ell^0(a_j^0) \upharpoonright [n_k, n_{k+1}) : j \in X^*\}| > 2^{n_{k+1} - n_k - 1}$  and  $|\{\mathcal{Z}_\ell^1(a_j^1) \upharpoonright [n_k, n_{k+1}) : j \in Y^*\}| > 2^{n_{k+1} - n_k - 1}$ .

*Proof of the Claim.* Let

$$K^X = \{k \in [m_{2^i+\ell}, m_{2^i+\ell+1}) : |\{\mathcal{Z}_\ell^0(a_j^0) \upharpoonright [n_k, n_{k+1}) : j \in X^*\}| \leq 2^{n_{k+1} - n_k - 1}\}$$

and

$$K^Y = \{k \in [m_{2^i+\ell}, m_{2^i+\ell+1}) : |\{\mathcal{Z}_\ell^1(a_j^1) \upharpoonright [n_k, n_{k+1}) : j \in Y^*\}| \leq 2^{n_{k+1} - n_k - 1}\}.$$

Assume towards contradiction that  $|K^X| \geq \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$ . Then

$$|X^*| = |\{\mathcal{Z}_\ell^0(a_j^0) : j \in X^*\}| \leq 2^{-1/2(m_{2^i+\ell+1} - m_{2^i+\ell})} \cdot 2^{|J_i|} < 2^{|J_i|} \cdot 2^{-4N(2^i+\ell)}.$$

(Remember 3.3(\*)<sub>1</sub>.) Hence  $|X_{\ell-1}| \leq 2^{|J_i|-4N(2^i+\ell)+1}$ . If  $\ell = 0$  then we get  $2^{|J_i|-2N(2^i)} < j^* \leq 2^{|J_i|-4N(2^i)+1}$ , which is impossible. If  $\ell > 0$ , then by the inductive hypothesis (iii) we know that  $|X_{\ell-1}| \geq j^* \cdot 2^{N(2^i)-N(2^i+\ell)-\ell} > 2^{|J_i|-i-N(2^i+\ell)-\ell}$ , so  $3N(2^i+\ell) - 1 < i + \ell$ , a clear contradiction. Consequently  $|K^X| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$ , and similarly  $|K^Y| < \frac{1}{2}(m_{2^i+\ell+1} - m_{2^i+\ell})$ . Pick  $k \in [m_{2^i+\ell}, m_{2^i+\ell+1})$  such that  $k \notin K^X \cup K^Y$ .  $\square$

Now, let  $k_\ell \in [m_{2^i+\ell}, m_{2^i+\ell+1})$  be as given by Claim 3.5.3. Necessarily the sets  $\{\rho \in [n_{k_\ell}, n_{k_\ell+1})2 : (\exists j \in X^*)((\mathcal{Z}_\ell^0(a_j^0) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) \otimes_c^{k_\ell} \rho = \eta \upharpoonright [n_{k_\ell}, n_{k_\ell+1}))\}$  and  $\{\mathcal{Z}_\ell^1(a_j^1) \upharpoonright [n_{k_\ell}, n_{k_\ell+1}) : j \in Y^*\}$  have non-empty intersection. Therefore, we may find  $j_X \in X^*$  and  $j_Y \in Y^*$  such that

$$(\mathcal{Z}_\ell^0(a_{j_X}^0) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) \otimes_c^{k_\ell} (\mathcal{Z}_\ell^1(a_{j_Y}^1) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) = \eta \upharpoonright [n_{k_\ell}, n_{k_\ell+1}).$$

Set

$$X_\ell = \{j \in X_{\ell-1} : \mathcal{Z}_\ell^0(a_j^0) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}_\ell^0(a_{j_X}^0) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1))\},$$

and

$$Y_\ell = \{j \in Y_{\ell-1} : \mathcal{Z}_\ell^1(a_j^1) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1)) = \mathcal{Z}_\ell^1(a_{j_Y}^1) \upharpoonright [N(2^i+\ell), N(2^i+\ell+1))\}.$$

By the definition of  $X^*, Y^*$  and by the inductive hypothesis (iii) we have

$$|X_\ell| \geq |X_{\ell-1}| \cdot 2^{N(2^i+\ell)-N(2^i+\ell+1)-1} \geq j^* \cdot 2^{N(2^i)-\ell-N(2^i+\ell+1)-1}$$

and similarly for  $Y_\ell$ . Consequently,  $X_\ell, Y_\ell$  and  $k_\ell$  satisfy the inductive demands (i)–(iii).

After the above construction is completed fix any  $j_0 \in X_{2^i-1}, j_1 \in Y_{2^i-1}$  and consider  $a^0 = a_{j_0}$  and  $a^1 = a_{j_1}$ . For each  $\ell < 2^i$  we have  $j_0 \in X_\ell, j_1 \in Y_\ell$  so

$$(\mathcal{Z}_\ell^0(a^0) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) \otimes_c^{k_\ell} (\mathcal{Z}_\ell^1(a^1) \upharpoonright [n_{k_\ell}, n_{k_\ell+1})) = \eta \upharpoonright [n_{k_\ell}, n_{k_\ell+1}).$$

Hence  $a^1, a^2 \in H_i$  are as required.  $\square$

**3.3. The  $\ast$ -Silver forcing notion.** The consistency result of the next section will be obtained using CS product of the following forcing notion  $\mathbb{S}_\ast$ .

**Definition 3.6.** (1) We define the  $\ast$ -Silver forcing notion  $\mathbb{S}_\ast$  as follows.

**A condition** in  $\mathbb{S}_\ast$  is a partial function  $p : \text{dom}(p) \rightarrow \omega$  such that  $\text{dom}(p) \subseteq \omega$  is coinfinite and  $p(m) \leq m$  for each  $m \in \text{dom}(p)$ .

**The order**  $\leq_{\mathbb{S}_\ast}$  of  $\mathbb{S}_\ast$  is the inclusion, i.e.,  $p \leq q$  if and only if  $p \subseteq q$ .

- (2) For  $p \in \mathbb{S}_\ast$  and  $1 \leq n < \omega$  we let  $u(n, p)$  be the set of the first  $n$  elements of  $\omega \setminus \text{dom}(p)$  (in the natural increasing order). Then for  $p, q \in \mathbb{S}_\ast$  we let  $p \leq_n q$  if and only if  $p \leq q$  and  $u(n, q) = u(n, p)$ .

We also define  $p \leq_0 q$  as equivalent to  $p \leq q$ .

- (3) Let  $p \in \mathbb{S}_\ast$ . We let  $S(n, p)$  be the set of all functions  $s : u(n, p) \rightarrow \omega$  with the property that  $s(m) \leq m$  for all  $m \in u(n, p)$ .

(4) We let  $\eta$  to be the canonical  $\mathbb{S}_*$ -name such that

$$\Vdash \eta = \bigcup \{p : p \in G_{\mathbb{S}_*}\}.$$

*Remark 3.7.* The forcing notion  $\mathbb{S}_*$  may be represented as a forcing of the type  $\mathbb{Q}_{w\infty}^*(K, \Sigma)$  for some finitary creating pair  $(K, \Sigma)$  which captures singletons, see Rosłanowski and Shelah [8, Definition 2.1.10]. It is a close relative of the Silver forcing notion and, in a sense, it lies right above all  $\mathbb{S}_n$ 's studied for instance in Rosłanowski [7] and Rosłanowski and Steprāns [9].

- Lemma 3.8.**
- (1)  $(\mathbb{S}_*, \leq_{\mathbb{S}_*})$  is a partial order of size  $\mathfrak{c}$ . If  $p \in \mathbb{S}_*$  and  $s \in S(n, p)$  then  $p \cup s \in \mathbb{S}_*$  is a condition stronger than  $p$ .
  - (2)  $\Vdash_{\mathbb{S}_*} \eta \in \prod_{m < \omega} (m+1)$  and  $p \Vdash_{\mathbb{S}_*} p \subseteq \eta$  (for  $p \in \mathbb{S}_*$ ).
  - (3) If  $p \in \mathbb{S}_*$  and  $1 \leq n < \omega$ , then the family  $\{p \cup s : s \in S(n, p)\}$  is an antichain pre-dense above  $p$ .
  - (4) The relations  $\leq_n$  are partial orders on  $\mathbb{S}_*$ ,  $p \leq_{n+1} q$  implies  $p \leq_n q$ .
  - (5) Assume that  $\tau$  is an  $\mathbb{S}_*$ -name for an ordinal,  $p \in \mathbb{S}_*$ ,  $1 \leq n, m < \omega$ . Then there is a condition  $q \in \mathbb{S}_*$  such that  $p \leq_n q$ ,  $\max(u(n+1, q)) > m$  and for all  $s \in S(n, q)$  the condition  $q \cup s$  decides the value of  $\tau$ .
  - (6) The forcing notion  $\mathbb{S}_*$  satisfies Axiom A of Baumgartner [2, §7] as witnessed by the orders  $\leq_n$ , it is  ${}^\omega\omega$ -bounding and, moreover, every meager subset of  ${}^\omega 2$  in an extension by  $\mathbb{S}_*$  is included in a  $\Sigma_2^0$  meager set coded in the ground model.

*Proof.* Straightforward - the same as for the Silver forcing notion. □

**Definition 3.9.** Assume  $\kappa^{\aleph_0} = \kappa \geq \aleph_2$ .

- (1)  $\mathbb{S}_*(\kappa)$  is the CS product of  $\kappa$  many copies of  $\mathbb{S}_*$ . Thus
  - a condition**  $p$  in  $\mathbb{S}_*(\kappa)$  is a function with a countable domain  $\text{dom}(p) \subseteq \kappa$  and with values in  $\mathbb{S}_*$ , and
  - the order**  $\leq$  of  $\mathbb{S}_*(\kappa)$  is such that  $p \leq q$  if and only if  $\text{dom}(p) \subseteq \text{dom}(q)$  and  $(\forall \alpha \in \text{dom}(p))(p(\alpha) \leq_{\mathbb{S}_*} q(\alpha))$ .
- (2) Suppose that  $p \in \mathbb{S}_*(\kappa)$  and  $F \subseteq \text{dom}(p)$  is a finite non-empty set and  $\mu : F \rightarrow \omega \setminus \{0\}$ . Let  $v(F, \mu, p) = \prod_{\alpha \in F} u(\mu(\alpha), p(\alpha))$  and  $T(F, \mu, p) = \prod_{\alpha \in F} S(\mu(\alpha), p(\alpha))$ .
  - If  $\sigma \in T(F, \mu, p)$  then let  $p|\sigma$  be the condition  $q \in \mathbb{S}_*(\kappa)$  such that  $\text{dom}(q) = \text{dom}(p)$  and  $q(\alpha) = p(\alpha) \cup \sigma(\alpha)$  for  $\alpha \in F$  and  $q(\alpha) = p(\alpha)$  for  $\alpha \in \text{dom}(q) \setminus F$ .
  - We let  $p \leq_{F, \mu} q$  if and only if  $p \leq q$  and  $v(F, \mu, p) = v(F, \mu, q)$ .
  - If  $\mu$  is constantly  $n$  then we may write  $n$  instead of  $\mu$ .



- (3) Suppose that  $p \in \mathbb{S}_*(\kappa)$  and  $\bar{\tau} = \langle \tau_n : n < \omega \rangle$  is a sequence of names for ordinals. We say that  $p$  *determines*  $\bar{\tau}$  *relative to*  $\bar{F}$  if
- $\bar{F} = \langle F_n : n < \omega \rangle$  is a sequence of finite subsets of  $\text{dom}(p)$ , and
  - $p$  forces a value to  $\tau_0$  and for  $1 \leq n < \omega$  and  $\sigma \in T(F_n, n, p)$  the condition  $p|\sigma$  decides the value of  $\tau_n$ .

- Lemma 3.10.** (1) *The forcing notion  $\mathbb{S}_*(\kappa)$  satisfies  $\mathfrak{c}^+$ -chain condition.*
- (2) *Suppose that  $p \in \mathbb{S}_*(\kappa)$ ,  $F \subseteq \text{dom}(p)$  is finite non-empty,  $\mu : F \rightarrow \omega \setminus \{0\}$  and  $\tau$  is a name for an ordinal. Then there is a condition  $q \in \mathbb{S}_*(\kappa)$  such that  $p \leq_{F, \mu} q$  and for every  $\sigma \in T(F, \mu, q)$  the condition  $q|\sigma$  decides the value of  $\tau$ .*
- (3) *Suppose that  $p \in \mathbb{S}_*(\kappa)$  and  $\bar{\tau} = \langle \tau_n : n < \omega \rangle$  is a sequence of  $\mathbb{S}_*(\kappa)$ -names for objects from the ground model  $\mathbf{V}$ . Then there is a condition  $q \geq p$  and a  $\subseteq$ -increasing sequence  $\bar{F} = \langle F_n : n < \omega \rangle$  of finite subsets of  $\text{dom}(q)$  such that  $q$  determines  $\bar{\tau}$  relative to  $\bar{F}$ .*
- (4) *Assume  $p, \bar{\tau}$  are as in (3) above and  $p \Vdash \text{“}\bar{\tau} \text{ is a sequence of elements of } \omega^2 \text{ with disjoint domains”}$ . Then there are a condition  $q \geq p$  and an increasing sequence  $\bar{F}$  of finite subsets of  $\text{dom}(q)$  and a function  $f = (f_0, f_1) : \bigcup_{1 \leq n < \omega} T(F_n, n, q) \rightarrow \omega \times \omega^2$  such that  $q|\sigma \Vdash \tau_{f_0(\sigma)} = f_1(\sigma)$  (for all  $\sigma \in \text{dom}(f)$ ) and the elements of  $\langle \text{dom}(f_1(\sigma)) : \sigma \in \bigcup_{n < \omega} T(F_n, n, q) \rangle$  are pairwise disjoint.*

*Proof.* The same as for the CS product of Silver or Sacks forcing notions, see e.g. Baumgartner [3, §1].  $\square$

**Corollary 3.11.** *Assume  $\kappa = \kappa^{\aleph_0} \geq \aleph_2$ . The forcing notion  $\mathbb{S}_*(\kappa)$  is proper and every meager subset of  $\omega^2$  in an extension by  $\mathbb{S}_*(\kappa)$  is included in a  $\Sigma_2^0$  meager set coded in the ground model.*

*If CH holds, then  $\mathbb{S}_*(\kappa)$  preserves all cardinals and cofinalities and  $\Vdash_{\mathbb{S}_*(\kappa)} 2^{\aleph_0} = \kappa$ .*

#### 4. MEAGER NON-NULL

The goal of this section is to present a model of ZFC in which every meager subgroup of  $\mathbb{R}$  or  $\omega^2$  is also Lebesgue null.

**Theorem 4.1.** *Assume CH. Let  $\kappa = \kappa^{\aleph_0} \geq \aleph_2$ . Then*

- (1)  $\Vdash_{\mathbb{S}_*(\kappa)} \text{“} 2^{\aleph_0} = \kappa \text{ and each meager subgroup of } (\omega^2, +_2) \text{ is Lebesgue null.} \text{”}$
- (2)  $\Vdash_{\mathbb{S}_*(\kappa)} \text{“ every meager subgroup of } (\mathbb{R}, +) \text{ is Lebesgue null.} \text{”}$

*Proof.* For  $\alpha < \kappa$  let  $\eta_\alpha$  be the canonical name for the  $\mathbb{S}_*$ -generic function in  $\prod_{m < \omega} (m+1)$  added on the  $\alpha^{\text{th}}$  coordinate of  $\mathbb{S}_*(\kappa)$ .

(1) Suppose towards contradiction that for some  $p_0 \in \mathbb{S}_*(\kappa)$  and a  $\mathbb{S}_*(\kappa)$ -name  $\bar{H}$  we have

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \bar{H} \text{ is a meager non-null subgroup of } (\omega 2, +_2) \text{.”}$$

By Corollary 3.11 (or, actually, Lemma 3.10(4)) we may pick a condition  $p_1 \geq p_0$ , a strictly increasing sequence  $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$  and a function  $f \in \omega 2$  such that

$$(*)_0 \ p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \bar{H} \subseteq \{x \in \omega 2 : (\forall^\infty j < \omega)(x \upharpoonright [n_j, n_{j+1}] \neq f \upharpoonright [n_j, n_{j+1}])\} \text{.”}$$

Let  $\bar{m} = \bar{m}[\bar{n}]$ ,  $\bar{N} = \bar{N}[\bar{n}]$ ,  $\bar{J} = \bar{J}[\bar{n}]$ ,  $\bar{H} = \bar{H}[\bar{n}]$ ,  $\pi = \pi[\bar{n}]$  and  $\mathbf{F} = \mathbf{F}[\bar{n}]$  be as defined in Definition 3.3 for the sequence  $\bar{n}$ . Also let  $A = \{|H_i| - 1 : i < \omega\}$  and  $r^+ \in \mathbb{S}_*$  be such that  $\text{dom}(r^+) = \omega \setminus A$  and  $r^+(k) = 0$  for  $k \in \text{dom}(r^+)$ .

Since, by Lemma 3.4, we have  $\Vdash \text{“ } \mathbf{F}(\eta_\alpha) \subseteq \omega 2 \text{ is a measure one set ”}$ , we know that  $p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } (\forall \alpha < \kappa)(\mathbf{F}(\eta_\alpha) \cap \bar{H} \neq \emptyset) \text{”}$ . Consequently, for each  $\alpha < \kappa$ , we may choose a  $\mathbb{S}_*(\kappa)$ -name  $\rho_\alpha$  for an element of  $\omega 2$  such that

$$p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \rho_\alpha \in \bar{H} \ \& \ \rho_\alpha \in \mathbf{F}(\eta_\alpha) \text{”}.$$

Let us fix  $\alpha \in \kappa \setminus \text{dom}(p_1)$  for a moment. Let  $p_1^\alpha \in \mathbb{S}_*(\kappa)$  be a condition such that  $\text{dom}(p_1^\alpha) = \text{dom}(p_1) \cup \{\alpha\}$ ,  $p_1^\alpha(\alpha) = r^+$  and  $p_1 \subseteq p_1^\alpha$ . Using the standard fusion based argument (like the one applied in the classical proof of Lemma 3.10(3) with 3.10(2) used repeatedly), we may find a condition  $q^\alpha \in \mathbb{S}_*(\kappa)$ , a sequence  $\bar{F} = \langle F_n^\alpha : n < \omega \rangle$  of finite sets, a sequence  $\langle \mu_n^\alpha : n < \omega \rangle$  and an integer  $i^\alpha < \omega$  such that the following demands  $(*)_1$ – $(*)_6$  are satisfied.

- $(*)_1$   $q^\alpha \geq p_1^\alpha$ ,  $\text{dom}(q^\alpha) = \bigcup_{n < \omega} F_n^\alpha$ ,  $F_n^\alpha \subseteq F_{n+1}^\alpha$  and  $F_0^\alpha = \{\alpha\}$ .
- $(*)_2$   $\mu_n^\alpha : F_n^\alpha \rightarrow \omega$ ,  $\mu_n^\alpha(\alpha) = n + 1$ ,  $\mu_n^\alpha(\beta) = n$  for  $\beta \in F_n^\alpha \setminus \{\alpha\}$ .
- $(*)_3$   $\min(\omega \setminus \text{dom}(q^\alpha(\alpha))) > |H_{i^\alpha}|$  and  
if  $\max(u(n+1, q^\alpha(\alpha))) = |H_i| - 1$  and  $n \geq 1$ , then  $|T(F_n, n, q^\alpha)|^2 < 2^i$ ,
- $(*)_4$   $q^\alpha \Vdash (\forall i \geq i^\alpha)(\rho_\alpha \upharpoonright J_i \in \pi_i(\eta_\alpha(|H_i| - 1)))$ , and
- $(*)_5$   $q^\alpha$  determines  $\rho_\alpha$  relative to  $\bar{F}$ , moreover
- $(*)_6$  if  $\sigma \in T(F_n^\alpha, \mu_n^\alpha, q^\alpha)$  and  $\max(u(n+1, q^\alpha(\alpha))) = |H_i| - 1$ , then  $q^\alpha \upharpoonright \sigma$  decides the value of  $\rho_\alpha \upharpoonright J_i$ .

Unfixing  $\alpha$  and using a standard  $\Delta$ -system argument with CH we may find distinct  $\gamma, \delta \in \kappa \setminus \text{dom}(p_1)$  such that  $\text{otp}(\text{dom}(q^\gamma)) = \text{otp}(\text{dom}(q^\delta))$  and if  $g : \text{dom}(q^\gamma) \rightarrow \text{dom}(q^\delta)$  is the order preserving bijection, then the following demands  $(*)_7$ – $(*)_9$  hold true.

- $(*)_7$   $i^\gamma = i^\delta$ ,  $g \upharpoonright (\text{dom}(q^\gamma) \cap \text{dom}(q^\delta))$  is the identity,  $g(\gamma) = \delta$ ,
- $(*)_8$   $q^\gamma(\beta) = q^\delta(g(\beta))$  for each  $\beta \in \text{dom}(q^\gamma)$ , and  $g[F_n^\gamma] = F_n^\delta$ ,
- $(*)_9$  if  $F \subseteq \text{dom}(q^\delta)$  is finite,  $\mu : F \rightarrow \omega \setminus \{0\}$ ,  $i < \omega$ ,  $\sigma \in T(F, \mu, q^\delta)$ , then

$$q^\delta \upharpoonright \sigma \Vdash \rho_\delta \upharpoonright J_i = z \quad \text{if and only if} \quad q^\gamma \upharpoonright (\sigma \circ g) \Vdash \rho_\gamma \upharpoonright J_i = z.$$

Clearly  $q^* \stackrel{\text{def}}{=} q^\gamma \cup q^\delta$  is a condition stronger than both  $q^\gamma$  and  $q^\delta$ . Let  $F_n^* = F_n^\gamma \cup F_n^\delta$  for  $n < \omega$ .

Let  $\langle k_\ell : \ell < \omega \rangle$  be the increasing enumeration of  $\omega \setminus \text{dom}(q^\gamma(\gamma)) = \omega \setminus \text{dom}(q^\delta(\delta))$ . Note that by the choice of  $r^+$  and  $p_1^\gamma$ , we have  $\omega \setminus \text{dom}(q^\gamma(\gamma)) \subseteq A$ , so each  $k_\ell$  is of the form  $|H_i| - 1$  for some  $i$ . Now we will choose conditions  $r_\delta, r_\gamma \in \mathbb{S}_*$  so that

$$\text{dom}(r_\delta) = \text{dom}(r_\gamma) = \text{dom}(q^\delta(\delta)) \cup \{k_{2\ell} : \ell < \omega\},$$

$q^\delta(\delta) \leq r_\delta$ ,  $q^\gamma(\gamma) \leq r_\gamma$  and the values of  $r_\delta(k_{2\ell}), r_\gamma(k_{2\ell})$  are picked as follows.

Let  $i$  be such that  $k_{2\ell} = |H_i| - 1$ . If  $x \in \{\gamma, \delta\}$  and  $\sigma \in T(F_{2\ell}^x, \mu_{2\ell}^x, q^x)$  then  $q^x \upharpoonright \sigma$  decides the value of  $\rho_x \upharpoonright J_i$  (by  $(*)_6$ ) and this value belongs to  $\pi_i(\sigma(x)(k_{2\ell}))$  (by  $(*)_4 + (*)_3$ ). Consequently, for  $x \in \{\gamma, \delta\}$  and  $\tau \in T(F_{2\ell}^*, 2\ell, q^*)$  we may define a function  $\mathcal{Z}_\tau^x : H_i \rightarrow {}^{J_i}2$  so that

- $(*)_{10}$  if  $a \in H(i)$ ,  $\mu : F_{2\ell}^* \rightarrow \omega$  is such that  $\mu(x) = 2\ell + 1$  and  $\mu(\alpha) = 2\ell$  for  $\alpha \neq x$ , and  $\tau_a \in T(F_{2\ell}^*, \mu, q^*)$  is such that  $\tau_a(\alpha) = \tau(\alpha)$  for  $\alpha \in F_{2\ell}^* \setminus \{x\}$  and  $\tau_a(x) = \tau(x) \cup \{(k_{2\ell}, a)\}$ ,  
**then**  $q^* \upharpoonright \tau_a \Vdash_{\mathbb{S}_*(\kappa)} \rho_x \upharpoonright J_i = \mathcal{Z}_\tau^x(a)$  and  $\mathcal{Z}_\tau^x(a) \in a$ .

Since  $|T(F_{2\ell}^*, 2\ell, q^*)| \leq |T(F_{2\ell}^\gamma, 2\ell, q^\gamma)|^2 < 2^i$  (remember  $(*)_3$ ), we may use Lemma 3.5 to find  $r_\delta(k_{2\ell}), r_\gamma(k_{2\ell}) \leq k_{2\ell}$  such that

- $(*)_{11}$  for every  $\tau \in T(F_{2\ell}^*, 2\ell, q^*)$  there is  $k \in [m_{2^i}, m_{2^{i+1}})$  satisfying

$$(\mathcal{Z}_\tau^\gamma(\pi_i(r_\gamma(k_{2\ell}))) \upharpoonright [n_k, n_{k+1})) +_2 (\mathcal{Z}_\tau^\delta(\pi_i(r_\delta(k_{2\ell}))) \upharpoonright [n_k, n_{k+1})) = f \upharpoonright [n_k, n_{k+1}).$$

(Remember,  $f$  was chosen in  $(*)_0$ .)

This completes the definition of  $r_\gamma$  and  $r_\delta$ . Let  $q^+ \in \mathbb{S}_*(\kappa)$  be such that  $\text{dom}(q^+) = \text{dom}(q^*) = \text{dom}(q^\gamma) \cup \text{dom}(q^\delta)$  and  $q^+(\alpha) = q^*(\alpha)$  for  $\alpha \in \text{dom}(q^+) \setminus \{\gamma, \delta\}$  and  $q^+(\gamma) = r_\gamma$  and  $q^+(\delta) = r_\delta$ . Then  $q^+$  is a (well defined) condition stronger than both  $q^\gamma$  and  $q^\delta$  and such that

$$(\clubsuit) q^+ \Vdash (\exists^\infty k < \omega) \left( (\rho_\gamma \upharpoonright [n_k, n_{k+1})) +_2 (\rho_\delta \upharpoonright [n_k, n_{k+1})) = f \upharpoonright [n_k, n_{k+1}) \right)$$

(by  $(*)_{10} + (*)_{11}$ ). Consequently, by  $(*)_0$ ,

$$(\heartsuit) q^+ \Vdash \text{“ } \rho_\gamma, \rho_\delta \in \underline{H} \text{ and } \rho_\gamma +_2 \rho_\delta \notin \underline{H} \text{ and } (\underline{H}, +_2) \text{ is a group”},$$

a contradiction.

(2) The proof is a small modification of that for the first part, so we describe the new points only. Assume towards contradiction that for some  $p_0 \in \mathbb{S}_*(\kappa)$  and a  $\mathbb{S}_*(\kappa)$ -name  $\underline{H}^*$  we have

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underline{H}^* \text{ is a meager non-null subgroup of } (\mathbb{R}, +) \text{”}.$$

Let  $\underline{H}_0, \underline{H}_1$  be  $\mathbb{S}_*$ -names for subsets of  $D_0^\infty$  such that

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underline{H}_0 = \mathbf{E}^{-1}[\underline{H}^* \cap [0, 1/2)] \text{ and } \underline{H}_1 = \mathbf{E}^{-1}[\underline{H}^* \cap [0, 1)] \text{”}.$$

Necessarily  $p_0 \Vdash \underline{H}^* \cap [0, 1/2)$  is not null", so it follows from 2.2(1) that

$$p_0 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underline{H}_0 \notin \mathcal{N} \text{ and } \underline{H}_1 \in \mathcal{M} \text{ and } \underline{H}_0 \subseteq \underline{H}_1 \text{”}.$$

Clearly we may pick a condition  $p_1 \geq p_0$ , a sequence  $\bar{n} = \langle n_j : j < \omega \rangle \subseteq \omega$  and a function  $f \in {}^\omega 2$  such that

$$\begin{aligned} (\oplus)_0 & n_{j+1} > n_j + j + 1 \text{ for each } j, \\ (\oplus)_1 & f(n_{j+1} - 1) = 0 \text{ for each } j, \text{ and} \\ (\oplus)_2 & p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \underline{H}_1 \subseteq \{x \in {}^\omega 2 : (\forall^\infty j < \omega)(x \upharpoonright [n_j, n_{j+1}-1] \neq f \upharpoonright [n_j, n_{j+1}-1])\} \text{”} \\ & \text{(Note: “} [n_j, n_{j+1} - 1] \text{” not “} [n_j, n_{j+1}] \text{”)} \end{aligned}$$

Like in part (1), let  $\bar{m} = \bar{m}[\bar{n}]$ ,  $\bar{N} = \bar{N}[\bar{n}]$ ,  $\bar{J} = \bar{J}[\bar{n}]$ ,  $\bar{H} = \bar{H}[\bar{n}]$ ,  $\pi = \pi[\bar{n}]$  and  $\mathbf{F} = \mathbf{F}[\bar{n}]$ . Let  $A = \{|H_i| - 1 : i < \omega\}$  and  $r^+ \in \mathbb{S}_*$  be such that  $\text{dom}(r^+) = \omega \setminus A$  and  $r^+(k) = 0$  for  $k \in \text{dom}(r^+)$ . Then each  $\alpha < \kappa$  fix a  $\mathbb{S}_*(\kappa)$ -name  $\rho_\alpha$  such that  $p_1 \Vdash_{\mathbb{S}_*(\kappa)} \text{“ } \rho_\alpha \in \underline{H}_0 \cap \mathbf{F}(\eta_\alpha) \text{”}$ .

Now repeat the arguments of the first part (with  $(*)_1$ - $(*)_{11}$  there applied to our  $\bar{n}$ ,  $f$ ,  $\rho_\alpha$  and the operation  $\otimes_0$  here) to find  $q^+ \geq p_1$  and  $\gamma, \delta \in \text{dom}(q^+)$  such that

$$(\diamond) q^+ \Vdash \text{“ } (\exists^\infty k < \omega)((\rho_\gamma \upharpoonright [n_k, n_{k+1}]) \otimes_0 (\rho_\delta \upharpoonright [n_k, n_{k+1}]) = f \upharpoonright [n_k, n_{k+1}]) \text{”}.$$

Let  $G \subseteq \mathbb{S}_*(\kappa)$  be a generic over  $\mathbf{V}$  such that  $q^+ \in G$  and let us work in  $\mathbf{V}[G]$ . Let  $\eta \in D_0^\infty$  be such that  $\mathbf{E}(\rho_\gamma^G) + \mathbf{E}(\rho_\delta^G) = \mathbf{E}(\eta)$  (remember  $\mathbf{E}(\rho_\gamma^G), \mathbf{E}(\rho_\delta^G) < 1/2$ ). We know from  $(\diamond)$  that there are infinitely many  $k < \omega$  satisfying

$$(\blacklozenge) (\rho_\gamma^G \upharpoonright [n_k, n_{k+1}]) \otimes_0 (\rho_\delta^G \upharpoonright [n_k, n_{k+1}]) = f \upharpoonright [n_k, n_{k+1}].$$

Since  $f(n_{k+1} - 1) = 0$  (see  $(\oplus)_1$ ), we get from 3.2(3) that for each  $k$  as in  $(\blacklozenge)$  we also have

$$\begin{aligned} & ((\rho_\gamma^G \upharpoonright [n_k, n_{k+1}]) \otimes_0 (\rho_\delta^G \upharpoonright [n_k, n_{k+1}])) \upharpoonright [n_k, n_{k+1} - 1] = \\ & ((\rho_\gamma^G \upharpoonright [n_k, n_{k+1}]) \otimes_1 (\rho_\delta^G \upharpoonright [n_k, n_{k+1}])) \upharpoonright [n_k, n_{k+1} - 1] = f \upharpoonright [n_k, n_{k+1} - 1]. \end{aligned}$$

Therefore (by 3.2(4)) for each  $k$  satisfying  $(\blacklozenge)$  we have  $\eta \upharpoonright [n_k, n_{k+1} - 1] = f \upharpoonright [n_k, n_{k+1} - 1]$ , so

$$(\exists^\infty k < \omega)(\eta \upharpoonright [n_k, n_{k+1} - 1] = f \upharpoonright [n_k, n_{k+1} - 1]).$$

Consequently, by  $(\oplus)_2$ , we have that  $\eta \notin \underline{H}_1^G$ , i.e.,  $\mathbf{E}(\eta) \notin (\underline{H}^*)^G \cap [0, 1)$ . This contradicts the fact that  $\mathbf{E}(\rho_\gamma^G), \mathbf{E}(\rho_\delta^G) \in (\underline{H}^*)^G$ ,  $\mathbf{E}(\eta) = \mathbf{E}(\rho_\gamma^G) + \mathbf{E}(\rho_\delta^G)$  and  $(\underline{H}^*)^G$  is a subgroup of  $(\mathbb{R}, +)$ .  $\square$

*Remark 4.2.* Instead of the CS product of forcing notions  $\mathbb{S}_*$  we could have used their CS iteration of length  $\omega_2$ . Of course, that would restrict the value of the continuum in the resulting model.

## 5. PROBLEMS

Both theorems 2.3(1) and 4.1(1) can be repeated for other product groups. We may consider a sequence  $\langle H_n : n < \omega \rangle$  of finite groups and their coordinate-wise product  $H = \prod_{n < \omega} H_n$ . Naturally,  $H$  is equipped with product topology of discrete  $H_n$ 's and the product probability measure. Then there exists a null non-meager subgroup of  $H$  but it is consistent that there is no meager non-null such subgroup. It is natural to ask now:

- Problem 5.1.** (1) Does every locally compact group (with complete Haar measure) admit a null non-meager subgroup?  
 (2) Is it consistent that no locally compact group has a meager non-null subgroup?

In relation to Theorem 4.1, we still should ask:

- Problem 5.2.** Is it consistent that there exists a translation invariant Borel hull for the meager ideal on  ${}^\omega 2$ ? On  $\mathbb{R}$ ?

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