Saturated null and meager ideal

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Abstract

We prove that the meager ideal and the null ideal could both be somewhere $\aleph_1$-saturated.

1 Introduction

In [**?**], starting with a measurable cardinal, Komjáth constructed a model of ZFC in which there is a non meager set of reals which cannot be partitioned into uncountably many non meager sets. In [**?**], starting with a measurable cardinal, Shelah constructed a model of ZFC in which there is a non null set of reals which cannot be partitioned into uncountably many non null sets. Our main result is that the two consistency results can be combined.

Theorem 1.1. *Suppose there is a measurable cardinal. Then there is a ccc forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is a set $X \subseteq \mathbb{R}$ such that X is neither null nor meager, X cannot be partitioned into uncountably many non null sets and X cannot be partitioned into uncountably many non meager sets.*

Let us briefly point out why other boolean combinations are also possible. Ulam showed that if there is an $\aleph_1$-saturated sigma ideal $\mathcal{I}$ on some set X such that $\mathcal{I}$ contains every countable set, then there is a weakly inaccessible cardinal below $|X|$. It follows that, under the continuum hypothesis, every non meager (resp. non null) set of reals can be partitioned into uncountably many non meager (resp. non null) sets.

Suppose X is a non meager set of reals that cannot be partitioned into uncountably many non meager sets. Let $\mathbb{P}$ be the forcing for adding $\aleph_1$ Cohen reals. Then in $V^{\mathbb{P}}$, X continues to be non meager and it is easy to check that it still cannot be partitioned into uncountably many non meager sets. Also, in $V^{\mathbb{P}}$, the real line can be covered by $\aleph_1$ null sets. It follows that every non null set in $V^{\mathbb{P}}$ can be partitioned into uncountably many non null sets.

Similarly, if X is a non null set of reals that cannot be partitioned into uncountably many non null sets, then adding $\aleph_1$ random reals gives us a model where X remains non null, it cannot be partitioned into uncountably many non null sets and every non meager set can be partitioned into uncountably many non meager sets.

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On notation: A subset $W \subseteq 2^\omega$ is fat if for every clopen set C, either $W \cap C = \emptyset$ or $\mu(W \cap C) > 0$. A subtree $T \subseteq {}^{<\omega}2$ is fat if $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \restriction n \in T)\}$ is fat. For a clopen subset $C \subseteq 2^\omega$, define $\mathsf{supp}(C)$ to be the smallest (finite) set F such that $(\forall x, y \in 2^\omega)((x \restriction F = y \restriction F) \implies (x \in C \iff y \in C))$. Random denotes the random real forcing. Note that $\{[T] : T \subseteq {}^{<\omega}2$ is a fat tree$\}$ is dense in Random. Cohen denotes Cohen forcing. Its conditions are members of ${}^{<\omega}\omega$ ordered by end extension. In forcing we use the convention that a larger condition is the stronger one; so $p \geq q$ means p extends q. If $\mathbb{P}, \mathbb{Q}$ are forcing notions and $\mathbb{Q} \subseteq \mathbb{P}$, we write $\mathbb{Q} \lessdot \mathbb{P}$ if every maximal antichain in $\mathbb{Q}$ is also a maximal antichain in $\mathbb{P}$. For $x, y \in \omega^\omega$, define $x \oplus y \in \omega^\omega$ by $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$.

2 Eventually different forcing

Suppose $\bar{Y} = \langle y_i : i < \theta \rangle$ where each $y_i \in \omega^\omega$. Define a forcing notion $\mathbb{E} = \mathbb{E}(\bar{Y})$ as follows. $p \in \mathbb{E}$ iff $p = (\sigma_p, F_p) = (\sigma, F)$ where $\sigma \in {}^{<\omega}\omega$ and $F \in [\theta]^{<\aleph_0}$. For $p, q \in \mathbb{E}$, $p \leq q$ iff $\sigma_p \preceq \sigma_q$, $F_p \subseteq F_q$ and for every $k \in [|\sigma_p|, |\sigma_q|)$, for every $i \in F_p$, $\sigma_q(k) \neq y_i(k)$. It is easy to see that $\mathbb{E}$ is a sigma-centered forcing that makes the set $\{y_i : i < \theta\}$ meager since it adds the real $\tau_{\mathbb{E}} = \bigcup\{\sigma_p : p \in G_{\mathbb{E}}\}$ which satisfies $(\forall i < \theta)(\forall^\infty k)(y_i(k) \neq \tau_{\mathbb{E}}(k))$. The following lemma is well known. We include a short proof for completeness.

Lemma 2.1. *Let $\bar{Y}$, $\mathbb{E} = \mathbb{E}(\bar{Y})$ and $\tau_{\mathbb{E}}$ be as above. Let $\mathring{x} \in 2^\omega \cap V^{\mathbb{E}}$. Then there is a Borel function $B : \omega^\omega \to 2^\omega$ such that $\Vdash_{\mathbb{E}} B(\tau_{\mathbb{E}}) = \mathring{x}$.*

Proof of Lemma **??**: For each $n < \omega$ and $i < 2$, choose $\mathcal{A}_{i,n} \subseteq \mathbb{E}$ such that for every $p \in \mathcal{A}_{i,n}$, $p \Vdash \mathring{x}(n) = i$ and $\mathcal{A}_{0,n} \cup \mathcal{A}_{1,n}$ is a maximal antichain in $\mathbb{E}$. Define $B : \omega^\omega \to 2^\omega$ as follows. Given $z \in \omega^\omega$ and $n < \omega$, look for unique $i < 2$ and $(\sigma, F) \in \mathcal{A}_{i,n}$ such that $\sigma \subseteq z$ and for every $k \in [|\sigma|, \omega)$ and $\gamma \in F$, $z(k) \neq y_\gamma(k)$ and define $B(z)(n) = i$. If there are no such unique i and (σ, F), define $B(z)(n) = 0$. Note that if $(\sigma_0, F_0), (\sigma_1, F_1) \in \mathcal{A}_{0,n} \cup \mathcal{A}_{1,n}$ are incompatible, then either σ_0 and σ_1 are incomparable or (say) $\sigma_0 \prec \sigma_1$ and for some $k \in [|\sigma_0|, |\sigma_1|)$ and $\gamma \in F_0$, we have $\sigma_1(k) \neq y_\gamma(k)$. Hence $\Vdash_{\mathbb{E}} B(\tau_{\mathbb{E}}) = \mathring{x}$. □

Note that if $Y = \emptyset$, then $\mathbb{E}(Y)$ is Cohen forcing. We can think of $\mathbb{E}(Y)$ as adding a "partial Cohen" real with memory Y which becomes decreasingly Cohen-like with increasing memory.

3 Background ideas

Let us describe some of the ideas that led to the model witnessing Theorem **??**. By a result of Solovay, we must start with a measurable cardinal κ. Let $\mathcal{I}$ be a witnessing normal prime ideal. We are going to construct a ccc forcing $\mathbb{P}$ that adds two sets of reals $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$ such that (**??**) and (**??**) below hold. Let $\mathcal{J} = \{W \subseteq \kappa : (\exists W' \in \mathcal{I})(W \subseteq W')\}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Since $\mathbb{P}$ is ccc, $\mathcal{J}$ is an $\aleph_1$-saturated κ-additive ideal on κ. We would like to have for every $W \subseteq \kappa$,

$$W \in \mathcal{J} \iff \{x_\alpha : \alpha \in W\} \text{ is meager} \tag{A}$$

$$W \in \mathcal{J} \iff \{y_\alpha : \alpha \in W\} \text{ is null} \tag{B}$$

This would suffice for Theorem **??** since if N is a dense G_δ null subset of $\mathbb{R}$, then the set $(N \cap X) \cup ((\mathbb{R} \setminus N) \cap Y)$ is both non meager and non null and it cannot be partitioned into uncountably many non meager or non null sets.

In [**?**], Komjáth starts by adding κ Cohen reals $X = \{x_\alpha : \alpha < \kappa\}$. So every meager subset of X is currently countable. Using a finite support product, he then makes every subset of X of the form $\{x_\alpha : \alpha \in W\}$ (where $W \in \mathcal{I}$) meager. He finally invokes the properties of product forcing to show that X remains non meager in the final model. Note that the analogous construction fails for the null ideal: If we start by adding a set Y of κ random reals and then, using a finite support iteration (for ccc), add null sets containing some subsets of Y, then we inevitably add Cohen reals at stages of cofinality ω which makes all of Y null. To get around this difficulty, Shelah [**?**] proceeds as follows. Let $\langle X_\alpha : \alpha < \lambda \rangle$ be a list where each member of $\mathcal{I}$ occurs $\lambda = 2^\kappa$ times. First add λ Cohen reals $\langle c_\alpha : \alpha < \lambda \rangle$. Each c_α codes a null G_δ-set N_α in a natural way. We now do a finite support iteration of length κ adding a "partial random" y_ξ at stage $\xi < \kappa$ whose memory is $V_1 = V[\langle c_\alpha : \xi \notin X_\alpha \rangle][\langle y_\eta : \eta < \xi \rangle]$. This means that y_ξ is Random^{V_1}-generic. The expectation is that if $\xi \in X_\alpha$, then $y_\xi \in N_\alpha$ (although showing this requires some work) and that $Y = \{y_\xi : \xi < \kappa\}$ would be the desired set of reals in the final model.

To combine these two construction via a single forcing, we first reverse Komjáth construction as follows. Let $\langle X_\alpha : \alpha < \lambda \rangle$ be the list mentioned above. First add λ Cohen reals $\langle c_\alpha : \alpha < \lambda \rangle$. Each c_α codes an F_σ-meager set - namely, $M_\alpha = \{y \in \omega^\omega : (\forall^\infty k)(y(k) \neq c_\alpha(k))\}$. Now do a finite support iteration of length κ adding a "partial Cohen" real x_ξ at stage $\xi < \kappa$ with memory $C_\xi = \{c_\alpha : \xi \in X_\alpha\}$. This means that x_ξ is $\mathbb{E}(C_\xi)$-generic. Note that if $\xi \in X_\alpha$, then $x_\xi \in M_\alpha$. It is not difficult to check that $X = \{x_\xi : \xi < \kappa\}$ is a non meager set on which the meager ideal is $\aleph_1$-saturated.

The next section begins by describing iterations $\bar{\mathbb{P}}_\lambda = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda + \kappa \rangle$ for $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ (where $\lambda_0 = 2^\kappa$) which combine partial Cohen and partial random reals. The reason behind considering $\bar{\mathbb{P}}_\lambda$ for various λ's and not just for $\lambda = \lambda_0$ will become clear during the proof of Lemma **??** where we use automorphisms of $\mathbb{P}_{\lambda+\kappa}$ for $\lambda > \lambda_0$ to construct certain finitely additive measures on $\mathcal{P}(\omega) \cap V^{\mathbb{P}_{\lambda_0+\xi}}$ for $\xi < \kappa$.

4 Forcing

Suppose κ is measurable and $\mathcal{I}$ is a normal prime ideal on κ. Put $\lambda_0 = 2^\kappa$. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, define the following.

(1) $\langle X_\alpha : \alpha < \lambda_0^{+\omega} \rangle$ is a sequence of members of $\mathcal{I}$.

(2) For every $n < \omega$ and $X \in \mathcal{I}$, $|\{\alpha < \lambda_0^{+n} : X_\alpha = X\}| = \lambda_0^{+n}$.

(3) For $\xi < \kappa$, $C^\lambda_{\lambda+\xi} = C_{\lambda+\xi} = \{\alpha < \lambda : \xi \in X_\alpha\}$. This is the memory of the partial Cohen real to be added at stage $\lambda + \xi$ (see item (7) below).

(4) For $\xi < \kappa$, $A^\lambda_{\lambda+\xi} = A_{\lambda+\xi} = \{\alpha < \lambda : \xi \notin X_\alpha\} \cup [\lambda, \lambda + \xi)$. This is the memory of the partial random real to be added at stage $\lambda + \xi$ (see item (7) below).

(5) $\bar{\mathbb{P}}_\lambda = \langle \mathbb{P}_{\lambda,\alpha}, \mathbb{Q}_{\lambda,\alpha} : \alpha < \lambda + \kappa \rangle$ is a finite support iteration with limit $\mathbb{P}_{\lambda,\lambda+\kappa}$. In the contexts where the value of λ is constant, we drop the λ in the subscript and just write $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$.

(6) For $\alpha < \lambda$, $\mathbb{Q}_\alpha = \mathsf{Cohen}$ with generic real $\tau_\alpha \in \omega^\omega$.

(7) For $\xi < \kappa$, $\mathbb{Q}_{\lambda+\xi} = \mathbb{Q}^1_{\lambda+\xi} \times \mathbb{Q}^2_{\lambda+\xi}$ where $\mathbb{Q}^1_{\lambda+\xi} = (\mathsf{Random})^{V[\langle \tau_i : i \in A_{\lambda+\xi}\rangle]}$ with generic partial random $\tau^1_{\lambda+\xi} \in 2^\omega$ and $\mathbb{Q}^2_{\lambda+\xi} = \mathbb{E}(\langle \tau_\alpha : \alpha \in C_{\lambda+\xi}\rangle)$ with generic $\tau^2_{\lambda+\xi} \in \omega^\omega$. Let $\tau_{\lambda+\xi} = \tau^1_{\lambda+\xi} \oplus \tau^2_{\lambda+\xi}$.

(8) Define $\mathbb{P} = \mathbb{P}_{\lambda_0,\lambda_0+\kappa}$.

The model for Theorem **??** will be $V^{\mathbb{P}}$. The verification of this will conclude with the proof of Lemma **??**. In the remainder of this section, we establish some basic facts about these iterations.

The following claim is easily proved by induction on $\xi \leq \kappa$ using Lemma **??** and the standard properties of Cohen and random forcings.

Claim 4.1. *For every $\xi \leq \kappa$, $\mathring{x} \in 2^\omega \cap V^{\mathbb{P}_{\lambda+\xi}}$, there are a Borel function $B : \omega^\omega \to 2^\omega$ and $\langle (n_k, \gamma_k) : k < \omega\rangle$ such that every $\gamma_k < \lambda + \xi$ and $n_k < \omega$, and $\Vdash_{\mathbb{P}} \mathring{x} = B(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle)$.*

Definition 4.2. *Let $\mathbb{P}'_{\lambda,\lambda+\kappa} = \mathbb{P}'_{\lambda+\kappa}$ be the set of conditions $p \in \mathbb{P}_{\lambda+\kappa}$ satisfying the following requirements.*

(a) For each $\alpha \in \lambda \cap \mathsf{dom}(p)$, $p(\alpha) = \sigma_\alpha \in {}^{<\omega}\omega$. In this case, define $\mathsf{supp}(p(\alpha)) = \emptyset$.

(b) For every $\alpha \in \mathsf{dom}(p) \cap [\lambda, \lambda+\kappa)$, letting $p(\alpha) = (p(\alpha)(1), p(\alpha)(2))$, we have the following.

(i) There exist $\langle (n_k, \gamma_k) : k < \omega\rangle$, $\rho \in {}^{<\omega}2$ and a Borel function B such that for every k, $n_k < \omega$, $\gamma_k \in A_\alpha$, the range of B consists of fat trees in ${}^{<\omega}2$ and $\Vdash_{\mathbb{P}_\alpha} p(\alpha)(1) = [B(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle)]$ is a subset of $[\rho]$ of relative measure more than $1 - 2^{-(n-j+10)}$ where $n = |\mathsf{dom}(p) \cap [\lambda, \lambda+\kappa)|$ and $j = |\mathsf{dom}(p) \cap [\lambda, \alpha)|$. Recall that for $X \subseteq Y \subseteq 2^\omega$, the relative measure of X in Y is $\mu(X)/\mu(Y)$.

(ii) $\Vdash_{\mathbb{P}_\alpha} p(\alpha)(2) = (\nu, F)$ where $F \in [C_\alpha]^{<\aleph_0}$, $\nu \in {}^{<\omega}\omega$, $F \subseteq \mathsf{dom}(p)$ and for each $\beta \in F$, $|\sigma_\beta| \geq |\nu|$.

(iii) In this case, define $\mathsf{supp}(p(\alpha)) = \{\gamma_k : k < \omega\} \cup F$.

(c) Define $\mathsf{supp}(p) = \mathsf{dom}(p) \cup \bigcup\{\mathsf{supp}(p(\alpha)) : \alpha \in \mathsf{dom}(p)\}$.

For $\xi < \kappa$, $\mathbb{P}'_{\lambda,\lambda+\xi} = \mathbb{P}'_{\lambda+\xi} \subseteq \mathbb{P}_{\lambda+\xi}$ is defined analogously.

Using Claim **??** and the Lebesgue density theorem, it is easily checked that $\mathbb{P}'_{\lambda+\xi}$ is dense in $\mathbb{P}_{\lambda+\xi}$ for every $\xi \leq \kappa$.

Suppose $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $\xi_\star \leq \kappa$. Let $h : \lambda + \xi_\star \to \lambda + \xi_\star$ be a bijection satisfying the following.

(1) $h \restriction [\lambda, \lambda + \xi_\star)$ is the identity.

(2) For every $\xi < \xi_\star$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi}$ iff $h(\alpha) \in A_{\lambda+\xi}$ (equivalently, $\alpha \in C_{\lambda+\xi}$ iff $h(\alpha) \in C_{\lambda+\xi}$).

Define $\hat{h} : \mathbb{P}'_{\lambda+\xi_\star} \to \mathbb{P}'_{\lambda+\xi_\star}$ as follows. For $p \in \mathbb{P}'_{\lambda+\xi_\star}$, put $\hat{h}(p) = p'$ where

(a) $\mathsf{dom}(p') = \{h(\alpha) : \alpha \in \mathsf{dom}(p)\}$,

(b) for $\alpha \in \mathsf{dom}(p) \cap \lambda$, $p'(h(\alpha)) = p(\alpha)$ and

(c) for $\alpha \in \mathsf{dom}(p) \cap [\lambda, \lambda+\xi_\star)$, put $p'(\alpha)(1) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)$ where B, $\langle (n_k, \gamma_k) : k < \omega\rangle$ are as in Definition **??**(b)(i) for αth coordinate of p and $p'(\alpha)(2) = (\nu, h[F])$ where $(\nu, F) = p(\alpha)(2)$.

Claim 4.3. *$\hat{h}$ is an automorphism of $\mathbb{P}'_{\lambda+\xi_\star}$.*

Proof: By induction on $\xi_\star$. □

Definition 4.4. *For $\lambda_0 \le \lambda < \lambda_0^{+\omega}$ and $A \subseteq \lambda + \kappa$, define $\mathbb{P}'_{\lambda,A} = \mathbb{P}'_A = \{p \in \mathbb{P}'_{\lambda+\kappa} : \mathit{supp}(p) \subseteq A\}$.*

The following lemma describes a sufficient condition on $A \subseteq \lambda + \xi$ for ensuring that $\mathbb{P}'_A \lessdot \mathbb{P}_{\lambda+\xi}$. It is used in the proofs of Corollary **??** and Claim **??**.

Lemma 4.5. *Let $\xi_\star \le \kappa$, $A \subseteq \lambda + \xi_\star$ and $[\lambda, \lambda + \xi_\star) \subseteq A$. Suppose for every countable $B \subseteq \lambda$, there is a bijection $h : \lambda + \xi_\star \to \lambda + \xi_\star$ such that*

(a) $h \restriction ((B \cap A) \cup [\lambda, \lambda + \xi_\star))$ is the identity,

(b) for every $\xi < \xi_\star$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi}$ iff $h(\alpha) \in A_{\lambda+\xi}$,

(c) $h[B] \subseteq A$ and

(d) $h[B \cap A_{\lambda+\xi_\star}] \subseteq A \cap A_{\lambda+\xi_\star}$.

Then $\mathbb{P}'_A \lessdot \mathbb{P}'_{\lambda+\xi_\star}$.

Proof of Lemma **??**: By induction on $\xi_\star$. If $\xi_\star = 0$ or limit this is clear. So assume $\xi_\star = \xi + 1$ and put $\alpha = \lambda + \xi$. By inductive hypothesis, $\mathbb{P}'_{A\cap\alpha} \lessdot \mathbb{P}'_\alpha$ so it suffices to check the following: If $\{p_n : n < \omega\} \subseteq \mathbb{P}'_A$, $p \in \mathbb{P}'_{A\cap\alpha}$ and $p \Vdash_{\mathbb{P}'_{A\cap\alpha}} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\}$ is predense in $(\mathsf{Random})^{V[\langle \tau_\beta : \beta \in A\cap A_\alpha\rangle]} \times \mathbb{E}(\langle \tau_\beta : \beta \in A \cap C_\alpha\rangle)$, then $p \Vdash_{\mathbb{P}'_\alpha} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_\alpha}\}$ is predense in $(\mathsf{Random})^{V[\langle \tau_\beta : \beta \in A_\alpha\rangle]} \times \mathbb{E}(\langle \tau_\beta : \beta \in C_\alpha\rangle)$.

Suppose this fails for some $\{p_n : n < \omega\} \subseteq \mathbb{P}'_A$ and $p \in \mathbb{P}'_{A\cap\alpha}$. Choose $q \in \mathbb{P}'_\alpha$, ν, F, D, $\langle (n_k, \gamma_k) : k < \omega\rangle$ such that

- $q \ge p$,
- $\nu \in {}^{<\omega}\omega$, $F \in [C_\alpha]^{<\aleph_0}$, D is a Borel function on ω^ω whose range consists of fat trees, each $\gamma_k \in A_\alpha$ and
- $q \Vdash_{\mathbb{P}'_\alpha} r = [D(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle)] \wedge (r, (\nu, F))$ is incompatible with every member of $\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in \mathbb{G}_{\mathbb{P}'_\alpha}\}$.

Let W be the union of the following sets: $\mathsf{dom}(q)$, $\mathsf{supp}(q)$, $\mathsf{supp}(p)$, $\bigcup\{\mathsf{dom}(p_n) : n < \omega\}$, $\bigcup\{\mathsf{supp}(p_n) : n < \omega\}$ and $\{\gamma_k : k < \omega\} \cup F$. Using the hypothesis on A, we can find a bijection $h : \alpha \to \alpha$ such that

- $h \restriction ((B \cap A) \cup [\lambda, \alpha))$ is the identity,
- $(\forall \eta < \xi)(\forall \beta < \lambda)(\beta \in A_{\lambda+\eta} \iff h(\beta) \in A_{\lambda+\eta})$,

- $h[B] \subseteq A$ and
- $h[B \cap A_\alpha] \subseteq A \cap A_\alpha$.

So $\hat{h}$ is an automorphism of $\mathbb{P}'_\alpha$. As $h[B] \subseteq A$, $\hat{h}(q) \in \mathbb{P}'_{A\cap\alpha}$. Since $h \restriction (B \cap A)$ is the identity, it follows that $\hat{h}(p) = p$ and for every $n < \omega$, $\hat{h}(p_n) = p_n$. Since $\{\gamma_k : k < \omega\} \cup F \subseteq W$ and $h[B \cap A_\alpha] \subseteq A \cap A_\alpha$, we have that $\Vdash_{\mathbb{P}'_\alpha} r' = \hat{h}(r) = [D(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)] \in (\mathsf{Random})^{V[\langle \tau_\beta : \beta \in A\cap A_\alpha\rangle]}$ and $\Vdash_{\mathbb{P}'_\alpha} (\nu, h[F]) \in \mathbb{E}[\langle \tau_\beta : \beta \in A \cap C_\alpha\rangle]$. It follows that $\hat{h}(q) \Vdash_{\mathbb{P}'_\alpha} (r', (\nu, h[F]))$ is incompatible with every condition in $\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_\alpha}\}$. Since $P'_{A\cap\alpha} \lessdot P'_\alpha$, we also get that $\hat{h}(q) \Vdash_{\mathbb{P}'_{A\cap\alpha}} (r', (\nu, h[F]))$ is incompatible with every condition in $\left\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\right\}$. But since $p = \hat{h}(p) \leq \hat{h}(q)$ and $p \Vdash_{\mathbb{P}'_{A\cap\alpha}} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\}$ is predense in $(\mathsf{Random})^{V[\langle \tau_\beta : \beta\in A\cap A_\alpha\rangle]} \times \mathbb{E}(\langle \tau_\beta : \beta \in A \cap C_\alpha\rangle)$, we get a contradiction. □

Corollary 4.6. *For every $\xi_\star < \kappa$, $\mathbb{P}'_{A_{\lambda+\xi_\star}} \lessdot \mathbb{P}'_{\lambda+\xi_\star}$.*

Proof of Corollary **??**: Let $B \subseteq \lambda$ be countable. By Lemma **??** clauses (a)-(d), it suffices to construct a bijection $h : \lambda + \xi_\star \to \lambda + \xi_\star$ such that $h \restriction ((B \cap A_{\lambda+\xi_\star}) \cup [\lambda, \lambda + \xi_\star))$ is identity, $(\forall \xi < \xi_\star)(\forall \alpha < \lambda)(\alpha \in A_{\lambda+\xi} \iff h(\alpha) \in A_{\lambda+\xi})$ and $h[B] \subseteq A_{\lambda+\xi_\star}$. For each $x \subseteq \xi_\star$, let $W_x = \{\alpha < \lambda : X_\alpha \cap \xi_\star = x\}$. Let $W_{x,0} = \{\alpha \in W_x : \xi_\star \notin X_\alpha\}$ and $W_{x,1} = \{\alpha \in W_x : \xi_\star \in X_\alpha\}$ so $W_x = W_{x,0} \sqcup W_{x,1}$. Note that for every $x \subseteq \xi_\star$, $|W_{x,0}| = |W_{x,1}| = \lambda$. So for each $x \subseteq \xi_\star$, we can choose a bijection $h_x : W_x \to W_x$ such that $h_x[W_{x,0} \cap B] \subseteq W_{x,1}$ and $h_x \restriction (W_{x,1} \cap B)$ is identity. Put $h \restriction \lambda = \bigcup\{h_x : x \subseteq \xi_\star\}$. □

5 Meager ideal

Recall that $\mathbb{P} = \mathbb{P}_{\lambda_0,\lambda_0+\kappa}$. Throughout this section and the next, we fix $\lambda = \lambda_0 = 2^\kappa$. In $V^{\mathbb{P}}$, let $\mathcal{J} = \{Y \subseteq \kappa : (\exists X \in \mathcal{I})(Y \subseteq X)\}$ be the ideal generated by $\mathcal{I}$. Since $\mathbb{P}$ is ccc, $\mathcal{J}$ is an $\aleph_1$-saturated κ-additive ideal over κ. The next lemma says that the meager ideal restricted to $\{\tau^2_{\lambda+\xi} : \xi < \kappa\}$ is isomorphic to $\mathcal{J}$ and is, therefore, $\aleph_1$-saturated. Its proof will conclude at the end of Section **??**.

Lemma 5.1. *In $V^{\mathbb{P}}$, for every $Y \subseteq \kappa$, $\{\tau^2_{\lambda+\xi} : \xi \in Y\}$ is meager iff $Y \in \mathcal{J}$.*

Proof of Lemma **??**: Suppose $\mathring{Y} \in \mathcal{J}$. Since $\mathbb{P}$ is ccc, we can find $X \in \mathcal{I}$ such that $\Vdash \mathring{Y} \subseteq X$. Choose $\alpha < \lambda$ such that $X = X_\alpha$. Note that $\Vdash (\forall \xi \in X_\alpha)(\forall^\infty k)(\tau^2_{\lambda+\xi}(k) \neq \tau_\alpha(k))$. Hence $\{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\}$ is meager in $V^{\mathbb{P}}$.

Next suppose $\mathring{Y} \notin \mathcal{J}$. Towards a contradiction, WLOG, suppose $p \in \mathbb{P}'$ forces that $\{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\}$ is nowhere dense in ω^ω. Let $\mathring{T} \subseteq {}^{<\omega}\omega$ be a nowhere dense subtree such that $p \Vdash \{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\} \subseteq [\mathring{T}]$. For each $\sigma \in {}^{<\omega}\omega$, let $\mathcal{A}_\sigma$ be a maximal antichain of conditions in $\mathbb{P}'$ deciding $\sigma \in \mathring{T}$. Put $W = \bigcup\{\mathsf{supp}(p) : p \in \mathcal{A}_\sigma, p \in \mathcal{A}_\sigma\}$.

Choose $\xi < \kappa$ and $p' \in \mathbb{P}'$ such that $p' \geq p$, $\xi \notin \bigcup\{X_\alpha : \alpha \in W \cap \lambda\}$, $\lambda + \xi > \sup(W)$ and $p' \Vdash \xi \in \mathring{Y}$ and hence $p' \Vdash \tau^2_{\lambda+\xi} \in [\mathring{T}]$. By extending p', we can assume that $\lambda + \xi \in \mathsf{dom}(p')$. Let $q \in \mathbb{P}'$ be such that $\mathsf{dom}(q) = \mathsf{dom}(p') \cap (\lambda + \xi + 1)$, $q \restriction (\lambda + \xi) = p' \restriction (\lambda + \xi)$, $q(\lambda + \xi)(1) = 2^\omega$ and $q(\lambda + \xi)(2) = (\emptyset, \emptyset)$. Since $\mathring{T} \in V^{\mathbb{P}_{\lambda+\xi}}$, $q \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^2_{\lambda+\xi} \in [\mathring{T}]$.

Put $\mathsf{dom}(q)\cap\lambda = \{\alpha_j : j < m_\star\}\sqcup\{\beta_j : j < r_\star\}$ where $\{\beta_j : j < r_\star\} = \{\beta \in \mathsf{dom}(q)\cap\lambda : \xi \notin X_\beta\}$ and α_j's and β_j's are increasing with j. Note that $W\cap\{\alpha_j : j < m_\star\} = \emptyset$. Put $\mathsf{dom}(q)\cap[\lambda,\lambda+\xi) = \{\lambda+\xi_j : j < n_\star\}$ where ξ_j's are increasing with j. For $j < r_\star$, let $q(\beta_j) = \eta_j$. For $j < n_\star$, let $q(\lambda+\xi_j)(2) = (\nu_j, F_j)$ and let $\rho_j \in {}^{<\omega}2$ be such that $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of relative measure more than $1-2^{-(n_\star-j+10)}$. By extending q, we can also assume that

- $q(\lambda+\xi)(2) = (\nu_\star, F_\star)$ where $\nu_\star \in {}^{l_\star}\omega$,
- $F_\star = \{\alpha_j : j < m_\star\}$ is non empty,
- for every $j < m_\star$, $q(\alpha_j) = \sigma_j \in {}^{l_\star}\omega$; so $|\sigma_j| = |\nu_\star|$ and
- for each $j < n_\star$, $\nu_j \in {}^{l_\star}\omega$.

To produce such a q, first extend each $q(\alpha)$ for $\alpha \in \mathsf{dom}(q)\cap\lambda$ such that they all have the same sufficiently large length $l_\star$. Let $K \subseteq \omega$ be the finite set of values these $q(\alpha)$'s take. Next for each $j < n_\star$, extend each ν_j to a member of ${}^{l_\star}\omega$ with new values from $\omega\setminus K$. Finally extend $q(\lambda+\xi)(2)$ to $(\nu_\star, F_\star)$ where $\nu_\star \in {}^{l_\star}\omega$ and $F_\star = \{\alpha_j : j < m_\star\}$. This is permissible because $\xi \in X_{\alpha_j}$ for every $j < m_\star$.

For $\alpha < \lambda$, define $\mathring{S}_{\nu_\star,\alpha} = \{\nu \in {}^{<\omega}\omega : (\nu_\star \preceq \nu) \wedge (\forall k \in [|\nu_\star|, |\nu|)(\nu(k) \neq \tau_\alpha(k))\}$.

Claim 5.2. $q \restriction (\lambda+\xi) \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{T} \supseteq \bigcap\limits_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_j}$

Proof of Claim **??**: Suppose not. Choose $q \restriction (\lambda+\xi) \leq q_1 \in \mathbb{P}'_{\lambda+\xi}$, $\nu_\star \preceq \nu_1 \in {}^{<\omega}\omega$ such that $q_1 \Vdash_{\mathbb{P}_{\lambda+\xi}} \nu_1 \in \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_j} \wedge \nu_1 \notin \mathring{T}$. Let $q_2 \geq q_1$, $q_2 \in \mathbb{P}'_{\lambda+\xi+1}$ be such that $q_2 \restriction (\lambda+\xi) = q_1 \restriction (\lambda+\xi)$ and $q_2(\lambda+\xi)(2) = (\nu_1, F_\star)$. Then $q_2 \geq q$ and $q_2 \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \nu_1 \subseteq \tau^2_{\lambda+\xi}$. Hence $q_2 \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^2_{\lambda+\xi} \notin \mathring{T}$: Contradiction. □

Choose $\langle \alpha_{i,j} : i < \lambda, j < m_\star\rangle$ such that the following hold.

- For every $i < \lambda$ and $j < m_\star$, $\alpha_{i,j} \in \lambda\setminus(W \cup \mathsf{dom}(q))$.
- For every $i_1, i_2 < \lambda$ and $j_1, j_2 < m_\star$, $\alpha_{i_1,j_1} = \alpha_{i_2,j_2}$ iff $(i_1,j_1) = (i_2,j_2)$.
- For every $i < \lambda$ and $j < m_\star$, $X_{\alpha_{i,j}} = X_{\alpha_j}$.

For $i < \lambda$, the map $h_i : \lambda+\xi \to \lambda+\xi$ defined by

$$h_i(\alpha) = \begin{cases} \alpha_{i,j} & \text{if } j < m_\star \text{ and } \alpha = \alpha_j \\ \alpha_j & \text{if } j < m_\star \text{ and } \alpha = \alpha_{i,j} \\ \alpha & \text{otherwise} \end{cases}$$

induces an automorphism $\hat{h}_i$ of $\mathbb{P}'_{\lambda+\xi}$ that fixes $\mathring{T}$. Let $q_i = \hat{h}_i(q \restriction (\lambda+\xi))$. Then for each $i < \lambda$, we have the following.

(1) $\mathsf{dom}(q_i) = \{\alpha_{i,j} : j < m_\star\} \sqcup \{\beta_j : j < r_\star\} \sqcup \{\lambda+\xi_j : j < n_\star\}$.

(2) For every $j < m_\star$, $q_i(\alpha_{i,j}) = q(\alpha_j) = \sigma_j \in {}^{l_\star}\omega$.

(3) For every $j < r_\star$, $q_i(\beta_j) = q(\beta_j) = \eta_j$.

(4) For every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q_i(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of fractional measure more than $1 - 2^{-(n_\star - j + 10)}$.

(5) For every $j < n_\star$, $q_i(\lambda+\xi_j)(2) = (\nu_j, F_{i,j})$ where $\nu_j \in {}^{l_\star}\omega$ and $F_{i,j} = h_i[F_j]$.

(6) $q_i \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{T} \supseteq \bigcap\limits_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_{i,j}}$.

Since λ is uncountable, by a Δ-system argument we can further assume that for some $\langle F^\star_j : j < n_\star\rangle$, for every $j < n_\star$, $\langle F_{i,j} : i < \omega\rangle$ forms a Δ-system with root $F^\star_j$.

For $i < \omega$, define $g(i) : [l_\star, l_\star + i) \to \omega$ such that for every $k \in \mathsf{dom}(g(i))$, $g(i)(k) = i$.

Definition 5.3. *For each $i < \omega$ define $q^\star_i \in \mathbb{P}'_{\lambda+\xi}$ by $\mathsf{dom}(q^\star_i) = \mathsf{dom}(q_i)$ and*

$$q^\star_i(\alpha) = \begin{cases} \sigma_j \cup g(i) & \text{if } j < m_\star \text{ and } \alpha = \alpha_{i,j}, \\ q_i(\alpha) & \text{otherwise.} \end{cases}$$

Let $\bar{q}^\star = \langle q^\star_i : i < \omega\rangle$.

The next claim provides a sufficient condition to complete the proof of Lemma **??**.

Claim 5.4. *Suppose there exists $q_\star \in \mathbb{P}$ such that*

$$q_\star \Vdash (\exists^\infty i)(q^\star_i \in G_{\mathbb{P}}).$$

Then $q_\star \Vdash [\mathring{T}]$ has non empty interior.

Proof of Claim **??**: Let G be $\mathbb{P}$-generic over V with $q_\star \in G$. Suppose $\nu_\star \preceq \nu \in {}^{<\omega}\omega$. Choose $i < \omega$ such that $(\forall k \in \mathsf{dom}(\nu))(\nu(k) < i)$ and $q^\star_i \in G$. Since $q^\star_i(\alpha_{i,j}) = \sigma_j \cup g(i)$ for every $j < m_\star$, it follows that $\nu \in \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_{i,j}}$. By Claim **??**, it follows that $\nu \in \mathring{T}$. Hence $q_\star \Vdash [\nu_\star] \subseteq [\mathring{T}]$. $\square$

So to complete the proof of Lemma **??**, it is sufficient to construct $q_\star \in \mathbb{P}$ satisfying the hypothesis of Claim **??**. This will be done in Section **??**.

6 Null ideal

Definition 6.1. *For each $n < \omega$, let $\langle C^n_k : k < \omega\rangle$ be a one-one listing of all clopen subsets of 2^ω of measure 2^{-n}. For $\alpha < \lambda$, define $\mathring{N}_\alpha = \bigcap_k \bigcup_{n>k} C^n_{\tau_\alpha(n)}$. So $\mathring{N}_\alpha$ is a null G_δ-set coded by τ_α.*

The next claim says that the null ideal restricted to $\{\tau^1_{\lambda+\xi} : \xi < \kappa\}$ is isomorphic to $\mathcal{J}$ and is, therefore, $\aleph_1$-saturated. Its proof will be completed at the end of Section **??**.

Lemma 6.2. *In $V^{\mathbb{P}}$, for every $Y \subseteq \kappa$, $\{\tau^1_{\lambda+\xi} : \xi \in Y\}$ is null iff $Y \in \mathcal{J}$.*

Proof of Lemma **??**: Towards a contradiction, suppose $p \Vdash \mathring{Y} \notin \mathcal{J} \wedge \{\tau^1_{\lambda+\xi} : \xi \in \mathring{Y}\}$ is null. Let $\mathring{N}$ be a null Borel set in $V^{\mathbb{P}}$ such that $p \Vdash \mathring{N} \supseteq \{\tau^1_{\lambda+\xi} : \xi \in \mathring{Y}\}$. Choose a Borel function B coded in V, and $\langle (n_k, \gamma_k) : k < \omega\rangle$ such that for every $k < \omega$, $\gamma_k < \lambda + \kappa$, $n_k < \omega$ and $\Vdash B(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle) = \mathring{N}$. Let $A = \bigcup\{X_{\gamma_k} : k < \omega \wedge \gamma_k < \lambda\}$. Then $A \in \mathcal{I}$. Choose $q \geq p$ and

$\xi < \kappa$ such that $q \Vdash \xi \in \mathring{Y} \setminus A$ and $\lambda + \xi > \sup(\{\gamma_k : k < \omega\})$. Since $\mathring{N}$ is coded in $V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi}\rangle]$ (as $\{\gamma_k : k < \omega\} \subseteq A_{\lambda+\xi}$), it follows that $q \Vdash \tau^1_{\lambda+\xi} \notin \mathring{N}$: Contradiction.

Next suppose $\mathring{Y} \in \mathcal{J}$. Since $\mathbb{P}$ is ccc, we can find $X \in \mathcal{I}$ such that $\Vdash \mathring{Y} \subseteq X$. We'd like to show that $\{\tau^1_{\lambda+\xi} : \xi \in X\}$ is null. Choose $\alpha < \lambda$ such that $X = X_\alpha$. It is clearly enough to show that for every $\xi \in X_\alpha$, $\Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^1_{\lambda+\xi} \in \mathring{N}_\alpha$. Suppose this fails and fix $\xi \in X_\alpha$, $p \in \mathbb{P}'_{\lambda+\xi+1}$ and $k_\star < \omega$ such that $p \Vdash (\forall k \geq k_\star)(\tau^1_{\lambda+\xi} \notin C^k_{\tau_\alpha(k)})$. We can assume that $\alpha \in \mathsf{dom}(p)$ and $p(\alpha) = \sigma_\star \in {}^{l_\star}\omega$ for some $l_\star > k_\star$. Choose a Borel function B and $\langle (n_j, \gamma_j) : j < \omega\rangle$ such that $\gamma_j \in A_{\lambda+\xi}$, range of B consists of fat trees and $\Vdash_{\mathbb{P}_{\lambda+\xi}} B(\langle \tau_{\gamma_j}(n_j) : j < \omega\rangle) = \mathring{T}$ and $[\mathring{T}] = p(\lambda + \xi)(1)$. It follows that $p \restriction (\lambda + \xi) \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_\star)([\mathring{T}] \cap C^k_{\tau_\alpha(k)} = \emptyset)$. Let $W = \{\gamma_j : j < \omega\}$ and note that $\alpha \notin W$.

Put $\mathsf{dom}(p) \cap \lambda = \{\alpha\} \sqcup \{\beta_j : j < r_\star\}$ and $\mathsf{dom}(p) \cap [\lambda, \lambda + \xi) = \{\lambda + \xi_j : j < n_\star\}$ where β_j and ξ_j are increasing with j. For $j < r_\star$, let $p(\beta_j) = \eta_j$. For $j < n_\star$, let $p(\lambda + \xi_j)(2) = (\nu_j, F_j)$ and let $\rho_j \in {}^{<\omega}2$ be such that $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p(\lambda + \xi_j)(1)$ is a fat subset of $[\rho_j]$ of relative measure more than $1 - 2^{-(n_\star - j + 10)}$. By possibly extending p, we can assume that for every $j < n_\star$, $\nu_j \in {}^{l_\star}\omega$. Choose $\langle \alpha_i : i < \lambda\rangle$ such that the following hold.

- For all $i < j < \lambda$, $\alpha_i < \alpha_j < \lambda$.
- $X_{\alpha_i} = X_\alpha$ (so $\alpha_i \notin W$).
- $\alpha_i \notin \mathsf{supp}(p)$.

For $i < \lambda$, the map $h_i : \lambda + \xi \to \lambda + \xi$ defined by

$$h_i(\gamma) = \begin{cases} \alpha & \text{if } \gamma = \alpha_i \\ \alpha_i & \text{if } \gamma = \alpha \\ \gamma & \text{otherwise} \end{cases}$$

induces an automorphism $\hat{h}_i$ of $\mathbb{P}'_{\lambda+\xi}$ that fixes $\mathring{T}$. Let $p_i = \hat{h}_i(p \restriction (\lambda + \xi))$. Then for each $i < \lambda$, we have the following.

(1) $\mathsf{dom}(p_i) = \{\alpha_i\} \sqcup \{\beta_j : j < r_\star\} \sqcup \{\lambda + \xi_j : j < n_\star\}$.

(2) $p_i(\alpha_i) = p(\alpha) = \sigma_\star \in {}^{l_\star}\omega$.

(3) For every $j < r_\star$, $p_i(\beta_j) = p(\beta_j) = \eta_j$.

(4) For every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda + \xi_j)(1)$ is a fat subset of $[\rho_j]$ of fractional measure more than $1 - 2^{-(n_\star - j + 10)}$.

(5) For every $j < n_\star$, $p_i(\lambda + \xi_j)(2) = (\nu_j, F_{i,j})$ where $\nu_j \in {}^{l_\star}\omega$ $F_{i,j} = h_i[F_j]$.

(6) $p_i \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_\star)([\mathring{T}] \cap C^k_{\tau_{\alpha_i}(k)} = \emptyset)$.

As before, by thinning out we can assume that for some $\langle F^\star_j : j < n_\star\rangle$, for every $j < n_\star$, $\langle F_{i,j} : i < \omega\rangle$ forms a Δ-system with root $F^\star_j$.

For each $i < \omega$, we'll extend p_i on the α_ith coordinate to get p'_i as follows.

Definition 6.3. *For each $n < \omega$, let $K_n = \{k < \omega : \mathsf{supp}(C^{l_\star}_k) \subseteq n\}$. Note that for all $n \geq l_\star$, $|K_n| = \binom{2^n}{2^{n-l_\star}}$. Define $\bar{k} = \langle k_n : n < \omega\rangle$ by: $k_0 = 0$, $k_{n+1} - k_n = \binom{2^n}{2^{n-l_\star}}$. Let $f : \omega \to \omega$ be such that $f[[k_n, k_{n+1})] = K_n$. For each $i < \omega$, $\gamma \in \mathsf{dom}(p_i)$ define*

$$p'_i(\gamma) = \begin{cases} p_i(\gamma) & \text{if } \gamma \neq \alpha_i \\ \sigma_\star \cup \{(l_\star, f(i))\} & \text{if } \gamma = \alpha_i \end{cases}$$

Lemma 6.4. *Suppose $K < \omega$, $F \subseteq [\lambda, \lambda{+}\kappa)$ is finite, $\langle \rho_\theta : \theta \in F\rangle$ is a sequence in ${}^{<\omega}2$, $\langle a_\theta : \theta \in F\rangle$ is a sequence in $(1/2, 1)$ and $\langle q_j : j < K\rangle$ is a sequence of conditions in $\mathbb{P}'$ such that for every $j < K$, $\mathsf{dom}(q_j) = F$, for each $\theta \in F$, $\Vdash_{\mathbb{P}_\theta} q_j(\theta)(1)$ is a subset of $[\rho_\theta]$ of relative measure $\geq a_\theta$ and $q_j(\theta)(2)$ is the empty condition. Then there exists $q^\star \in \mathbb{P}'$ with $\mathsf{dom}(q^\star) = F$ such that for every $\theta \in F$, $\Vdash_{\mathbb{P}_\theta} q^\star(\theta)(1)$ is a fat subset of $[\rho_\theta]$ of relative measure $\geq 2a_\theta - 1$ and $q^\star(\theta)(2)$ is the empty condition and*

$$q^\star \Vdash_{\mathbb{P}} |\{j < K : q_j \in G_{\mathbb{P}}\}| \geq K 2^{-|F|} \prod_{\theta \in F} a_\theta.$$

Proof of Lemma **??**: By induction on $|F|$. Suppose $F = \{\theta\}$. Work in $V^{\mathbb{P}_\theta}$. Define $\phi = \sum_{j<K} 1_{q_j(\theta)(1)}$ where $1_{q_j(\theta)(1)}$ is the characteristic function of $q_j(\theta)(1)$. Put $A = \{x \in [\rho_\theta] : \phi(x) \geq \frac{Ka_\theta}{2}\}$. It suffices to show that $\mu(A) > \mu([\rho_\theta])(2a_\theta - 1)$. We have

$$Ka_\theta \mu([\rho_\theta]) \leq \int \phi d\mu = \int_A \phi d\mu + \int_{2^\omega \setminus A} \phi d\mu \leq K\mu(A) + (\mu([\rho_\theta]) - \mu(A))\frac{Ka_\theta}{2}.$$

Solving gives $\dfrac{\mu(A)}{\mu([\rho_\theta])} \geq \dfrac{a_\theta}{2 - a_\theta} > 2a_\theta - 1$.

Now suppose $|F| \geq 2$ and β is the largest member of F. Let $F' = F \setminus \{\beta\}$, $q'_j = q \restriction F'$. Choose $q' \in \mathbb{P}'$ with domain F' such that for every $\theta \in F'$, $\Vdash_{\mathbb{P}_\theta} q'(\theta)(1)$ is a subset of $[\rho_\theta]$ of relative measure $\geq 2a_\theta - 1$, $q'(\theta)(2)$ is the empty condition and $q' \Vdash_{\mathbb{P}} |\{j < K : q'_j \in G_{\mathbb{P}}\}| \geq K2^{-|F'|} \prod_{\theta \in F'} a_\theta$. Let $\mathring{W} = \{j < K : q'_j \in G_{\mathbb{P}}\}$. Let $\{W_i : i < N\}$ list all subsets of K of size $\geq K2^{-|F'|} \prod_{\theta \in F'} a_\theta$. Choose a maximal antichain $\{r_i : i < N\}$ in $\mathbb{P}'_\beta$ above q' such that each $r_i \Vdash_{\mathbb{P}} \mathring{W} = W_i$. Work in $V^{\mathbb{P}_\beta}$. For each $i < N$, arguing as above, we can get a condition $s_i \in \mathbb{Q}^1_\beta$ such that $r_i \Vdash_{\mathbb{P}_\beta} \mu(s_i) \geq 2a_\beta - 1$ and $s_i \Vdash_{\mathbb{Q}_\beta} |\{j \in W_i : q_j(\beta)(1) \in G_{\mathbb{Q}^1_\beta}\}| \geq \dfrac{|W_i| a_\beta}{2}$. Choose $q^\star$ such that $q^\star(\theta) = q'(\theta)$ if $\theta \in F'$ and for each $i < N$, $r_i \Vdash_{\mathbb{P}_\beta} q^\star(\beta)(1) = s_i$. □

For $i < \omega$, let p''_i be defined by $\mathsf{dom}(p''_i) = \{\lambda + \xi_j : j < n_\star\}$ and for every $j < n_\star$, $p''_i(\lambda + \xi_j)(1) = p'_i(\lambda + \xi_j)(1)$ and $p''_i(\lambda + \xi_j)(2)$ is the empty condition. Note that $p''_i \in \mathbb{P}'_{\lambda+\xi}$. For each $n < \omega$, apply Lemma **??** to the sequence $\langle p''_i : i \in [k_n, k_{n+1})\rangle$ to obtain $q^\star_n$ such that the following hold.

Definition 6.5. *(a) $q^\star_n \in \mathbb{P}'_{\lambda+\xi}$ and $\mathsf{dom}(q^\star_n) = \{\lambda + \xi_j : j < n_\star\}$.*

(b) For every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q^\star_n(\lambda{+}\xi_j)$ is a subset of $[\rho_j]$ of relative measure $\geq 2(1{-}2^{-(n_\star - j + 10)}) - 1 = 1 - 2^{-(n_\star - j + 9)}$.

(c) $q^\star_n \Vdash_{\mathbb{P}_{\lambda+\xi}} |\{i \in [k_n, k_{n+1}) : p''_i \in \mathbb{G}_{\mathbb{P}_{\lambda+\xi}}\}| \geq (k_{n+1} - k_n)2^{-n_\star} \prod_{j<n_\star}(1 - 2^{-(n_\star - j + 10)}) > (k_{n+1} - k_n)4^{-n_\star}$.

Definition 6.6. *For each $n<\omega$ and $i\in[k_n,k_{n+1})$ define $p^\star_i\in\mathbb{P}'_{\lambda+\xi}$ by $\mathsf{dom}(p^\star_i)=\mathsf{dom}(p_i)$ and*

$$p^\star_i(\alpha)=\begin{cases}\sigma_\star\cup\{(l_\star,f(i))\}\cup\{(k,i):k\in[l_\star+1,i+l_\star+1)\} & \text{if } \alpha=\alpha_i\\ p'_i(\beta_j)=\eta_j & \text{if } j<r_\star \text{ and } \alpha=\beta_j\\ (p'_i(\alpha)(1)\cap q^\star_n(\alpha)(1),(\nu_j,F_{i,j})) & \text{if } j<n_\star \text{ and } \alpha=\lambda+\xi_j\end{cases}$$

Let $\bar p^\star=\langle p^\star_i:i<\omega\rangle$.

Note that for every $j<n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p^\star_i(\lambda+\xi_j)(1)$ is a subset of $[\rho_j]$ of relative measure more than $1-2^{-(n_\star-j+8)}$. The next claim provides a sufficient condition to complete the proof of Lemma **??**.

Claim 6.7. *Suppose for some $p_\star\in\mathbb{P}$ and $\varepsilon>0$,*

$$p_\star\Vdash(\exists^\infty n)\frac{|\{i\in[k_n,k_{n+1}):p^\star_i\in G_{\mathbb{P}}\}|}{k_{n+1}-k_n}\geq\varepsilon.$$

Then, $p_\star\Vdash[\mathring T]$ is finite.

Proof of Claim **??**: For $n<\omega$, let $\mathring W_n=\{i\in[k_n,k_{n+1}):p^\star_i\in G_{\mathbb{P}}\}$ and $\mathring a_n=|\mathring T\cap{}^n2|$. Note that $p_\star\Vdash(\forall i\in\mathring W_n)(\forall\sigma\in\mathring T\cap{}^n2)(C^{l_\star}_{f(i)}\cap[\sigma]=\emptyset)$ because $l_\star>k_\star$, $p^\star_i(\alpha_i)(l_\star)=f(i)$ and $p^\star_i\Vdash(\forall k\geq k_\star)(C^k_{\tau_{\alpha_i}(k)}\cap[\mathring T]=\emptyset)$. It follows that $|\mathring W_n|\leq\binom{2^n-\mathring a_n}{2^{n-l_\star}}$. Hence

$$\frac{|\mathring W_n|}{k_{n+1}-k_n}\leq\frac{\binom{2^n-\mathring a_n}{2^{n-l_\star}}}{\binom{2^n}{2^{n-l_\star}}}=\prod_{j=1}^{\mathring a_n}\left(1-\frac{2^{n-l_\star}}{2^n-\mathring a_n+j}\right)\leq\left(1-\frac{2^{n-l_\star}}{2^n}\right)^{\mathring a_n}.$$

Therefore

$$\frac{|\mathring W_n|}{k_{n+1}-k_n}\leq(1-2^{-l_\star})^{\mathring a_n}.$$

As $\mathring a_n$ is increasing with n, it follows that $p_\star$ forces that $\lim_n\mathring a_n<\infty$ and hence that $[\mathring T]$ is finite. □

To complete the proof of Theorem **??**, it suffices to construct conditions $q_\star,p_\star$ satisfying the hypotheses of Claims **??** and **??**. Let us try to illustrate the main difficulty in doing this for $\bar p^\star$.

Let

$$\mathring A=\{i<\omega:(\exists n<\omega)(i\in[k_n,k_{n+1})\wedge(\forall k\in[k_n,k_{n+1}))(p^\star_k\restriction\lambda\in\mathbb{G}_{\mathbb{P}})\}$$

and for each $\xi<\kappa$, let

$$\mathring B_\xi=\{i<\omega:p^\star_i\restriction(\lambda+\xi)\in G_{\mathbb{P}}\}.$$

Put $\mathsf{dom}(p_\star)=\{\beta_j:j<r_\star\}\sqcup\{\lambda+\xi_j:j<n_\star\}$ and $p_\star\restriction\{\beta_j:j<r_\star\}=p^\star_i\restriction\{\beta_j:j<r_\star\}$ (this doesn't depend on $i<\omega$). For $j<n_\star$, define $p_\star(\lambda+\xi_j)(2)=(\nu_j,F^\star_j)$.

Note that $p_\star\restriction\lambda\Vdash\mathring A$ is infinite. It is clearly necessary to choose the random coordinates $p_\star(\lambda+\xi_j)(1)$ for $j<n_\star$ such that $p_\star\Vdash\mathring A\cap\mathring B_{\xi_{n_\star-1}+1}$ is infinite. Suppose we have constructed $p_\star\restriction(\lambda+\xi_j)$ such that $p_\star\restriction(\lambda+\xi_j)\Vdash\mathring A\cap\mathring B_{\xi_j}$ is infinite and we would like to choose $p_\star(\lambda+\xi_j)(1)\in\mathsf{Random}^{V[\langle\tau_\alpha:\alpha\in A_{\lambda+\xi_j}\rangle]}$ (recall that $p_\star(\lambda+\xi_j)(2)=(\nu_j,F^\star_j)$) such that $p_\star\restriction(\lambda+\xi_{j+1})\Vdash\mathring A\cap\mathring B_{\xi_{j+1}}$

is infinite. The problem is that we do not have access to $\mathring{B}_{\xi_j} \in V^{\mathbb{P}_{\lambda+\xi_j}}$ in $V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_j}\rangle]$ and hence it is unclear how to proceed.

To get around this difficulty, we will construct an auxiliary finitely additive measure $\mathring{\mathfrak{m}}$ on $\mathcal{P}(\omega)\cap V^{\mathbb{P}}$ which carries enough information about the partial randoms appearing at stages $\{\lambda+\xi_j : j < n_\star\}$ to allow us to choose appropriate $p_\star(\lambda+\xi_j)(1)$'s. Definition **??** lists a sufficient set of requirements on $\mathring{\mathfrak{m}}$ for this. The construction of $\mathring{\mathfrak{m}}$ in Lemma **??** is inductive and uses Lemma **??** to code enough information about the partial randoms to allow the inductive step to proceed. The class of blueprints in Definition **??** is general enough to allow a Lowenheim-Skolem type argument (Claim **??**) in the proof of Lemma **??**.

7 Measures and blueprints

An algebra $\mathcal{A}$ is a family of subsets of ω that contains all finite subsets of ω and is closed under complementation and finite union. A finitely additive measure on an algebra $\mathcal{A}$ is a function $\mathfrak{m} : \mathcal{A} \to [0,1]$ that satisfies the following.

- For every finite $F \subseteq \omega$, $\mathfrak{m}(F) = 0$.
- $\mathfrak{m}(\omega) = 1$.
- If $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \emptyset$, then $\mathfrak{m}(A_1 \cup A_2) = \mathfrak{m}(A_1) + \mathfrak{m}(A_2)$.

Suppose $\mathfrak{m} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure and $f : \omega \to [0,1]$. Following Lebesgue, define

$$\int f d\mathfrak{m} = \lim_{n\to\infty} \sum_{k=0}^{2^n} \frac{k a_k}{2^n}$$

where $a_k = \mathfrak{m}(\{n < \omega : k/2^n \leq f(n) < (k+1)/2^n\})$.

The following is a standard application of the Hahn-Banach theorem.

Lemma 7.1. *Suppose $\mathfrak{m} : \mathcal{A} \to [0,1]$ is a finitely additive measure on an algebra $\mathcal{A}$ and $X \subseteq \omega$. Let $a \in [0,1]$ be such that for every $A, B \in \mathcal{A}$, if $A \subseteq X \subseteq B$, then $\mathfrak{m}(A) \leq a \leq \mathfrak{m}(B)$. Then, there exists a finitely additive measure $\mathfrak{m}' : \mathcal{P}(\omega) \to [0,1]$ that extends $\mathfrak{m}$ and $\mathfrak{m}'(X) = a$.*

The proofs of the next two lemmas can be found in [**?**].

Lemma 7.2. *Suppose $\mathfrak{m} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure. For $i \in \{1,2\}$, let $\mathbb{R}_i$ be a forcing notion and $\mathring{\mathfrak{m}}_i \in V^{\mathbb{R}_i}$ be such that $\Vdash_{\mathbb{R}_i} \mathring{\mathfrak{m}}_i : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure extending $\mathfrak{m}$. Then, there exists $\mathring{\mathfrak{m}}_3 \in V^{\mathbb{R}_1\times\mathbb{R}_2}$ such that $\Vdash_{\mathbb{R}_1\times\mathbb{R}_2} \mathring{\mathfrak{m}}_3 : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure extending both $\mathring{\mathfrak{m}}_1$ and $\mathring{\mathfrak{m}}_2$.*

Lemma 7.3. *Suppose that $\mathfrak{m} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure. Let $\mathbb{B} = \mathsf{Random}$, $r \in \mathbb{B}$. Define $\mathring{\mathfrak{m}}_r \in V^{\mathbb{B}}$ as follows. For $\mathring{X} \in \mathcal{P}(\omega)\cap V^{\mathbb{B}}$, define*

$$\mathring{\mathfrak{m}}_r(\mathring{X}) = \sup\left\{\inf\left\{\int \frac{\mu(q\cap[[n\in\mathring{X}]]_{\mathbb{B}}}{\mu(q)} d\mathfrak{m} : q \geq p\right\} : p \geq r, p \in G_{\mathbb{B}}\right\}.$$

Then, the following hold.

(1) $r \Vdash \mathring{\mathfrak{m}}_r : \mathcal{P}(\omega) \to [0,1]$ *is a finitely additive measure extending* $\mathfrak{m}$.

(2) *If* $\mathring{X} \in \mathcal{P}(\omega) \cap V^{\mathbb{B}}$ *and* $a > 0$ *satisfy for every* $n < \omega$, $\dfrac{\mu(r \cap [[n \in \mathring{X}]]_{\mathbb{B}})}{\mu(r)} \geq a$, *then there exists* $s \geq r$ *such that* $s \Vdash \mathring{\mathfrak{m}}_r(\mathring{X}) \geq a$.

The next definition introduces blueprints. Their role is clarified in Claim **??**. Note the return of variable $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ here.

Definition 7.4. *For* $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, *let* $\mathcal{T}_\lambda$ *be the set of tuples*

$$t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) = (\bar{\alpha}^t, m^t, \bar{\sigma}^t, \bar{\beta}^t, r^t, \bar{\eta}^t, \bar{\xi}^t, n^t, \bar{\rho}^t, \bar{\nu}^t, \bar{F}^t, l^t, \bar{\varepsilon}^t)$$

where

(i) $l, m, n, r < \omega$,

(ii) $\bar{\alpha} = \langle \alpha_{i,j} : i < \omega, j < m \rangle$ *where each* $\alpha_{i,j} < \lambda$,

(iii) *for every* $i_1, i_2 < \omega$ *and* $j_1, j_2 < m$, $\alpha_{i_1,j_1} = \alpha_{i_2,j_2}$ *iff* $(i_1, j_1) = (i_2, j_2)$,

(iv) $\bar{\sigma} = \langle \sigma_{i,j} : i < \omega, j < m \rangle$ *where each* $\sigma_{i,j} \in {}^{<\omega}\omega$,

(v) $\bar{\beta} = \langle \beta_j : j < r \rangle$ *is a sequence of pairwise distinct member of* $\lambda \setminus \{\alpha_{i,j} : i < \omega, j < m\}$,

(vi) $\bar{\eta} = \langle \eta_j : j < r \rangle$ *where each* $\eta_j \in {}^{<\omega}\omega$,

(vii) $\bar{\xi} = \langle \xi_j : j < n \rangle$ *is an increasing sequence in* κ,

(viii) $\bar{\rho} = \langle \rho_j : j < n \rangle$ *where each* $\rho_j \in {}^{<\omega}2$,

(ix) $\bar{\nu} = \langle \nu_j : j < n \rangle$ *where each* $\nu_j \in {}^{l}\omega$,

(x) $\bar{F} = \langle F_{i,j} : i < \omega, j < n \rangle$ *where each* $F_{i,j} \in [C_{\lambda+\xi_j}]^{<\aleph_0}$ *and for every* $j < n$, $\langle F_{i,j} : i < \omega \rangle$ *forms a* Δ*-system with root* F_j *and*

(xi) $\bar{\varepsilon} = \langle \varepsilon_j : j < n \rangle$, *where* $\varepsilon_{n-1} \in (0, 2^{-8})$ *and* $2\varepsilon_j \leq \varepsilon_{j+1}$ *for every* $j < n-1$.

We call members of $\mathcal{T}_\lambda$ blueprints. They are intended to code information about certain sequences of conditions in $\mathbb{P}'_\lambda$ that look like $\bar{q}^\star$ and $\bar{p}^\star$ from Definitions **??**, **??** in the following sense.

Definition 7.5. *Suppose* $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$ *and* $\bar{p} = \langle p_i : i < \omega \rangle$ *is a sequence in* $\mathbb{P}'_\lambda$. *We say that* $\bar{p}$ *is of type* t *if the following hold.*

(a) *For every* $i < \omega$, $\mathbf{dom}(p_i) = \{\alpha_{i,j} : j < m\} \sqcup \{\beta_j : j < r\} \sqcup \{\lambda + \xi_j : j < n\}$.

(b) *For every* $i < \omega$ *and* $j < m$, $p_i(\alpha_{i,j}) = \sigma_{i,j}$.

(c) *For every* $i < \omega$ *and* $j < r$, $p_i(\beta_j) = \eta_j$.

(c) *For every* $i < \omega$ *and* $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda + \xi_j)(1)$ *is a subset of* $[\rho_j]$ *of relative measure more than* $1 - \varepsilon_j$.

(d) *For every* $i < \omega$ *and* $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda + \xi_j)(2) = (\nu_j, F_{i,j})$.

Definition 7.6. *Let $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$.*

(1) We say that t is q-like for every $i < \omega$ and $j < m$, $|\sigma_{i,j}| = l+i$ and $(\forall k \in [l, l+i))(\sigma_{i,j}(k) = i)$.

*(2) We say that t is p-like if for every $n < \omega$, $i \in [k_n, k_{n+1})$ and $j < m$, $|\sigma_{i,j}| = l + 1 + i$, $\langle \sigma_{i,j}(l) : i \in [k_n, k_{n+1}) \rangle$ are pairwise distinct and $(\forall k \in [l+1, l+1+i))(\sigma_{i,j}(k) = i)$ where $\langle k_n : n < \omega \rangle$ is as in Definition **??**.*

Note that if $\bar{q}^\star$ is of type t, then t is q-like and if $\bar{p}^\star$ is of type t, then t is p-like.

For $t \in \mathcal{T}_\lambda$, $\xi < \kappa$, we write $t \restriction \xi$ for the blueprint which is obtained by restricting the sequence $\bar{\xi}^t$ to ordinals below ξ and modifying $\bar{\rho}^t, \bar{\nu}^t, \bar{F}^t, \bar{\varepsilon}^t$ and n^t accordingly. The next definition relates finitely additive measures in $V^{\mathbb{P}_{\lambda+\kappa}}$ and blueprints in $\mathcal{T}_\lambda$.

Definition 7.7. *Suppose $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$, $\bar{k} = \langle k_n : n < \omega \rangle$ is an increasing sequence in ω with $k_0 = 0$, $\xi_{n-1} < \xi \leq \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$. We say that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$ if the following hold.*

(1) $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure.

(2) For every $j < n$, letting $V_j = V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_j} \rangle]$, we have $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V_j) \in V_j$.

(3) For every $\bar{p} = \langle p_i : i < \omega \rangle$ of type t, there exists $p_{\bar{p}} \in \mathbb{P}'_{\lambda+\xi}$ such that the following hold.

(a) $\mathrm{dom}(p_{\bar{p}}) = \{\beta_j : j < r\} \sqcup \{\lambda + \xi_j : j < n\}$.

(b) For every $j < r$, $p_{\bar{p}}(\beta_j) = \eta_j$.

(c) For every $X \in \mathcal{P}(\omega) \cap V$ that satisfies $(\forall n < \omega)(|X \cap [k_n, k_{n+1})| \leq 1)$, we have

$$p_{\bar{p}} \restriction \lambda \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(X) = 0.$$

(d) $p_{\bar{p}} \restriction \lambda \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{A}_{\bar{p},\bar{k}}) = 1$ where

$$\mathring{A}_{\bar{p},\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}.$$

(e) For every $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_{\bar{p}}(\lambda + \xi_j)(1) \subseteq [\rho_j]$ and $p_{\bar{p}}(\lambda + \xi_j)(2) = (\nu_j, F_j)$.

(f) For every $j < n$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{Y}_{\bar{p},\bar{k},j}) = 1$ where $i \in \mathring{Y}_{\bar{p},\bar{k},j}$ iff letting $N < \omega$ be such that $i \in [k_N, k_{N+1})$, we have $p_i(\lambda + \xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}$ and

$$|\{i' \in [k_N, k_{N+1}) : p_{i'}(\lambda + \xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}\}| \geq k_{N+1} - k_N - m^t.$$

(g) For every $j < n$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{X}_{\bar{p},j}) \geq 1 - 2\varepsilon_j > 0$ where

$$\mathring{X}_{\bar{p},j} = \{i < \omega : p_i \restriction [\lambda, \lambda + \xi_j + 1) \in \mathbb{G}_{\mathbb{P}}\}.$$

The next claim provides a sufficient condition for the existence of $q_\star$ and $p_\star$ satisfying the hypotheses of Claims **??** and **??** respectively.

Claim 7.8. *Suppose for every $t \in \mathcal{T}_\lambda$, if t is either q-like or p-like, then there are $\xi^t_{n-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$ where $\bar{k}$ is as in Definition **??**. Then there exist $q_\star$ and $p_\star$ satisfying the hypotheses of Claims **??** and **??** respectively.*

Proof of Claim **??**: Choose $t \in \mathcal{T}_\lambda$ such that $\bar{q}^\star$ from Definition **??** is of type t. Choose $\xi_{n_\star-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $q_\star = p_{\bar{q}^\star}$ be as in Clause (3) of Definition **??**. Let $\mathring{X}_{\bar{q}^\star,n_\star-1} = \{i < \omega : q_i^\star \restriction [\lambda, \lambda+\xi) \in G_{\mathbb{P}}\}$ and let

$$\mathring{A}_{\bar{q}^\star,\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(q_k^\star \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}$$

Then $q_\star$ forces that $\mathring{\mathfrak{m}}(\mathring{X}_{\bar{q}^\star,n_\star-1}) > 0$ and $\mathring{\mathfrak{m}}(\mathring{A}_{\bar{q}^\star,\bar{k}}) = 1$ and hence that $\mathring{A}_{\bar{q}^\star,\bar{k}} \cap \mathring{X}_{\bar{q}^\star,n_\star-1}$ is infinite. It follows that $q_\star \Vdash (\exists^\infty i)(q_i^\star \in G_{\mathbb{P}})$. Hence $q_\star$ satisfies the hypothesis of Claim **??**.

Next choose $t \in \mathcal{T}_\lambda$ such that $\bar{p}^\star$ from Definition **??** is of type t. Choose $\xi_{n_\star-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $p_\star = p_{\bar{p}^\star}$ be as in Clause (3) of Definition **??**.

Let $\mathring{X}_{\bar{p}^\star,n_\star-1} = \{i < \omega : p_i^\star \restriction [\lambda, \lambda+\xi) \in G_{\mathbb{P}}\}$. For $j < n_\star$, let $\mathring{Y}_{\bar{p}^\star,\bar{k},j}$ be defined by $i \in \mathring{Y}_{\bar{p}^\star,\bar{k},j}$ iff $p_i^\star(\lambda+\xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}$ and for some $N < \omega$, $i \in [k_N, k_{N+1})$ and $|\{i' \in [k_N, k_{N+1}) : p_{i'}^\star(\lambda+\xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}\}| \geq k_{N+1} - k_N - 1$ (recalling $m^t = 1$ for the blueprint of $\bar{p}^\star$). Finally let

$$\mathring{A}_{\bar{p}^\star,\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k^\star \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}$$

Then $p_\star$ forces that $\mathring{\mathfrak{m}}(\mathring{A}_{\bar{p}^\star,\bar{k}}) = 1$, $\mathring{\mathfrak{m}}(\mathring{X}_{\bar{p}^\star,n_\star-1}) > 0$ and for every $j < n_\star$, $\mathring{\mathfrak{m}}(\mathring{Y}_{\bar{p}^\star,\bar{k},j}) = 1$. Hence it also forces that

$$\mathring{A}_{\bar{p}^\star,\bar{k}} \cap \mathring{X}_{\bar{p}^\star,n_\star-1} \cap \bigcap_{j<n_\star} \mathring{Y}_{\bar{p}^\star,\bar{k},j}$$

is infinite. Let i be member of this set and fix n such that $i \in [k_n, k_{n+1})$. The set $\{i' \in [k_n, k_{n+1}) : p_{i'}^\star \notin G_{\mathbb{P}}\}$ has size at most $n_\star + (k_{n+1} - k_n)(1 - 4^{-n_\star})$. The first contribution comes from Definition **??**(3)(f) (noting $m^t = 1$) and the second comes from the partial random coordinates (see Definitions **??** and **??**(c)). It follows that

$$p_\star \Vdash (\exists^\infty n) \frac{|\{i \in [k_n, k_{n+1}) : p_i^\star \in G_{\mathbb{P}}\}|}{k_{n+1} - k_n} \geq 4^{-(n_\star+1)}$$

Hence $p_\star$ satisfies the hypothesis of Claim **??**. □

The following lemma finishes the proof of Theorem **??**.

Lemma 7.9. *Suppose $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$, $\xi_{n-1} < \xi < \kappa$ and $\bar{k} = \langle k_n : n < \omega \rangle$ is as in Definition **??**. Assume that t is either q-like or p-like. Then there exists $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$.*

Proof of Lemma **??**: By induction on $n = n^t = |\bar{\xi}|$.

Suppose $n = 0$. Fix $\xi < \kappa$. Since $n = 0$, there is a unique $\bar{p}$ of type t. Put $p_{\bar{p}} = \{(\beta_j, \eta_j) : j < r\}$. Define $\mathring{X}_{\bar{p}} = \{i : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k \in G_{\mathbb{P}_{\lambda+\xi}}))\}$. Let $\mathcal{W} = \{X : X \in \mathcal{P}(\omega) \cap V \wedge (\forall n < \omega)(|X \cap [k_n, k_{n+1})| \leq 1)\}$. Since $\lim_n(k_{n+1} - k_n) = \infty$, it follows that for every finite $\mathcal{F} \subseteq \mathcal{W}$, $p_{\bar{p}} \Vdash_{\mathbb{P}_\lambda} \mathring{X}_{\bar{p}} \setminus \bigcup \mathcal{F}$ is infinite. Hence we can choose $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure and for every $X \in \mathcal{F}$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{X}_{\bar{p}} \setminus X) = 1$. It follows that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$.

Next fix $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n+1, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$ such that t is either q-like or p-like. It suffices to construct $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi_{n+1}}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $\mathcal{T}'_{\lambda^+} = \{t' \in \mathcal{T}_{\lambda^+} : t' = (\bar{\alpha}^{t'}, m, \bar{\sigma}, \bar{\beta}^{t'}, r, \bar{\eta}, \bar{\xi} \restriction n, n, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n \rangle, l, \bar{\varepsilon} \restriction n)\}$.

By inductive assumption, for every $t' \in \mathcal{T}'_{\lambda^+}$, there exists $\mathring{\mathfrak{m}}^{t'} \in V^{\mathbb{P}_{\lambda^+,\lambda^++\xi_n}}$ such that $\mathring{\mathfrak{m}}^{t'}$ satisfies $(t', \bar{k})$. Fix such a map $t' \mapsto \mathring{\mathfrak{m}}^{t'}$ on $\mathcal{T}'_{\lambda^+}$.

Claim 7.10. *There exists* $\mathring{\mathfrak{m}} \in V^{\mathbb{P}'_{\lambda,\lambda+\xi_n}}$ *that satisfies* $(t \restriction \xi_{n+1}, \bar{k})$ *where* $t \restriction \xi_{n+1} = (\alpha, \sigma, \bar{\beta}, r, \bar{\xi} \restriction n, n, \bar{\sigma}, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$ *and* $\Vdash_{\mathbb{P}'_{\lambda,\lambda+\xi_n}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda,A}}) \in V^{\mathbb{P}'_{\lambda,A}}$ *where* $A = A^\lambda_{\lambda+\xi_n}$.

Proof of Claim **??**: Let χ be sufficiently large. Choose M_0, M_1 elementary submodels of $(\mathcal{H}_\chi, \in, <_\chi)$ such that $M_0 \in M_1$, $|M_0| = |M_1| = \lambda$, and for $l \in \{0,1\}$, $\bar{\mathbb{P}}_{\lambda^+}$, $\mathcal{T}'_{\lambda^+}$ and the map $t' \mapsto \mathring{\mathfrak{m}}^{t'}$ are in M_l, $\lambda + 1 \subseteq M_l$ and ${}^{\leq\kappa}M_l \subseteq M_l$. Note that if $B_j \in \{\lambda \cap A^\lambda_{\lambda+\xi_j}, \lambda \setminus A^\lambda_{\lambda+\xi_j}\}$ for $j < n+1$, then $|\bigcap_{j<n+1} B_j| = \lambda$. Also, if $D_j \in \{\lambda^+ \cap A^{\lambda^+}_{\lambda^++\xi_j}, \lambda^+ \setminus A^{\lambda^+}_{\lambda^++\xi_j}\}$ for $j < n+1$, then $|M_0 \cap \bigcap_{j<n+1} D_j| = \lambda$ and $|(M_1 \setminus M_0) \cap \bigcap_{j<n+1} D_j| = \lambda$. So we can choose a bijection $h : \lambda + \xi_n \to M_1 \cap (\lambda^+ + \xi_n)$ such that

(i) For every $\xi < \xi_n$, $h(\lambda + \xi) = \lambda^+ + \xi$

(ii) For every $j < n$ and $\alpha < \lambda$, $\alpha \in A^\lambda_{\lambda+\xi_j}$ iff $h(\alpha) \in A^{\lambda^+}_{\lambda^++\xi_j}$; hence also $\alpha \in C^\lambda_{\lambda+\xi_j}$ iff $h(\alpha) \in C^{\lambda^+}_{\lambda^++\xi_j}$

(iii) For every $\alpha < \lambda$, $\alpha \in A^\lambda_{\lambda+\xi_n}$ iff $h(\alpha) \in M_0$

Let $t' = (\langle h(\alpha_{i,j}) : i < \omega, j < m\rangle, m, \bar{\sigma}, \langle h(\beta_j) : j < r\rangle, r, \bar{\eta}, \bar{\xi} \restriction n, n, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle h[F_{i,j}] : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$. As ${}^\omega M_1 \subseteq M_1$, $t' \in M_1$. Hence also $\mathring{\mathfrak{m}}^{t'} \in M_1$.

Define $\hat{h} : \mathbb{P}'_{\lambda,\lambda+\xi_n} \to \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}$ as follows: $\hat{h}(p) = p'$ where $\mathsf{dom}(p') = \{h(\alpha) : \alpha \in \mathsf{dom}(p)\}$. If $\alpha \in \mathsf{dom}(p) \cap \lambda$, then $p'(h(\alpha)) = p(\alpha)$. If $\alpha \in \mathsf{dom}(p) \cap [\lambda, \lambda + \xi_n)$, then $p'(\alpha)(1) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)$ where B, $\langle (n_k, \gamma_k) : k < \omega\rangle$ are as in Definition **??**(b)(i) for coordinate α and $p'(\alpha)(2) = (\nu, h[F])$ where $(\nu, F) = p(\alpha)(2)$.

Subclaim 7.11. *The following hold.*

(1) $\hat{h} : \mathbb{P}'_{\lambda,\lambda+\xi_n} \to \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}$ *is an isomorphism*

(2) $\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_0} \lessdot \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1} \lessdot \mathbb{P}'_{\lambda^+,\lambda^++\xi_n}$

(3) For $j < n$, *put* $A_j = A^{\lambda^+}_{\lambda^++\xi_j} \cap M_1$. *Then* $\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_j}} \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,A_j}}) \in V^{\mathbb{P}'_{\lambda^+,A_j}}$

(4) For $l \in \{0,1\}$, $\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_n}} \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_l}}) \in V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_l}}$

Proof of Subclaim **??**: (1) and (4) should be clear. For (2), use Lemma **??**. For (3), use the fact that $\mathring{m}^{t'}$ satisfies $(t', \bar{k})$. □

Choose $\mathring{\mathfrak{m}}' \in V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}}$ such that $\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_n}} \mathring{\mathfrak{m}}' = \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}})$ and define $\mathring{\mathfrak{m}} \in V^{\mathbb{P}'_{\lambda,\lambda+\xi_n}}$ by $\hat{h}(\mathring{\mathfrak{m}}) = \mathring{\mathfrak{m}}'$.

By Subclaim **??**, $\mathring{\mathfrak{m}}$ satisfies $(t \restriction \xi_{n+1}, \bar{k})$ where $t \restriction \xi_{n+1} = (\alpha, \sigma, \bar{\beta}, r, \bar{\xi} \restriction n, n, \bar{\sigma}, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$ and, moreover, $\Vdash_{\mathbb{P}'_{\lambda,\lambda+\xi_n}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda,A}}) \in V^{\mathbb{P}'_{\lambda,A}}$ where $A = A^\lambda_{\lambda+\xi_n}$. This completes the proof of Claim **??**. □

To complete the proof of Lemma **??**, we would like to extend $\mathring{\mathfrak{m}}$ to $\mathring{\mathfrak{m}}_1 \in V^{\mathbb{P}_{\lambda+\xi_{n+1}}}$ such that $\mathring{\mathfrak{m}}_1$ satisfies $(t, \bar{k})$. We do this in two steps.

Let $q = \langle(\beta_j, \eta_j) : j < r\rangle$. Note that for every $X \in \mathcal{P}(\omega) \cap V$, if $(\forall n < \omega)(|X \cap [k_n, k_{n+1})| \leq 1)$, then $q \Vdash_{\mathbb{P}_{\lambda+\xi_n}} \mathring{m}(X) = 0$.

Claim 7.12. *q forces that the following holds in $V^{\mathbb{P}_{\lambda+\xi_n}}$: Letting $\mathbb{Q} = \mathbb{Q}^2_{\lambda+\xi_n}$, there exists a $\mathbb{Q}$-name $\mathring{\mathfrak{m}}_2$ such that $\Vdash_{\mathbb{Q}} \mathring{\mathfrak{m}}_2 : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure that extends $\mathring{\mathfrak{m}}$ and $(\nu_n, F_n) \Vdash_{\mathbb{Q}} \mathring{\mathfrak{m}}_2(\mathring{Y}) = 1$ where $i \in \mathring{Y}$ iff for some $N < \omega$, $i \in [k_N, k_{N+1})$, $(\nu_n, F_{i,n}) \in G_{\mathbb{Q}}$ and*

$$|\{i' \in [k_N, k_{N+1}) : (\nu_n, F_{i',n}) \in G_{\mathbb{Q}}\}| \geq k_{N+1} - k_N - m$$

Proof of Claim **??**: Work in $V_1 = V^{\mathbb{P}_{\lambda+\xi_n}}$ above q. By Lemma **??**, it suffices to show that for every $A \in \mathcal{P}(\omega) \cap V_1$ satisfying $\mathring{\mathfrak{m}}(A) > 0$, $(\nu_n, F_n) \Vdash_{\mathbb{Q}} A \cap \mathring{Y} \neq \emptyset$. Towards a contradiction, suppose this fails. Choose $(\nu, F) \in \mathbb{Q}$ and $A \in V_1$, such that $(\nu_n, F_n) \leq (\nu, F)$, $\mathring{\mathfrak{m}}(A) > 0$ and $(\nu, F) \Vdash_{\mathbb{Q}} A \cap \mathring{Y} = \emptyset$. We can assume $|\nu| > |\nu_n| = l$. Choose $q_1 \in \mathbb{P}'_{\lambda+\xi_n}$, $q_1 \geq q$ that forces this.

First suppose t is q-like. Then, for every $i < \omega$ and $j < m$, $|\sigma_{i,j}| = l + i$ and $(\forall k \in [l, l+i))(\sigma_{i,j}(k) = i)$. Let H be $\mathbb{P}_{\lambda+\xi_n}$-generic over V with $q_1 \in H$. Work in $V[H]$. Since $\mathring{m}(A) > 0$, A is infinite. Choose $N < \omega$ and $i \in [k_N, k_{N+1}) \cap A$ such that $k_N > |\nu|$, $(\forall k \in \mathsf{dom}(\nu))(k_N > \nu(k))$ and for every $i' \in [k_N, k_{N+1})$, $F_{i',n} \setminus F_n \subseteq \{\alpha_{i',j} : j < m\}$. It follows that $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n})$ extends $(\nu_n, F_{i',n})$ for every $i' \in [k_N, k_{N+1})$ and hence $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n}) \Vdash_{\mathbb{Q}} i \in \mathring{Y} \cap A$: Contradiction.

Next suppose t is p-like. Then, for every $N < \omega$, $i \in [k_N, k_{N+1})$ and $j < m$, $|\sigma_{i,j}| = l + 1 + i$, $\langle \sigma_{i,j}(l) : i \in [k_N, k_{N+1})\rangle$ are pairwise distinct and $(\forall k \in [l+1, l+1+i))(\sigma_{i,j}(k) = i)$. Let $X = \{i < \omega : (\exists n < \omega)(\exists j < m)(i \in [k_n, k_{n+1}) \wedge \nu(l) = \sigma_{i,j}(l))\}$. Then for every $n < \omega$, $|X \cap [k_n, k_{n+1})| \leq m$ hence $q \Vdash_{\mathbb{P}'_{\lambda+\xi_n}} \mathring{m}(X) = 0$. Let H be $\mathbb{P}_{\lambda+\xi_n}$-generic over V with $q_1 \in H$. Work in $V[H]$. Since $\mathring{m}(A \setminus X) > 0$, $A \setminus X$ is infinite. Choose $N < \omega$ and $i \in [k_N, k_{N+1}) \cap (A \setminus X)$ such that $k_N > |\nu|$, $(\forall k \in \mathsf{dom}(\nu))(k_N > \nu(k))$ and for every $i \in [k_N, k_{N+1})$, $F_{i,n} \setminus F_n \subseteq \{\alpha_{i,j} : j < m\}$. It follows the set of $i' \in [k_N, k_{N+1})$ for which $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n})$ does not extend $(\nu_n, F_{i',n})$ has size at most m and hence $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n}) \Vdash_{\mathbb{Q}} i \in \mathring{Y} \cap A$: Contradiction. □

Claim 7.13. *The following holds in $V^{\mathbb{P}_{\lambda+\xi_n}}$: Let $\mathbb{B} = \mathbb{Q}^1_{\lambda+\xi_n}$. There exist $s \in \mathbb{B}$ and a $\mathbb{B}$-name $\mathring{\mathfrak{m}}_3$ such that $s \geq [\rho_n]$, $\Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_3 : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure extending $\mathring{\mathfrak{m}}$ and $s \Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_3(\{i < \omega : p_i(\lambda + \xi_n) \in G_{\mathbb{B}}\}) \geq 1 - \varepsilon_n$.*

Proof of Claim **??**: Put $V_n = V^{\mathbb{P}'_{\lambda, A^\lambda_{\lambda+\xi_n}}}$ so that $\mathbb{B} = (\mathsf{Random})^{V_n}$. Working in V_n, apply Lemma **??** to $\mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V_n)$, with $r = [\rho_n]$ to obtain the extension $\mathring{\mathfrak{m}}_r \in (V_n)^{\mathbb{B}}$ as defined there. By Lemma **??**(2), we can choose $s \in \mathbb{B}$, $s \geq [\rho_n]$ such that $s \Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_r(\{i < \omega : p_i(\lambda+\xi_n) \in G_{\mathbb{B}}\}) \geq 1 - \varepsilon_n$.

Since $\mathbb{P}'_{\lambda, A^\lambda_{\lambda+\xi_n}} \lessdot \mathbb{P}'_{\lambda+\xi_n}$, we can write $V^{\mathbb{P}'_{\lambda+\xi_n}} = (V_n)^{\mathbb{R}}$ for some $\mathbb{R} \in V_n$. By Lemma **??**, it follows that $\mathring{m}_r \in (V_n)^{\mathbb{B}}$ and $\mathring{m} \in (V_n)^{\mathbb{Q}}$ have a common extension $\mathring{m}_3 \in (V_n)^{\mathbb{Q} \times \mathbb{B}} = V^{\mathbb{P}_{\lambda+\xi_n} \star \mathbb{Q}^1_{\lambda+\xi_n}}$. So s and $\mathring{m}_3$ are as required. □

Since $\mathbb{P}_{\lambda+\xi_n+1} = \mathbb{P}_{\lambda+\xi_n} \star (\mathbb{Q}^1_{\lambda+\xi_n} \times \mathbb{Q}^2_{\lambda+\xi_n})$, using Lemma **??** again, we can find a common extension $\mathring{\mathfrak{m}}_1 \in V^{\mathbb{P}_{\lambda+\xi_n+1}}$ of $\mathring{\mathfrak{m}}_2$ and $\mathring{\mathfrak{m}}_3$.

Let us check that $\mathring{\mathfrak{m}}_1$ satisfies $(t, \bar{k})$. So fix $\bar{p} = \langle p_j : j < \omega\rangle$ of type t and construct $p_{\bar{p}}$ as follows. Put $\bar{q} = \langle p_j \restriction (\lambda + \xi_n) : j < \omega\rangle$. Since $\mathring{\mathfrak{m}}$ satisfies $t \restriction \xi_{n+1}$, we can find $p_{\bar{q}} \in \mathbb{P}'_{\lambda+\xi_n}$ satisfying clauses (3)(a)-(f) in Defintion **??** for $\bar{q}$.

Define $p_{\bar{p}}$ by $p_{\bar{p}} \restriction (\lambda + \xi_n) = p_{\bar{q}}$, $p_{\bar{p}}(\lambda + \xi_n)(1) = s$ and $p_{\bar{p}}(\lambda + \xi_n)(2) = (\nu_n, F_n)$.

For $j \leq n$, put $\mathring{X}_{\bar{p},j} = \{i < \omega : p_i \restriction [\lambda, \lambda + \xi_j + 1) \in G_{\mathbb{P}}\}$. Clause (3)(f) in Defintion **??** follows from Claim **??**. For clause (3)(g), we need to check that $p_{\bar{p}} \Vdash \mathring{m}_1(\mathring{X}_{\bar{p},n}) \geq 1 - 2\varepsilon_n$. Since $p_{\bar{q}} \Vdash \mathring{m}(\mathring{X}_{\bar{p},n-1}) \geq 1 - 2\varepsilon_{n-1}$, $\varepsilon_n \geq 2\varepsilon_{n-1}$ and $p_{\bar{p}} \Vdash \mathring{m}_1(\{i < \omega : p_i \restriction \{\lambda + \xi_j\} \in G_{\mathbb{P}}\} \geq 1 - \varepsilon_n$ (using Claims **??** and **??**), it follows that $p_{\bar{p}} \Vdash \mathring{m}_1(\mathring{X}_{\bar{p},n}) \geq 1 - 2\varepsilon_{n-1} - \varepsilon_n \geq 1 - 2\varepsilon_n$. Hence $\mathring{m}_1$ satisfies $(t, \bar{k})$. This completes the proof of Lemma **??** and therefore of Theorem **??**. □

References

[1] T. Bartozynski, Measure and category, Handbook of set theory (M. Foreman, A. Kanamori, ed.) Vol. 2 Springer (2010), 885-1147

[2] P. Komjáth, On second-category sets, Proc. Amer. Math. Soc., 107 (1989), 653-654

[3] S. Shelah, The null ideal restricted to some non-null set may be $\aleph_1$-saturated, Fund. Math. 179 (2003), 97-129

[4] S. Shelah, Covering of the null ideal may have countable cofinality, Fund. Math. 166 (2000), 109-136