REGRESSIVE RAMSEY NUMBERS ARE ACKERMANNIAN

MENACHEM KOJMAN AND SAHARON SHELAH

Abstract. We give an elementary proof of the fact that regressive Ramsey numbers are Ackermannian. This fact was first proved by Kanamori and McAloon with mathematical logic techniques.

Nous vivons encore sous le règne de la logique, voilà, bien entendu, à quoi je voulais en venir. Mais les procédés logiques, de nos jours, ne s'appliquent plus qu'à la résolution de problèmes d'intérêt secondaire. [1, 1924, p. 13]

1. Introduction

Definition 1. (1) *let A be a set of natural numbers. A coloring $c : [A]^e \to \mathbb{N}$ of unordered e-tuples from A is* regressive *if $c(x) < \min x$ for all $x \in [A]^e$.*
(2) *A subset $B \subseteq A$ is* min-homogeneous *for a coloring c of $[A]^e$ if for all $x \in [A]^e$ the color $c(x)$ depends only on $\min x$.*

Theorem 2 (Kanamori and McAloon)**.** (1) *For every k and e there exists N such that for every regressive coloring of e-tuples from $\{1, 2, \dots, N\}$ there exists a min-homogeneous subset of size k.*
(2) *The statement in (1) cannot be proved from the axioms of Peano Arithmetic (although it can be* phrased *in the language of PA)*
(3) *Let $\nu(k)$ be the least N which satisfies 1 for $e = 2$. The function ν eventually dominates every primitive recursive function.*

Part (3) of Kanamori and McAloon's result [3] was proved with methods from mathematical logic. We present below an elementary proof of 3.

2. The lower bound

For every function $f : \mathbb{N} \to \mathbb{N}$ and n, $f^{(n)}$ is defined by $f^{(0)}(x) = x$ and $f^{(n+1)}(x) = f(f^{(n)}(x))$ for all $x \in \mathbb{N}$.

The first author was partially supported by NSF grant No. DMS-9622579.
The second author was partially supported by the Binational Science Foundation.
Number 649 in list of publications.

Define a sequence of (strictly increasing) integer functions $f_i : \mathbb{N} \to \mathbb{N}$ for $i \geq 1$ as follows:

$$f_1(n) = n + 1 \tag{1}$$

$$f_{i+1}(n) = f_i^{(\lfloor \sqrt{n}/2 \rfloor)}(n) \tag{2}$$

Fix an integer $k > 2$. Define a sequence of semi-metrics $\langle d_i : i \in \mathbb{N} \rangle$ on $\{n : n \geq 4k^2\}$ by putting, for $m, n \geq 4k^2$,

$$d_i(m, n) = |\{l \in \mathbb{N} : m < f_i^{(l)}(4k^2) \leq n\}| \tag{3}$$

For $n > m \geq 4k^2$ let $I(m, n)$ be the greatest i for which $d_i(m, n)$ is positive, and $d(m, n) = d_{I(m,n)}(m, n)$.

Claim 3. *For all $n \geq m \geq 4k^2$, $d(m, n) \leq \sqrt{m}/2$.*

Proof. Let $i = I(m, n)$. Since $d_{i+1}(m, n) = 0$, there exists t and l such that $t = f_{i+1}^{(l)}(4k^2) \leq m \leq n < f_{i+1}^{(l+1)}(4k^2) = f_{i+1}(t)$. But $f_{i+1}(t) = f_i^{(\lfloor \sqrt{t}/2 \rfloor)}(t)$ and therefore $\sqrt{t}/2 \geq d_i(t, f_{i+1}(t)) \geq d(m, n)$. □

Let us fix the following (standard) pairing function Pr on $\mathbb{N}^2$

$$\Pr(m, n) = \binom{m + n + 1}{2} + n$$

Pr is a bijection between $[\mathbb{N}]^2$ and $\mathbb{N}$ and is monotone in each variable. Observe that if $m, n \leq l$ then $\Pr(m, n) < 4l^2$ for all $l > 2$.

Define a pair coloring c on $\{n : n \geq 4k^2\}$ as follows:

$$c(\{m, n\}) = \Pr(I(m, n), d(m, n)) \tag{4}$$

Claim 4. *For every $i \in \mathbb{N}$, every sequence $x_0 < x_1 < \cdots < x_i$ that satisfies $d_i(x_0, x_i) = 0$ is not min-homogeneous for c.*

Proof. The claim is proved by induction on i. If $i = 1$ then there are no $x_0 < x_1$ with $d_1(x_0, x_1) = 0$ at all. Suppose to the contrary that $i > 1$, that $x_0 < x_1 < \cdots < x_i$ form a min-homogeneous sequence with respect to c and that $d_i(x_0, x_i) = 0$. Necessarily, $I(x_0, x_i) = j < i$. By min-homogeneity, $I(x_0, x_1) = j$ as well, and $d_j(x_0, x_i) = d_j(x_0, x_1)$. Hence, $\{x_1, x_2, \ldots x_i\}$ is min-homogeneous with $d_j(x_1, x_i) = 0$ — contrary to the induction hypothesis. □

Claim 5. *The coloring c is regressive on the interval $[4k^2, f_k(4k^2))$.*

Proof. Clearly, $d_{k+1}(m,n) = 0$ for $4k^2 \leq m < n < f_k(4k^2)$ and therefore $I(m,n) < k \leq \sqrt{m}/2$. From Claim 3 we know that $d(m,n) \leq \sqrt{m}/2$. Thus, $c(\{m,n\}) \leq \Pr(\lfloor\sqrt{m}/2\rfloor, \lfloor\sqrt{m}/2\rfloor)$, which is $< m$, since $\sqrt{m} > 2$. □

We show that $f_k(4k^2)$ grows eventually faster than every primitive recursive function by comparing the functions f_i with the usual approximations of Ackermann's function. It is well known that every primitive recursive function is dominated by some approximation of Ackermann's function (see, e.g. [2]).

Let $A_i(n)$ be defined as follows:

$$A_1(n) = n + 1 \tag{5}$$

$$A_{i+1}(n) = A_i^{(n)}(n) \tag{6}$$

The A_i-s are the usual approximations to Ackermann's function, which is defined by $Ack(n) = A_n(n)$.

Claim 6. (1) *for all $n \geq 16$ and $i \geq 7$,*
(a) $16n^2 \leq f_i(n)$;
(b) $f_i(16n^2) \leq f_i^{(2)}(n)$.
(2) $f_i(n) \leq f_{i+6}^{(2)}(n)$ *for all $i \geq 1$ and $n \geq 16$.*

Proof. Inequality (a) is verified directly.

Inequality (b) follows from (a) by substituting $f_i(n)$ for $16n^2$ in $f_i(16n^2)$, since f_i is increasing.

We prove 2 by induction on i. For $i = 1$ it holds that $n+1 < f_7^{(2)}(n)$ for all $n \geq 16$ by (a).

Suppose the inequality holds for i and all $n \geq 16$, and let $n \geq 16$ be given. Since $A_i(n) \leq f_{i+6}^{(2)}(n)$ for all $n \geq 16$, it follows by monotonicity of A_i that $A_i^{(n)}(n) \leq f_{i+6}^{(2n)}(n)$. The latter term is smaller than $f_{i+6}^{(2n)}(16n^2)$ by monotonicity, which equals $f_{i+7}(16n^2)$ by (2). Inequality (b) implies that $A_i^{(n)}(n) \leq f_{i+7}^{(2)}(n)$. Finally, $A_i^{(n)}(n) = A_{i+1}(n)$ by (6). □

Claim 7. *For all $i \geq 7$ and $n \geq 16$ it holds that $A_i(n) \leq f_{i+7}(n)$.*

Proof. By 2 in the previous claim, $A_i(n) \leq f_{i+6}^{(2)}(n)$ for $n \geq 16$. If $n \geq 16$, then $\sqrt{n}/2 \geq 2$ and hence, by (2), $f_{i+7}(n) \stackrel{\text{def}}{=} f_{i+6}^{\lfloor(\sqrt{n}/2\rfloor)} \geq f_{i+6}^{(2)}(n)$. □

Corollary 8. *The function $\nu(k)$ eventually dominates every primitive recursive function.*

3. Discussion

3.1. **Other Ramsey numbers.** Paris and Harrington [8] published in 1976 the first finite Ramsey-type statement that was shown to be independent over Peano Arithmetic. Soon after the discovery of the Paris-Harrington result, Erdős and Mills studied the Ramsey-Paris-Harrington numbers in [7]. Denoting by $R^e_c(k)$ the Ramsey-Paris-Harrington number for exponent e and c many colors, Erdős and Mills showed that $R^2_2(k)$ is double exponential in k and that $R^2_c(k)$ is Ackermannian as a function of k and c. In the same paper, several small Ramsey-Paris-Harrington numbers were computed. Later Mills tightened the double exponential upper bound for $R^2_2(k)$ in [5].

Canonical Ramsey numbers for pair colorings were treated in [4] and were also found to be double exponential.

The second author showed that van der Waerden numbers are primitive recursive, refuting the conjecture that they were Ackermannian, in [9] (see also [6]).

We remark that an upper bound for regressive Ramsey numbers for pairs is $R^3_2(k)$ — the Ramsey-Paris-Harrington number for *triples*. Let N be large enough and suppose that c is regressive on $\{1, 2, \dots, N-1\}$. Color a triple $x < y < z$ red if $c(x, y) = c(x, z)$ and blue otherwise. Find a homogeneous set A of size at least k and so that $|A| > \min A+1$. The homogeneous color on A cannot be blue for $k > 5$, and therefore A is min-homogeneous for c.

3.2. **Problems.** The following two problems about regressive Ramsey numbers remain open:

Problem 9.
1. *Find a concrete upper bound for regressive Ramsey numbers.*
2. *Compute small regressive Ramsey numbers*

References

[1] André Breton. Manifeste du surréalisme. In *Manifestes du Surréalisme*, pages 11–66. Gallimard, 1972.

[2] Calude, Cristian. *Theories of computational complexity.* Annals of Discrete Mathematics, 35. Amsterdam etc.: North-Holland. XII, 487 p., 1988.

[3] Akihiro Kanamori and Kenneth McAloon. On Gödel incompleteness and finite combinatorics. *Ann. Pure Appl. Logic*, 33(1):23–41, 1987.

[4] Hanno Leffman and Vojtěch Rödl. On canonical Ramsey numbers for complete graphs versus paths. *Journal of Combinatorial theory, Series B*, 58:1–13, 1993.

[5] George Mills. Ramsey–Paris–Harrington numbers for graphs. *Journal of Combinatorial theory, Series A*, 38:30–37, 1985.

[6] Alon Nilli. Shelah's proof of the Hales-Jewett theorem. In Jaroslav Nešetřil and Vojtěch Rödl, editors, *Mathematics of Ramsey Theory*, volume 5 of *Algorithms and Combinatorics*, pages 151–152. Springer, Berlin, 1990.

[7] Paul Erdős and George Mills. Some bounds for the Ramsey-Paris-Harrington numbers. *Journal of Combinatorial theory, Series A*, 30:53–70, 1981.

[8] J. Paris and L. Harrington. A mathematical incompleteness in Peano arithmetic. In J. Barwise, editor, *Handbook of Mathematical Logic*. North-Holland, 1977.

[9] Saharon Shelah. Primitive recursive bounds for van der Waerden numbers. *Journal of the American Mathematical Society*, 1:683–697, 1988.

Department of Mathematics and Computer Science, Ben Gurion University of the Negev, Beer Sheva, Israel

Department of Mathematical Sciences, Carnegie-Mellon University, Pittsburgh, PA

Email address: kojman@cs.bgu.ac.il

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: shelah@math.huji.ac.il