The f-Factor Problem for Graphs and the Hereditary Property*

Frank Niedermeyer, Bonn
Saharon Shelah, Jerusalem
Karsten Steffens, Hannover

Abstract

If P is a hereditary property then we show that, for the existence of a perfect f-factor, P is a sufficient condition for countable graphs and yields a sufficient condition for graphs of size $\aleph_1$. Further we give two examples of a hereditary property which is even necessary for the existence of a perfect f-factor. We also discuss the $\aleph_2$-case.

We consider graphs $G = (V, E)$, where $V = V(G)$ is a nonempty set of vertices and $E = E(G) \subseteq \{ e \subseteq V\colon |e| = 2\}$ is the set of edges of G. If x is a vertex of G and $F \subseteq E$, then we denote by $d_F(x)$ the cardinal $|\{e \in F\colon x \in e\}|$. $d_F(x)$ is called the **degree of x with respect to F** and $d_E(x)$ the **degree of x**. ON denotes the class of ordinals, CN the class of cardinals. Greek letters $\alpha, \beta, \gamma, \ldots$ always denote ordinals, whereas the middle letters $\kappa, \lambda, \mu, \nu, \ldots$ are reserved for infinite cardinals.

Let $G = (V, E)$ be a graph, $f\colon V \to CN$ be a function and $F \subseteq E$. F is said to be an **f-factor** of G if $d_F(x) \leq f(x)$ for all $x \in V$. We call an f-factor F of G **perfect** if $d_F(x) = f(x)$ for all $x \in V$. For $\kappa \in CN$ we denote $f^{-1}(\kappa) := \{x \in V : f(x) = \kappa\}$.

Let $\mathcal{C}$ be the class of all ordered pairs (G, f), such that $G = (V, E)$ is a graph, $f\colon V \to CN$ is a function, and $f(x) \leq d_E(x)$ for all $x \in V$.

This paper discusses the problem to find a necessary and sufficient condition for the existence of a perfect f-factor of a graph. In [5], Tutte published a criterion for finite graphs, and in [4] Niedermeyer solved the problem for countable graphs and functions $f : V \longrightarrow \omega$. We present a solution for graphs of size $\aleph_0$ and functions $f : V \longrightarrow \omega \cup \{\aleph_0\}$, a solution for graphs of size $\aleph_1$, and discuss the $\aleph_2$-case.

If $H \subseteq E$, then denote by $G - H$ the graph $(V, E \setminus H)$, and if $e \in E$, then let $G - e$ be the graph $G - \{e\}$. If $x, y \in V$, denote by $f_{x,y}\colon V \to CN$ the function defined by

$$f_{x,y}(v) := \begin{cases} f(v) - 1 & \text{if } v \in \{x, y\} \text{ and } 1 \leq f(v) < \aleph_0 \\ f(v) & \text{else} \end{cases}.$$

Now let P be a formula with two free variables. $P(G, f)$ means that $(G, f) \in \mathcal{C}$ and (G, f) has the property P. P is said to be **hereditary** if for every (G, f) with $P(G, f)$, for every vertex $x \in V(G)$ with $f(x) > 0$ there exists a vertex $y \in V(G)$ with $f(y) > 0$, $\{x, y\} \in E(G)$, and $P(G - \{x, y\}, f_{x,y})$.

*This paper was supported by the Volkswagen Stiftung

Remark Let P be a hereditary property, let $(G,f) \in \mathcal{C}$ such that $P(G,f)$, and let $W \subseteq V(G)$ be finite. Then there exists a finite f-factor F of G such that $P(G-F, f-d_F)$, $d_F(x) = f(x)$ for every $x \in W$ with $f(x) < \aleph_0$, and $d_F(x) > 0$ for every $x \in W$ with $f(x) \geq \aleph_0$.

Example 1 Let $P_1(G,f)$ be the property "G possesses a perfect f-factor". Obviously P_1 is a hereditary property.

Definition Let $(G,f) \in \mathcal{C}$. By recursion on $\alpha \in ON$ we define the property that (G,f) is an α**-obstruction**. Let $G = (V,E)$.
If there is an $x \in V$ with $f(x) > 0$ such that $f(y) = 0$ for all $y \in V$ with $\{x,y\} \in E$, then (G,f) is a 0**-obstruction**.
If there is a vertex $x \in V$ such that $f(x) > 0$ and

(i) for every $y \in V$ with $\{x,y\} \in E$ and $f(y) > 0$ there is an ordinal β_y such that $(G - \{x,y\}, f_{x,y})$ is a β_y-obstruction and

(ii) $\alpha = \sup\{\beta_y + 1 \colon \{x,y\} \in E, f(y) > 0\}$,

then (G,f) is an α**-obstruction**.

Example 2 Let $P_2(G,f)$ be the property "(G,f) is not an α-obstruction for every $\alpha \in ON$". Then we can prove the following

Lemma 1

(i) P_2 is a hereditary property.

(ii) If P is a hereditary property, then $P_2(G,f)$ is necessary for $P(G,f)$. Therefore P_2 is a necessary condition for the existence of a perfect f-factor.

Proof

(i) Assume $P_2(G,f)$, that means that for all $\alpha \in ON$, (G,f) is not an α-obstruction. Let $G = (V,E)$ and $x \in V$ with $f(x) > 0$. To get a contradiction let us assume that, for each $y \in V$ with $\{x,y\} \in E$ and $f(y) > 0$, there is an ordinal β_y such that $(G - \{x,y\}, f_{x,y})$ is a β_y-obstruction. If $\alpha = \sup\{\beta_y + 1 \colon \{x,y\} \in E, f(y) > 0\}$, then (G,f) is an α-obstruction which contradicts our assumption.

(ii) By induction on $\alpha \in ON$ we prove for any $(G,f) \in \mathcal{C}$ with $P(G,f)$ that (G,f) is not an α-obstruction.

Since P is heriditary, (G,f) is obviously not a 0-obstruction.

Now let $\alpha > 0$. Assume that (G,f) is an α-obstruction. Let $G = (V,E)$. By definition, there is a vertex $x \in V$ with $f(x) > 0$ such that for each $y \in V$ with $f(y) > 0$ and $\{x,y\} \in E$ there is an ordinal $\beta_y < \alpha$ such that $(G - \{x,y\}, f_{x,y})$ is a β_y-obstruction. On the other hand, since $P(G,f)$, P is hereditary, and $f(x) > 0$, there is an edge $\{x,y\} \in E$ such that $P(G - \{x,y\}, f_{x,y})$. By inductive hypothesis $(G - \{x,y\}, f_{x,y})$ is *not* a β_y-obstruction. This contradiction proves (ii).

For a hereditary property P, it must not be true that $P_2(G,f)$ is sufficient for $P(G,f)$. This is demonstrated by the following example.

Example 3 Let $P_3(G,f)$ be the property "G possesses a perfect f-factor without cycles".

P_3 also shows that not every hereditary property is a necessary condition for the existence of a perfect f-factor.

Definition Let $(G,f) \in \mathcal{C}$. For $0 < k \leq \omega$ we call a sequence $T = (v_i)_{0 \leq i < k}$ of vertices of G a **trail** if $\{v_{i-1}, v_i\} \in E(G)$ for $0 < i < k$ and $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ for $i \neq j$. For any f-factor F, a trail $T = (v_i)_{0 \leq i < k}$ is called F**-augmenting** if

(i) $k > 1$

(ii) $\{v_{i-1}, v_i\} \in F$ iff $i > 0$ is even

(iii) $d_F(v_0) < f(v_0)$

(iv) $k = \omega$
 or
 $k < \omega$ is even, $v_0 \neq v_{k-1}$ and $d_F(v_{k-1}) < f(v_{k-1})$
 or
 $k < \omega$ is even, $v_0 = v_{k-1}$ and $d_F(v_{k-1}) + 1 < f(v_{k-1})$

Example 4 Let $P_4(G,f)$ be the property "for every f-factor F of G and every vertex $x \in V(G)$ with $d_F(x) < f(x)$ there exists an F-augmenting trail starting at x". Further let $P_4'(G,f)$ be the property "$P_4(G,f)$ and $ran(f) \subseteq \omega$".

Lemma 2 If $(G,f) \in \mathcal{C}$ and G possesses a perfect f-factor, then $P_4(G,f)$.

Proof For the convenience of the reader, we present the easy proof. Let $G = (V,E)$, let F be an f-factor of G and H be a perfect f-factor of G. For all $x \in V$ with $d_F(x) < f(x)$, we construct by induction an F-augmenting trail starting at x. Let $v_0 = x$. Since $d_F(v_0) < f(v_0) = d_H(v_0)$ there is an edge $\{v_0, y\} \in H \setminus F$. Let $v_1 = y$. Let the trail $T = (v_j)_{0 \leq j \leq i}$ be defined such that

(1) $\{v_{j-1}, v_j\} \in F \setminus H$ iff $j > 0$ is even.

(2) $\{v_{j-1}, v_j\} \in H \setminus F$ iff j is odd.

If i is odd, $v_i \neq v_0$, and $d_F(v_i) < f(v_i)$, let $k = i+1$.
If i is odd, $v_i = v_0$, and $d_F(v_i) + 1 < f(v_i)$, let again $k = i+1$.
If i is odd and $v_i \neq v_0$, $d_F(v_i) = f(v_i)$ or $v_i = v_0$, $d_F(v_i) + 1 \geq f(v_i)$, then there is an edge $\{v_i, y\} \in F \setminus H$ which is not an edge of T. Let $v_{i+1} = y$.
Finally, if i is even, there is an edge $\{v_i, y\} \in H \setminus F$ which is not an edge of the trail T. Let $v_{i+1} = y$.

Much more difficult is the proof of Lemma 3 which is Corollary 4 of [4].

Lemma 3 P_4' is a hereditary property.

It is not true that every hereditary property P is a sufficient condition for the existence of a perfect f-factor of a given graph. This demonstrates the property P_4, applied to the complete bipartite graph $K_{\aleph_0,\aleph_1}$ and the function $f \equiv 1$. But we have the following

Theorem 1 Let $(G, f) \in \mathcal{C}$ and $|V(G)| = \aleph_0$. If P is a hereditary property and $P(G, f)$ then G possesses a perfect f-factor.

Proof Let $v_0, v_1, v_2, \dots$ be an enumeration of the vertices of G such that, for every $x \in V$ with $f(x) = \aleph_0$, the set $\{i < \omega \colon x = v_i\}$ is infinite. Since $P(G, f)$ and P is hereditary, one can define recursively finite f-factors $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that $(G - F_k, f - d_{F_k})$ fulfills property P and the following is true: If $f(v_0) = \aleph_0$, then $F_0 = \{\{x, v_0\}\}$, if $f(v_k) = \aleph_0$, $k > 0$, then $F_k \setminus F_{k-1} = \big\{\{x, v_k\}\big\}$ for some $x \in V$, and if $f(v_k) < \aleph_0$, then $d_{F_k}(v_k) = f(v_k)$. By construction, $F := \bigcup\{F_k \colon k < \omega\}$ is a perfect f-factor.

Corollary 1 Let $(G, f) \in \mathcal{C}$ and $|V(G)| = \aleph_0$.

(1) G has a perfect f-factor iff $P_2(G, f)$.

(2) If $ran(f) \subseteq \omega$, then G has a perfect f-factor iff $P_4(G, f)$.

Tutte's condition ([3], [5]) for the existence of a perfect 1-factor for finite graphs is necessary but not sufficient for countable graphs. Thus Theorem 1 shows that not every necessary condition for the existence of a perfect f-factor is a hereditary property. The property "G has a perfect f-factor with cycles" tells us that a sufficient condition for the existence of a perfect f-factor for G is not necessarily hereditary.

Definition Let $(G, f) \in \mathcal{C}$, $G = (V, E)$, and $|V| = \kappa^+$ for some infinite cardinal κ. Let $(A_\alpha)_{\alpha<\kappa^+}$ be an increasing continuous sequence of subsets of V such that $|A_\alpha| < \kappa^+$ for all $\alpha < \kappa^+$ and $V = \bigcup\{A_\alpha \colon \alpha < \kappa^+\}$. For $\alpha < \kappa^+$ we define

$$\begin{aligned} V_\alpha &:= (V \setminus A_\alpha) \cup f^{-1}(\kappa^+) \\ E_\alpha &:= \big\{\{x, y\} \in E \colon x \in V_\alpha,\, y \in V \setminus A_\alpha\big\} \\ G_\alpha &:= (V_\alpha, E_\alpha) \\ f_\alpha &:= f \restriction V_\alpha \end{aligned}$$

For any property P, $(A_\alpha)_{\alpha<\kappa^+}$ is said to be a P-**destruction** of (G, f) if

$$S = \{\alpha < \kappa^+ \colon (G_\alpha, f_\alpha) \text{ does not fulfill } P\}$$

is stationary in κ^+. (G, f) is called P-**destructed** if there is a P-destruction of (G, f).

Lemma 4 *(Transfer Lemma)* Let $P(G, f)$ be a necessary condition for the existence of a perfect f-factor of a graph G. If $(G, f) \in \mathcal{C}$, $|V(G)| = \kappa^+$ for an infinite cardinal κ, and if G possesses a perfect f-factor, then (G, f) is not P-destructed.

Proof Let F be a perfect f-factor of G and assume that there is a P-destruction $(A_\alpha)_{\alpha<\kappa^+}$ of (G,f). Define $V_\alpha, E_\alpha, G_\alpha, f_\alpha, S$ as above and let $\alpha \in S$. (G_α, f_α) does not fulfill P, and by the hypothesis of the Lemma, G_α has not a perfect f_α-factor. In particular $F_\alpha := F \cap E_\alpha$ is not a perfect f_α-factor of G_α. Therefore there is a vertex $x_\alpha \in V_\alpha$ such that $d_{F_\alpha}(x_\alpha) < f_\alpha(x_\alpha) = f(x_\alpha)$. Since F is a perfect f-factor, there exists, for some vertex y_α, an edge $\{x_\alpha, y_\alpha\} \in F \setminus F_\alpha$. Using the fact $|A_\alpha| < \kappa^+$ we know that $d_{F_\alpha}(x) = d_F(x) = \kappa^+ = f(x)$ for any $x \in f^{-1}(\kappa^+)$. So $x_\alpha \in V_\alpha \setminus f^{-1}(\kappa^+)$ and $y_\alpha \in A_\alpha \setminus f^{-1}(\kappa^+)$. .

If $\alpha \in S$ is a limit ordinal, let $\beta(\alpha) < \alpha$ be an ordinal with $y_\alpha \in A_{\beta(\alpha)}$. By Fodor's Theorem (cf. [1] or [2], Theorem 1.8.8), there is an ordinal $\gamma < \kappa^+$ such that

$$|\{\alpha \in S \colon \alpha \text{ limit ordinal}, \beta(\alpha) = \gamma\}| = \kappa^+.$$

Since $|A_\gamma| < \kappa^+$, there is a vertex $y^* \in A_\gamma$ such that

$$|\{\alpha \in S \colon \alpha \text{ limit ordinal}, y_\alpha = y^*\}| = \kappa^+.$$

If $x \in A_{\alpha_0} \setminus f^{-1}(\kappa^+)$ for some $\alpha_0 < \kappa^+$, then $x \notin V_\alpha$ for all $\alpha > \alpha_0$ and thus

$$|\{\alpha \in S \colon x_\alpha = x\}| < \kappa^+.$$

It follows that $f(y^*) = d_F(y^*) = \kappa^+$, so $y^* \in f^{-1}(\kappa^+)$. On the other hand $y^* \in A_\alpha \setminus f^{-1}(\kappa^+)$ for every ordinal α with $y^* = y_\alpha$. This contradiction proves the lemma.

Theorem 2 Let $(G,f) \in \mathcal{C}$ and $|V(G)| = \aleph_1$. If P is a hereditary property such that $P(G,f)$ and if (G,f) is not P-destructed then G possesses a perfect f-factor.

Proof Let $(A_\alpha)_{\alpha<\omega_1}$ be an increasing continuous sequence of countable subsets of V such that $V = \bigcup_{\alpha<\omega_1} A_\alpha$. Define $V_\alpha, E_\alpha, G_\alpha, f_\alpha$ as above. Since $(A_\alpha)_{\alpha<\omega_1}$ is not a P-destruction, there is a closed unbounded set $K \subseteq \omega_1$ such that (G_α, f_α) fulfills P for every $\alpha \in K$. We can assume w. l. o. g. that $K = \omega_1$, because otherwise we could consider the sequence $(A_\alpha)_{\alpha\in K}$ instead of $(A_\alpha)_{\alpha<\omega_1}$. Since (G,f) fulfills P we can further assume that $A_0 = \emptyset$.

To obtain a perfect f-factor of G, we now construct an increasing continuous function $i \colon \omega_1 \to \omega_1$ and an increasing sequence $(F_\varepsilon)_{\varepsilon<\omega_1}$ of f-factors of G with the following properties:

(i) $\bigcup F_\varepsilon \subseteq A_{i(\varepsilon)}$

(ii) $\forall x \in A_{i(\varepsilon)} \big(f(x) \le \aleph_0 \Rightarrow d_{F_\varepsilon}(x) = f(x)\big)$

(iii) $\forall x \in A_{i(\varepsilon)} \big(f(x) = \aleph_1 \Rightarrow d_{F_{\varepsilon+1}\setminus F_\varepsilon}(x) = \aleph_0\big)$

Then $F := \bigcup_{\varepsilon<\omega_1} F_\varepsilon$ obviously is a perfect f-factor of G.

The function i and the sequence $(F_\varepsilon)_{\varepsilon<\omega_1}$ will be defined by transfinite recursion. Let $i(0) := 0$ and $F_0 := \emptyset$. Now let $\varepsilon > 0$ and let us assume that, for each $\delta < \varepsilon$, $i(\delta)$ and F_δ are already defined. If ε is a limit ordinal, let $i(\varepsilon) := \bigcup_{\delta<\varepsilon} i(\delta)$ and $F_\varepsilon := \bigcup_{\delta<\varepsilon} F_\delta$.

Now let $\varepsilon = \delta+1$. By induction on m we define an increasing sequence $(H_m)_{m<\omega}$ of finite $f_{i(\delta)}$-factors of $G_{i(\delta)}$, an increasing function $\varrho \colon \omega \longrightarrow \omega_1$, and, for any $n \ge m$, vertices $x_{m,n} \in V_{i(\delta)}$ such that for every m

(a) $\{x_{m,n} : n \geq m\} = A_{\varrho(m+1)} \setminus (A_{\varrho(m)} \setminus f^{-1}(\aleph_1))$

(b) $\bigcup H_m \subseteq A_{\varrho(m)}$

(c) $d_{H_{m+1}}(x_{k,m}) = f_{i(\delta)}(x_{k,m})$ for all $k \leq m$ with $f_{i(\delta)}(x_{k,m}) < \aleph_0$

(d) $d_{H_{m+1} \setminus H_m}(x_{k,m}) > 0$ for all $k \leq m$ with $f_{i(\delta)}(x_{k,m}) \geq \aleph_0$

(e) $P(G_{i(\delta)} - H_m, f_{i(\delta)} - d_{H_m})$.

Then let $F_\varepsilon := F_\delta \cup \bigcup\{H_m : m < \omega\}$ and $i(\varepsilon) := \bigcup\{\varrho(m) : m < \omega\}$. By construction, (i), (ii), (iii) are fulfilled.

m = 0: Let $\varrho(0) := i(\delta)$, $H_0 := \emptyset$.

m = m + 1: Now suppose that for $m < \omega$ the ordinal $\varrho(m)$, the finite $f_{i(\delta)}$-factor H_m of $G_{i(\delta)}$, and, for all $k < m$ and $n \geq k$, the vertices $x_{k,n} \in V_{i(\delta)}$ are already defined such that (a) - (e) are fulfilled.

The set $W_m := \{x_{k,n} : k \leq n < m\}$ is finite. Since P is hereditary, there exists a finite $f_{i(\delta)}$-factor $H_{m+1} \supseteq H_m$ of $G_{i(\delta)}$ such that $P(G_{i(\delta)} - H_{m+1}, f_{i(\delta)} - d_{H_{m+1}})$ and $d_{H_{m+1}}(x) = f_{i(\delta)}(x)$ whenever $x \in W_m$ and $f_{i(\delta)}(x) < \aleph_0$, or $d_{H_{m+1} \setminus H_m}(x) > 0$ whenever $x \in W_m$ and $f_{i(\delta)}(x) \geq \aleph_0$.

Let $\varrho(m+1) > \varrho(m)$ be the least ordinal such that $\bigcup H_{m+1} \subseteq A_{\varrho(m+1)}$. For $n \geq m$ choose $x_{m,n}$ with $\{x_{m,m}, x_{m,m+1}, x_{m,m+2}, \ldots\} = A_{\varrho(m+1)} \setminus (A_{\varrho(m)} \setminus f^{-1}(\aleph_1))$.

Corollary 2 Let $(G, f) \in \mathcal{C}$ and $|V(G)| = \aleph_1$.

(i) G possesses a perfect f-factor if and only if (G, f) is not P_2-destructed.

(ii) If $ran(f) \subseteq \omega$ then G possesses a perfect f-factor if and only if (G, f) is not P_4-destructed.

To handle the cases of higher cardinality, we introduce the notion of a κ-perfect f-factor.

Definition Let $(G, f) \in \mathcal{C}$ and let κ be an infinite cardinal. An f-factor F of G is said to be **κ-perfect** if $d_F(x) = f(x)$ for all vertices x with $f(x) \leq \kappa$ and $d_F(x) > 0$ for all vertices x with $f(x) > \kappa$.

Theorem 3 Let κ be an infinite cardinal, $(G, f) \in \mathcal{C}$, and $|V(G)| = \kappa^+$. G possesses a perfect f-factor if and only if there is an increasing continuous sequence $(A_\alpha)_{\alpha < \kappa^+}$ of subsets of $V(G)$ such that

(i) $A_0 = \emptyset$, $V(G) = \bigcup\{A_\alpha : \alpha < \kappa^+\}$,

(ii) $|A_{\alpha+1} \setminus A_\alpha| = \kappa$ for all $\alpha < \kappa^+$,

(iii) for all $\alpha < \kappa^+$ there exists an κ-perfect g_α-factor of $(B_\alpha, \{\{x, y\} \in E : x \in B_\alpha, y \in A_{\alpha+1} \setminus A_\alpha\})$, where $B_\alpha = (A_{\alpha+1} \setminus (A_\alpha \setminus f^{-1}(\kappa^+)))$ and $g_\alpha := f \restriction B_\alpha$.

Proof Let $(A_\alpha)_{\alpha<\kappa^+}$ be an increasing continuous sequence of subsets of V and, for $\alpha < \kappa^+$, let F_α be a κ-perfect g_α-factor with the properties (i), (ii), (iii). Then $F_{\alpha_1} \cap F_{\alpha_2} = \emptyset$ if $\alpha_1 \neq \alpha_2$. Let $F := \bigcup\{F_\alpha\colon \alpha < \kappa^+\}$. We will show that F is a perfect f-factor of G. Let $x \in V$ and let α be the smallest ordinal such that $x \in A_{\alpha+1}$. If $f(x) \leq \kappa$ then $d_F(x) = d_{F_\alpha}(x) = f(x)$. If on the other hand $f(x) > \kappa$, we have $d_{F_\beta}(x) > 0$ for all $\beta \geq \alpha$ since F_β is κ-perfect. Thus $d_F(x) = \kappa^+$.

To prove the converse, let F be a perfect f-factor of G and $A_0 := \emptyset$. Let $(P_\delta\colon \delta < \kappa^+)$ be a partition of V such that $|P_\delta| = \kappa$ for all $\delta < \kappa^+$. Now assume that $A_\delta \subseteq V$ is defined for all $\delta < \alpha$. If α is a limit ordinal, then let $A_\alpha = \bigcup\{A_\delta\colon \delta < \alpha\}$. If $\alpha = \delta+1$, we define by induction an increasing sequence $(C_n)_{n<\omega}$ of subsets of V. Let $C_0 \subseteq V$ such that $A_\delta \cup P_\delta \subseteq C_0$ and $|C_0 \setminus A_\delta| = \kappa$. If C_n is defined let C_{n+1} be a "κ-neighborhood" of C_n: If $x \in C_n$ and $f(x) \leq \kappa$ let $N(x) = \{y \in V\colon \{y, x\} \in F\}$, and if $f(x) = \kappa^+$ choose $y_x \in V \setminus C_n$ with $\{y_x, x\} \in F$ and let $N(x) = \{y_x\}$. Then let $C_{n+1} = C_n \cup \bigcup\{N(x)\colon x \in C_n\}$ and $A_\alpha := \bigcup\{C_n\colon n < \omega\}$. By construction, $(A_\alpha)_{\alpha<\kappa^+}$ is an increasing continuous sequence of subsets of V with the properties (i), (ii), (iii).

Remark If $\kappa^+ = \aleph_2$, $g_\alpha := f \restriction V_\alpha \cap A_{\alpha+1}$ and $X_\alpha := A_{\alpha+1} \cap f^{-1}(\aleph_2)$, then there is an $\aleph_1$-perfect g_α-factor of $(V_\alpha \cap A_{\alpha+1}, \{\{x,y\} \in E\colon x \in V_\alpha \cap A_{\alpha+1}, y \in A_{\alpha+1} \setminus A_\alpha\})$ if and only if there exists a function $h_\alpha\colon A_{\alpha+1} \cap f^{-1}(\aleph_2) \to \omega \cup \{\aleph_0, \aleph_1\}$ such that there is a perfect $(g_\alpha \setminus (g_\alpha{\restriction}X_\alpha)) \cup h_\alpha$-factor of $(V_\alpha \cap A_{\alpha+1}, \{\{x,y\} \in E\colon x \in V_\alpha \cap A_{\alpha+1}, y \in A_{\alpha+1} \setminus A_\alpha\})$.

Corollary 3 Let $(G, f) \in \mathcal{C}$ and $|V(G)| = \aleph_2$. G possesses a perfect f-factor if and only if there is an increasing continuous sequence $(A_\alpha)_{\alpha<\omega_2}$ of subsets of $V(G)$, such that

(i) $A_0 = \emptyset$, $V(G) = \bigcup_{\alpha<\omega_2} A_\alpha$.

(ii) $|A_{\alpha+1} \setminus A_\alpha| = \aleph_1$ for all $\alpha < \omega_2$.

(iii) For each $\alpha < \omega_2$ there is a function $h_\alpha\colon A_{\alpha+1} \cap f^{-1}(\aleph_2) \to \omega \cup \{\aleph_0, \aleph_1\}$ such that the graph

$$(A_{\alpha+1} \setminus (A_\alpha \setminus f^{-1}(\aleph_2)), \{\{x,y\} \in E\colon x \in A_{\alpha+1} \setminus (A_\alpha \setminus f^{-1}(\aleph_2)), y \in A_{\alpha+1} \setminus A_\alpha\})$$

together with

$$\big(f \restriction A_{\alpha+1} \setminus (A_\alpha \setminus f^{-1}(\aleph_2)\,) \setminus f \restriction (A_{\alpha+1} \cap f^{-1}(\aleph_2))\,\big) \cup h_\alpha$$

is not P_2-destructed.

References

[1] G. Fodor, *Eine Bemerkung zur Theorie der regressiven Funktionen*, Acta Sci. Math. (Szeged) 17 (1956), 139-142.

[2] M. Holz, K. Steffens, E. Weitz, *Introduction to Cardinal Arithmetic*, Birkhäuser, Advanced Texts, Basel, Boston, Berlin 1999.

[3] L. Lovász, M. D. Plummer, *Matching Theory*, Annals of Discrete Mathematics, 29, North-Holland 1986.

[4] F. Niedermeyer, *f-optimal factors of infinite graphs*, Discrete Mathematics 95 (1991), 231-254, North Holland.

[5] W. T. Tutte, *The Factors of Graphs*, Canad. J. Math. 4 (1952), 314-328.