A PARTIAL ORDER WHERE ALL MONOTONE MAPS ARE DEFINABLE

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Abstract. It is consistent that there is a partial order $(P, \leq)$ of size $\aleph_1$ such that every monotone function $f : P \to P$ is first order definable in $(P, \leq)$

It is an open problem whether there can be an infinite lattice L such that every monotone function from L to L is a polynomial. Kaiser and Sauer [KS93] showed that such a lattice would have to be bounded, and cannot be countable.

Sauer then asked the weaker question if there can be an infinite partial order $(P, \leq)$ such that all monotone maps from P to P are at least definable. (Throughout the paper, "definable" means "definable with parameters by a first order formula in the structure $(P, \leq)$.)

Since every infinite partial order P admits at least $\mathfrak{c} = 2^{\aleph_0}$ many monotone maps from P to P, our partial order must have size (at least) continuum.

We show:

0.1. **Theorem.** *The statement*

> *There is a partial order $(P, \leq)$ of size $\mathfrak{c} = \omega_1$ such that all monotone functions $f : P \to P$ are definable in P*

is consistent relative to ZFC. Moreover, the statement holds in any model obtained by adding (iteratively) ω_1 Cohen reals to a model of CH.

We do not know if Sauer's question can be answered outright (i.e., in ZFC), or even from CH.

Structure of the paper. In section 1 we give four conditions on a partial order on $(P, \sqsubseteq)$ of size κ and we show that they are sufficient to ensure the conclusion of the theorem. This section is very elementary. The two main conditions of section 1 are

1. a requirement on small sets, namely that they should be definable
2. two requirements on large sets (among them: "there are no large antichains")

Here, "small" means of size $< |P|$, and "large" means of size $= |P|$.

In section 2 we show how to take care of requirement 1 in an inductive construction of our partial order in $\mathfrak{c}$ many steps. Each definability requirement will be satisfied at some stage $\alpha < \mathfrak{c}$.

Finally, in section 4 we deal with the problem of avoiding large antichains. Here the inductive construction is not so straightforward, as we have to "anticipate"

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potential large sets and ensure that in the end they will not contradict our requirement. The standard tool for dealing with such a problem is $\diamondsuit$. This combinatorial principle has been used for a related construction in [She85], and it is possible to use the techniques of [She83, section 5] to combine it with the requirement on small sets to show that the conclusion of our theorem actually follows from $\diamondsuit$. However, we use instead a forcing construction which seems to be somewhat simpler. We will work in an iterated forcing extension, and use a Δ-system argument to ensure that the requirements about large sets are met. This argument is carried out in section 4, which therefore requires a basic knowledge of forcing.

1. Four conditions

1.1. **Theorem.** *Assume that $(P, \sqsubseteq)$ is a partial order, $\kappa := |P|$ a regular cardinal. We will call subsets of cardinality $< \kappa$ "small." Assume that the following conditions hold:*

- *C1 Every antichain is small (an antichain is a set of pairwise incomparable elements).*
- *C2 Whenever $g : (P, \sqsubseteq) \to (P, \sqsubseteq)$ is monotone, then there is a small set $A \subseteq P$ such that for all $\alpha \in P$, $f(\alpha) \in A \cup \{\alpha\}$.*
- *C3 For all $\alpha \in P$ the set $\{\beta \in P : \beta \sqsubseteq \alpha\}$ is small.*
- *C4 Every small subset of $P \times P$ is definable in the structure $(P, \sqsubseteq)$.*

Then every monotone map from $(P, \sqsubseteq)$ to itself is definable in $(P, \sqsubseteq)$.

We will prove this theorem below.

1.2. **Lemma.** *Let $\sqsubseteq$ be as above. Then for any set $A \subseteq P$ there is a small set $A' \subseteq A$ such that $\forall \alpha \in A\, \exists \gamma \in A' : \gamma \sqsubseteq \alpha$.*

Proof. Let $B \subseteq A$ be a maximal antichain in A. So B is small, by C1. Let A' be the downward closure of B in A, i.e., $A' := \bigcup_{\beta \in B} \{\gamma \in A : \gamma \sqsubseteq \beta\}$. Then A' is still small because of C3. Clearly A' is as required. □

1.3. **Fact.** Let $(P, \sqsubseteq)$ be any partial order, $g : P \to P$ a monotone function. Then g is definable iff the set $\{(x, y) : g(x) \sqsubseteq y\}$ is definable.

Proof. Let $B := \{(x, y) : g(x) \sqsubseteq y\}$. Clearly B is definable from g. Conversely, g can be defined from B as follows:

$$g(x) = z \;\Leftrightarrow\; (\forall y\, (x, y) \in B \Rightarrow z \sqsubseteq y) \text{ and } ((x, z) \in B)$$

□

Proof of 1.1: Let $g : P \to P$ be monotone. Let $P_0 \subseteq P$ be small such that

- If $g(\alpha) \neq \alpha$ then $g(\alpha) \in P_0$.
- $\beta \sqsubseteq \alpha \in P_0$ implies $\beta \in P_0$.

(Such a set P_0 can be found using C2 and C3). For every $j \in P_0$ let $D_j := \{\alpha \in P \setminus P_0 : g(\alpha) = j\}$, and let $D := \{\alpha \in P \setminus P_0 : g(\alpha) = \alpha\}$. We can find D' and D'_j as in 1.2. Let $P_1 = \bigcup_j D'_j \cup D'$.

Claim 1. For $\alpha \in P \setminus P_0$ we have:

$$g(\alpha) = \alpha \iff \exists \beta \in P_1 : \beta \sqsubseteq \alpha, g(\beta) = \beta$$

The direction $\Rightarrow$ follows immediately from the definition of D'. For the other direction, assume that $\beta \in P_1$, $\beta \sqsubseteq \alpha$, $g(\beta) = \beta$. Since $g(\beta) \sqsubseteq g(\alpha)$, and $g(\beta) = \beta \notin P_0$, we also have $g(\alpha) \notin P_0$. Hence $g(\alpha) = \alpha$.

Claim 2. For $\alpha \in P \setminus P_0$, $g(\alpha) \neq \alpha$ we have:

$$g(\alpha) \sqsubseteq i \iff \forall \gamma \in P_1 : \gamma \sqsubseteq \alpha \Rightarrow g(\gamma) \sqsubseteq i$$

The implication $\Rightarrow$ follows from the monotonicity of g. For the converse direction, assume that $g(\alpha) = j$. Since $D'_j \subseteq P_1$, we can find $\gamma \in P_1$, $\gamma \sqsubseteq \alpha$ and $g(\gamma) = j$. By assumption, $g(\gamma) \sqsubseteq i$, so $g(\alpha) \sqsubseteq i$.

Claims 1 and 2, together with 1.3 now imply that g can be defined from the graphs of the functions $g{\restriction}P_0$ and $g{\restriction}P_1$.

2. Coding small sets

In this section we will show how to build a partial order satisfying C3 and C4. Throughout this section we assume the continuum hypothesis.

2.1. **Definition.** (1) We call $\mathfrak{M}$ a "creature", if $\mathfrak{M} = (M, \sqsubseteq, F, H)$, where
- $\sqsubseteq$ is a partial order on M
- F is a partial symmetric function, $\operatorname{dom}(F) \subseteq M \times M$, $\operatorname{ran}(F) \subseteq M$
- if $F(x, y) = z$, then $x < z$, $y < z$, and there is no $z' < z$ with $x \leq z'$ and $y \leq z'$ (i.e., z is a minimal upper bound of x, y, and x, y are incomparable).
- F is locally finite, i.e.: For any finite $A \subseteq M$ there is a finite set B, $A \subseteq B \subseteq M$ such that for any $(x, y) \in \operatorname{dom} F \cap (B \times B)$, $F(x, y) \in B$.
- $H \subseteq M \times M \times M$. We define $H(x, y) := \{z : H(x, y, z)\}$
- If $H(x, y, z)$ then z is a minimal upper bound of x, y, and x, y are incomparable
- $H(x, y) = H(y, x)$
- If $H(x, y, z)$ then $(x, y) \notin \operatorname{dom}(F)$.

(2) If $\mathfrak{M}_1$, $\mathfrak{M}_2$ are creatures, then $\mathfrak{M}_1 \leq \mathfrak{M}_2$ ($\mathfrak{M}_2$ is an extension of $\mathfrak{M}_1$) means that $M_1 \subseteq M_2$, and the relations/functions of $\mathfrak{M}_1$ are just the restrictions of the corresponding relations/functions of $\mathfrak{M}_2$. (so in particular, $\operatorname{dom}(F^{\mathfrak{M}_2}) \cap (M_1 \times M_2) = \operatorname{dom}(F^{\mathfrak{M}_1})$).

(3) We say that $\mathfrak{M}_2$ is an "end extension" of $\mathfrak{M}_1$ (or that $\mathfrak{M}_1$ is an "initial segment" of $\mathfrak{M}_2$) if $\mathfrak{M}_1 \leq \mathfrak{M}_2$ and $\mathfrak{M}_1$ is downward closed in $\mathfrak{M}_2$, i.e., $\mathfrak{M}_2 \models x \sqsubseteq y$, $y \in M_1$ implies $x \in M_1$.

(4) We use the above terminology also if one or both of the structures $\mathfrak{M}_1$, $\mathfrak{M}_2$ is just a partial order without an additional function F or relation H.

2.2. **Notation.** When we consider several creatures $\mathfrak{M}$, $\mathfrak{M}_1$, $\mathfrak{M}^*$, ..., then it is understood that their universes are called M, M_1, M^*, ..., their partial orders are $\sqsubseteq$, $\sqsubseteq_1$, $\sqsubseteq^*$, ..., etc. Instead of writing $x \sqsubseteq_1 y$ we may write $\mathfrak{M}_1 \models x \sqsubseteq y$, etc.

We will use the letters $\mathfrak{M}$ and $\mathfrak{N}$ to denote possibly infinite creatures, and $\mathfrak{p}$, $\mathfrak{q}$ to denote finite creatures.

2.3. **Definition.** If $(M, \sqsubseteq)$ is a partial order, then we let $\mathcal{F}(\sqsubseteq)$ be the partial binary function F satisfying

$$F(x,y) = \begin{cases} \text{the unique minimal upper bound of } x \text{ and } y, \text{ if it} \\ \text{exists and if } x \text{ and } y \text{ are incomparable} \\ \\ \text{undefined, otherwise} \end{cases}$$

2.4. **Setup.** Let $(M_\delta : \delta < \mathfrak{c})$ be a continuous increasing sequence of infinite sets with $|M_{\delta+1} \setminus M_\delta| = |M_\delta| < \mathfrak{c}$. Let $(R_\delta : \delta < \mathfrak{c})$ be a sequence of relations with $R_\delta \subseteq M_\delta \times M_\delta$ such that for any $R \subseteq M_\delta \times M_\delta$ there is a $\delta' > \delta$ such that $R = R_{\delta'}$. (Such a sequence can be found since we are assuming CH).

For each δ we fix some element $e_\delta \in M_{\delta+1} \setminus M_\delta$. For each $(\alpha, \beta) \in M_\delta \times M_\delta$ we pick disjoint sets $A^\delta_{\alpha\beta}$, $B^\delta_{\alpha\beta}$, $C^\delta_{\alpha\beta}$, $\Delta^\delta_{\alpha\beta}$, $\Gamma^\delta_{\alpha\beta}$ (we will omit the superscript δ if it is clear from the context) satisfying

$$|A_{\alpha\beta}| = 2,\ |B_{\alpha\beta}| = 3,\ |C_{\alpha\beta}| = 1 \tag{1}$$

$$|\Delta_{\alpha\beta}| = 3,\ \Delta_{\alpha\beta} = \{a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}\},\ \Gamma_{\alpha\beta} = \{\gamma_{\alpha\beta}\} \tag{2}$$

where e_δ is not in any of these sets and the sets

$$\Omega_{\alpha\beta} := A_{\alpha\beta} \cup B_{\alpha\beta} \cup C_{\alpha\beta} \cup \Delta_{\alpha\beta} \cup \Gamma_{\alpha\beta}$$

satisfy $(\alpha, \beta) \neq (\alpha', \beta') \Rightarrow \Omega_{\alpha\beta} \cap \Omega_{\alpha'\beta'} = \emptyset$. For $x \in \Omega_{\alpha\beta}$ we define $\Omega(x) := \Omega_{\alpha\beta} \cup \{\alpha, \beta\}$.

Moreover, we choose the sets $\Omega_{\alpha\beta}$ such that $M_{\delta+1} \setminus \bigcup_{\alpha,\beta \in M_\delta} \Omega_{\alpha\beta}$ still has size $|M_\delta|$. For $x \in M_{\delta+1} \setminus \bigcup_{\alpha,\beta \in M_\delta} \Omega_{\alpha\beta}$ we let $\Omega(x) = \emptyset$.

2.5. **Definition.**
(1) Let $\mathfrak{M}$ be a creature. We say that $\Delta \in [M]^3$ is a *triangle* (also called an $\mathfrak{M}$-triangle or F-triangle if $F = F^{\mathfrak{M}}$) iff $\mathrm{dom}(F^{\mathfrak{M}}) \supseteq [\Delta]^2$.
(2) Let Δ be an F-triangle. We say that $a \in \Delta$ is a *base point* for Δ if there is a unique (unordered) pair $\{b, c\}$ with $F(b, c) = a$.
(3) Let Δ be a triangle. We say that b is *anchor* for Δ if there is a c such that $F(b, c)$ is a base point for Δ.
(4) If $\mathfrak{M}_1 \leq \mathfrak{M}_2$ then we say that $\mathfrak{M}_2$ is a "separated" extension of $\mathfrak{M}_1$ iff
   - every triangle in $\mathfrak{M}_2$ is contained in M_1 or in $M_2 \setminus M_1$, and
   - if $\Delta \subseteq M_2 \setminus M_1$ is a triangle, then Δ is not anchored at any point in M_1.
   - If $\mathfrak{M}_2 \models H(x, y, z)$, and $x, y \in M_1$ then also $z \in M_1$.

2.6. **Note.** Our goal is to define $F_{\delta+1}$ such that R_δ will become definable (and at the same time make it possible for $F_{\delta+1}$ to be of the form $\mathcal{F}(\sqsubseteq)$ for some end extension $\sqsubseteq$ of $\sqsubseteq_\delta$). We will achieve this by "attaching" triangles to all pairs (α, β) in R_δ in a way that (α, β) can be reconstructed from the triangle. We also need to ensure that the only triangles in $\mathfrak{M}_{\delta+1}$ (and also in any $\mathfrak{M}_{\delta'}$, $\delta' > \delta$) are the ones we explicitly put there.

The particular way of coding pairs by triangles is rather arbitrary.

2.7. **Overview of the construction.** By induction on $\delta \leq \mathfrak{c}$ we will define creatures $\mathfrak{M}_\delta = (M_\delta, \sqsubseteq_\delta, F_\delta, H_\delta)$ such that

(A) $\gamma < \delta$ implies that $\mathfrak{M}_\delta$ is a separated end extension of $\mathfrak{M}_\gamma$

(B) $F_\delta = \mathcal{F}(\sqsubseteq_\delta)$. (See 2.3)

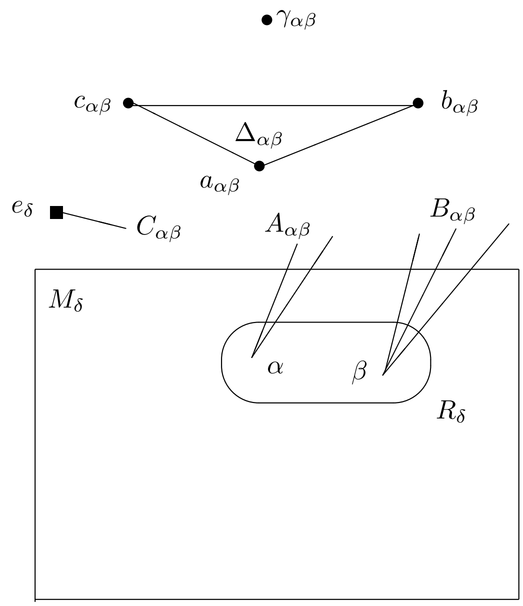
(C) R_δ is definable in $(M_{\delta+1}, \sqsubseteq_{\delta+1}, F_{\delta+1})$ and hence also in $(M_{\delta+1}, \sqsubseteq_{\delta+1})$, and the definition of R_δ is absolute for any separated end extension of $\mathfrak{M}_{\delta+1}$.

For limit δ we let $\sqsubseteq_\delta = \bigcup_{\gamma<\delta} \sqsubseteq_\gamma$, $F_\delta = \bigcup_{\gamma<\delta} F_\gamma$, $H_\delta = \bigcup_{\gamma<\delta} H_\gamma$.

We will construct $\mathfrak{M}_{\delta+1}$ from $\mathfrak{M}_\delta$ in two steps. First we define a function $F_{\delta+1}$ such that R_δ becomes definable in $(M_{\delta+1}, F_{\delta+1})$. Then we show that we can find a partial order $\sqsubseteq_{\delta+1}$ (end-extending $\sqsubseteq_\delta$) such that $F_{\delta+1} = \mathcal{F}(\sqsubseteq_{\delta+1})$.

Construction of $F_{\delta+1}$. $F_{\delta+1}$ will be defined as follows:

* $F_{\delta+1} \restriction (M_\delta \times M_\delta) = F_\delta$
* If $(\alpha, \beta) \in R_\delta$ then
  * $F(\alpha, x) = a_{\alpha\beta}$ for $x \in A_{\alpha\beta}$
  * $F(\beta, x) = b_{\alpha\beta}$ for $x \in B_{\alpha\beta}$
  * $F(e_\delta, x) = c_{\alpha\beta}$ for $x \in C_{\alpha\beta}$
  * $F(x, y) = \gamma_{\alpha\beta}$ for $\{x, y\} \in [\Delta_{\alpha\beta}]^2$.
* Except where required by the above (and by symmetry), $F_{\delta+1}$ is undefined.



This completes the definition of $F_{\delta+1}$. The diagram above is supposed to illustrate this definition. Pairs (x, y) on which $F_{\delta+1}$ is defined are connected by a line; the value of $F_{\delta+1}$ at such pairs is the small black disk above the pair.

2.8. **Fact.** (1) $(M_{\delta+1}, F_{\delta+1})$ is a separated extension of $\mathfrak{M}_\delta$.

(2) R_δ is definable in $(M_{\delta+1}, F_{\delta+1})$.

(3) Let $(M_{\delta+1}, F_{\delta+1}) \leq (M, F)$ and assume that
  * (M, F) is a separated extension of $(M_{\delta+1}, F_{\delta+1})$
  * If $F(x, y) \in M_{\delta+1}$ then $x, y \in M_{\delta+1}$. (This is certainly true if $F = \mathcal{F}(\sqsubseteq)$, where $\sqsubseteq$ is an end extension of $\sqsubseteq_{\delta+1}$.)

Then R_δ is definable in (M, F).

Proof. (1) is clear. (2) is a special case of (3). Let (M, F) be as in (3). Then R_δ is the set of all pairs (α, β) such that there is a triangle Δ with a unique base, anchored at e_δ such that

$$\begin{aligned} \text{either } & \alpha \neq \beta \text{ and } |\{x : F(\alpha, x) \in \Delta\}| = 2,\ |\{x : F(\beta, x) \in \Delta\}| = 3. \\ \text{or } & \alpha = \beta \text{ and } |\{x : F(\alpha, x) \in \Delta\}| = 5, \end{aligned}$$

because the only triangles anchored at e_δ will be the sets $\Delta^\delta_{\alpha\beta}$. Clearly this is a definition in first order logic with the parameter e_δ. $\square$

Construction of $\sqsubseteq_{\delta+1}$. We will define $\sqsubseteq_{\delta+1}$ from a (sufficiently) generic filter for a forcing notion $\mathbb{Q}_\delta = \mathbb{Q}(\mathfrak{M}_\delta, F_{\delta+1})$.

2.9. **Definition.** Assume that $\mathfrak{M}_\delta$, $F_{\delta+1}$ are as above. We define the forcing notion $\mathbb{Q}_\delta$ as the set of all $\mathfrak{p}$ such that

- $\mathfrak{p}$ is a finite creature, $\mathfrak{p} = (p, \sqsubseteq_\mathfrak{p}, F_\mathfrak{p}, H_\mathfrak{p})$
- $\mathfrak{p} \restriction M_\delta \leq \mathfrak{M}_\delta$
- $p \subseteq M_{\delta+1}$,
- For all $x \in p \setminus M_\delta$, $\Omega(x) \subseteq p$.
- $F_\mathfrak{p} = F_{\delta+1} \restriction (p \times p)$
- $\mathfrak{p}$ is a separated end extension of $\mathfrak{p} \restriction M_\delta$.

Clearly $(\mathbb{Q}_\delta, \leq)$ is a partial order. **Note: We force "upwards," i.e., $\mathfrak{p} \leq \mathfrak{q}$ means that $\mathfrak{q}$ is a stronger condition than $\mathfrak{p}$.** But we still call the generic set "filter" and not "ideal."

If $G \subseteq Q_\delta$ is a filter then there is a smallest creature which extends all $\mathfrak{p} \in G$. We call this creature $\mathfrak{M}_G$.

2.10. **Fact.** The following sets are dense in $\mathbb{Q}_\delta$.

(a) $D_x := \{\mathfrak{p} : x \in p\}$, for any $x \in M_{\delta+1}$.

(b) $D_{\mathfrak{p},x,y} := \{\mathfrak{q} : \mathfrak{q} \perp \mathfrak{p} \text{ or } \mathfrak{q} \models \exists z \notin p : H(x, y, z)\}$, whenever $x, y \in p$ are incomparable and $(x, y) \notin \mathrm{dom}(F_\mathfrak{p})$.

(c) $E_{\mathfrak{p},x,y,z} := \{\mathfrak{q} : \mathfrak{q} \perp \mathfrak{p} \text{ or } \mathfrak{q} \models \exists z' \sqsubseteq z : x \sqsubseteq z', y \sqsubseteq z', z \neq z'\}$, whenever $x, y \in p$, $(x, y) \in \mathrm{dom}(F_\mathfrak{p})$, $x \sqsubseteq z$, $y \sqsubseteq z$, $z \neq F(x, y)$.

2.11. **Fact.** If G meets all the dense sets above, then $\mathfrak{M}_{\delta+1} := \mathfrak{M}_G$ is a separated end extension of $\mathfrak{M}_\delta$, $F_{\delta+1} = \mathcal{F}(\sqsubseteq_{\mathfrak{M}_G})$.

Proof. We only prove the last statement. Let $F = \mathcal{F}(\sqsubseteq_{\mathfrak{M}_G})$. First assume $F_{\delta+1}(x, y) = z^*$. Clearly z^* is a minimal upper bound of x, y, so to prove $F(x, y) = z^*$ we only have to show that z^* is the *unique* minimal upper bound. If $z \neq z^*$ is also a minimal upper bound then we can find $p \in G$ containing $\{x, y, z, z^*\}$. Now use 2.10(c) to find $z' \neq z$ in $M_{\delta+1}$, $x, y \sqsubseteq z' \sqsubseteq z$, which is a contradiction.
Now assume that $F_{\delta+1}(x, y)$ is undefined. We have to show that also $F(x, y)$ is undefined. Applying 2.10(b) twice we can find two distinct elements z_1, z_2 such that $H(x, y, z_1)$ and $H(x, y, z_2)$ both hold in $\mathfrak{M}_{\delta+1}$, hence there is no unique minimal upper bound of x and y, so $F(x, y)$ is undefined. $\square$

2.12. **Fact.** $\mathbb{Q}_\delta$ is countable, hence satisfies the ccc.

2.13. **Conclusion.** *CH implies that a sufficiently generic filter exists. Thus we have completed the definition of $\sqsubseteq_{\delta+1}$ and $H_{\delta+1}$. Clearly $\mathfrak{M}_{\delta+1}$ will be as required.*

In the last section we will show how to embed this construction into an iterated forcing argument.

3. Amalgamation

Starting from a model of GCH we will construct an iterated forcing notion $(\mathbb{P}_\delta, Q_\delta : \delta < \kappa)$ such that each partial order Q_δ will be some $\mathbb{Q}(\mathfrak{M}_\delta, F_{\delta+1})$ as in the previous section. This will define us a model $\mathfrak{M}_\kappa$. An additional argument is then needed to show that $\mathfrak{M}_\kappa$ will satisfy C1 and C2. In this section we prepare some tools for this additional argument by collecting some facts about amalgamation.

3.1. **Definition.** Let $\mathfrak{p}$ and $\mathfrak{q}$ be (finite) creatures, $x \in p$, $y \in q$. We define $\mathfrak{p} \oplus \mathfrak{q}$ and $\mathfrak{p} \oplus_{x,y} \mathfrak{q}$ as follows:

$\mathfrak{p} \oplus \mathfrak{q} = (p \cup q, \sqsubseteq_{\mathfrak{p}\oplus\mathfrak{q}}, F_{\mathfrak{p}\oplus\mathfrak{q}}, H_{\mathfrak{p}\oplus\mathfrak{q}})$, where $\sqsubseteq_{p\oplus q}$ is the transitive closure of $\sqsubseteq_{\mathfrak{p}} \cup \sqsubseteq_{\mathfrak{q}}$, and $F_{\mathfrak{p}\oplus\mathfrak{q}} = F_{\mathfrak{p}} \cup F_{\mathfrak{q}}$, $H_{\mathfrak{p}\oplus\mathfrak{q}} = H_{\mathfrak{p}} \cup H_{\mathfrak{q}}$.

$\mathfrak{p} \oplus_{x,y} \mathfrak{q} = (p \cup q, \sqsubseteq_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}, F_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}, H_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}})$, where $\sqsubseteq_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}$ is the transitive closure of $\sqsubseteq_{\mathfrak{p}} \cup \sqsubseteq_{\mathfrak{q}} \cup \{(x,y)\}$, and $F_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}} = F_{\mathfrak{p}} \cup F_{\mathfrak{q}}$. $H_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}} = H_{\mathfrak{p}} \cup H_{\mathfrak{q}}$.

3.2. **Fact.** (1) Assume that $\sqsubseteq_{\mathfrak{p}}$ and $\sqsubseteq_{\mathfrak{q}}$ agree on $p \cap q$, and similarly $F_{\mathfrak{p}}$ and $F_{\mathfrak{q}}$. Then $\mathfrak{p} \oplus \mathfrak{q}$ is a creature, $\mathfrak{p} \sqsubseteq \mathfrak{p} \oplus \mathfrak{q}$, $\mathfrak{q} \sqsubseteq \mathfrak{p} \oplus \mathfrak{q}$.

(2) If $\mathfrak{p}$ and $\mathfrak{q}$ are as above, and moreover: $\mathfrak{p}$ and $\mathfrak{q}$ are separated end extensions of $\mathfrak{r} := \mathfrak{p} \cap \mathfrak{q}$, and $\mathrm{type}_{\mathfrak{p}}(x, \mathfrak{r}) = \mathrm{type}_{\mathfrak{q}}(y, \mathfrak{r})$ (that is, for any $z \in r$ we have $\mathfrak{p} \models z \sqsubseteq x$ iff $\mathfrak{q} \models z \sqsubseteq y$), then also $\mathfrak{p} \oplus_{x,y} \mathfrak{q}$ is a model extending $\mathfrak{q}$ and end-extending $\mathfrak{p}$, and

$$(*) \qquad \mathfrak{p} \oplus_{x,y} \mathfrak{q} \models a \sqsubseteq b \quad \Leftrightarrow \quad \begin{array}{l} \mathfrak{p} \models a \sqsubseteq b \\ \text{or } \mathfrak{q} \models a \sqsubseteq b \\ \text{or } \mathfrak{p} \models a \sqsubseteq x, \ \mathfrak{q} \models y \sqsubseteq b \end{array}$$

Proof. We leave (1) to the reader. Let $\sqsubseteq^*$ be the relation defined in $(*)$. First we have to check that $\sqsubseteq^*$ is transitive. Note that the third clause in the definition of $a \sqsubseteq^* b$ can only apply if $a \in p$ and $b \in q \setminus r = q \setminus p$.

Let $a \sqsubseteq^* b \sqsubseteq^* c$. *A priori*, there are 9 possible cases:

$$\begin{array}{llcl}
a \sqsubseteq_{\mathfrak{p}} b & b \sqsubseteq_{\mathfrak{p}} c & \Rightarrow & a \sqsubseteq_{\mathfrak{p}} c \\
a \sqsubseteq_{\mathfrak{p}} b & b \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & b \in p \cap q, \ a \sqsubseteq_{\mathfrak{q}} b, \text{ so } a \sqsubseteq_{\mathfrak{q}} c \\
a \sqsubseteq_{\mathfrak{p}} b & b \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c \\
a \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{p}} c & \Rightarrow & b \in p \cap q, \ a \sqsubseteq_{\mathfrak{p}} b, \text{ so } a \sqsubseteq_{\mathfrak{p}} c \\
a \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & a \sqsubseteq_{\mathfrak{q}} c \\
a \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & b \in p \cap q, \ a \sqsubseteq_{\mathfrak{p}} b, \ a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c \\
a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{p}} c & \Rightarrow & \text{impossible} \\
a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c \\
a \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} b & b \sqsubseteq_{\mathfrak{p}} x, \ y \sqsubseteq_{\mathfrak{q}} c & \Rightarrow & \text{impossible}
\end{array}$$

So we see that in any case we get $a \sqsubseteq^* c$.

Clearly $\mathfrak{p} \oplus_{x,y} \mathfrak{q}$ is an end extension of $\mathfrak{p}$. We now check that $\mathfrak{q} \leq \mathfrak{p} \oplus_{x,y} \mathfrak{q}$. Let $a, b \in q$, $\mathfrak{p} \oplus_{x,y} \mathfrak{q} \models a \sqsubseteq b$. The only nontrivial case is that $\mathfrak{p} \models a \sqsubseteq x$, $\mathfrak{q} \models y \leq b$. Since x and y have the same type over $\mathfrak{p} \cap \mathfrak{q}$, we also have $\mathfrak{q} \models a \sqsubseteq y$. Hence $\mathfrak{q} \models a \sqsubseteq b$, so $\mathfrak{q} \leq \mathfrak{p} \oplus_{x,y} \mathfrak{q}$.

Finally we check that $\mathfrak{p} \oplus_{x,y} \mathfrak{q}$ is a creature. It is clear that $F_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}$ is locally finite.
To check that the conditions on $H_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}$ and $F_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}$ (in particular, minimality) are still satisfied, the following key fact is sufficient:

> if $\{a,b\} \subseteq p$ or $\{a,b\} \subseteq q$ (in particular, if $(a,b) \in \mathrm{dom}(H_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}}) \cup \mathrm{dom}(F_{\mathfrak{p}\oplus_{x,y}\mathfrak{q}})$),
> then $\mathfrak{p} \oplus_{x,y} \mathfrak{q} \models a,b \sqsubseteq c$ holds iff at least one of the following is satisfied:
> (1) $\mathfrak{p} \models a,b \sqsubseteq c$
> (2) $\mathfrak{q} \models a,b \sqsubseteq c$
> (3) $\mathfrak{p} \models a,b \sqsubseteq x$, $\mathfrak{q} \models y \sqsubseteq c$.

We leave the details of the argument to the reader. $\square$

3.3. **Corollary.** *Assume that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ are as above, $\mathfrak{q} \in \mathbb{Q}_\delta$, $\mathfrak{p} \leq \mathfrak{M}_\delta$.*
Then $\mathfrak{p} \oplus \mathfrak{q} \in \mathbb{Q}_\delta$ and $\mathfrak{p} \oplus_{x,y} \mathfrak{q} \in \mathbb{Q}_\delta$.

3.4. **Definition.** Let $(P, \sqsubseteq)$ be a partial order of size κ, κ a regular cardinal. We say that P has the strong chain condition if:

> Whenever $(\mathfrak{X}_\alpha : \alpha < \kappa)$ is a sequence of finite suborderings of $(P, \sqsubseteq)$, and $x_\alpha \in X_\alpha$ for all $\alpha < \kappa$, then there are $\alpha < \beta < \kappa$ such that:
> $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ agree on $X_\alpha \cap X_\beta$, $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ are separated end extensions of $\mathfrak{X}_\alpha \cap \mathfrak{X}_\beta$, x_α and y_α have the same type over $\mathfrak{X}_\alpha \cap \mathfrak{X}_\beta$, and
> $$\mathfrak{X}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{X}_\beta \leq (P, \sqsubseteq)$$

3.5. **Remark.** By the Δ-system lemma we know that for any sequence $(X_\alpha : \alpha < \kappa)$ of finite sets there is a set $A \subseteq \kappa$ of size κ such that the sets $(X_\alpha : \alpha \in A)$ form a Δ-system. If $(P, \sqsubseteq)$ moreover satisfies C3, then we may additionally assume that for any $\alpha < \beta < \kappa$ the models $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ are separated end extensions of $X_\alpha \cap \mathfrak{X}_\beta$. So the important part of the above definition is really the last clause.

3.6. **Lemma.** *Assume that $(P, \sqsubseteq)$ has the strong chain condition. Then $(P, \sqsubseteq)$ satisfies conditions C1 and C2.*

Proof. To show C1, consider any family $(x_\alpha : \alpha < \kappa)$. Let $\mathfrak{X}_\alpha := \{x_\alpha\}$. Since $(P, \sqsubseteq)$ has the strong chain condition, we can find $\alpha < \beta$ such that $\mathfrak{X}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{X}_\beta \leq (P, \sqsubseteq)$, but this means that $x_\alpha \sqsubseteq x_\beta$.

Finally we show C2. Let $f : (P, \sqsubseteq) \to (P, \sqsubseteq)$ be monotone and assume that 1.1(C2) does not hold. This means that for all small sets A there is some x such that $f(x) \neq x$ and $f(x) \notin A$. So we can find a sequence $(x_\alpha, y_\alpha : \alpha < \kappa)$ such that $\forall \alpha \, x_\alpha \neq f(x_\alpha) = y_\alpha$ and the sets $\{x_\alpha, y_\alpha\}$ are pairwise disjoint. Let $\mathfrak{X}_\alpha := \{x_\alpha, y_\alpha\} \leq (P, \sqsubseteq)$. Wlog we either have $x_\alpha \sqsubseteq y_\alpha$ for all α or for no α, similarly $y_\alpha \sqsubseteq x_\alpha$ for all α or for no α.

Now find α and β such that $\mathfrak{X}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{X}_\beta \leq \mathfrak{M}$. Now $\mathfrak{X}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{X}_\beta \models x_\alpha \sqsubseteq x_\beta$, but clearly $\mathfrak{X}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{X}_\beta \models y_\alpha \not\sqsubseteq y_\beta$, so f is not monotone, a contradiction. $\square$

4. Forcing

In this section we will carry out the forcing construction that will prove the theorem.

We start with a universe V_0 satisfying GCH. Let $\kappa = \omega_1$, and let $(M_\delta : \delta \le \kappa)$, $A^\delta_{\alpha\beta}, \ldots \Gamma^\delta_{\alpha\beta}$ be as in 2.4.

We define sequences $(\mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa)$, $(\underset{\sim}{\mathfrak{M}}_\alpha : \alpha < \kappa)$, $(\underset{\sim}{R}_\alpha : \alpha < \kappa)$ satisfying the following for all $\alpha < \kappa$:

(1) $\mathbb{P}_0 = \{\emptyset\}$, $\mathfrak{M}_0 = \emptyset$.
(2) $\underset{\sim}{R}_\alpha$ is a $\mathbb{P}_\alpha$-name of a subset of $M_\alpha \times M_\alpha$.
(3) In $V^{\mathbb{P}_\alpha}$, $F_{\alpha+1}$ is constructed from R_α as in 2.7–2.8.
(4) $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name, $\Vdash_{\mathbb{P}_\alpha} \underset{\sim}{\mathbb{Q}}_\alpha = \mathbb{Q}(\mathfrak{M}_\alpha, F_{\alpha+1})$.
(5) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$.
(6) $\mathbb{P}_{\alpha+1} \Vdash \mathfrak{M}_{\alpha+1}$ is the creature defined by the generic filter on $\underset{\sim}{\mathbb{Q}}_\alpha$, as in section 2.
(7) If α is a limit, then $\mathbb{P}_\alpha$ is the finite support limit of $(\mathbb{P}_\beta : \beta < \alpha)$, and $\mathfrak{M}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{M}_\beta$.

We let $\mathbb{P}_\kappa$ be the finite support limit of this iteration. Since all forcing notions involved satisfy the countable chain condition, we may (using the usual bookkeeping arguments) assume that:

(∗) Whenever $\underset{\sim}{R}$ is a $\mathbb{P}_\kappa$-name of a subset of $M_\kappa \times M_\kappa$ of size $< \kappa$, then there is some α such that $\underset{\sim}{R} = \underset{\sim}{R}_\alpha$.

4.1. **Lemma.** *Let $A \subseteq M_\delta$ be finite. Define a set $\bar{\mathbb{P}}_\delta(A)$ as the set of all $\mathfrak{p} \in \mathbb{P}_\delta$ satisfying*

$$\exists \mathfrak{M},\ \mathfrak{M} \text{ a finite creature}, A \subseteq M, \forall \alpha \in \mathrm{dom}(\mathfrak{p}) : \mathfrak{p}{\restriction}\alpha \Vdash \mathfrak{p}(\alpha) = \mathfrak{M}{\restriction}M_{\alpha+1}$$

Then $\bar{\mathbb{P}}_\delta(A)$ is dense in $\mathbb{P}_\delta$. In particular, $\bar{\mathbb{P}}_\delta := \bar{\mathbb{P}}_\delta(\emptyset)$ is dense in $\mathbb{P}_\alpha$.

Proof. By induction on δ. Limit steps are easy.

Let $\mathfrak{p} \in \mathbb{P}_{\alpha+1}$. By strengthening $\mathfrak{p}{\restriction}\alpha$ we may assume that there is a creature $\mathfrak{N}$ such that $\mathfrak{p}{\restriction}\alpha \Vdash \mathfrak{p}(\alpha) = \mathfrak{N}$. By 2.10(c) we may assume that $A \subseteq N$. By inductive assumption we may assume that there is a creature $\mathfrak{M}$ witnessing $\mathfrak{p}{\restriction}\alpha \in \bar{\mathbb{P}}_\alpha(N \cap M_\delta)$. Clearly $\mathfrak{M}$ and $\mathfrak{N}$ agree on $M \cap N$. Define $\mathfrak{M}' := \mathfrak{M} \oplus \mathfrak{N}$. So $\mathfrak{p}{\restriction}\alpha \Vdash \mathfrak{p}(\alpha) \le \mathfrak{M}'$. Define $\mathfrak{p}'$ by demanding $\mathfrak{p}'{\restriction}\alpha = p{\restriction}\alpha$, $\mathfrak{p}'(\alpha) = \mathfrak{M}'$. By 3.3, $\mathfrak{p}'{\restriction}\alpha \Vdash \mathfrak{p}'(\alpha) \in \mathbb{Q}_\delta$. Clearly $\mathfrak{p}' \ge \mathfrak{p}$, and $\mathfrak{p}' \in \mathbb{P}_{\alpha+1}$. □

In $V^{\mathbb{P}_\kappa}$, let $\mathfrak{M}_\kappa = \bigcup_{\alpha<\kappa} \mathfrak{M}_\alpha$. We claim that the structure $(\kappa, \sqsubseteq_{\mathfrak{M}_\kappa})$ satisfies the four conditions from 1.1. The argument in section 2 shows that we have C3 and C4. So, by 3.6 we only have to check that $\mathfrak{M}$ has the strong chain condition.

Let $\underset{\sim}{\mathfrak{X}} := (\underset{\sim}{\mathfrak{X}}_\alpha : \alpha < \kappa)$ be a sequence of names for finite creatures, and let $(\underset{\sim}{x}_\alpha : \alpha < \kappa)$ be forced to satisfy $\underset{\sim}{x}_\alpha \in X_\alpha$. Assume that $\mathfrak{p}$ is a condition forcing that $\underset{\sim}{\mathfrak{X}}$ witnesses the failure of 3.4. For each $\alpha < \kappa$ we find $\mathfrak{p}_\alpha \ge \mathfrak{p}$ which decides $\mathfrak{X}_\alpha$ and x_α. Let $e_\alpha = \mathrm{supp}(\mathfrak{p}_\alpha)$, $\delta_\alpha := \max(e_\alpha) + 1$, so $\mathfrak{p}_\alpha \in \mathbb{P}_{\delta_\alpha}$.

By 4.1 we may assume that there are finite creatures $\mathfrak{P}_\alpha$ such that $\forall \varepsilon \in e_\alpha\ p_\alpha{\restriction}\varepsilon \Vdash p(\varepsilon) = \mathfrak{P}_\alpha{\restriction}M_{\varepsilon+1}$.

We may also assume that $X_\alpha \subseteq P_\alpha$, and hence, (since $p_\alpha \Vdash \mathfrak{X}_\alpha \le \mathfrak{M}_\kappa$, $p_\alpha \Vdash \mathfrak{P}_\alpha \le \mathfrak{M}_\kappa$) $\mathfrak{X}_\alpha \le \mathfrak{P}_\alpha$.

We may assume that the creatures $\mathfrak{P}_\alpha$ form a Δ-system, say with heart $\mathfrak{P}^\Delta$ (so in particular $\mathfrak{P}^\Delta \le \mathfrak{P}_\alpha$). Moreover, we may assume that each $\mathfrak{P}_\alpha$ is a separated end extension of $\mathfrak{P}^\Delta$. Also, since the x_α are wlog all different, we may assume $x_\alpha \in P_\alpha \setminus P^\Delta$. Finally, we may assume that each x_α has the same type over $\mathfrak{P}^\Delta$.

Now pick any $\alpha < \beta$. We will find a condition $\mathfrak{q} \geq \mathfrak{p}_\alpha$, $\mathfrak{q} \geq \mathfrak{p}_\beta$ such that $\mathfrak{q} \Vdash \mathfrak{P}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{P}_\beta \leq \mathfrak{M}_\kappa$.

Let $\mathfrak{Q} = \mathfrak{P}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{P}_\beta$. First note that $\mathfrak{P}_\alpha \leq \mathfrak{Q}$, $\mathfrak{P}_\beta \leq \mathfrak{Q}$ by 3.2. Let ε^* be such that $x_\beta \in M_{\varepsilon^*+1} \setminus M_\varepsilon$.

Now note that

$$\mathfrak{Q} \restriction M_{\varepsilon+1} = \begin{cases} \mathfrak{P}_\alpha \restriction M_{\varepsilon+1} & \text{if } \varepsilon < \varepsilon^* \\ \mathfrak{P}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{P}_\beta \restriction M_{\varepsilon+1} & \text{if } \varepsilon \geq \varepsilon^* \end{cases}$$

We now define a condition $\mathfrak{q}$ with $\operatorname{supp}(\mathfrak{q}) = \operatorname{supp}(\mathfrak{p}_\alpha) \cup \operatorname{supp}(\mathfrak{p}_\beta)$, by stipulating $\mathfrak{q}(\varepsilon) = \mathfrak{Q} \restriction M_{\varepsilon+1}$.

By 3.3 we know that $\mathfrak{q} \restriction \varepsilon \Vdash \mathfrak{Q} \restriction M_{\varepsilon+1} \in \mathbb{Q}_\varepsilon$, so by induction it is clear that $\mathfrak{q}$ is indeed a condition. Clearly $\mathfrak{q} \Vdash \mathfrak{P}_\alpha \oplus_{x_\alpha, x_\beta} \mathfrak{P}_\beta \leq \mathfrak{M}_\kappa$.

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