A ZFC Dowker space in $\aleph_{\omega+1}$: an application of pcf theory to topology

Menachem Kojman
Department Of Mathematics
Carnegie-Mellon University

Saharon Shelah*
Institute of Mathematics
The Hebrew University of Jerusalem
and
Department of Mathematics
Rutgers University

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Abstract

The existence of an $\aleph_{\omega+1}$-Dowker space is proved in ZFC using pcf theory.

1 Introduction

A Dowker space is a normal Hausdorff topological space whose product with the unit interval is not normal. The problem of existence of such spaces was raised by Dowker in 1951. Dowker characterized Dowker spaces as normal Hausdorff and not countably paracompact (see below).

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Exactly two Dowker spaces were constructed in ZFC so far. The existence of a Dowker space in ZFC was first proved by M. E. Rudin in 1971 [5], and her space was the only known Dowker space in ZFC for over two decades. Rudin's space is a subspace of $\prod_{n\geq 1}(\aleph_n + 1)$ and has cardinality $\aleph_\omega^{\aleph_0}$. The problem of finding a Dowker space of smaller cardinality in ZFC was referred to as the "small dowker space problem".

Z. T. Balogh constructed recently [1] a dowker space in ZFC whose cardinality is $2^{\aleph_0}$.

While both Rudin's and Balogh's spaces are constructed in ZFC, their respective cardinalities are not decided in ZFC, as is well known by the independence results of Cohen and Easton: both $2^{\aleph_0}$ and $\aleph_\omega^{\aleph_0}$ have no bound in ZFC, (and may be equal to each other).

The problem of which is the first $\aleph_\alpha$ in which ZFC proves the existence of a Dowker space remains thus unanswered by Rudin's and Balogh's results.

In this paper we prove that there is a Dowker space of cardinality $\aleph_{\omega+1}$. A non-exponential bound is thus provided for the cardinality of the smallest ZFC Dowker space. We do this by exhibiting a Dowker subspace of Rudin's space of that cardinality. Our construction avoids the exponent which appears in the cardinality of Rudin's space by working with only a fraction of $\aleph_\omega^{\aleph_0}$. It remains open whether $\aleph_{\omega+1}$ is the *first* cardinal at which there is a ZFC Dowker space.

We shall describe shortly the cardinal arithmetic developments which enable this result. The next three paragraphs are not necessary for understanding the proof.

In the last decade there has been a considerable advance in understanding of the infinite exponents of singular cardinals, in particular the exponent $\aleph_\omega^{\aleph_0}$. This exponent is the product of two factors: $2^{\aleph_0} \times \operatorname{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$. The second factor, the cofinality of the partial ordering of inclusion over all countable subsets of $\aleph_\omega$, is the least number of countable subsets of $\aleph_\omega$ needed to cover every countable subset of $\aleph_\omega$; the first factor is the number of subsets of a single countable set. Since $\aleph_\omega^{\aleph_0}$ is the number of countable subsets of $\aleph_\omega$, the equality $\aleph_\omega = 2^{\aleph_0} \times \operatorname{cf}\langle[\aleph_\omega]^{\aleph_0}\rangle$ is clear.

While for $2^{\aleph_0}$ it is consistent with ZFC to equal any cardinal of uncountable cofinality, the second author's work on Cardinal Arithmetic provides a ZFC bound of $\aleph_{\omega_4}$ on the factor $\operatorname{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$.

This is done by approximating $\operatorname{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$ by an interval of regular cardinals, whose first element is $\aleph_{\omega+1}$ and whose last element is $\operatorname{cf}\langle[\aleph_\omega]^{\aleph_0}, \subseteq\rangle$, and so that every regular cardinal λ in this interval is the *true cofinality* of a reduced product $\prod B_\lambda / J_{<\lambda}$ of a set $B_\lambda \subseteq \{\aleph_n : n < \omega\}$ modulo an ideal $J_{<\lambda}$ over ω. The theory of reduced products of small sets of regular cardinals, known now as *pcf theory*[1], is used to put a bound of ω_4 on the length of this interval.

Back to topology now, it turns out that the pcf approximations to $\aleph_\omega^{\aleph_0}$ are concrete enough to "commute" with Rudin's construction of a Dowker space. Rudin define a topology on a subspace of the functions space $\prod_{n>1}(\aleph_n + 1)$. What is gotten by restricting Rudin's definition to the first approximation of $\aleph_\omega^{\aleph_0}$ is a *closed* and *cofinal* Dowker subspace X of the Rudin space X^R of cardinality $\aleph_{\omega+1}$. The fact that X is Dowker follows readily from the closure and cofinality of X in X^R, and Rudin's proof that X^R is Dowker. The proof presented here is, however, self contained and does not use the fact X^R is Dowker, mostly because proving that X is collectionwise normal is essentially the same as showing that X is closed in X^R. The reader who is fluent with the proof in [5] can be content with reading Case 2.2 in the proof of collectionwise normality and Claim 20 below.

Hardly any background is needed to state the pcf theorem we are using here. However, an interested reader can find presentations of pcf theory in either [2], the second author's [6] or the first author's [4]. The pcf theorem used here is covered in detail in each of those three sources.

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2 Notation and pcf

In this section we present a few simple definitions needed to state the pcf theorem used in proving the existence of an $\aleph_{\omega+1}$-Dowker space.

Suppose $B \subseteq \omega$ is a subset of the natural numbers.

Definition 1. *1.* $\prod_{n\in B} \aleph_n = \{f : \mathit{dom}\, f = B \wedge f(n) < \aleph_n \text{ for } n \in B\}$

2. $\prod_{n\in B}(\aleph_n + 1) = \{f : \mathit{dom}\, f = B \wedge f(n) \leq \aleph_n \text{ for } n \in B\}$

[1] pcf means *possible cofinalities*

3. *for* $f, g \in \prod_{n\in B}(\aleph_n + 1)$ *let:*
   (a) $f < g$ *iff* $\forall n \in B\,[f(n) < g(n)]$
   (b) $f \leq g$ *iff* $\forall n \in B\,[f(n) \leq g(n)]$
   (c) $f \leq^* g$ *iff* $\{n : f(n) > g(n)\}$ *is finite*
   (d) $f <^* g$ *iff* $\{n : f(n) \geq g(n)\}$ *is finite*
   (e) $f =^* g$ *iff* $\{n : f(n) \neq g(n)\}$ *is finite*
4. *A sequence* $\langle f_\alpha : \alpha < \lambda\rangle$ *of functions in* $\prod_{n\in B}\aleph_n$ *is* increasing in $<$ $(\leq, <^*, \leq^*)$ *iff* $\alpha < \beta < \lambda \Rightarrow f_\alpha < f_\beta$ ($f_\alpha \leq f_\beta$, $f_\alpha <^* f_\beta$, $f_\alpha \leq^* f_\beta$)
5. $g \in \prod_{n\in B}(\aleph_n + 1)$ *is an* upper bound *of* $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B}\aleph_n$ *if and only if* $f_\alpha \leq^* g$ *for all* $\alpha < \delta$
6. $g \in \prod_{n\in B}(\aleph_n + 1)$ *is a* least upper bound *of* $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B}\aleph_n$ *if and only if* g *is an upper bound of* $\{f_\alpha : \alpha < \delta\} \subseteq \prod_{n\in B}\aleph_n$ *and if* g' *is an upper bound of* $\{f_\alpha : \alpha < \delta\}$ *then* $g \leq^* g$

Theorem 1. *(Shelah) There is a set* $B = B_{\aleph_{\omega+1}} \subseteq \omega$ *and a sequence* $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ *of functions in* $\prod_{n\in B}\aleph_n$ *such that:*

- $\overline{f}$ *is increasing in* $<^*$
- $\overline{f}$ *is cofinal: for every* $f \in \prod_{n\in B}\aleph_n$ *there is* $\alpha < \aleph_{\omega+1}$ *so that* $f <^* f_\alpha$

A sequence as in the theorem above will be referred to as an "$\aleph_{\omega+1}$-scale".

By Theorem 1 we can find $B \subseteq \omega$ and an $\aleph_{\omega+1}$-scale $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n\in B}\aleph_n$. The set B is clearly infinite. Removing from every $g_\alpha \in \overline{g}$ a fixed finite set of coordinates does not matter, so we assume without loss of generality that $0, 1 \notin B$. For notational simplicity we pretend that $B = \omega - \{0, 1\}$; if this is not the case, we need to replace $\aleph_n$ in what follows by the n-th element of B. We sum up our assumptions in the following:

Claim 2. *We can assume without loss of generality that there is an* $\aleph_{\omega+1}$*-scale* $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ *in* $\prod_{n>1}\aleph_n$.

Claim 3. *There is an* $\aleph_{\omega+1}$*-scale* $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ *in* $\prod_{n>1}\aleph_n$ *so that for every* $\delta < \aleph_{\omega+1}$*, if* $cf\delta > \aleph_0$ *and a least upper bound of* $\overline{f}\restriction\delta$ *exists, then* f_δ *is a least upper bound of* $f\restriction\delta$.

Proof. Fix an $\aleph_{\omega+1}$-scale $\overline{g} = \langle g_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in $\prod_{n>1} \aleph_n$ as guranteed by Claim 2. Define f_α by induction on $\alpha < \aleph_{\omega+1}$ as follows: If α is successor or limit of countable cofinality let f_α be g_β for the first $\beta \in (\alpha, \aleph_{\omega+1})$ for which $g_\beta >^* f_\beta$ for all $\beta < \alpha$. If $\operatorname{cf}\alpha > \aleph_0$ then let g_α be a least upper bound to $\overline{f}\restriction\delta := \langle f_\beta : \beta < \alpha\rangle$, if such least upper bound exists; else, define f_α as in the previous cases.

The sequence $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ is increasing cofinal in $\prod_{n>1} \aleph_n$ and by its definition satisfies the required condition. $\square$

Claim 4. *Suppose θ is regular uncountable and $\langle \alpha(\zeta) : \zeta < \theta\rangle$ is strictly increasing with $\sup\{\alpha(\zeta) : \zeta < \theta\} = \delta < \aleph_{\omega+1}$. Suppose $\langle g_\zeta : \zeta < \theta\rangle$ is a sequence of functions in $\prod_{n>1} \aleph_n$ which is increasing in $<$, and that $g_\zeta =^* f_{\alpha(\zeta)}$ for every $\zeta < \theta$. Then*

- *$g := \sup\{g_\zeta : \zeta < \theta\rangle$ is a least upper bound of $\overline{f}\restriction\delta$*
- *$\operatorname{cf} g(n) = \theta$ for all $n > 1$*
- *$g =^* f_\delta$*

Proof of Claim. Let $g := \sup\{g_\zeta : \zeta < \theta\}$. Since $\langle g_\zeta : \zeta < \theta\rangle$ is increasing in $<$, necessarily $\operatorname{cf} g(n) = \theta$ for all $n > 1$.

Suppose that $\gamma < \delta$ is arbitrary. There exists $\zeta < \theta$ such that $\gamma < \alpha(\zeta)$ and thus $f_\gamma <^* f_{\alpha(\zeta)} =^* g_\zeta \le g$. Thus g is an upper bound or $\overline{f}\restriction\delta$.

To show that g is a least upper bound suppose that $g' \in \prod_{n>1} \aleph_n$ is an upper bound of $\overline{f}\restriction\delta$. Let $X := \{n < \omega : g'(n) < g(n)\}$. For every $n \in X$ find $\zeta(n) < \theta$ such that $g_{\zeta(n)}(n) > g'(n)$. Such $\zeta(n)$ can be found because $g = \sup\{g_\zeta : \zeta < \theta\}$. Let $\zeta^* := \sup\{\zeta(n) : n > 1\}$. Since $\operatorname{cf}\theta > \aleph_0$, $\zeta^* < \theta$. Since $\langle g_\zeta : \zeta < \theta\rangle$ is increasing in $<$, it holds that $f_{\zeta(*)} \ge f_{\zeta(n)}(n) > g'(n)$ for every $n \in X$. But g' is an upper bound of $\overline{f}\restriction\delta$, so $f_{\zeta(*)} \le^* g'$ and X is finite.

By the definition of $\overline{f}$ we conclude that f_δ is a least upper bound of $\overline{f}\restriction\delta$. Since both g and f_δ are least upper bounds of $\overline{f}\restriction\delta$ it follows that $g =^* f_\delta$. $\square$

3 The Space

Theorem 2. *There is a Dowker space of cardinality $\aleph_{\omega+1}$.*

Problem 5. *Is $\aleph_{\omega+1}$ the* first *cardinal in which one can prove the existence of a Dowker space in ZFC?*

Proof of the Theorem. Let $\overline{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be as provided by Claim 3. We use this scale to extract a closed Dowker subspace of cardinality $\aleph_{\omega+1}$ from Rudin's space.

Definition 6. *Let* $X^R = \{h \in \prod_{n>1}(\aleph_n + 1) : \exists m \forall n\, [\aleph_0 < cf h(n) < \aleph_m]\}$.

The space is X^R is *the Rudin space* from [5], and is Dowker with the topology defined by letting, for every $f < g$ in $\prod_{n>1}(\aleph_n + 1)$,

$$(f, g] := \{h \in X^R : f < h \leq g\} \tag{1}$$

be a basic open set (see [5]).

It is straightforward to verify that (1) this gives a Hausdorff topology on X^R.

Definition 7. 1. *For* $\alpha < \aleph_{\omega+1}$ *let* $F_\alpha = \{h \in X^R : h =^* f_\alpha\}$

2. *let* $X = \{h : \exists \alpha < \aleph_{\omega+1}[h \in F_\alpha$ *and* $cf h(n)$ *is constant on a co-finite subset of* $\omega]\}$

Since for every function $f_\alpha \in \overline{f}$ the set F_α has cardinality $\aleph_{\omega+1}$ we see that $|X| = \aleph_{\omega+1}$.

Observe next that since every $h \in X$ is $=^*$ to some function $f_\alpha \in \overline{f}$ and $\overline{f}$ is totally ordered by $<^*$, the space X is totally quasi ordered by $<^*$:

$$\forall h, k \in X\, [h <^* k \ \vee \ k <^* h \ \vee \ h =^* k] \tag{2}$$

We put on X the induced topology from Rudin's space X^R above.

Claim 4 translates to a property of the space X:

Claim 8. *If* θ *has uncountable cofinality,* $\langle h_\zeta : \zeta < \theta\rangle$ *is a sequence of elements of* X *which is increasing in* $<$ *and* $h = \sup\langle h_\zeta : \zeta < \theta\rangle$*, then* $h \in X$*, and* h *is in the closure of every cofinal subsequence of* $\langle h_\alpha : \zeta < \theta\rangle$.

Proof. For every $\zeta < \theta$ there is a unique $\alpha(\zeta) < \aleph_{\omega+1}$ for which $h_\zeta =^* f_{\alpha(\zeta)}$. Since $\langle h_\zeta : \zeta < \theta\rangle$ is increasing in $<$, the sequence $\langle \alpha(\zeta) : \zeta < \theta\rangle$ is strictly increasing. Let $\delta = \sup\{\alpha(\zeta) : \zeta < \theta\}$. Clearly, $\mathrm{cf}\,\delta = \theta$. By Claim 4 the function $h = \sup\{h_\zeta : \zeta < \theta\rangle$ is a least upper bound of $\overline{f}\restriction\delta$, and therefore $g =^* f_\delta$. Furthermore, $\mathrm{cf}\, h(n) = \theta$ for all $n < \omega$ and is therefore constant and greater than $\aleph_0$. Thus $h \in X$.

Let $(f, h]$ be a basic open set containing h. For every $n < \omega$ there is some $\zeta(n) < \theta$ for which $h_\zeta(n) > f(n)$. Since $\mathrm{cf}\,\theta > \aleph_0$, it follows that

$\zeta := \sup\{\zeta(n) : n < \omega\} < \theta$. If $\zeta < \xi < \theta$ then because $h_\xi > h_{\zeta(n)}$ for all n we conclude that $h_\zeta \in (f, h]$. This proves that an end segment of $\langle h_\zeta : \zeta < \theta\rangle$ is contained in $(f, h]$. $\square$

Claim 9. *If $U_n \subseteq X$ is open for $n < \omega$ then $\bigcap_n U_n$ is open.*

Proof. Suppose that $h \in \bigcap_n U_n$ and for every n let $(f_n, h] \subseteq U_n$ be basic open. Then $f := \sup\{f_n : n < \omega\} < t$ because $\operatorname{cf} t(n) > \aleph_0$ for all n and thus $(f, h] \subseteq \bigcap_n U_n$. $\square$

We show next that X is Dowker.

Recall that a normal Hausdorff space is *countably paracompact* iff for every decreasing sequence $\langle D_n : n < \omega\rangle$ of closed sets such that $\bigcap D_n = \emptyset$ there are open sets $U_n \supseteq D_n$ with $\bigcap U_n = \emptyset$.

Let $D_n = \{f \in X : \exists m \geq n\, [f(m) = \aleph_m]\}$. It is straightforward that D_n is closed and that $\bigcap_n D_n = \emptyset$.

Claim 10. *X is collectionwise normal.*

Claim 11. *If $U_n \subseteq X$ is open, and $D_n \subseteq U_n$ for all n, then $\bigcap U_n$ is not empty.*

Proof of Claim 10. We prove that X is collectionwise normal in the same fashion Rudin proves collectionwise normality of X^R in [5].

Suppose that $\mathcal{H}$ is a disjoint collection of closed subsets of X with $\operatorname{cl} \bigcup \mathcal{J} = \bigcup \mathcal{J}$ for every subcollection $\mathcal{J} \subseteq \mathcal{H}$. We need to find a disjoint collection $\mathcal{C}$ of open sets that separates $\mathcal{H}$, namely for all $D \in \mathcal{H}$ there exists $U \in \mathcal{C}$ such that $D \subseteq U$.

ßLet $H = \bigcup \mathcal{H}$ and for every $U \subseteq X$ let t_U denote $\sup U$. ß

Define by induction on $\alpha < \omega_1$ an increasing continuous sequence of trees $T(\alpha)$ with tree order being reverse inclusion such that:

1. $U \in T(\alpha) \Rightarrow U$ is an open subset of X and $X \subseteq \bigcup T(0)$
2. If $U \in T(\alpha)$ then $U = \bigcup\{V : V \in T(\alpha) \wedge V \subseteq U\}$
3. If B is a branch of $T(\alpha)$ then $\bigcap B \in T(\alpha + 1)$
4. If $U \in \bigcup_{\beta<\alpha} T(\beta)$ then U is a leaf of $T(\alpha)$ if and only if U meets at most one member of $\mathcal{H}$.
5. If $U, V \in T(\alpha)$ and $V \subseteq U$ then either V is a leaf or $t_V \neq t_U$

Suppose first that the inductive definition is carried out for all $\alpha < \omega_1$. Let $T = \bigcup_{\alpha<\omega_1} T(\alpha)$. The function $U \mapsto t_U$ is decreasing along branches of T and therefore every branch of T is countable. Thus every branch of T terminates at a leaf, which meets at most one member of $\mathcal{H}$. For every $h \in T$ the set $B(h) := \{U \in T : h \in U\}$ is a branch of T, and therefore every $h \in$ belongs to a leaf $U(h) \in T$. Thus $\mathcal{C} = \{U : U \in T \text{ is a leaf }\}$ is an open disjoint cover of H with the property that every $U \in \mathcal{C}$ meets at most one member of $\mathcal{H}$. Letting $U_D = \bigcup\{U \in \mathcal{C} : D \cap U \neq \emptyset\}$ for each $D \in \mathcal{H}$ it is immediate to verify that $\{U_D : D \in \mathcal{H}\}$ separates $\mathcal{H}$.

Let us define $T(\alpha)$ by induction on $\alpha < \omega_1$. Let $T(0) = \{X\}$. If $\alpha < \omega_1$ is limit, let $T(\alpha) \bigcup_{\beta<\alpha} T(\beta)$ and let $T(\alpha+1) = T(\alpha) \cup \{\bigcup B : B \text{ is a branch of } T(\alpha)\}$. By Lemma 9 every $U \in T(\alpha+1)$ for limit $\alpha < \omega_1$ is open and the remaining conditions are easy to verify as well.

We are left with the case $\alpha = \beta + 1$ and β is successor. For every leaf U of $T(\beta)$ we define the set of immediate successors of U in $T(\alpha)$. Let U be a leaf of $T(\beta)$ and let $t = t_U$.

Case 0: U meets at most one member of $\mathcal{H}$. In this case let U have an empty set of successors in $T(\alpha)$, and is a leaf of $T(\alpha)$.

Case 1: There is some n such that $\operatorname{cf} t(n) \leq \aleph_0$. Since for every $h \in V$ we have $\operatorname{cf} h(n) \geq \aleph_0$ the case that $\tau(n)$ is a successor ordinal or 0 is impossible, thus $\operatorname{cf} t(n) = \omega_0$ and we can fix an increasing cofinal sequence ζ_n in $t(n)$ with $\zeta(0) = 0$. Let $U_n = \{h \in U : h(n) \in (\zeta_n, \zeta_{n+1}]\}$. This is a disjoint collection of open subsets of V and since $h(n) \neq t(n)$ for all $h \in V$ the union of this collection is U. Let $\{U_n : n < \omega\}$ be the set of immediate successors of U in $T(\alpha)$.

Case 2: $\operatorname{cf} t(n) > \aleph_0$ for all $n > 1$. In this case we prove:

Claim 12. *There is some function $f < t$ such that $U \cap (f, t]$ meets at most one member of $\mathcal{H}$.*

Once the claim is satisfied by some $f < t$ we proceed to define U_M for every $M \subseteq \omega$ as follows: $U_M = \{h \in U : h\restriction M \leq f\restriction M \wedge h\restriction(\omega - M) > f\restriction(\omega - M)\}$. Each U_M is obviously open, $\{U_M : M \subseteq \omega\}$ is disjoint, $\bigcup\{U_M : M \subseteq \omega\} = U$ and if $t_{U_M} = t_u$ then $M = \emptyset$ and $U_M \subseteq (f, t]$ and thus meets at most one member of $\mathcal{H}$. Let $\{U_M : M \subseteq \omega\}$ be the immediate successors of U in $T(\alpha)$.

Let us prove the claim then. Suppose the claim does not hold. Then for every $f < t$ in $\prod_{n>1} \aleph_n + 1$ there are $h, k \in (f, t]$ which belong to two different members of $\mathcal{H}$.

Let us say that a set $A \subseteq X$ is *dense below* t if for every $f < t$ there is $h \in X \cap (f, t]$.

Case 2.1 The number of sets $D \in \mathcal{H}$ for which $D \cap U$ is dense below t is ≤ 1.

In this case we define a sequence $\langle h_\zeta : \zeta < \omega_1 \rangle$, a sequence $\langle D_\zeta : \zeta < \omega_1 \rangle$ and a sequence $\langle f_\zeta : \zeta < \omega_1 \rangle$ by induction on $\zeta < \omega_1$ as follows: Find $h_\zeta < t$ such that $h_\zeta > \sup\{\xi < \zeta : f_\xi\}$ and $h_\zeta \in D$ for some $D \in \mathcal{H}$ which is not dense below t. This is possible because $\sup\{f_\xi : \xi < \zeta\} < t$ by the assumption on t in the present case, and because there are at least two members of $\mathcal{H}$ that meet $U \cap (\sup_{\xi<\zeta} f_\xi, t]$, but at most one which is dense below t.

Let D_ζ be the unique $D \in \mathcal{H}$ to which h_ζ belongs and let $f_\zeta < t$ be large enough so that $(f_\zeta, t] \cap U \cap D_\zeta$ is empty.

Since $\langle h_\zeta : \zeta < \omega_1 \rangle$ is clearly increasing in $<$, the function $h = \sup_{\zeta<\omega_1} h_\zeta$ belongs to $\mathrm{cl}\, \{h_\zeta : \zeta < \omega_1\} \subseteq \mathrm{cl} \bigcup_{\zeta<\omega_1} D_\zeta$ by Claim 8. But $h \notin D_\zeta$ for every $\zeta < \omega_1$ because $h > f_\zeta$. This contradicts the fact that $\bigcup_{\zeta<\omega_1} D_\zeta$ is closed, by the assumption on $\mathcal{H}$.

Case 2.2: There are two different sets, D_1 and D_2 in $\mathcal{H}$, such that $D_1 \cap U$ and $D_2 \cap U$ are dense below t.

In this case it is convenient to use the following:

Definition 13. *For functions f, g on ω let $E(f, g) := \{n < \omega : f(n) = g(n)\}$.*

Definition 14. *For $i \in \{1, 2\}$ let us define*

$$\mathbf{W}_i = \Big\{ w \subseteq \omega : \forall f < t \exists h \in (D_i \cap U) \Big[h \in (f, t] \wedge E(h, t) = w \Big] \Big\}$$

Fact 15. *If $w \in \mathbf{W}_i$ then w is finite or w is co-finite.*

Proof. Suppose $w \in \mathbf{W_i}$ is infinite and co-infinite. Find $h \in D_i$ such that $E(h, t) = w$. By the definition of $\mathbf{W}_i$ find $k \in D_i \cap U$ such that $E(k, t) = w$ and $h \restriction (\omega - w) < k \restriction (\omega - w)$. This is contradicts the trichotomy (2) because neither $k =^* h$, nor $k <^* h$ nor $h <^* k$. $\square$

Fact 16. $\mathbf{W_i} \neq \emptyset$ *for $i \in \{1, 2\}$.*

Proof. Suppose that for every $w \subseteq \omega$ which is finite or co-finite, $w \notin \mathbf{W_i}$. This is equivalent to assuming $\mathbf{W}_i$ is empty by the previous fact. Then for every w finite or co-finite there is some $f_w < t$ such that $h \in D_i \cap U \cap (f_w, t] \Rightarrow E(h,t) \neq w$. Let $f = \sup\{f_w : w \subseteq \omega \wedge \min\{|w|, |(\omega - w)| < \aleph_0\}$. Since $\operatorname{cf} t(n)$ is uncountable for all n, it follows that $f < t$.

Since $D_i \cap U$ is dense below t, we can find $h \in (f,t] \cap D_i$. Since $h > f \geq f_w$ for all w finite or co-finite, we conclude that $E(h,t)$ is infinite and co-infinite. Find $k \in D_i \cap U$ such that $k > \max\{h, f\}$.

This is again contradictory, because $h \restriction (\omega - w) < k \restriction (\omega - w)$ while $k \restriction w \leq t \restriction w = h \restriction w$, and both sets are infinite, thus violating the trichotomy (2). $\square$

Fact 17. $\mathbf{W}_1 \cap \mathbf{W_2} = \emptyset$.

Proof. Suppose to the contrary that $w \in \mathbf{W}_1 \cap \mathbf{W}_2$.

By induction on $\zeta < \omega_1$ define a sequence of functions $h_\zeta \in U$ so that:

1. $E(h_\zeta, t) = w$
2. $\xi < \zeta \Rightarrow h_\xi \restriction (\omega - w) < h_\zeta \restriction (\omega - w)$
3. if ζ is odd then $h_\zeta \in D_1$ and if ζ is even then $h_\zeta \in D_2$.

This is possible by $w \in \mathbf{W}_1 \cap \mathbf{W}_2$ and because $\operatorname{cf} t(n) > \aleph_0$ for all n, and thus $\sup\{h_\xi : \xi < \zeta\}$ is bounded below t on $(\omega - w)$ for all $\zeta < \omega_1$.

If w is finite, then by Claim 8 the function $h := \sup\{h_\zeta : \zeta < \omega_1\}$ belongs to X and to the closure of both D_1 and D_2. If w is co-finite and $f < h$, it is straightforward to find $\zeta < \omega_1$ such that $f < f_\zeta < f_{\zeta+1}$, showing again that h is in the closure of both D_1 and D_2, which is of course contradictory. $\square$

We need one more fact before we derive a contradiction:

Fact 18. *Fix $i \in \{1,2\}$ and suppose that $v \in \mathbf{W}_i$. If $w \neq \omega$ and m is the least for which $\exists n \in (\omega - w)\,[cft(n) = \aleph_n]$ then $v \cup \{n \in (\omega - w) : cft(n) = \aleph_m\} \in \mathbf{W}_i$.*

Proof. Denote $u := \{n \in (\omega - v) : \operatorname{cf} t(n) = \aleph_m\}$ for m as above. By induction on $\zeta < \omega_n$ find functions $h_\zeta \in D_i \cap U$ with $E(h_\zeta, t) = v$ which are strictly increasing on u with $\sup\{h_\zeta(n) : \zeta < \omega_n\} = t(n)$ for all $n \in u$. Again, if u is infinite use Claim 8 and otherwise the definition of X to show that $h := \sup\{h_\zeta : \zeta < \omega_n\} \in D_i$. It is clear that $E(h,t) = v \cup u$. $\square$

We obtain contradiction by proving:

Fact 19. $\mathbf{W_1} \cap \mathbf{W_2}$ *is not empty.*

Proof. Choose $w_i \in \mathbf{W}_i$. If w_i is finite then the set $\{\operatorname{cf} t(n) : n \in w_i\}$ is finite, and if w_i is co-finite then by the definition of X the set $\{\operatorname{cf} t(n) : n \in w_i\}$ is finite as well.

Let m be the least in $\omega - (\{\operatorname{cf} t(n) : n \in w_1\} \cup \{\operatorname{cf} t(n) : n \in w_2\})$. By iterated use of the previous fact we obtain that $\{n \in \omega : \operatorname{cf} t(n) < \aleph_m\} \in \mathbf{W}_1 \cap \mathbf{W}_2$. $\square$

Fact 17 and Fact 19 provide the desired contradiction. Thus Case 2 and the proof of collectionwise normality are done. $\square$

Claim 20. *X is a closed subspace of X^R.*

Proof. Suppose $t \in X^R$ and for every $f < t$ there is $h \in X \cap (f, t]$. Define

$$\mathbf{W} = \Big\{ w \subseteq \omega : \forall f < t \exists h \in X \Big[E(h,t) = w \wedge h \in (f,t] \Big] \Big\}$$

As X is closed (in X) and dense below t, all the facts we proved in case 2.2 of the proof of collectionwise normality apply for $\mathbf{W}$ substituted for $\mathbf{W}_i$. Since $t \in X^R$, the set $\{\operatorname{cf} t(n) : n < \omega\}$ is finite. Finitely many iterations of Fact 18 produce $h \in X$ with $E(t,h) = \omega - \{0,1\}$. Thus $t \in X$. $\square$

Discussion. Collectionwise normality of X follows, of course, from collectionwise normality in X^R and the closure of X in X^R. However, since proving that X is closed is basically the same as the case 2.2 in the proof of collectionwise normality of X, we chose to present it directly.

We prove now that X is not countably paracompact. The truth of the matter is that this follows trivially from the analogous property in X^R: the definition of D_n above is absolute between X and X^R and Rudin's proof in [5] shows that if $D_n \subseteq U_n$ and U_n is open for all n then there is some $f \in \prod_{n>1} \aleph_n$ such that $h \in \bigcap_n U_n$ for all $h > f$ in X^R. Since for every such f there is $h \in X$ with $h > f$, we see that $\bigcap U_n \cap X$ is not empty.

For the sake of completeness, though (but not less, for the reader's amusement) we shall prove this property directly for X using elementary submodels.

Proof of Claim 11. We suppose that $U_n \supseteq D_n$ is open, where $D_n = \{h \in X : \exists m \geq n[h(m) = \aleph_m]\}$. We need to prove that $\bigcap_n U_n$ is not empty.

We shall prove that there is some $f \in \prod_{n>1} \aleph_n$ such that every $h > f$ in X belongs to this intersection.

It suffices to show that for each n separately there is some $f_n \in \prod_{n>1} \aleph_n$ such that $h > f_n \Rightarrow f \in U_n$, because then $f = \sup\{f_n : n < \omega\}$ is as required.

Suppose to the contrary that m is fixed and for every function $f \in \prod_{n>1} \aleph_n$ there is some function $h_f > f$ in $X - U_m$.

For a given f, let $g_f = \sup\{h_{f'} : (m, \omega) \subseteq E(f', f)\}$.

Let $\langle M_\zeta : \zeta \leq \omega_1 \rangle$ be an elementary chain of submodels of $H(\theta)$ for large enough regular θ so that:

- $\overline{f}$, X and the functions $f \mapsto h_f$ and $f \mapsto g_f$ belong to M_0
- M_ζ has cardinality $\aleph_1$ and $\langle M_\xi : \xi < \zeta \rangle \in M_{\zeta+1}$ for all ζ.

For every ζ let $\chi_\zeta(n) := \sup(M_\zeta \cap \aleph_n)$ for all $n > 1$. Since $|M_\zeta| = \aleph_1$, it follows that $\chi_\zeta(n) < \aleph_n$ for all n and hence $\chi_\zeta \in \prod_{n>1} \aleph_n$.

Since $\chi_\zeta \in M_\xi$ for all $\zeta < \xi < \omega_1$, by elementarity also h_{χ_ζ} and g_{χ_ζ} belong to M_ξ under the same assumptions. Similarly, $f_{\alpha(\zeta)} \in M_\xi$ for $\zeta < \chi < \omega_1$, where $\alpha(\zeta)$ is the first below $\aleph_{\omega+1}$ for which $\chi_\zeta \leq^* f_{\alpha(\zeta)}$.

By Claim 8, $\chi_{\omega_1} \in X$. Let $\chi' = \{\langle n, \aleph_n \rangle : n \leq m\} \cup \chi_{\omega_1} \restriction (m, \omega)$. So $\chi' \in D_n \subseteq U_n$ and therefore $(f, \chi'] \subseteq U_n$ for some $f < \chi'$.

Find some $\zeta < \omega_1$ such that $f \restriction (m, \omega) < \chi_\zeta$. Let now $f' = f \restriction (m+1) \cup \chi_\zeta \restriction (m, \omega)$. By the definition of g_{χ_ζ} we see that $f' < h_{f'} \restriction (m, \omega) \leq g_{\chi_\zeta} \restriction (m, \omega)$ and, of course, $h_{f'} \notin U_m$. This contradicts $h \in (f, \chi'] \subseteq U_m$. $\square$

$\square$

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