# ON INVARIANTS FOR $\omega_1$-SEPARABLE GROUPS

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ABSTRACT. We study the classification of $\omega_1$-separable groups using Ehrenfeucht-Fraïssé games and prove a strong classification result assuming PFA, and a strong non-structure theorem assuming $\diamondsuit$.

## INTRODUCTION

An $\omega_1$-separable (or $\aleph_1$-separable) group is an abelian group such that every countable subset is contained in a free direct summand of the group. In particular, therefore, an $\omega_1$-separable group is $\aleph_1$-free, i.e., every countable subgroup is free. The structure of $\omega_1$-separable groups of cardinality $\aleph_1$ was investigated in [1] and [8]; most of the results proved there required set-theoretic assumptions beyond ZFC. (See also [2, Chap. VIII] for an exposition of these results.) Specifically, assuming Martin's Axiom (MA) plus $\neg$CH or the stronger Proper Forcing Axiom (PFA), one can prove nice structure and classification results; these results are not theorems of ZFC since counterexamples exist assuming CH or "prediction principles" like $\diamondsuit$. In [1, Remark 3.3] it is asserted that a construction given there under the assumption of CH (or even $2^{\aleph_0} < 2^{\aleph_1}$) of two non-isomorphic $\omega_1$-separable groups

> "is strong evidence for the claim that in a model of CH there is no possible meaningful classification of all $\omega_1$-separable groups. It is difficult to see what conceivable scheme of classification could distinguish between [the groups constructed here]."

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But, in fact, the Helsinki school of model theory provides a scheme for distinguishing between such groups. It is our aim here to use the methodology of the Helsinki school — which involves Ehrenfeucht-Fraïssé games (cf. [9], [11] or [12]) — to strengthen the dichotomy referred to above: that is, to obtain strong classification results assuming PFA, and a strong "non-structure theorem" assuming $\diamondsuit$.

We begin by describing the Ehrenfeucht-Fraïssé (or EF) games, after which we can state our results more precisely. If $\alpha$ is an ordinal and $A$ and $B$ are any structures, the game $EF_\alpha(A, B)$ is played between two players $\forall$ and $\exists$ who take turns choosing elements of $A \cup B$ through $\alpha$ rounds. Specifically, in each round $\forall$ picks first an element of either $A$ or $B$; and then $\exists$ picks an element of the other structure. The result is, at the end, two sequences $(a_\nu)_{\nu<\alpha}$ and $(b_\nu)_{\nu<\alpha}$ of elements of, respectively, $A$ and $B$. Player $\exists$ wins if and only if the function $f$ which takes $a_\nu$ to $b_\nu$ is a partial isomorphism; otherwise $\forall$ wins. If $A$ and $B$ have cardinality $\kappa$, $\exists$ has a winning strategy for $EF_\kappa(A, B)$ if and only if $A$ and $B$ are isomorphic. (Let $\forall$ list all the elements of $A \cup B$ during his moves.)

We consider variations of these games defined using trees. Given any tree $T$, we define the game $EF(A, B; T)$: the game is played as before except that player $\forall$ must also, whenever it is his turn, pick a node of the tree strictly above his previous choices (thus his successive choices will form a branch — a linearly ordered subset — of the tree). The game ends when $\forall$ can no longer pick a node above his previous choices; the criterion for winning is as before, that is, $\exists$ wins if and only if the function $f$ defined by the play is a partial isomorphism. We write $A \equiv^T B$ if $\exists$ has a winning strategy in the game $EF(A, B; T)$. For the purposes of motivation consider first the case $\alpha = \omega$. (Our interest is in the case $\alpha = \omega_1$.) In this case, we consider only well-founded trees, i.e., trees without infinite branches; then for every such $T$, each play of the game $EF(A, B; T)$ is finite. (So $EF(A, B; T)$ may be regarded as an approximation to the game $EF_\omega(A, B)$.) Scott's Theorem implies that for each countable $A$ there is a countable ordinal $\beta$ such that if $T_\beta$ is any tree of rank $\beta$, then for any countable $B$, $B$ is isomorphic to $A$ if and only if $A \equiv^{T_\beta} B$ . In terms of infinitary languages, $A$ is determined up to isomorphism (among countable structures) by a sentence of $L_{\infty\omega}$ of rank $\beta$.

For structures of cardinality $\aleph_1$, it is natural to look at approximations to the EF game of length $\omega_1$ and use trees which may have countably infinite branches, but do not have branches of cardinality $\aleph_1$; we call these *bounded trees.* For such $T$, each play of the game

$EF(A,B;T)$ will end after countably many moves. We will say $A$ is $T$-*equivalent* to $B$ if $A \equiv^T B$. This relation provides a possible way of distinguishing between the $\omega_1$-separable groups constructed in [1] under the assumption of CH (cf. the remark after the quotation above).

By a theorem of Hyttinen [3], the entire class of bounded trees determines $A$ up to isomorphism; that is, if $A$ and $B$ are of cardinality $\aleph_1$ and $A \equiv^T B$ for all bounded trees, then $A$ is isomorphic to $B$. The structure of the class of bounded trees is much more complicated than that of the class of well-founded trees (cf. [12]). However, in contrast to the situation for countable structures, there is not always a single tree which suffices to describe $A$ up to isomorphism. Specifically, Hyttinen and Tuuri [4] proved (assuming CH) that there is a linear order $A$ of cardinality $\aleph_1$ such that for every bounded tree $T$ there is a linear order $B_T$ of cardinality $\aleph_1$ such that $A \equiv^T B_T$ but $A$ is not isomorphic to $B_T$. They call this result a *non-structure theorem* for $A$. It can be translated in terms of infinitary languages and says that there is no complete description of $A$ in a certain strong language $M_{\omega_2\omega_1}$ (which we shall not define here).

A similar non-structure theorem for $p$-groups was proved by Mekler and Oikkonen [10]; their theorem is proved by carrying over to $p$-groups, by means of a Hahn power construction, the result of Hyttinen and Tuuri. Whether the analogous result for $\aleph_1$-free groups is a theorem of ZFC + CH remains open, but when we consider the question for $\aleph_1$-separable groups, we obtain an independence result, which is the subject of this paper. In the first section we prove (with the help of the structural results referred to above) that assuming PFA

> *if $A$ and $B$ are $\omega_1$-separable groups of cardinality $\aleph_1$ such that $A \equiv^{\omega^2+\omega} B$, then they are isomorphic (where $\omega^2+\omega$ is the countable ordinal regarded as a — linearly ordered — tree).*

See Theorem 6. Thus a single, simple, tree contains enough information to classify any $\omega_1$-separable group — in the precise sense that a single sentence of $M_{\omega_2\omega_1}$ of "tree rank" $\omega^2+\omega$ completely describes $A$.

In section 2 we show, assuming $\diamondsuit$, that not only does $\omega^2+\omega$ not have the property above, but for *any* bounded tree $T$, there are non-isomorphic $\omega_1$-separable groups $A^T$ and $B^T$ of cardinality $\aleph_1$ which cannot be separated by $T$, in the sense that $A^T \equiv^T B^T$. (See Theorem 7.) The construction in section 2 is strengthened in section 3 to obtain a non-structure theorem (Theorem 8.):

> *there is an $\omega_1$-separable group $A$ of cardinality $\aleph_1$ such that for every bounded tree $T$ there is an $\omega_1$-separable*

*group $B^T$ of cardinality $\aleph_1$ which is not isomorphic to $A$ but is $T$-equivalent* to $A$.

(Note that $A$ does not depend on $T$.)

We shall make use, at times, of the following simple lemma, where $A^*$ denotes the dual of $A$, i.e. $\mathrm{Hom}(A, \mathbb{Z})$.

**Lemma 1.** *Suppose $A \subseteq B$ and $A' \subseteq B' \subseteq C'$ where $C'/B'$ is $\aleph_1$-free, $B/A$ is countable and $(B/A)^* = 0$. If $\theta : B \to C'$ such that $\theta[A] \subseteq A'$, then $\theta[B] \subseteq B'$.*

*Proof.* $\theta$ induces a homomorphism: $B/A \to C'/A'$. By the hypotheses, the composition of this map with the canonical surjection: $C'/A' \to C'/B'$ must be zero; that is, $\theta[B] \subseteq B'$. □

## 1. A structure theorem

An $\aleph_1$-separable group $A$ of cardinality $\aleph_1$ is characterized by the property that it has a *filtration*, that is, a continuous chain $\{A_\nu : \nu < \omega_1\}$ of countable free subgroups whose union is $A$ and is such that $A_0 = 0$ and for all $\nu$, $A_{\nu+1}$ is a direct summand of $A$. We say that two $\aleph_1$-separable groups $A$ and $B$ are *quotient-equivalent* if and only if they have filtrations, $\{A_\nu : \nu < \omega_1\}$ and $\{B_\nu : \nu < \omega_1\}$, respectively, such that for every $\alpha < \omega_1$, $A_{\alpha+1}/A_\alpha$ is isomorphic to $A'_{\alpha+1}/A'_\alpha$. We say that $A$ and $B$ are *filtration-equivalent* if and only if they satisfy the stronger condition that for every $\alpha < \omega_1$ there is a *level-preserving isomorphism* $\theta_\alpha : A_{\alpha+1} \to B_{\alpha+1}$, i.e., an isomorphism such that for every $\nu \leq \alpha$, $\theta[A_\nu] = B_\nu$. Under the assumption of MA + ¬CH, filtration-equivalence implies isomorphism.

In [8] (see also [2, Chap. VIII]) it is proved under the hypothesis of the Proper Forcing Axiom, PFA, that $\aleph_1$-separable groups of cardinality $\aleph_1$ have a nice structure theory. More precisely, it is shown that, under PFA, every $\aleph_1$-separable group of cardinality $\aleph_1$ is *in standard form*. (Roughly, this means that they have a "classical" construction. We will give a definition below.) Our goal in this section is to use that theory to prove the following:

**Theorem 2.** *(PFA) $\omega^2+\omega$ is a universal equivalence tree for the class of $\aleph_1$-separable abelian groups of cardinality $\aleph_1$. That is, any two $\aleph_1$-separable abelian groups of cardinality $\aleph_1$ which are $\omega^2+\omega$-equivalent are isomorphic.*

We shall see in the next section that this is not a theorem of ZFC. We begin with a weaker result.

**Proposition 3.** *If $A$ and $A'$ are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are $\omega 2$-equivalent, then they are quotient equivalent.*

*Proof.* Suppose that $\tau$ is a w.s. for $\exists$. Let $C$ be a cub such that if $\alpha \in C$ then for any $n \in \omega$, as long as the first $n$ moves of $\forall$ are in $A_\alpha \cup A'_\alpha$, the replying moves of $\exists$ given by $\tau$ are also in $A_\alpha \cup A'_\alpha$. If $A$ and $A'$ are not quotient-equivalent, there exists $\alpha \in C$ such that $A_{\alpha+1}/A_\alpha \oplus \mathbb{Z}^{(\omega)}$ is not isomorphic to $A'_{\alpha+1}/A'_\alpha \oplus \mathbb{Z}^{(\omega)}$. Now let $\forall$ play the game so that during the first $\omega$ moves he makes sure that all elements of $A_\alpha \cup A'_\alpha$ are played; the result, since $\tau$ is a w.s., is that an isomorphism $f : A_\alpha \to A'_\alpha$ is obtained.

Then in the next $\omega$ moves, $\forall$ plays so that all, and only, the elements of $A_\beta \cup A'_\beta$ are played for some $\beta \geq \alpha + 1$. This is possible by using a bijection of $\omega$ with $\omega \times \omega$. The result is an extension of $f$ to an isomorphism $f' : A_\beta \to A'_\beta$. Then, since $A_\beta/A_{\alpha+1}$ and $\mathrm{A}'_\beta/A'_{\alpha+1}$ are free, we have $A_\beta/A_\alpha \cong A_{\alpha+1}/A_\alpha \oplus A_\beta/A_{\alpha+1}$ and similarly on the other side. Since $f'$ induces an isomorphism of $A_\beta/A_\alpha$ with $A'_\beta/A'_\alpha$, we obtain a contradiction of the choice of $\alpha$. $\square$

Suppose $A$ is an $\aleph_1$-separable group of cardinality $\aleph_1$ with a filtration $\{A_\nu \colon \nu \in \omega_1\}$, and let $E = \{\delta : A_\delta$ is not a direct summand of $A\}$; $A$ is said to be *in standard form* if:

(1) it has a coherent system of projections $\{\pi_\nu \colon \nu \notin E\}$, i.e., projections $\pi_\nu \colon A \to A_\nu$ with the property that for all $\nu < \tau$ in $\omega_1 \setminus E$, $\pi_\nu \circ \pi_\tau = \pi_\nu$; and

(2) for every $\delta \in E$ there is a ladder $\eta_\delta$ on $\delta$ and a subset $Y_\delta$ of $A_{\delta+1}$ such that $A_{\delta+1} = A_\delta + \langle Y_\delta \rangle$ and

> (†) for all $y \in \langle Y_\delta \rangle$ and all $\nu < \delta$ with $\nu \notin E$, $\pi_\nu(y) = \sum_{\alpha \in S}(\pi_{\alpha+1}(y) - \pi_\alpha(y))$ where $S = \{\alpha \in \mathrm{rge}(\eta_\delta) \colon \alpha < \nu\}$.

(Here a ladder on $\delta$ means a strictly increasing function $\eta_\delta : \omega \to \delta$ with $\mathrm{rge}(\eta_\delta) \subseteq \omega_1 \setminus E$ and $\sup \mathrm{rge}(\eta_\delta) = \delta$.) This property is actually stronger than the usual definition of standard form (because of the assertion about the ladder); it can be shown that the Proper Forcing Axiom (PFA) implies that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ has this property (by essentially the same proof as in [2, Thm. VIII.3.3]).

Let $K_\alpha = \ker(\pi_\alpha)$ and let $K_{\alpha,\alpha+1} = K_\alpha \cap A_{\alpha+1}$. Notice that we can replace any $y$ in $Y_\delta$ by $y + u$ where $u \in K_{\alpha,\alpha+1}$ for some $\alpha \in \delta \setminus E$, and we will still have a generating set of $A_{\delta+1}$ over $A_\delta$ which satisfies (†).

Also we can, and will, assume that $A_{\nu+1}/A_\nu$ has infinite rank for every $\nu \notin E$.

**Lemma 4.** *Suppose $A$ is in standard form. Then there is a filtration $\{A_\nu : \nu \in \omega_1\}$ of $A$ and for each $\delta \in E = \{\delta : A_\delta$ is not a direct summand of $A\}$, there are: a ladder $\eta_\delta$ on $\delta$; and a subset $\bar{y}_\delta = \{y_{\delta,i} : i \in I\}$ of $A_{\delta+1}$ which is linearly independent mod $A_\delta$ such that if $\beta_n = \eta_\delta(n)$:*

1. *for all $n \in \omega$, $\beta_n \notin E$; and*
2. *$A_{\delta+1}$ is generated mod $A_\delta$ by a set of elements of the form*

$$\frac{t(\bar{y}_\delta) - a}{d} \tag{1}$$

*where $t(\bar{y}_\delta)$ is a linear combination of the elements of $\bar{y}_\delta$, $d \in \mathbb{Z}$, and $a \in \bigoplus_{n\in\omega} K_{\beta_n,\beta_n+1}$.*

*Moreover, given $\mu < \delta$, we can choose $\eta_\delta$ such that $\eta_\delta(0) > \mu$.*

*Proof.* Let $Y_\delta$ and $\eta_\delta$ be as in the definition of standard form above. Let $\bar{y}_\delta = \{y_{\delta,i} : i \in I\}$ be a maximal linearly independent subset of $Y_\delta$. By the remark preceding the lemma we can (by replacing $y_{\delta,i}$ by $y_{\delta,i}+u$ for some $u$) assume that $\eta_\delta(0) > \mu$.

If $d$ divides $t(\bar{y}_\delta)$ mod $A_{\nu+1}$ for some integer $d$ and linear combination $t(\bar{y}_\delta)$, then $d$ divides $t(\bar{y}_\delta) - a$ where $a = \pi_{\nu+1}(\ t(\bar{y}_\delta)) = \sum_{\beta\in S} \pi_{\beta,\beta+1}(\ t(\bar{y}_\delta))$ for some finite subset $S \subseteq \mathrm{rge}(\eta_\delta)$. □

**Proposition 5.** *Let $G$ and $G'$ be $\aleph_1$-separable groups such that $G$ is in standard form. Suppose that they have filtrations $\{G_\nu : \nu \in \omega_1\}$ and $\{G'_\nu : \nu \in \omega_1\}$ respectively such that the filtration of $G$ attests that $G$ is in standard form and $E = \{\nu \in \omega_1 : G_\nu$ is not a summand of $G\} = \{\nu \in \omega_1 : G'_\nu$ is not a summand of $G'\}$. Suppose also that for all limit ordinals $\delta$, given a ladder $\eta_\delta$ on $\delta$, there is an isomorphism $\theta_\delta : G_{\delta+1} \to G'_{\delta+1}$ such that for all $n \in \omega$, $\theta_\delta[G_{\eta_\delta(n)}] = G'_{\eta_\delta(n)}$ and $\theta_\delta[G_{\eta_\delta(n)+1}] = G'_{\eta_\delta(n)+1}$. Then $G$ and $G'$ are filtration-equivalent.*

*Proof.* We can assume that the filtration of $G$ is as in Lemma 4. We prove by induction on $\nu$ the following:

> if $\mu < \nu$ and $\mu, \nu \in \omega_1 \setminus E$ and $f : G_\mu \to G'_\mu$ is a level-preserving isomorphism, then $f$ extends to a level-preserving isomorphism $g : G_\nu \to G'_\nu$.

If $\nu = \tau + 1$ where $\tau \notin E$, then the result follows easily by induction and the fact that $G_\nu/G_\tau$ and $G'_\nu/G'_\tau$ are free. If $\nu$ is a limit ordinal, choose a ladder $\zeta_\nu$ on $\nu$ such that $\zeta_\nu(0) > \mu$ and for all $n$, $\zeta_\nu(n) \notin E$, and extend $f$ successively, by induction, to $g_n : G_{\zeta_\nu(n)} \to G'_{\zeta_\nu(n)}$, and let $g = \cup_n g_n$.

The crucial case is when $\nu = \delta + 1$ where $\delta \in E$. Let $\eta_\delta$ be as in Lemma 4 with $\eta_\delta(0) > \mu$, and let $\theta_\delta$ be the corresponding isomorphism given by the hypothesis of this Proposition. Let $C_{\delta,n} = K_{\beta_n,\beta_n+1}$. By induction, extend $f$ to a level-preserving isomorphism $f_0 : G_{\eta_\delta(0)} \to G'_{\eta_\delta(0)}$ and then extend it to $g_0 : G_{\eta_\delta(0)+1} \to G'_{\eta_\delta(0)+1}$ by letting $g_0 \restriction C_{\delta,0} = \theta_\delta \restriction C_{\delta,0}$. Clearly $g_0$ is level-preserving. By induction extend $g_0$ to a level-preserving $f_1 : G_{\eta_\delta(1)} \to G'_{\eta_\delta(1)}$ and then to $g_1 : G_{\eta_\delta(1)+1} \to G'_{\eta_\delta(1)+1}$ by letting $g_1 \restriction C_{\delta,1} = \theta_\delta \restriction C_{\delta,1}$. Continuing in this way we obtain level-preserving isomorphisms $g_n : G_{\eta_\delta(n)+1} \to G'_{\eta_\delta(n)+1}$ for each $n$. Let $\tilde{g} = \cup_n g_n : G_\delta \to G'_\delta$.

By Lemma 4, $G_{\delta+1}$ is generated mod $G_\delta$ by a set of elements of the form

$$\frac{t(\bar{y}_\delta) - a}{d}$$

where $a \in \oplus_{n\in\omega} C_{\delta,n}$; hence $G'_{\delta+1}$ is generated mod $G'_\delta$ by elements

$$\frac{t(\theta_\delta(\bar{y}_\delta)) - \theta_\delta(a)}{d}.$$

But then since $\tilde{g}(a) = \theta_\delta(a)$ for each such $a$ by construction, we can extend $\tilde{g}$ to $g : G_{\delta+1} \to G'_{\delta+1}$ by sending each $y_{\delta,i}$ in $\bar{y}_\delta$ to $\theta_\delta(y_{\delta,i})$. Since $\bar{y}_\delta$ is linearly independent over $G_\delta$ this is a well-defined homomorphism. □

**Theorem 6.** *Suppose $A$ and $A'$ are $\aleph_1$-separable groups of cardinality $\aleph_1$ and at least one of them is in standard form. If $A$ and $A'$ are $\omega^2 + \omega$-equivalent, then they are filtration-equivalent.*

*Proof.* We can suppose that $A$ is in standard form, and that we have chosen a filtration, $\{A_\nu \colon \nu \in \omega_1\}$ which attests to that fact. Moreover, we can assume that if $\delta \in E = \{\delta : A_\delta \text{ is not a direct summand of } A\}$, then $(A_{\delta+1}/A_\delta)^* = 0$. (Use Stein's Lemma [2, Exer. 3, p. 112], and replace $A_{\delta+1}$ by a direct summand, if necessary.)

Since $A$ is quotient-equivalent to $A'$ by Proposition 3 , we can assume that there is a filtration $\{A'_\nu \colon \nu \in \omega_1\}$ of $A'$ such that $E = \{\delta : A'_\delta$ is not a direct summand of $A'\}$ and for $\delta \in E$, $A'_{\delta+1}/A'_\delta \cong A_{\delta+1}/A_\delta = 0$, so in particular $(A'_{\delta+1}/A'_\delta)^* = 0$.

Fix a bijection $\psi_{\alpha\beta} : \omega \to (A_\beta \setminus A_\alpha) \cup (A'_\beta \setminus A'_\alpha)$ for each $\alpha < \beta$. Let $\psi = \{\psi_{\alpha\beta} : \alpha < \beta < \omega_1\}$.

Whenever we talk about moves in a game, we refer to the game $\mathrm{EF}_{\omega^2+\omega}(A, A')$. Given a strictly increasing finite sequence of countable ordinals $\alpha_1 < \alpha_2 < ... < \alpha_n$, we will say that $\forall$ *plays according to* $\psi$ *and* $\langle \alpha_1, \alpha_2, ..., \alpha_n \rangle$ *for the first* $\omega n$ *moves* if the $\omega k + \ell$ move of player $\forall$ is $\psi_{\alpha_k \alpha_{k+1}}(\ell)$ for $k = 0, ..., n-1$ and $\ell \in \omega$ (where $\alpha_0 = 0$).

Suppose that $\tau$ is a w.s. for $\exists$ in the game $\mathrm{EF}_{\omega^2+\omega}(A, A')$. Let $C$ be the set of all $\delta < \omega_1$ such that for any integers $n > 0$ and $m \geq 0$ and any ordinals $\alpha_1 < \alpha_2 < ... < \alpha_n < \delta$, if $\forall$ plays according to $\psi$ and $\langle \alpha_1, \alpha_2, ..., \alpha_n \rangle$ for the first $\omega n$ moves and then plays any elements of $A_\delta$ for the next $m$ moves, then the responses of $\exists$ using $\tau$ are all in $A_\delta \cup A'_\delta$.

Then $C$ is a cub: for the proof of unboundedness, note that there are only countably many possibilities that one has to close under: choice of $n$ and $m$, choice of $\alpha_1 < \alpha_2 < ... < \alpha_n$, and choice of moves $\omega n, \omega n + 1, ...\omega n + m - 1$. (The earlier moves are determined by the $\psi_{\alpha_k \alpha_{k+1}}$ and by $\tau$.)

There is a continuous strictly increasing function $\tilde{h} : \omega_1 \to \omega_1$ whose range is $C$. Define $h : \omega_1 \to \omega_1$ by

$$h(\beta) = \begin{cases} \tilde{h}(\beta) + 1 & \text{if } \beta \text{ is a successor and } \tilde{h}(\beta) \in E \\ \tilde{h}(\beta) & \text{otherwise} \end{cases}$$

Let $G_\alpha = A_{h(\alpha)}$ and $G'_\alpha = A'_{h(\alpha)}$. Then for successor $\beta$, $G_\beta$ is a summand of $A$ and for limit $\delta$ $G_\delta = A_{\tilde{h}(\delta)} = \cup_{\beta<\delta} A_{h(\beta)} = \cup_{\beta<\delta} G_\beta$, so $\{G_\alpha : \alpha \in \omega_1\}$ (resp. $\{G'_\alpha : \alpha \in \omega_1\}$) is a filtration of $A$ (resp. $A'$). Given a limit ordinal $\delta$ and a ladder $\eta_\delta$ on $\delta$, it follows — from Lemma 1 and the definition of $C$ — that there is an isomorphism $\theta_\delta : G_{\delta+1} \to G'_{\delta+1}$ such that for all $n \in \omega$, $\theta_\delta[G_{\eta_\delta(n)}] = G'_{\eta_\delta(n)}$ and $\theta_\delta[G_{\eta_\delta(n)+1}] = G'_{\eta_\delta(n)+1}$. In fact, $\theta_\delta$ is the partial isomorphism which results because $\exists$ wins the game where the $\omega k + \ell$ move of $\forall$ is

$$\psi_{h(\eta_\delta(n)), h(\eta_\delta(n)+1)}(\ell)$$

when $k = 2n$, and is

$$\psi_{h(\eta_\delta(n)+1), h(\eta_\delta(n+1))}(\ell)$$

when $k = 2n + 1$, and the $\omega^2 + m$ move of $\forall$ is $\psi_{h(\delta), h(\delta+1)}(m)$.

Thus we have satisfied the hypotheses of Proposition 5 so we conclude that $A$ and $A'$ are filtration-equivalent. $\square$

Now we can prove Theorem 2. PFA implies that every strongly $\aleph_1$-free abelian groups of cardinality $\aleph_1$ is $\aleph_1$-separable and in standard form. Moreover, assuming PFA, filtration-equivalent $\aleph_1$-separable groups of cardinality $\aleph_1$ are isomorphic. Thus the result follows from Theorem 6.

## 2. A diamond construction: one tree

The result to be proved in this section is the following:

**Theorem 7.** *Assume $\diamondsuit$. For any bounded tree $T_1$ there exist non-isomorphic $\aleph_1$-separable groups $G^0$ and $G^1$ of cardinality $\aleph_1$ which are $T_1$-equivalent (and filtration-equivalent) and are both in standard form.*

*Proof.* We will present the proof in layers of increasing detail.

**(I)** Fix a stationary subset $E$ of $\omega_1$ consisting of limit ordinals and such that $E$ is the disjoint union of two uncountable subsets $E_0$ and $E_1$ such that $\diamondsuit(E_1)$ holds.

Given a bounded tree $T$ (which in practice will be determined by, but not equal to, $T_1$), we shall identify its nodes with countable ordinals in such a way that if $\nu <_T \mu$ (in the tree ordering), then $\nu < \mu$ (as ordinals).

By induction on $\alpha < \omega_1$ we will define the following data:

(1) continuous chains $\{G^\ell_\nu : \nu < \alpha\}$ of countable free groups (for $\ell = 0, 1$) such that for all $\nu < \mu < \alpha$, $G^\ell_\mu / G^\ell_\nu$ is free if $\nu \notin E_1$, and if $\nu \in E_1$, then $G^\ell_{\nu+1}/G^\ell_\nu$ has rank at most 1.

(2) homomorphisms $\pi^\ell_{\nu,\mu} : G^\ell_\mu \to G^\ell_\nu$ for $\nu \leq \mu < \alpha$ and $\nu \notin E_1$ such that: $\pi^\ell_{\nu,\mu}$ is the identity on $G^\ell_\nu$ ; for $\nu \leq \mu < \rho$, $\pi^\ell_{\nu,\mu} \subseteq \pi^\ell_{\nu,\rho}$; and for $\tau < \nu \leq \mu$, $\pi^\ell_{\tau,\nu} \circ \pi^\ell_{\nu,\mu} = \pi^\ell_{\tau,\mu}$
(i.e., $\pi^\ell_{\nu,\mu}$ is a projection and the system of projections is coherent);

(3) for each $\nu$ with $\nu + 1 < \alpha$ an isomorphism $f^0_\nu : G^0_{\nu+1} \to G^1_{\nu+1}$ satisfying:

$$\text{if } \nu_1 <_T \nu_2, \text{ then } f^0_{\nu_2} \restriction G^0_{\nu_1+1} = f^0_{\nu_1}.$$

(These partial isomorphisms will give $\exists$ her winning strategy.)

For convenience we will use $f^1_\nu$ to denote $(f^0_\nu)^{-1} : G^1_{\nu+1} \to G^0_{\nu+1}$.

Define $G^\ell = \cup_{\nu<\omega_1} G^\ell_\nu$. (It depends on $T$, but we suppress that in the notation.) Now we will indicate how we choose $T$ so that $G^0$ and $G^1$ are $T_1$-equivalent.

Let $T_2 = {}^{<\omega_1}\omega_1 \setminus \emptyset$, i.e., the tree of non-empty countable sequences of countable ordinals, partially ordered by inclusion (so it has $\aleph_1$ nodes of height 0). Let $T$ be the product $T_1 \otimes T_2$, i.e., the (bounded) tree whose nodes are elements $(s, \sigma) \in T_1 \times T_2$, where $s$ and $\sigma$ have the same height, and the partial ordering is defined coordinate-wise. (As above, we identify the nodes of $T$ with ordinals.)

Suppose we are able to carry out the construction outlined above for this $T$. Then since the $G^\ell_\nu$ are free, $G^\ell$ is $\aleph_1$-free. Moreover, for $\nu \notin E_1$ $\bigcup_{\mu<\omega_1} \pi^\ell_{\nu,\mu} : G^\ell \to G^\ell_\nu$ is a projection which shows that $G^\ell_\nu$ is a direct summand of $G^\ell$ ; so $G^\ell$ is $\aleph_1$-separable (and has a coherent system of projections; the fact that it is in standard form will follow from the details of the construction — see part (V)).

We claim that $G^0$ and $G^1$ are $T_1$-equivalent. In fact, here is $\exists$'s winning strategy in the $T_1$-game. If in his first move $\forall$ plays $s_0 \in T_1$ (which we may assume has height 0), and $y_0 \in G^{\ell_0}_{\gamma_0}$, $\exists$ chooses $\alpha_0$ such that $(s_0, \langle\alpha_0\rangle) \in T$ is the element $\nu_0$ in the enumeration of $T$, where $\nu_0 \geq \gamma_0$; and she plays $f^{\ell_0}_{\nu_0}(y_0) \in G^{1-\ell_0}_{\nu_0+1}$. (Note that the domain of $f^{\ell_0}_{\nu_0}$ is $G^{\ell_0}_{\nu_0+1} \supseteq G^{\ell_0}_{\gamma_0}$.) Suppose that after $\beta$ moves $\forall$ has chosen $s_0 <_{T_1} s_1 <_{T_1} ... <_{T_1} s_\iota <_{T_1} ...$ in the tree and $y_0, y_1, ..., y_\iota, ...$ in the groups where $y_\iota \in G^{\ell_\iota}$ $(\iota < \beta)$, and $\exists$ has responded to the $\iota$th move with $f^{\ell_\iota}_{\nu_\iota}(y_\iota)$ where $\nu_\iota = (s_\iota, \langle\alpha_0, ..., \alpha_\iota\rangle)$. Now if $\forall$ plays $s_\beta >_{T_1} s_\iota$ $(\iota < \beta)$ — which we can assume has height $\beta$ — and $y_\beta \in G^{\ell_\beta}_{\gamma_\beta}$, then $\exists$ chooses $\alpha_\beta$ such that $(s_\beta, \langle\alpha_0, ..., \alpha_\beta\rangle)$ is $\nu_\beta \geq \gamma_\beta$, and plays $f^\ell_{\nu_\beta}(y_\beta)$. Notice that $\nu_\beta >_T \nu_\iota$, so $f^\ell_{\nu_\beta}$ extends $f^\ell_{\nu_\iota}$ for $\ell = 0, 1$. Therefore the sequence of moves determines a partial isomorphism, so $\exists$ will win.

**(II)** Of course, we also want to do the construction so that $G^0$ and $G^1$ are not isomorphic. This will be achieved by our construction of $G^\ell_{\delta+1}$ for $\delta \in E_1$ (plus the requirement 4 below); when $\delta \in E_1$ we will make use of the "guess" provided by $\diamondsuit(E_1)$ of an isomorphism: $G^0_\delta \to G^1_\delta$.

Our construction will be such that when $\alpha = \mu + 1$ where $\mu \notin E$, then

$$G^\ell_\alpha = G^\ell_\mu \oplus \mathbb{Z}x^\ell_{\mu,0} \oplus \mathbb{Z}x^\ell_{\mu,1}$$

When $\alpha = \sigma + 1$ where $\sigma \in E_0$, then

$$G^\ell_\alpha = G^\ell_\sigma \oplus \bigoplus_{n\in\omega} \mathbb{Z}u^\ell_{\sigma,n} \oplus \mathbb{Z}v^\ell_{\sigma,n}.$$

We define

$$w_{\sigma,n} = 2u^0_{\sigma,n+1} - u^0_{\sigma,n}.$$

Notice that $\{w_{\sigma,n} : n \in \omega\}$ generates a pure subgroup of $\bigoplus_{n\in\omega} \mathbb{Z}u^0_{\sigma,n}$ which is not a direct summand. Hence there is no isomorphism of $\bigoplus_{n\in\omega} \mathbb{Z}u^0_{\sigma,n} \oplus \mathbb{Z}v^0_{\sigma,n}$ with $\bigoplus_{n\in\omega} \mathbb{Z}u^1_{\sigma,n} \oplus \mathbb{Z}v^1_{\sigma,n}$ which takes each $w_{\sigma,n}$ to $v^1_{\sigma,n}$. In order to carry out the inductive construction we will define in addition:

4. subsets $W_\alpha[\Theta]$ of $G^0_\alpha$ for every non-empty finite subset $\Theta$ of $\alpha$ which is an antichain in $T$, satisfying:
(a) for all $\alpha < \beta$, $W_\alpha[\Theta] \subseteq W_\beta[\Theta]$;
(b) every element of $W_\alpha[\Theta]$ is of the form $w_{\sigma,n}$ for some $\sigma \in E_0$, and $n \in \omega$.

The functions $f^0_\alpha$ will be required to satisfy:

(c) for all $\mu \leq \alpha$, $j \in \{0,1\}$ $f^0_\alpha(x^0_{\mu,j}) = x^1_{\mu,j}$; moreover, if $w_{\sigma,n} \in W_{\alpha+1}[\Theta]$ and $\Theta \cap \{\nu : \nu \leq_T \alpha\} \neq \emptyset$, then $f^0_\alpha(w_{\sigma,n}) = v^1_{\sigma,n}$.

For any finite antichain $\Theta$ in $T$, let $W[\Theta] = \bigcup_\alpha W_\alpha[\Theta]$.

Now we will outline how we do the construction so that $G^0$ and $G^1$ are not isomorphic. Before we start, we choose a function $\Upsilon$ with domain $E_0$ which maps onto the set of all $\omega$-sequences $\langle \Theta_n : n \in \omega \rangle$ of finite subsets of $T$ such that $\bigcup_{n\in\omega} \Theta_n$ is an antichain; we also require that if $\Upsilon(\sigma) = \langle \Theta^\sigma_n : n \in \omega \rangle$, then each $\Theta^\sigma_n \subseteq \sigma$.

Suppose now that we have defined $G^\ell_\nu$ for $\nu \leq \alpha$. If $\alpha = \sigma \in E_0$, then $G^\ell_{\sigma+1}$ will be defined as indicated above and is such that (as we will prove)

(II.1) for all $e \in \{1,-1\}$, there is no isomorphism of $G^0_{\sigma+1}$ with $\bigoplus_{n\in\omega} \mathbb{Z}v^1_{\sigma,n} \oplus C$ for any $C$, which for all $n \in \omega$ takes $w_{\sigma,n}$ to $ev^1_{\sigma,n}$.

Moreover $w_{\sigma,n}$ will be put into $W_{\sigma+1}[\Theta^\sigma_n]$. (This is the only way that an element becomes a member of a $W_\alpha[\Theta]$.)

If $\alpha = \delta \in E_1$ and $\beta < \delta$, we introduce the notation $A_{\beta,\delta} = \{t : t$ is $<_T$-minimal in $\delta \setminus \beta\}$ — so $A_{\beta,\delta}$ is an antichain. We fix finite subsets $\Theta^{\beta,\delta}_n$ of $A_{\beta,\delta}$ which form a chain such that $\cup_{n\in\omega}\Theta^{\beta,\delta}_n = A_{\beta,\delta}$. We consider the prediction given by $\diamondsuit(E_1)$ of an isomorphism $h : G^0_\delta \to G^1_\delta$ and we ask whether the following holds:

(II.2) $\exists\, \beta < \delta\ \forall\, e \in \{1,-1\}\ \forall\, n \in \omega\ \exists\, w_{\sigma,m} \in W_\delta[\Theta^{\beta,\delta}_n]$ such that $h(w_{\sigma,m}) \neq ev^1_{\sigma,m}$.

We will do the construction of $G^\ell_{\delta+1}$ so that:

(II.3) If (II.2) holds, then $G^\ell_{\delta+1}/G^\ell_\delta$ is non-free rank 1 and $h$ does not extend to a homomorphism: $G^0_{\delta+1} \to G^1_{\delta+1}$.

Assuming we can do all of this, let us see why $G^0$ is not isomorphic to $G^1$. Suppose, to the contrary, that there is an isomorphism $H : G^0 \to G^1$. Then there is a stationary set, $S$, of $\delta \in E_1$ where $\diamondsuit(E_1)$ guesses $h = H \restriction G^0_\delta$ and $H : G^0_\delta \to G^1_\delta$. Note that Lemma 1 implies that $H$ must map $G^0_{\delta+1}$ into $G^1_{\delta+1}$ because $G^0_{\delta+1}/G^0_\delta$ is non-free rank 1 but $G^1/G^1_{\delta+1}$ is $\aleph_1$-free by construction. If for any such $\delta$ (II.2) holds, then $H \restriction G^0_{\delta+1}$ would extend $h = H \restriction G^0_\delta$, contradicting (II.3).

Since (II.2) fails, for all $\delta \in S$ and all $\beta < \delta$ there exists $e \in \{1, -1\}$ and a finite subset $\Theta$ of $A_{\beta,\delta}$ such that $H(w_{\sigma,n}) = ev^1_{\sigma,n}$ for all $w_{\sigma,n} \in W_\delta[\Theta]$. Now there is a cub $C$ such that for all $\delta \in C$, all $e \in \{1, -1\}$, all $\beta < \delta$, and all finite subsets $\Theta$ of $A_{\beta,\delta}$, if $H(w_{\sigma,n}) \neq ev^1_{\sigma,n}$ for some $w_{\sigma,n} \in W[\Theta]$, then $H(w_{\sigma,n}) \neq ev^1_{\sigma,n}$ for some $w_{\sigma,n} \in W_\delta[\Theta]$. Thus for all $\delta \in C \cap S$ and all $\beta < \delta$, there exists $e \in \{1, -1\}$ and a finite subset $\Theta$ of $A_{\beta,\delta}$ such that $H(w_{\sigma,n}) = ev^1_{\sigma,n}$ for all $w_{\sigma,n} \in W[\Theta]$. Since $C \cap S$ is uncountable, it follows easily that there exists $e \in \{1, -1\}$, and an uncountable set $\{\Theta_\nu : \nu < \omega_1\}$ of pairwise disjoint finite antichains such that $H(w_{\sigma,n}) = ev^1_{\sigma,n}$ for all $w_{\sigma,n} \in W[\Theta_\nu]$ for all $\nu < \omega_1$. Since $T$ has no uncountable branches, by a standard argument (see, for example, [5, Lemma 24.2, p. 245]), there is a countably infinite subset $\{\nu_n : n \in \omega\}$ of $\omega_1$ such that $\bigcup\{\Theta_{\nu_n} : n \in \omega\}$ is an antichain. There exists $\sigma \in E_0$ such that $\Upsilon(\sigma) = \langle \Theta_{\nu_n} : n \in \omega \rangle$. Now $H \restriction G^0_{\sigma+1}$ is such that for all $n \in \omega$, $H(w_{\sigma,n}) = ev^1_{\sigma,n}$ since $w_{\sigma,n} \in W_{\sigma+1}[\Theta_{\nu_n}]$; this contradicts (II.1), since $\bigoplus_{n\in\omega} \mathbb{Z}v^1_{\sigma,n}$ is a direct summand of $G^1_{\sigma+1}$, and hence of $G^1$ (by 2).

**(III)** The next step is to describe in detail the recursive construction of the data satisfying the properties 1, 2, 3 and 4, as well as (II.1) and (II.3). So assume that we have defined $G^\ell_\nu$, and $W_\nu[\Theta]$ for $\nu < \alpha$ and $f^\ell_\nu$ for $\nu + 1 < \alpha$.

There are several cases to consider.

*Case 1: $\alpha$ is a limit ordinal.* We let $G^\ell_\alpha = \cup_{\nu<\alpha} G^\ell_\nu$, $W_a[\Theta] = \bigcup_{\nu<\alpha} W_\nu[\Theta]$. Clearly the desired properties are satisfied.

If $\alpha$ is a successor, $\alpha = \mu + 1$, we will define $G^\ell_\alpha$ so that

> (III.1) if $B = \{t : t <_T \mu\}$ and we define $g_B = \cup\{f^0_t : t \in B\}$, then $g_B$ (which is a function by 3.) extends to an isomorphism, $f^0_\mu$, of $G^0_\alpha$ onto $G^1_\alpha$ which satisfies 4(c), i.e. for all $\nu \leq \mu$, $j \in \{0, 1\}$ $f^0_\mu(x^0_{\nu,j}) = x^1_{\nu,j}$ and if $w_{\sigma,n} \in W_\alpha[\Theta]$ and $\Theta \cap \{\nu : \nu \leq_T \mu\} \neq \emptyset$, then $f^0_\mu(w_{\sigma,n}) = v^1_{\sigma,n}$.

Leaving the verification of (III.1) to the next part, we will show how to define the data at $\alpha$ (except for the definition of the $\pi^\ell_{\sigma,\alpha}$ which we defer to part (V)).

*Case 2: $\alpha = \mu + 1$ for some $\mu \notin E$.* As described above, define

$$G^\ell_\alpha = G^\ell_\mu \oplus \mathbb{Z}x^\ell_{\mu,0} \oplus \mathbb{Z}x^\ell_{\mu,1}.$$

Let $W_\alpha[\Theta] = W_\mu[\Theta]$ for every $\Theta \subseteq \mu$ ($= \emptyset$ if $\Theta$ is not a subset of $\mu$). Assuming (III.1), we have $f^0_\mu$ as desired.

*Case 3: $\alpha = \sigma + 1$, where $\sigma \in E_0$.* In this case, as stated before,

$$G^\ell_\alpha = G^\ell_\sigma \oplus \bigoplus_{n\in\omega} \mathbb{Z}u^\ell_{\sigma,n} \oplus \mathbb{Z}v^\ell_{\sigma,n}$$

and recall that $w_{\sigma,n}$ is defined to be $2u^0_{\sigma,n+1} - u^0_{\sigma,n}$. Say $\Upsilon(\sigma) = \langle \Theta^\sigma_n : n \in \omega\rangle$. Define

$$W_\alpha[\Theta] = \begin{cases} W_\sigma[\Theta] \cup \{w_{\sigma,n}\} & \text{if } \Theta = \Theta^\sigma_n \\ W_\sigma[\Theta] & \text{otherwise.} \end{cases}$$

Assuming (III.1) (with $\mu = \sigma$), we can define $f^0_\sigma$. Now let us see why (II.1) holds. Suppose to the contrary that there is an isomorphism $H : G^0 \to G^1$ contradicting (II.1). Now $\bigoplus_{n\in\omega} \mathbb{Z}v^1_{\sigma,n}$ is a direct summand of $G^1_\alpha$ and hence (by 2) a direct summand of $G^1$. Thus $H^{-1}[\bigoplus_{n\in\omega} \mathbb{Z}v^1_{\sigma,n}]$ is a direct summand of $G^0$. But by assumption on $H$, $H^{-1}[\bigoplus_{n\in\omega} \mathbb{Z}v^1_{\sigma,n}] = \bigoplus_{n\in\omega} \mathbb{Z}w_{\sigma,n}$ and the latter is *not* a direct summand of $G^0$ because the coset of $u^0_{\sigma,0}$ is a non-zero element of $G^0/\bigoplus_{n\in\omega} \mathbb{Z}w_{\sigma,n}$ which is divisible by all power of 2 by definition of the $w_{\sigma,n}$.

*Case 4: $\alpha = \delta + 1$, where $\delta \in E_1$.* If (II.2) fails, let $G^\ell_{\delta+1} = G^\ell_\delta$. Otherwise, let $\beta$ be as in (II.2). We introduce some *ad hoc* notation. For any finite subset $\Theta$ of $A_{\beta,\delta}$, let $f_\Theta$ be the function whose domain is the subgroup generated by $\{x^0_{\mu,j} : \mu \notin E,\ \mu < \delta,\ j \in \{0,1\}\} \cup W_\delta[\Theta]$ such that $f_\Theta(x^0_{\mu,j}) = x^1_{\mu,j}$ and $f_\Theta(w_{\sigma,n}) = v^1_{\sigma,n}$. Notice that for all $u \in \mathrm{dom}(f_\Theta)$ and all $\nu \in \Theta$, if $\nu \leq_T \rho$ and $u \in \mathrm{dom}(f^0_\rho)$, then $f_\Theta(u) = f^0_\rho(u)$ by 4(c). Let $\Theta^{\beta,\delta}_n$ be as before (finite subsets forming a chain whose union is $A_{\beta,\delta}$); for short, let $\Theta_n = \Theta^{\beta,\delta}_n$. We claim that:

> (III.2) given $m, m' \in \mathbb{Z} \setminus \{0\}$, $n \in \omega$, $y \in G^1_\delta$, for sufficiently large $\gamma < \delta$ there exists $k^0 \in \mathrm{dom}(f_{\Theta_n}) \cap G^0_{\gamma+2}$ such that $k^0$ is pure-independent mod $G^0_{\gamma+1}$ and is such that $mh(k^0) \neq m' f_{\Theta_n}(k^0) + y$. Moreover, $f_{\Theta_n}(k^0)$ is pure-independent mod $G^1_{\gamma+1}$.

Supposing this is true — we will prove it in part (IV) — let us define $G^\ell_{\delta+1}$. Fix a ladder $\eta_\delta$ on $\delta$. Also, enumerate in an $\omega$-sequence all triples $\langle r, d, v\rangle$ where $r \in \omega$, $d \in \mathbb{Z} \setminus \{0\}$, and $g \in G^1_\delta$ so that the $n$th triple $\langle r, d, g\rangle$ satisfies $n > r$. By (III.2) we can inductively define primes

$p_n$, ordinals $\gamma_n \geq \eta_\delta(n)$, and elements $k^0_{\delta,n} \in \mathrm{dom}(f_{\Theta_n}) \cap G^0_{\gamma_n+2}$ pure-independent over $G^0_{\gamma_n+1}$ such that (if the $n$th triple is $\langle r, d, g\rangle$), $p_n$ does not divide $mh(k^0_{\delta,n}) - m' f_{\Theta_n}(k^0_{\delta,n}) - y$ where

$$m = \textstyle\prod_{i=0}^{n-1} p_i$$
$$m' = d\textstyle\prod_{i=r}^{n-1} p_i$$
$$y = \textstyle\sum_{j=0}^{n}(\prod_{i=0}^{j-1} p_i)h(k^0_{\delta,j}) + g - d\sum_{j=r}^{n}(\prod_{i=r}^{j-1} p_i) f_{\Theta_j}(k^0_{\delta,j}).$$

(Note that since $G^1_\delta$ is free, every non-zero element is divisible by only finitely many primes, so we can take $p_n$ to be any sufficiently large prime.) Then we let $G^0_{\delta+1}$ be generated by $G^0_\delta \cup \{z^0_{\delta,n} : n \in \omega\}$ modulo the relations

$$p_n z^0_{\delta,n+1} = z^0_{\delta,n} + k^0_{\delta,n}$$

and $G^1_{\delta+1}$ is defined similarly, except that we impose the relations

$$p_n z^1_{\delta,n+1} = z^1_{\delta,n} + f_{\Theta_n}(k^0_{\delta,n}).$$

We need to show that $h$ does not extend to a homomorphism: $G^0_{\delta+1} \to G^1_{\delta+1}$. If it does, then $h(z^0_{\delta,0}) = dz^1_{\delta,r} + g$ for some $r \in \omega$, $d \in \mathbb{Z} \setminus \{0\}$, and $g \in G^1_\delta$. Let $n$ be such that $\langle r, d, g\rangle$ is the $n$th triple in the list. Now, in $G^0_{\delta+1}$ we have

$$(\prod_{i=0}^{n} p_i) z^0_{\delta,n+1} = z^0_{\delta,0} + \sum_{j=0}^{n}(\prod_{i=0}^{j-1} p_i) k^0_{\delta,j}$$

so, applying $h$, we conclude that $p_n$ divides

$$dz^1_{\delta,r} + g + \sum_{j=0}^{n}(\prod_{i=0}^{j-1} p_i) h(k^0_{\delta,j}).$$

On the other hand, in $G^1_{\delta+1}$ we have $p_n$ divides

$$dz^1_{\delta,r} + d\sum_{j=r}^{n}(\prod_{i=r}^{j-1} p_i) f_{\Theta_j}(k^0_{\delta,j})$$

so, subtracting, we obtain a contradiction since $p_n$ divides $mh(k^0_{\delta,n}) - m' f^0_{\Theta^\sigma_n}(k^0_{\delta,n}) - y$, where $m$, $m'$, and $y$ are as above.

We let $W_{\delta+1}[\Theta] = W_\delta[\Theta]$ for any subset $\Theta$ of $\delta$ (and $= \emptyset$ if $\Theta \not\subseteq \delta$). By (III.1) we can define $f^0_\delta$.

**(IV)** In this layer we will prove (III.1) and (III.2).

First let us prove (III.2) since for the purposes of proving (III.1) we will need more information about the nature of the elements $k^0_{\delta,n}$. Fix

$m, m', n, y, \gamma$ as in (III.2); there are several cases. In the first two cases we can use any $\gamma < \delta$.

*Case (i): $y \neq 0$.* If neither $x^0_{\gamma+1,0}$ nor $x^0_{\gamma+1,1}$ will serve for $k^0$, then $x^0_{\gamma+1,0} - x^0_{\gamma+1,1}$ will.

*Case (ii): $y = 0$, $m \neq \pm m'$.* Let $k^0 = x^0_{\gamma+1,0}$. Then by construction, $k^0$ generates a cyclic summand of $G^0_\delta$; hence $f_{\Theta_n}(k^0)$ and $h(k^0)$ both generate cyclic summands of $G^1_\delta$. Hence $mh(k^0) \neq m' f^0_{\Theta_n}(k^0)$.

*Case (iii): $y = 0$, $m = m'$.* Pick $\gamma$ sufficiently large so that there exists $w_{\sigma,j} \in G^0_{\gamma+1} \cap W_\delta[\Theta_n]$ such that $f_{\Theta_n}(w_{\sigma,j}) \neq h(w_{\sigma,j})$. If $x^0_{\gamma+1,0}$ will not serve for $k^0$ (i.e., $h(x^0_{\gamma+1,0}) = x^1_{\gamma+1,0}$), then we can take $k^0$ to be $x^0_{\gamma+1,0} + w_{\sigma,j}$.

*Case (iv): $y = 0$, $m = -m'$.* Similarly $k^0$ can be taken to be of the form $x^0_{\gamma+1,0}$ or $x^0_{\gamma+1,0} - w_{\sigma,j}$ where $f_{\Theta_n}(w_{\sigma,j}) \neq -h(w_{\sigma,j})$.

Now if we examine the construction in Case 4 of (III) and the proof above we see that

> (IV.1) each $k^0_{\delta,n}$ can be (and will be) taken to be of the form $x^0_{\mu_n,j_n} \pm \xi_{\delta,n}$ where $\xi_{\delta,n}$ is 0, $x^0_{\sigma,j}$ or $w_{\sigma,j}$ for some $\sigma, j$.

We will say that $w_{\sigma,j}$ *is a part of* $k^0_{\delta,n}$ in case $\xi_{\delta,n}$ is $w_{\sigma,j}$.

Before beginning the proof of (III.1), let us observe the following facts:

> (IV.2) Given $\sigma \in E_0$ and $N \in \omega$, there is an isomorphism $g' : \bigoplus_{n\in\omega} \mathbb{Z}u^0_{\sigma,n} \oplus \mathbb{Z}v^0_{\sigma,n} \to \bigoplus_{n\in\omega} \mathbb{Z}u^1_{\sigma,n} \oplus \mathbb{Z}v^1_{\sigma,n}$ such that $g'(w_{\sigma,n}) = v^1_{\sigma,n}$ for $n \leq N$ and $g'(u^0_{\sigma,n}) = u^1_{\sigma,n}$ for $n \geq N+1$.

Indeed, we can define $g'(u^0_{\sigma,n}) = 2g'(u^0_{\sigma,n+1}) - v^1_{\sigma,n}$ for $n \leq N$ (and the other values appropriately).

> (IV.3) Given an isomorphism $g : G^0_\delta \to G^1_\delta$ where $\delta \in E_1$, we can extend $g$ to an isomorphism $g' : G^0_{\delta+1} \to G^1_{\delta+1}$ provided that (using the notation of Case 4) $g(k^0_{\delta,n}) = f_{\Theta_n}(k^0_{\delta,n})$ for almost all $n \in \omega$.

Indeed, if $g(k^0_{\delta,n}) = f_{\Theta_n}(k^0_{\delta,n})$ for all $n \geq N$, we can define $g'(z^0_{\delta,n}) = z^1_{\delta,n}$ for $n \geq N$ and $g'(z^0_{\delta,n}) = p_n g'(z^0_{\delta,n+1}) - g(k^0_{\delta,n})$ for $n < N$ by "downward induction". We will apply (IV.3) to the situation of (III.1), with $g = g_B$, $\delta = \mu$, $\delta + 1 = \alpha$; if we are in Case 4, then the hypothesis on $g$ in

(IV.3) will hold if there exists $t \in B$ such that $t \geq \beta$ (where $\beta$ is as in Case 4).

We return to the notation of (III.1). Let $\tau = \sup\{t+1 : t \in B\}$; then $\mathrm{dom} g_B = G^0_\tau$. Assume first that $\tau = \mu$. In case $G^\ell_{\mu+1}/G^\ell_\mu$ is free there is no problem extending $g_B$; in the other case $\mu = \delta \in E_1$ and by the remarks above we can extend $g_B$ since there exists $t \in B$ such that $t \geq \beta$ (since $\sup B = \delta$).

We are left with the case when $\tau < \mu$. We will first define an extension of $g_B$ to a partial isomorphism $\tilde{g}_B$ whose domain is

$$\mathrm{dom}(g_B) + \left\langle \begin{array}{c} \{x^0_{\nu,j} : \nu \ \leq \mu,\ j = 0,1\} \cup \\ \{u^0_{\sigma,n} : \sigma \in E_0 \cap \mu+1,\ n \in \omega\} \cup \\ \{v^0_{\sigma,n} : \sigma \in E_0 \cap \mu+1,\ n \in \omega\} \end{array} \right\rangle$$

Notice that every $k^0_{\delta,n}$ for $\delta \leq \mu, n \in \omega$ belongs to the domain of $\tilde{g}_B$. We let $\tilde{g}_B(x^0_{\nu,j}) = x^1_{\nu,j}$ for all $\nu, j$. By enumerating in an $\omega$-sequence the set $(E_0 \cup E_1) \cap (\mu+1)$ we can define by recursion the values $\tilde{g}_B(u^0_{\sigma,n})$ and $\tilde{g}_B(v^0_{\sigma,n})$ so that:

- $\tilde{g}_B(w_{\sigma,n}) = v^1_{\sigma,n}$ whenever $w_{\sigma,n} \in W_{\mu+1}[\Theta]$ for some $\Theta$ with $B \cap \Theta \neq \emptyset$;
- for all $\sigma \in E_0$ with $\tau \leq \sigma \leq \mu$, for almost all $n \in \omega$, $\tilde{g}_B(u^0_{\sigma,n}) = u^1_{\sigma,n}$; and
- for all $\delta \in E_1$ with $\tau \leq \delta \leq \mu$, for almost all $n \in \omega$, if (for some $\sigma, m$) $w_{\sigma,m}$ is a part of $k^0_{\delta,n}$ , then $\tilde{g}_B(w_{\sigma,m}) = v^1_{\sigma,m}$.

The first condition is required by 4(c). In view of (IV.2), there is no conflict between the first two conditions because for any $\sigma \in E_0$, $\bigcup_{n\in\omega} \Theta^\sigma_n$ is an antichain, so there is at most one $n$ such that $\Theta^\sigma_n \cap B \neq \emptyset$.

To be sure that the third condition can indeed be satisfied, we need to consider the case that for some $\delta \in E_1$, there are infinitely many $n$ such that there exists $w_{\sigma_n,m_n}$ which is a part of $k^0_{\delta,n}$ and belongs to the domain of $g_B$. Say this is the case for $n$ belonging to the (infinite) set $Y \subseteq \omega$ (for a fixed $\delta$). Then for each $n \in Y$ $\exists t_n \in B$ such that $t_n \geq \sigma_n$. Suppose that the construction of $G^\ell_{\delta+1}$ uses $A_{\beta,\delta} = \cup_{n\in\omega}\Theta^{\beta,\delta}_n$. Selecting one $n_* \in Y$, we see that since $\Theta^{\beta,\delta}_{n_*} \subseteq \sigma_{n_*}$, $\sigma_{n_*} > \beta$ and hence $t_{n_*} \in A_{\beta,\delta}$. Therefore there exists $M$ such that for all $n \geq M$, $t_{n_*} \in \Theta^{\beta,\delta}_n$. But then, for $n \in Y$ with $n \geq M$, $t_n \geq \sigma_n \supseteq \Theta^{\beta,\delta}_n$, so $t_{n_*} \leq t_n$ and thus $t_{n_*} \leq_T t_n$. By the construction in Case 4 and by 4(c), $g_B(w_{\sigma_n,m_n}) = v^1_{\sigma_n,m_n}$ for $n \in Y$, $n \geq M$. Moreover, there is no conflict between the last two conditions because, by construction, if $\delta \in E_1$ and $\sigma \in E_0$, then $w_{\sigma,m} \in W_\delta[\Theta^{\beta,\delta}_n]$ if and only if $\Theta^{\beta,\delta}_n = \Theta^\sigma_m$, but the elements of $\{\Theta^\sigma_m : m \in \omega\}$ are disjoint and the $\Theta^{\beta,\delta}_n$ form a chain under $\subseteq$ .

It remains to extend $\tilde{g}_B$ to $f^0_\mu$ by defining $f^0_\mu(z^0_{\delta,n})$ for $\tau \le \delta \le \mu, n \in \omega$. This is possible by observation (IV.3) because of the construction of $\tilde{g}_B$.

**(V)** We will define the projections $\pi^\ell_{\nu,\mu}$ by induction on $\mu$ and then verify the conditions to be in standard form (see section 1 or [2, Def. 1.9(ii), p. 257]). We refer to the cases of the construction in part (III). In Case 1, we take unions. In Case 2, for $\nu < \mu+1$ we let $\pi^\ell_{\nu,\mu+1}$ be the extension of $\pi^\ell_{\nu,\mu}$ which sends each $x^\ell_{\mu,j}$ to 0 . (Here, $\pi^\ell_{\mu,\mu}$ is the identity.) In Case 3, for $\nu \le \sigma$ we let $\pi^\ell_{\nu,\sigma+1}$ be the extension of $\pi^\ell_{\nu,\sigma}$ which sends each $u^\ell_{\sigma,n}$ and each $v^\ell_{\sigma,n}$ to 0.

Finally, for Case 4, we use the notation of that case. We define $\pi^0_{\nu,\delta+1}(z^0_{\delta,n}) = -\sum_{j=n}^m d_{n,j} k^0_{\delta,j}$ where $m$ is maximal such that $\gamma_m + 2 \le \nu$ and $d_{n,j} = \prod_{i=n}^{j-1} p_i$ (and $d_{n,0} = 1$ ). (Compare [2, pp. 249f].) The definition of $\pi^1_{\nu,\delta+1}$ is similar, replacing $k^0_{\delta,j}$ by $f_{\Theta_j}(k^0_{\delta,j})$. Let $Y^\ell_\delta = \{z^\ell_{\delta,n} : n \in \omega\}$. Then we can easily verify the conditions of [2, Def. 1.9(ii), p. 257] using the information in the proof of (III.2) about the form of $k^0_{\delta,j}$.

This completes the proof of Theorem 7.

## 3. A non-structure theorem

Our goal is to generalize the construction in the previous section to prove:

**Theorem 8.** *Assume $\diamondsuit$. There exists an $\aleph_1$-separable group $G^0$ and for each bounded tree $T_1$ an $\aleph_1$-separable group $G^{T_1}$ which is $T_1$-equivalent to $G^0$ but not isomorphic to $G^0$. Moreover, all the groups are of cardinality $\aleph_1$ and in standard form.*

*Proof.* We assume familiarity with the previous proof and outline the modifications, in layers of increasing detail.

**(VI)** Fix a stationary subset $E$ of $\omega_1$ consisting of limit ordinals ($> 0$) and such that $E$ is the disjoint union of two subsets $E_0$ and $E_1$ such that cardinality $\diamondsuit(E_0)$ and $\diamondsuit(E_1)$ hold. ($\diamondsuit(E_0)$ is not essential, but convenient.)

We need only consider bounded trees $T$ on $\omega_1$ such that if $\nu <_T \mu$ (in the tree ordering), then $\nu < \mu$ (as ordinals). For each $\delta \in E_1$ (resp. $\sigma \in E_0$), diamond will give us a "prediction" $T_\delta = \langle \delta, <_\delta \rangle$ (resp. $T_\sigma$) of the restriction of a bounded tree to $\delta$ (resp. $\sigma$). If $\mu < \delta$ we write $T_\delta \restriction \mu$ for $\langle \mu, <_\delta \cap (\mu \times \mu) \rangle$.

By induction on $\delta \in \{0\} \cup E$ we will define the following data:

(1) continuous chains $\{G^\delta_\nu : \nu \leq \delta + 1\}$ of countable free groups such that for all $\nu < \mu \leq \delta + 1$, $G^\delta_\mu / G^\delta_\nu$ is free if $\nu \notin E_1$, and if $\nu \in E_1$, then $G^\delta_{\nu+1}/G^\delta_\nu$ has rank at most 1.

(2) projections $\pi^\delta_{\nu,\mu} : G^\delta_\mu \to G^\delta_\nu$ for $\nu \leq \mu \leq \delta + 1$ and $\nu \notin E_1$ such that: for $\nu \leq \mu < \rho$, $\pi^\delta_{\nu,\mu} \subseteq \pi^\delta_{\nu,\rho}$; and for $\tau < \nu \leq \mu$, $\pi^\delta_{\tau,\nu} \circ \pi^\delta_{\nu,\mu} = \pi^\delta_{\tau,\mu}$;

(3) for each $\delta \in E$ and each $\nu \leq \delta$ an isomorphism $f^\delta_\nu : G^0_{\nu+1} \to G^\delta_{\nu+1}$ satisfying:

if $\nu_1 <_\delta \nu_2$, then $f^\delta_{\nu_2} \restriction G^0_{\nu_1+1} = f^\delta_{\nu_1}$.

Moreover, we require that if $\delta < \delta'$ are elements of $E$ such that $T_\delta = T_{\delta'} \restriction \delta$, then $G^\delta_\nu = G^{\delta'}_\nu$ for $\nu \leq \delta+1$; $\pi^{\delta'}_{\nu,\mu} = \pi^\delta_{\nu,\mu}$ for $\nu \leq \mu \leq \delta+1$; and $f^{\delta'}_\nu = f^\delta_\nu$ for $\nu \leq \delta$.

Define $G^0 = \cup_{\nu<\omega_1} G^0_\nu$ and for each bounded tree $T$ on $\omega_1$ let $G^T = \bigcup\{G^\delta_\nu : T_\delta = T \restriction \delta,\ \nu \leq \delta + 1\}$. As before, given $T_1$ we can choose $T$ so that $G^0$ and $G^T$ are $T_1$-equivalent.

We indicate how to modify the previous construction so that $G^0$ and $G^T$ are not isomorphic. Our construction will be such that when $\alpha = \mu + 1$ where $\mu \notin E$, then

$$(*)\ G^0_\alpha = G^0_\mu \oplus \mathbb{Z}x^0_{\mu,0} \oplus \mathbb{Z}x^0_{\mu,1}$$

and

$$(**)\ G^\delta_\alpha = G^\delta_\mu \oplus \mathbb{Z}x^1_{\mu,0} \oplus \mathbb{Z}x^1_{\mu,1}$$

for $\delta \in E$, $\alpha < \delta$.
When $\alpha = \sigma + 1$ where $\sigma \in E_0$, then

$$(***)\ G^0_\alpha = G^0_\sigma \oplus \bigoplus_{n\in\omega} \mathbb{Z}u^0_{\sigma,n} \oplus \mathbb{Z}v^0_{\sigma,n}$$

and

$$(****)\ G^\delta_\alpha = G^\delta_\sigma \oplus \bigoplus_{n\in\omega} \mathbb{Z}u^1_{\sigma,n} \oplus \mathbb{Z}v^1_{\sigma,n}$$

for $\delta \in E$, $\sigma < \delta$.
We define

$$w_{\sigma,n} = 2u^0_{\sigma,n+1} - u^0_{\sigma,n}.$$

In order to carry out the inductive construction we will define in addition:

4. for $\delta \in E$ and $\alpha \leq \delta+1$, subsets $W^\delta_\alpha[\Theta]$ of $G^0_\alpha$ for every non-empty finite subset $\Theta$ of $\alpha$ which is an antichain in $T_\delta$, satisfying:

(a) for all $\alpha < \beta$, $W^\delta_\alpha[\Theta] \subseteq W^\delta_\beta[\Theta]$;

(b) every element of $W^\delta_\alpha[\Theta]$ is of the form $w_{\sigma,n}$ for some $n \in \omega$ and some $\sigma \in E_0$ such that $T_\delta \restriction \sigma = T_\sigma$.

The functions $f^\delta_\alpha$ will be required to satisfy (as before):

(c) for all $\mu \leq \alpha$, $j \in \{0,1\}$ $f^\delta_\alpha(x^0_{\mu,j}) = x^1_{\mu,j}$; moreover, if $w_{\sigma,n} \in W^\delta_{\alpha+1}[\Theta]$ and $\Theta \cap \{\nu : \nu \leq_\delta \alpha\} \neq \emptyset$, then $f^\delta_\alpha(w_{\sigma,n}) = v^1_{\sigma,n}$.

Moreover, in order to carry out the inductive construction we will also require the following for all $\delta \in E$, $\alpha \leq \delta$ :

(d) if $\sigma \in E_0$ with $\sigma \leq \alpha + 1$ and $T_\delta \restriction \sigma \neq T_\sigma$, then $f^\delta_\alpha(u^0_{\sigma,n}) = u^1_{\sigma,n}$ for all $n \in \omega$;

(e) for all pairs $\beta_1$, $\beta_2$ with $\sup\{t : t <_\delta \alpha\} \leq \beta_1 < \beta_2 \leq \alpha$, it is the case for almost all $n \in \omega$ that for all $w_{\sigma,m} \in W^\delta_{\alpha+1}[\Theta^{\beta_1,\beta_2}_n]$ we have $f^\delta_\alpha(w_{\sigma,m}) = v^1_{\sigma,m}$.

(The notation $\Theta^{\beta_1,\beta_2}_n$ is defined before (II.2).)

$\diamondsuit(E_0)$ gives us for each $\sigma \in E_0$ a "prediction" $\Upsilon(\sigma) = \langle \Theta^\sigma_n : n \in \omega\rangle$ of an $\omega$-sequence of finite subsets of $T_\sigma$ such that $\bigcup_{n\in\omega} \Theta^\sigma_n$ is an antichain in $T_\sigma$. The proof that $G^0$ and $G^T$ are not isomorphic will then work as before.

**(VII)** The next step is to describe in detail the inductive construction of the data satisfying the properties given above. Our construction is by induction on the elements of $E$. At stage $\delta \in E$ we will define $G^0_\alpha$ and $G^\delta_\alpha$ for any $\alpha \leq \delta + 1$ for which they are not already defined. We will have already defined $G^0_\nu$ for $\nu \leq \sup\{\delta' + 1 : \delta' \in E,\ \delta' < \delta\}$. By following the prescriptions in (*) and (***), we can assume that $G^0_\nu$ is defined for all $\nu \leq \delta$.

Let $\gamma = \sup\{\delta' + 1 : \delta' \in E \cap \delta,\ T_\delta \restriction \delta' = T_{\delta'}\}$. Then we need to define $G^\delta_\alpha$ for $\gamma < \alpha \leq \delta + 1$. We need to do this is such a way that we are able to define the partial isomorphisms $f^\delta_\alpha$. We shall leave the details of the latter to the next section and describe the construction of the groups here. There are two cases to consider.

*Case 1:* $\gamma = \delta \in E$. Then $G^\delta_\delta$ is already defined. If $\delta \in E_0$, follow the prescription in (***) and (****). If $\delta \in E_1$, $\diamondsuit(E_1)$ gives us an isomorphism $h : G^0_\delta \to G^\delta_\delta$; the construction of $G^0_{\delta+1}$ and $G^\delta_\delta$ is essentially the same as in the previous Theorem (Case 4 of (III)); in particular, if (II.2) holds, we use an antichain $A^\delta_{\beta,\delta} = \{t : t \text{ is } <_\delta\text{-minimal in } \delta \setminus \beta\}$; $G^0_{\delta+1}$ is generated by $G^0_\delta \cup \{z^0_{\delta,n} : n \in \omega\}$ subject to relations

$p_n z^0_{\delta,n+1} = z^0_{\delta,n} + k^0_{\delta,n}$ (which keep $h$ from extending) and $G^\delta_{\delta+1}$ is generated by $G^\delta_\delta \cup \{z^\delta_{\delta,n} : n \in \omega\}$ subject to relations $p_n z^\delta_{\delta,n+1} = z^\delta_{\delta,n} + k^\delta_{\delta,n}$ (where $k^\delta_{\delta,n} = f^\delta_{\Theta_n}(k^0_{\delta,n})$).

For the purposes of later stages of the construction we also define, for any $\delta_1 > \delta$ such that $\delta_1 \in E$ and $T_{\delta_1} \restriction \delta \neq T_\delta$, elements $k^{\delta_1}_{\delta,n} \in G^{\delta_1}_\delta$. We know that $k^0_{\delta,n}$ has the form $x^0_{\mu_n,j_n} \pm \xi_{\delta,n}$ where $\xi_{\delta,n}$ is either 0, $x^0_{\sigma,j}$, or $w_{\sigma,j}$ for some $\sigma$, $j$ (cf. (IV.1)). In case $\xi_{\delta,n}$ is 0, let $k^{\delta_1}_{\delta,n} = x^1_{\mu_n,j_n}$; in case $\xi_{\delta,n} = x^0_{\sigma,j}$ , let $k^{\delta_1}_{\delta,n} = x^1_{\mu_n,j_n} \pm x^1_{\sigma,j}$. Finally, if $\xi_{\delta,n} = w_{\sigma,j}$, let $k^{\delta_1}_{\delta,n} = x^1_{\mu_n,j_n} \pm \xi'_{\delta,n}$ where

$$\xi'_{\delta,n} = \begin{cases} w^1_{\sigma,j} & \text{if } T_{\delta_1} \restriction \sigma \neq T_\sigma \\ v^1_{\sigma,j} & \text{if } T_{\delta_1} \restriction \sigma = T_\sigma \end{cases}$$

and $w^1_{\sigma,j} = 2u^1_{\sigma,j+1} - u^1_{\sigma,j}$. We will be able to show (in the next section) the following:

> (VII.1) for any branch $B$ in $T_{\delta_1} \restriction \delta$ with $\delta = \sup\{t+1 : t \in B\}$, $g_B = \cup\{f^{\delta_1}_\alpha : \alpha \in B\}$ is such that for almost all $n$, $g_B(k^0_{\delta,n}) = k^{\delta_1}_{\delta,n}$.

(This is evidence of what, in view of (IV.3), will enable us to extend functions.)

*Case 2:* $\gamma < \delta$. We need to define $G^\delta_\alpha$ for $\gamma + 1 \leq \alpha \leq \delta + 1$ by induction on $\alpha$. If we have defined $G^\delta_\alpha$ for $\alpha \leq \rho < \delta$, and $\rho$ does not belong to $E_1$, we follow the prescription in (**) or (****). If $\rho \in E_1$, then $T_\delta \restriction \rho \neq T_\rho$ (by definition of $\gamma$). By induction $G^0_{\rho+1}$ is constructed as in Case 1 and we have $k^\delta_{\rho,n}$ as there (with $\delta$ playing the role of $\delta_1$ and $\rho$ playing the role of $\delta$). In particular, $G^0_{\rho+1}$ is generated by $G^0_\rho \cup \{z^0_{\rho,n} : n \in \omega\}$ subject to relations $p_n z^0_{\rho,n+1} = z^0_{\rho,n} + k^0_{\rho,n}$. We define $G^\delta_{\rho+1}$ to be generated by $G^\delta_\rho \cup \{z^\delta_{\rho,n} : n \in \omega\}$ subject to relations $p_n z^\delta_{\rho,n+1} = z^\delta_{\rho,n} + k^\delta_{\rho,n}$. Finally, we define $G^\delta_{\delta+1}$ as in Case 1.

The definition of the $W^\delta_\alpha[\Theta]$ will be as in (III); specifically, $W^\delta_{\alpha+1}[\Theta] = W^\delta_\alpha[\Theta]$ unless $\alpha = \sigma \in E_0$, $T_\delta \restriction \sigma = T_\sigma$ and $\Theta = \Theta^\sigma_n$ for some $n$, in which case $W^\delta_{\sigma+1}[\Theta] = W^\delta_\sigma[\Theta] \cup \{w_{\sigma,n}\}$.

**(VIII)** We have defined the groups and the sets $W^\delta_\alpha[\Theta]$; the last step is to show that the partial isomorphisms $f^\delta_\nu$ can be defined satisfying the conditions in 4.

First let us verify (VII.1). Let $\delta$ and $\delta_1$ be as in Case 1 of (VII) and suppose $B$ is a branch in $T_{\delta_1} \restriction \delta$ with $\delta = \sup\{t+1 : t \in B\}$. Then $g_B$ is an isomorphism : $G^0_\delta \to G^{\delta_1}_\delta$ and we want to show that $g_B(k^0_{\delta,n}) = k^{\delta_1}_{\delta,n}$

for almost all $n$. Recall that $k^0_{\delta,n}$ has the form $x^0_{\mu_n,j_n} \pm \xi_{\delta,n}$ where $\xi_{\delta,n}$ is either 0, $x^0_{\sigma,j}$, or $w_{\sigma,j}$ for some $\sigma$, $j$; the only case we need to worry about is when $\xi_{\delta,n} = w_{\sigma,j}$.

Let $\mu = \sup\{\alpha < \delta : T_\delta \restriction \alpha = T_{\delta_1} \restriction \alpha\}$; so $\mu < \delta$ and $G^{\delta_1}_\alpha = G^\delta_\alpha$ for $\alpha \leq \mu$. Suppose that $G^0_{\delta+1}$ and $G^\delta_{\delta+1}$ are defined using $A^\delta_{\beta,\delta} = \bigcup_{n\in\omega} \Theta^{\beta,\delta}_n$ as in Case 1 of (VII) and Case 4 of (III). We consider several cases. First, suppose that there exists $t \in A^\delta_{\beta,\delta}$ with $t \geq \mu$. Then for almost all $n$, $t \in \Theta^{\beta,\delta}_n$ and thus if $w_{\sigma,j} \in W^\delta_\delta[\Theta^{\beta,\delta}_n]$ then $\sigma > t \geq \mu$; hence $T_{\delta_1} \restriction \sigma \neq T_\delta \restriction \sigma$ and it follows from 4(d) that $g_B(k^0_{\delta,n}) = k^{\delta_1}_{\delta,n}$. If this case does not hold then $A^\delta_{\beta,\delta} \subseteq \mu$ so $A^\delta_{\beta,\delta} = A^\delta_{\beta,\mu}$ is an antichain in $T_{\delta_1} \restriction \mu = T_\delta \restriction \mu$. If there exists $t \in B$ with $\beta \leq t < \mu$, then there exists $t \in B$ with $t \in A^\delta_{\beta,\delta}$ and hence $t \in \Theta^{\beta,\delta}_n$ for almost all $n$; it follows easily that for almost all $n$ $g_B(k^0_{\delta,n}) = k^{\delta_1}_{\delta,n}$ (considering separately the cases when $\sigma \leq \mu$ and $\sigma > \mu$). In the remaining case, if $\alpha = \inf\{t \in B : t \geq \beta\}$, then $\alpha \geq \mu$ so we have $\sup\{t : t <_{\delta_1} \alpha\} \leq \beta < \mu \leq \alpha$ and we have the desired conclusion by 4(e) — again distinguishing between the cases when $\sigma \leq \mu$ and $\sigma > \mu$. This completes the proof of (VII.1).

Now we need to verify the analog of (III.1). Letting $\delta$ and $\gamma$ be as in (VII), we need to define $f^\delta_\alpha$ for $\gamma \leq \alpha \leq \delta$. Fix $\alpha$ and let $B = \{t < \gamma : t <_\delta \alpha\}$ and $g_B = \cup\{f^\delta_t : t \in B\}$. We can suppose that $\alpha$ is $<_\delta$-minimal among elements of $\{\beta : \gamma \leq \beta \leq \alpha\}$.

We will first define an extension of $g_B$ to a partial isomorphism $\tilde{g}_B$ whose domain is

$$\operatorname{dom}(g_B) + \left\langle \begin{array}{c} \{x^0_{\nu,j} : \nu \ \leq \alpha,\, j = 0,1\} \cup \\ \{u^0_{\sigma,n} : \sigma \in E_0 \cap (\alpha+1),\, n \in \omega\} \cup \\ \{v^0_{\sigma,n} : \sigma \in E_0 \cap (\alpha+1),\, n \in \omega\} \end{array} \right\rangle$$

Using an enumeration in an $\omega$-sequence of $Y_0 \cup Y_1$ where

$$Y_0 = \{\sigma \in E_0 : \sup B \leq \sigma < \gamma \text{ and } T_\delta \restriction \sigma = T_\sigma\}$$

and

$$Y_1 = \{\langle \beta_1, \beta_2 \rangle : \sup B \leq \beta_1 < \beta_2 \leq \alpha\}$$

we can define $\tilde{g}_B$ such that

(c′) for all $\nu \leq \alpha$, $j \in \{0,1\}$ $\tilde{g}_B(x^0_{\nu,j}) = x^1_{\nu,j}$; moreover, if $w_{\sigma,n} \in W^\delta_{\gamma+1}[\Theta]$ and $\Theta \cap B \neq \emptyset$, then $\tilde{g}_B(w_{\sigma,n}) = v^1_{\sigma,n}$;

(d′) if $\sigma \in E_0 \cap \alpha + 2$, then $\tilde{g}_B(u^0_{\sigma,n}) = u^1_{\sigma,n}$ for almost all $n$, and if $T_\delta \restriction \sigma \neq T_\sigma$, then $\tilde{g}_B(u^0_{\sigma,n}) = u^1_{\sigma,n}$ for all $n$; and (e′) for all pairs $\beta_1$, $\beta_2$ with $\sup B \le \beta_1 < \beta_2 \le \alpha$, it is the case for almost all $n \in \omega$ that for all $w_{\sigma,m} \in W^{\delta}_{\alpha+1}[\Theta^{\beta_1,\beta_2}_n]$ we have $\tilde{g}_B(w_{\sigma,m}) = v^1_{\sigma,m}$.

Now $\tilde{g}_B(k^0_{\rho,n})$ is defined for all $\rho \in E_1$ with $\rho \le \alpha$. We need to define $f^{\delta}_{\alpha}(z^{\delta}_{\rho,n})$ for all such $\rho \ge \sup B$. In view of (IV.3), we can do this provided that $\tilde{g}_B(k^0_{\rho,n}) = k^{\delta}_{\rho,n}$ for almost all $n \in \omega$. We consider separately the cases: $T_\delta \restriction \rho = T_\rho$; and $T_\delta \restriction \rho \ne T_\rho$. The first case is as in (IV); the last is as in the proof of (VII.1) (with $\delta$ playing the role of $\delta_1$, $\rho$ playing the role of $\delta$ and using (d′) and (e′)).

This completes the proof of Theorem 8.

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