THE AUTOMORPHISM TOWER PROBLEM REVISITED

WINFRIED JUST, SAHARON SHELAH, AND SIMON THOMAS

ABSTRACT. It is well-known that the automorphism towers of infinite centreless groups of cardinality κ terminate in less than $(2^\kappa)^+$ steps. But an easy counting argument shows that $(2^\kappa)^+$ is not the best possible bound. However, in this paper, we will show that it is impossible to find an explicit better bound using ZFC.

1. INTRODUCTION

If G is a centreless group, then there is a natural embedding of G into its automorphism group $\operatorname{Aut} G$, obtained by sending each $g \in G$ to the corresponding inner automorphism $i_g \in \operatorname{Aut} G$. In this paper, we will always work with the left action of $\operatorname{Aut} G$ on G. Thus $i_g(x) = gxg^{-1}$ for all $x \in G$. If $\pi \in \operatorname{Aut} G$ and $g \in G$, then an easy calculation shows that $\pi i_g \pi^{-1} = i_{\pi(g)}$. Hence the group of inner automorphisms $\operatorname{Inn} G$ is a normal subgroup of $\operatorname{Aut} G$; and $C_{\operatorname{Aut} G}(\operatorname{Inn} G) = 1$. In particular, $\operatorname{Aut} G$ is also a centreless group. This enables us to define the automorphism tower of G to be the ascending chain of groups

$$G = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots G_\alpha \trianglelefteq G_{\alpha+1} \trianglelefteq \dots$$

such that for each ordinal α

(a) $G_{\alpha+1} = \operatorname{Aut} G_\alpha$; and
(b) if α is a limit ordinal, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$.

(At each successor step, we identify G_α with $\operatorname{Inn} G_\alpha$ via the natural embedding.)

The automorphism tower is said to terminate if there exists an ordinal α such that $G_{\alpha+1} = G_\alpha$. This occurs if and only if there exists an ordinal α such that

The research of the first author was done during a visit to Rutgers University in September 1997, which was partially supported by NSF Grant DMS-9704477.

The research of the second author was partially supported by the U.S.-Israel Binational Science Foundation. This paper is number 654 in the cumulative list of the second author's publications.

The research of the third author was partially supported by NSF Grants.

$\operatorname{Aut} G_\alpha = \operatorname{Inn} G_\alpha$. A classical result of Wielandt [15] says that if G is finite, then the automorphism tower terminates after finitely many steps. Wielandt's theorem fails for infinite centreless groups. For example, consider the infinite dihedral group $D_\infty = \langle a, b \rangle$, where a and b are elements of order 2. Then $D_\infty = \langle a \rangle * \langle b \rangle$ is the free product of its cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$. It follows that D_∞ is a centreless group; and that D_∞ has an outer automorphism π of order 2 which interchanges the elements a and b. It is easily shown that $\operatorname{Aut} D_\infty = \langle \pi, i_a \rangle$. Thus $\operatorname{Aut} D_\infty$ is also an infinite dihedral group, and so $\operatorname{Aut} D_\infty \simeq D_\infty$. Hence for each $n \in \omega$, the n^{th} group in the automorphism tower of D_∞ is isomorphic to D_∞; and the automorphism tower of D_∞ does not terminate after finitely many steps.

In the 1970s, a number of special cases of the automorphism tower problem were solved. For example, Rae and Roseblade [8] proved that the automorphism tower of a centreless Černikov group terminates after finitely many steps; and Hulse [4] proved that the automorphism tower of a centreless polycyclic group terminates in countably many steps. But the problem was not solved in full generality until 1984, when Thomas [13] showed that the automorphism tower of an arbitrary centreless group eventually terminates; and that for each ordinal α, there exists a group whose automorphism tower terminates in exactly α steps.

Definition 1.1. If G is a centreless group, then the *height* $\tau(G)$ of the automorphism tower of G is the least ordinal α such that $G_{\alpha+1} = G_\alpha$.

This raises the question of finding bounds for $\tau(G)$ in terms of the cardinality of G. In his original paper [13], Thomas proved that if G is an infinite centreless group of cardinality κ, then $\tau(G) \le (2^\kappa)^+$. Soon afterwards, Thomas and Felgner independently noticed that an easy application of Fodor's Lemma yielded the following slightly better bound.

Theorem 1.2 (Thomas [14])**.** *If G is an infinite centreless group of cardinality κ, then $\tau(G) < (2^\kappa)^+$.*

□

Definition 1.3. If κ is an infinite cardinal, then τ_κ is the least ordinal such that $\tau(G) < \tau_\kappa$ for every centreless group G of cardinality κ.

Since there are only 2^κ centreless groups of cardinality κ up to isomorphism, it follows that $\tau_\kappa < (2^\kappa)^+$. On the other hand, Thomas [13] has shown that for each ordinal $\alpha < \kappa^+$, there exists a centreless group G of cardinality κ such that $\tau(G) = \alpha$. Thus $\kappa^+ \leq \tau_\kappa < (2^\kappa)^+$. It is natural to ask whether a better explicit bound on τ_κ can be proved in ZFC, preferably one which does not involve cardinal exponentiation.

The proof of Theorem 1.2 is extremely simple, and uses only the most basic results in group theory, together with some elementary properties of the infinite cardinal numbers. So it is not surprising that Theorem 1.2 does not give the best possible bound for τ_κ. In contrast, the proof of Wielandt's theorem is much deeper, and involves an intricate study of the subnormal subgroups of a finite centreless group. The real question behind the search for better explicit bounds for τ_κ is whether there exists a subtler, more informative, group-theoretic proof of the automorphism tower theorem for infinite groups. The main result of this paper says that no such bounds can be proved in ZFC, and thus can be interpreted as saying that no such proof exists. (It is perhaps worth mentioning that the proof of Theorem 1.2 yields that the automorphism tower of a finite centreless group terminates in *countably* many steps. However, there does not seem to be an easy reduction from countable to finite; and it appears that some form of Wielandt's analysis is necessary.)

Theorem 1.4. *Let $V \vDash GCH$ and let κ, $\lambda \in V$ be uncountable cardinals such that $\kappa < \operatorname{cf}(\lambda)$. Let α be any ordinal such that $\alpha < \lambda^+$. Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that the following statements are true in the corresponding generic extension $V^{\mathbb{P}}$.*

(a) $2^\kappa = \lambda$.

(b) *There exists a centreless group G of cardinality κ such that $\tau(G) = \alpha$.*

Thus it is impossible to find better explicit bounds for τ_κ when κ is an uncountable cardinal. However, our methods do not enable us to deal with countable groups; and it remains an open question whether or not there exists a countable centreless group G such that $\tau(G) \geq \omega_1$.

Most of this paper will be concerned with the problem of constructing centreless groups with extremely long automorphism towers. Unfortunately it is usually very difficult to compute the successive groups in an automorphism tower. We will get around this difficulty by reducing it to the much easier computation of the successive normalisers of a subgroup H of a group G.

Definition 1.5. If H is a subgroup of the group G, then the *normaliser tower* of H in G is defined inductively as follows.

(a) $N_0(H) = H$.

(b) If $\alpha = \beta + 1$, then $N_\alpha(H) = N_G(N_\beta(H))$.

(c) If α is a limit ordinal, then $N_\alpha(H) = \bigcup_{\beta<\alpha} N_\beta(H)$.

It is sometimes necessary for the notation to include an explicit reference to the ambient group G. In this case, we will write $N_\alpha(H) = N_\alpha(H, G)$.

As we will see in Section 2, if α is any ordinal, then it is easy to construct examples of pairs of groups, $H < G$, such that the normaliser tower of H in G terminates in exactly α steps. The following lemma, which was essentially proved in [13], will enable us to convert normaliser towers into corresponding automorphism towers.

Lemma 1.6. *Let K be a field such that $|K| > 3$ and let H be a subgroup of* $\operatorname{Aut} K$. *Let*

$$G = PGL(2, K) \rtimes H \leqslant P\Gamma L(2, K) = PGL(2, K) \rtimes \operatorname{Aut} K.$$

Then G is a centreless group; and for each α, $G_\alpha = PGL(2, K) \rtimes N_\alpha(H)$, where $N_\alpha(H)$ is the α^{th} group in the normaliser tower of H in $\operatorname{Aut} K$.

□

It is well-known that every group G can be realised as the automorphism group of a suitable graph Γ. Thus the following result implies that every group G can also be realised as the automorphism group of a suitable field K.

Lemma 1.7 (Fried and Kollár [2])**.** *Let $\Gamma = \langle X, E\rangle$ be any graph. Then there exists a field K_Γ of cardinality* $\max\{|X|, \omega\}$ *which satisfies the following conditions.*

(a) *X is an $\operatorname{Aut} K_\Gamma$-invariant subset of K_Γ.*

(b) *The restriction mapping, $\pi \mapsto \pi \restriction X$, is an isomorphism from $\operatorname{Aut} K_\Gamma$ onto $\operatorname{Aut} \Gamma$.*

□

Combining Lemmas 1.6 and 1.7, we obtain the following reduction of our problem.

Lemma 1.8. *Suppose that there exists a graph Γ of cardinality κ and a subgroup H of $\operatorname{Aut}\Gamma$ such that*

(a) $|H| \leq \kappa$*; and*

(b) *the normaliser tower of H in $\operatorname{Aut}\Gamma$ terminates in exactly α steps.*

Then there exists a centreless group T of cardinality κ such that $\tau(T) = \alpha$.

Proof. Let K_Γ be the corresponding field, which is given by Lemma 1.7, and let $T = PGL(2, K_\Gamma) \rtimes H$. By Lemma 1.6, $\tau(T) = \alpha$. □

From now on, fix a regular uncountable cardinal κ such that $\kappa^{<\kappa} = \kappa$ and an ordinal α. Roughly speaking, our strategy will be to

(1) first construct a pair of groups, $H < G$, such that $|H| \leq \kappa$ and the normaliser tower of H in G terminates in α steps; and

(2) then attempt to find a cardinal-preserving notion of forcing $\mathbb{P}$ which adjoins a graph Γ of cardinality κ such that $G \simeq \operatorname{Aut}\Gamma$.

Of course, there are many groups G for which such a notion of forcing $\mathbb{P}$ cannot possibly exist. For example, De Bruijn [1] has shown that the alternating group $\operatorname{Alt}(\kappa^+)$ cannot be embedded in $\operatorname{Sym}(\kappa)$. Consequently, there is no cardinal-preserving notion of forcing $\mathbb{P}$ which adjoins a graph Γ of cardinality κ such that $\operatorname{Alt}(\kappa^+) \simeq \operatorname{Aut}\Gamma$.

Our next definition singles out a combinatorial condition which is satisfied by all those groups G such that G is embeddable in $\operatorname{Sym}(\kappa)$. (See Proposition 1.11.) Conversely, in Theorem 1.12, we will show that if a group G satisfies this combinatorial condition, then there exists a cardinal-preserving notion of forcing $\mathbb{P}$ which adjoins a graph Γ of cardinality κ such that $G \simeq \operatorname{Aut}\Gamma$.

Definition 1.9. Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then a group G is said to satisfy the κ^+*-compatibility condition* if it has the following property. Suppose that H is a group such that $|H| < \kappa$. Suppose that $(f_i \mid i < \kappa^+)$ is a sequence of embeddings $f_i : H \to G$; and let $H_i = f_i[H]$ for each $i < \kappa^+$. Then there exist ordinals $i < j < \kappa^+$ and a surjective homomorphism $\varphi : \langle H_i, H_j \rangle \to H_i$ such that

(a) $\varphi \circ f_j = f_i$; and
(b) $\varphi \restriction H_i = id_{H_i}$.

Example 1.10. To get an understanding of Definition 1.9, it will probably be helpful to see an example of a group which *fails* to satisfy the κ^+-compatibility condition. So we will show that $\mathrm{Alt}(\kappa^+)$ does not satisfy the κ^+-compatibility condition. Let $H = \mathrm{Alt}(4)$. For each $3 \leq i < \kappa^+$, let $\Delta_i = \{0, 1, 2, i\}$ and let $f_i : \mathrm{Alt}(4) \to \mathrm{Alt}(\Delta_i)$ be an isomorphism. If $3 \leq i < j < \kappa^+$, then

$$\langle \mathrm{Alt}(\Delta_i), \mathrm{Alt}(\Delta_j) \rangle = \mathrm{Alt}(\Delta_i \cup \Delta_j) \simeq \mathrm{Alt}(5).$$

Since $\mathrm{Alt}(5)$ is a simple group, there does not exist a surjective homomorphism from $\langle \mathrm{Alt}(\Delta_i), \mathrm{Alt}(\Delta_j) \rangle$ onto $\mathrm{Alt}(\Delta_i)$.

Proposition 1.11. *Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$, and let $G \leqslant \mathrm{Sym}(\kappa)$. Then G satisfies the κ^+-compatibility condition.*

Proof. Let H be a group such that $|H| < \kappa$, and let $(f_i \mid i < \kappa^+)$ be a sequence of embeddings $f_i : H \to G$. For each $i < \kappa^+$, let $H_i = f_i[H]$; and let Z_i be a subset of κ chosen so that

(a) $|Z_i| < \kappa$;
(b) Z_i is H_i-invariant; and
(c) $g \restriction Z_i \neq id_{Z_i}$ for all $1 \neq g \in H_i$.

After passing to a suitable subsequence if necessary, we can suppose that the following conditions hold.

(1) There exists a fixed subset Z such that $Z_i = Z$ for all $i < \kappa^+$.
(2) For each $i < \kappa^+$, let $r_i : H_i \to \mathrm{Sym}(Z)$ be the restriction mapping, $g \mapsto g \restriction Z$; and let $\pi_i : H \to \mathrm{Sym}(Z)$ be the embedding defined by $\pi_i = r_i \circ f_i$. Then $\pi_i = \pi_j$ for all $i < j < \kappa^+$.

Fix any pair of ordinals i, j such that $i < j < \kappa^+$. Let $\rho : \langle H_i, H_j \rangle \to \mathrm{Sym}(Z)$ be the restriction mapping, $g \mapsto g \restriction Z$. Then $\rho\left[\langle H_i, H_j \rangle\right] = r_i\left[H_i\right]$, and so we can define a surjective homomorphism $\varphi : \langle H_i, H_j \rangle \to H_i$ by $\varphi = r_i^{-1} \circ \rho$. Clearly $\varphi \restriction H_i = id_{H_i}$; and it is easily checked that $\varphi \circ f_j = f_i$. $\square$

Theorem 1.12. *Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$, and let G be a group which satisfies the κ^+-compatibility condition. Then there exists a notion of forcing $\mathbb{P}$ such that*

(a) *$\mathbb{P}$ is κ-closed;*

(b) *$\mathbb{P}$ has the κ^+-c.c.; and*

(c) $\Vdash_{\mathbb{P}}$ *There exists a graph Γ of cardinality κ such that $G \simeq \operatorname{Aut}\Gamma$.*

Furthermore, if $|G| = \lambda$, then $|\mathbb{P}| = \max\{\kappa, \lambda^{<\kappa}\}$.

Combining Proposition 1.11 and Theorem 1.12, we see that if κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$ and G is an arbitrary subgroup of $\operatorname{Sym}(\kappa)$, then there exists a cardinal-preserving notion of forcing $\mathbb{P}$ and a graph $\Gamma \in V^{\mathbb{P}}$ such that $G \simeq \operatorname{Aut}\Gamma$. This result is *not* true of arbitrary subgroups of $\operatorname{Sym}(\omega)$; for Solecki [11] has shown that no uncountable free abelian group is the automorphism group of a countable first-order structure.

Theorem 1.12 will be proved in Section 3. It is now easy to explain the main points of the proof of Theorem 1.4. Assume that $V \vDash GCH$. Let κ be a regular uncountable cardinal, and let λ be a cardinal such that $\operatorname{cf}(\lambda) > \kappa$. Let α be any ordinal such that $\alpha < \lambda^+$. In Section 2, we will prove that there exists a notion of forcing $\mathbb{Q}$ such that

(1) $\mathbb{Q}$ is κ-closed;

(2) $\mathbb{Q}$ has the κ^+-c.c.;

and such that the following statements are true in the generic extension $V^{\mathbb{Q}}$.

(a) $2^\kappa = \lambda$.

(b) There exist groups $H < G < \operatorname{Sym}(\kappa)$ such that $|H| = \kappa$ and the normaliser tower of H in G terminates in exactly α steps.

By Proposition 1.11, G satisfies the κ^+-compatibility condition. Hence we can use Theorem 1.12 to generically adjoin a graph Γ of cardinality κ such that $G \simeq \operatorname{Aut}\Gamma$. A moment's thought shows that the normaliser tower of H in G is an absolute notion. Thus Lemma 1.8 yields a centreless group T of cardinality κ such that $\tau(T) = \alpha$. The case when κ is a singular cardinal requires a little more work, and will be dealt with in Section 4. The remainder of this section will be devoted to another two easy applications of Theorem 1.12.

Application 1.13. A well-known open problem asks whether there exists a countable structure $\mathcal{M}$ such that $\operatorname{Aut}\mathcal{M}$ is the free group on 2^ω generators. Using Theorem 1.12, it is easy to establish the consistency of the existence of a structure $\mathcal{N}$ of cardinality ω_1 such that $\operatorname{Aut}\mathcal{N}$ is the free group on 2^{ω_1} generators. It is not known whether the existence of such a structure can be proved in ZFC.

Theorem 1.14. *Let V be a transitive model of ZFC and let κ, λ, θ be cardinals such that $\kappa^{<\kappa} = \kappa < \lambda \leq \theta = \theta^\kappa$. Then there exists a notion of forcing $\mathbb{P}$, which preserves cofinalities and cardinalities, such that the following statements are true in $V^{\mathbb{P}}$.*

(a) $2^\kappa = \theta$*; and*

(b) *there exists a graph Γ of cardinality κ such that $\operatorname{Aut}\Gamma$ is the free group on λ generators.*

Proof. After performing a preliminary forcing if necessary, we can also suppose that $2^\kappa = \theta$. Let F be the free group on λ generators. Then it is enough to show that F satisfies the κ^+-compatibility condition. Let H be a (necessarily free) group such that $|H| < \kappa$, and let $(f_i \mid i < \kappa^+)$ be a sequence of embeddings $f_i : H \to F$. For each $i < \kappa^+$, let $H_i = f_i[H]$. Then $\langle H_i \mid i < \kappa^+\rangle$ is a free group of cardinality at most κ^+. By Theorem 5.1 [10], there exists an embedding of $\langle H_i \mid i < \kappa^+\rangle$ into $\operatorname{Sym}(\kappa)$. So Proposition 1.11 yields the existence of ordinals $i < j < \kappa^+$ and a surjective homomorphism $\varphi : \langle H_i, H_j\rangle \to H_i$ such that $\varphi \circ f_j = f_i$ and $\varphi \upharpoonright H_i = id_{H_i}$. □

Application 1.15. Theorem 1.6 [14] says that if G is a finitely generated centreless group, then the automorphism tower of G terminates in countably many steps. It is conceivable that a more general result holds; namely, that the automorphism tower of G terminates in countably many steps, whenever G is a countable centreless group such that $\operatorname{Aut} G$ is also countable. To see why this might be true, let G be such a group. Then, by Kueker [5], there exists a finite subset $F \subseteq G$ such that each automorphism $\pi \in \operatorname{Aut} G$ is uniquely determined by its restriction $\pi \upharpoonright F$. In terms of the automorphism tower of G, this says that there is a finite subset $F \subseteq G$ such that $C_{G_1}(F) = 1$. Suppose that the "rigidity" of F within $G = G_0$ is propagated along the automorphism tower of G; ie. that $C_{G_\alpha}(F) = 1$ for all ordinals $\alpha < \omega_1$. Then the proof of Theorem 1.6 [14] shows that the automorphism tower of G terminates in countably many steps.

Question 1.16. Let G be a centreless group such that $|\operatorname{Aut} G| = \omega$. Does there exist a finite subset $F \subseteq G$ such that $C_{G_\alpha}(F) = 1$ for all ordinals $\alpha < \omega_1$?

If $|G_\alpha| = \omega$ for all $\alpha < \omega_1$, then Fodor's Lemma implies that there exists an ordinal $\beta < \omega_1$ and a finite subset F_β of G_β such that $C_{G_\alpha}(F_\beta) = 1$ for all $\beta \leq \alpha < \omega_1$; and so $\tau(G) < \omega_1$. This observation suggests the following weak form of Question 1.16, which is also open.

Question 1.17. Does there exist a centreless group G such that $|\operatorname{Aut} G| = \omega$ and $|\operatorname{Aut}(\operatorname{Aut} G)| = 2^\omega$?

Of course, a positive answer to Question 1.16 implies a negative answer to Question 1.17. Using Theorem 1.12, it is easy to establish the consistency of the existence of a centreless group G of cardinality ω_1 such that $|\operatorname{Aut} G| = \omega_1$ and $|\operatorname{Aut}(\operatorname{Aut} G)| = 2^{\omega_1}$. Once again, it is not known whether the existence of such a group can be proved in ZFC.

Theorem 1.18. *Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then it is consistent that there exists a centreless group G of cardinality κ such that $|\operatorname{Aut} G| = \kappa$ and $|\operatorname{Aut}(\operatorname{Aut} G)| = 2^\kappa$.*

Proof. Let V be the ground model. For each α, $\xi < \kappa$, let $Z_\xi^\alpha = \langle z_\xi^\alpha \rangle$ be an infinite cyclic group. For each $\xi < \kappa$, let $A_\xi = \bigoplus_{\alpha<\kappa} Z_\xi^\alpha$; and let $B = \bigoplus_{\xi<\kappa} A_\xi$. Define an action of $\operatorname{Sym}(\kappa)$ on B by $\pi z_\xi^\alpha \pi^{-1} = z_{\pi(\xi)}^\alpha$ for all α, $\xi < \kappa$; and let $W = B \rtimes \operatorname{Sym}(\kappa)$ be the corresponding semidirect product. Let $H = \bigoplus_{\xi<\kappa} Z_\xi^\xi$. Then the members of the normaliser tower of H in W are

(a) $N_0(H) = H$;
(b) $N_1(H) = B$;
(c) $N_2(H) = W$.

Clearly W is embeddable in $\operatorname{Sym}(\kappa)$; and so W satisfies the κ^+-compatibility condition. Let $\mathbb{P}$ be the notion of forcing, given by Theorem 1.12, which adjoins a graph Γ of cardinality κ such that $W \simeq \operatorname{Aut}\Gamma$. Let $K_\Gamma \in V^{\mathbb{P}}$ be the corresponding field, which is given by Lemma 1.7. Then $G = PGL(2, K_\Gamma) \rtimes H$ is a group such that

$$|\operatorname{Aut} G| = |PGL(2, K_\Gamma) \rtimes B| = \kappa$$

and

$$|\operatorname{Aut}(\operatorname{Aut} G)| = |PGL(2, K_\Gamma) \rtimes W| = 2^\kappa.$$

$\square$

Our set-theoretic notation mainly follows that of Kunen [6]. Thus if $\mathbb{P}$ is a notion of forcing and $p,q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. We say that $\mathbb{P}$ is κ-closed if for every $\lambda < \kappa$, every descending sequence of elements of $\mathbb{P}$

$$p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \ldots, \qquad \xi < \lambda,$$

has a lower bound in $\mathbb{P}$. If V is the ground model, then we will denote the generic extension by $V^{\mathbb{P}}$ if we do not wish to specify a particular generic filter $H \subseteq \mathbb{P}$. If we want to emphasize that the term t is to be interpreted in the model M of ZFC, then we write t^M; for example, $\operatorname{Sym}(\lambda)^M$.

Our group-theoretic notation is standard. For example, the (restricted) wreath product of A by C is denoted by $A \operatorname{wr} C$; and the direct sum of the groups H_ξ, $\xi < \lambda$, is denoted by $\bigoplus_{\xi<\lambda} H_\xi$. If $\pi \in \operatorname{Sym}(\kappa)$, then $\operatorname{supp}(\pi) = \{\alpha \in \kappa \mid \pi(\alpha) \neq \alpha\}$; and if λ is an infinite cardinal such that $\lambda \leq \kappa$, then $\operatorname{Sym}_\lambda(\kappa) = \{\pi \in \operatorname{Sym}(\kappa) \mid |\operatorname{supp}(\pi)| < \lambda\}$.

2. Realising normaliser towers within infinite symmetric groups

Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. In this section, we will study the problem of realising long normaliser towers within $\operatorname{Sym}(\kappa)$. In particular, we will prove that if α is any ordinal, then there exists a generic extension $V^{\mathbb{Q}}$ such that a normaliser tower of height α can be realised in $\operatorname{Sym}(\kappa)^{V^{\mathbb{Q}}}$.

First for each ordinal α, we will construct a pair of groups, $H < G$, such that the normaliser tower of H in G terminates in exactly α steps.

Definition 2.1. The ascending chain of groups

$$W_0 \leqslant W_1 \leqslant \cdots \leqslant W_\alpha \leqslant W_{\alpha+1} \leqslant \ldots$$

is defined inductively as follows.

(a) $W_0 = C_2$, the cyclic group of order 2.

(b) Suppose that $\alpha = \beta + 1$. Then

$$W_\beta = W_\beta \oplus 1 \leqslant \left[W_\beta \oplus W^*_\beta\right] \rtimes \langle \sigma_{\beta+1} \rangle = W_{\beta+1}.$$

Here W^*_β is an isomorphic copy of W_β; and $\sigma_{\beta+1}$ is an element of order 2 which interchanges the factors $W_\beta \oplus 1$ and $1 \oplus W^*_\beta$ of the direct sum $W_\beta \oplus W^*_\beta$ via conjugation. Thus $W_{\beta+1}$ is isomorphic to the wreath product $W_\beta \mathrm{wr} C_2$.

(c) If α is a limit ordinal, then $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$.

Lemma 2.2. $|W_\alpha| \leq \max\{|\alpha|, \omega\}$ *for all ordinals* α.

Proof. This follows by an easy induction on α. □

Lemma 2.3. (a) *If* $1 \leq n < \omega$, *then the normaliser tower of* W_0 *in* W_n *terminates in exactly* $n + 1$ *steps.*

(b) *If* $\alpha \geq \omega$, *then the normaliser tower of* W_0 *in* W_α *terminates in exactly* α *steps.*

Proof. (a) It is easily checked that

$$N_1(W_0, W_n) = W_0 \oplus W^*_0 \oplus W^*_1 \oplus \cdots \oplus W^*_{n-1}$$

and that

$$N_2(W_0, W_n) = W_1 \oplus W^*_1 \oplus \cdots \oplus W^*_{n-1};$$

and that, in general, for each $0 \leq \ell \leq n - 1$,

$$N_{\ell+1}(W_0, W_n) = W_\ell \oplus \bigoplus_{\ell \leq m < n} W^*_m.$$

(b) For example, consider the case when $\alpha > \omega$. Then for each $\ell \in \omega$,

$$N_{\ell+1}(W_0, W_\alpha) = W_\ell \oplus \bigoplus_{\ell \leq \beta < \alpha} W^*_\beta;$$

and for each γ such that $\omega \leq \gamma < \alpha$,

$$N_\gamma(W_0, W_\alpha) = W_\gamma \oplus \bigoplus_{\gamma \leq \beta < \alpha} W^*_\beta.$$

□

Remark 2.4. Unfortunately, the group W_{κ^+} does not satisfy the κ^+-compatibility condition. To see this, let

$$H = C_2 \mathrm{wr} C_2 = [\langle a \rangle \oplus \langle b \rangle] \rtimes \langle c \rangle;$$

and for each limit ordinal $i < \kappa^+$, let $f_i : H \to W_{\kappa^+}$ be the embedding such that $f_i(a) = \sigma_{i+1}$ and $f_i(c) = \sigma_{i+2}$. (Here we are using the notation which was introduced in Definition 2.1.) Let i, j be any limit ordinals such that $i < j < \kappa^+$. Then

$$\langle H_i, H_j \rangle \simeq ((C_2 \mathrm{wr} C_2)\, \mathrm{wr} C_2)\, \mathrm{wr} C_2.$$

Suppose that there exists a a surjective homomorphism $\varphi : \langle H_i, H_j \rangle \to H_i$ such that

(a) $\varphi \circ f_j = f_i$; and
(b) $\varphi \restriction H_i = id_{H_i}$.

Then $\varphi(\sigma_{j+1}) = \varphi(\sigma_{i+1}) = \sigma_{i+1}$ and $\varphi(\sigma_{j+2}) = \varphi(\sigma_{i+2}) = \sigma_{i+2}$. Consider the element $x = \sigma_{j+1}\sigma_{j+2}\sigma_{j+1}\sigma_{j+2} \in H_j$. Then it is easily checked that

(1) x lies in the centre of H_j; and
(2) $\sigma_{j+1} y \sigma_{j+1}^{-1} = xyx^{-1}$ for all $y \in H_i$.

Thus $z = \varphi(x)$ lies in the centre of H_i. Since $\varphi(\sigma_{j+1}) = \sigma_{i+1}$, we find that

$$\sigma_{i+1} y \sigma_{i+1}^{-1} = zyz^{-1} = y$$

for all $y \in H_i$. But this contradicts the fact that σ_{i+1} is a noncentral element of H_i.

Thus if $\alpha \geq \kappa^+$, then W_α is not embeddable in $\mathrm{Sym}(\kappa)$. However, the above argument does not rule out the possibility that W_α is embeddable in the quotient group $\mathrm{Sym}(\kappa)/\mathrm{Sym}_\kappa(\kappa)$; and this is enough for our purposes.

Lemma 2.5. *Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Suppose that γ is an ordinal, and that there exists an embedding*

$$f : W_\gamma \to \mathrm{Sym}(\kappa)/\mathrm{Sym}_\kappa(\kappa).$$

Then for each ordinal $\alpha \leq \gamma$, there exist groups $H \leqslant G \leqslant \mathrm{Sym}(\kappa)$ such that $|H| = \kappa$ and the normaliser tower of H in G terminates in exactly α steps.

Proof. For each $\alpha \le \gamma$, let H^α be the subgroup of $\mathrm{Sym}(\kappa)$ such that $f[W_\alpha] = H^\alpha / \mathrm{Sym}_\kappa(\kappa)$. Since $|\mathrm{Sym}_\kappa(\kappa)| = \kappa^{<\kappa} = \kappa$, it follows that $|H^0| = \kappa$.

Claim 2.6. *For each ordinal β,*

$$f[N_\beta(W_0, W_\alpha)] = N_\beta(H^0, H^\alpha)/\mathrm{Sym}_\kappa(\kappa).$$

Proof. We will argue by induction on β. The result is clear when $\beta = 0$, and no difficulties arise when β is a limit ordinal. Suppose that $\beta = \xi + 1$ and that the result holds for ξ. Let $R = N_\xi(H^0, H^\alpha)$; and for each subgroup K such that $\mathrm{Sym}_\kappa(\kappa) \leqslant K \leqslant H^\alpha$, let $\overline{K} = K/\mathrm{Sym}_\kappa(\kappa)$. Then

$$f[N_{\xi+1}(W_0, W_\alpha)] = N_{\overline{H^\alpha}}(\overline{R});$$

and so we must show that

$$N_{\overline{H^\alpha}}(\overline{R}) = \overline{N_{H^\alpha}(R)}.$$

But this is an immediate consequence of the Correspondence Theorem for subgroups of quotient groups, together with the observation that the normaliser of any subgroup L is the largest subgroup M such that $L \trianglelefteq M$. □

It is now easy to complete the proof of Lemma 2.5. Applying Lemma 2.3 and Claim 2.6, we see that if $\alpha \ge \omega$, then the normaliser tower of H^0 in H^α terminates in exactly α steps; and that if $2 \le \alpha = n < \omega$, then the normaliser tower of H^0 in H^{n-1} terminates in exactly n steps. This just leaves the cases when $\alpha = 0, 1$. When $\alpha = 0$, we can take $H = G = \mathrm{Alt}(\kappa)$; and when $\alpha = 1$, we can take $H = \mathrm{Alt}(\kappa)$ and $G = \mathrm{Sym}(\kappa)$. □

The next result implies that if $\omega < \kappa^\kappa = \kappa$ and W is *any* group, then there exists a cardinal-preserving notion of forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, the group W is embeddable in $\mathrm{Sym}(\kappa)/\mathrm{Sym}_\kappa(\kappa)$. (If $|W|^\kappa > |W|$, then just embed W in a larger group L such that $|L|^\kappa = |L|$; and then apply Lemma 2.7 to L.)

Lemma 2.7. *Let V be a transitive model of ZFC and let κ, λ be cardinals such that $\omega < \kappa^{<\kappa} = \kappa < \lambda = \lambda^\kappa$. Let W be any group of cardinality λ. Then there exists a notion of forcing $\mathbb{Q}$ such that*

(1) $\mathbb{Q}$ *is κ-closed.*

(2) $\mathbb{Q}$ *has the κ^+-c.c.;*

and such that the following statements are true in the generic extension $V^{\mathbb{Q}}$.

(a) $2^\kappa = \lambda$.

(b) *There exists an isomorphic embedding*

$$f : W \to (\mathrm{Sym}(\kappa)/\,\mathrm{Sym}_\kappa(\kappa))^{V^{\mathbb{Q}}}.$$

Proof. Let $\Omega = \bigcup_{\alpha<\kappa}\{\alpha\} \times \alpha$. We will work with the symmetric group $\mathrm{Sym}(\Omega)$ rather than with $\mathrm{Sym}(\kappa)$. Let $\mathbb{Q}$ be the notion of forcing consisting of the conditions

$$p = (\delta_p, H_p, E_p)$$

such that the following hold.

(a) $\omega \leq \delta_p < \kappa$.

(b) H_p is a subgroup of W such that $|H_p| \leq |\delta_p|$.

(c) E_p is a function which assigns a permutation $e^p_{\pi,\xi} \in \mathrm{Sym}(\{\xi\} \times \xi)$ to each pair $(\pi, \xi) \in H_p \times \delta_p$.

We set $q = (\delta_q, H_q, E_q) \leq p = (\delta_p, H_p, E_p)$ if and only if

(1) $\delta_p \leq \delta_q$;

(2) $H_p \leqslant H_q$;

(3) $E_p \subseteq E_q$; and

(4) if $\delta_p \leq \xi < \delta_q$, then the restriction to H_p of the function, $\pi \mapsto e^q_{\pi,\xi}$, is an isomorphic embedding of H_p into $\mathrm{Sym}(\{\xi\} \times \xi)$.

Claim 2.8. $\mathbb{Q}$ *is* κ*-closed.*

Proof of Claim 2.8. This is clear. □

Claim 2.9. $\mathbb{Q}$ *has the* κ^+*-c.c..*

Proof of Claim 2.9. Suppose that $p_i = (\delta_{p_i}, H_{p_i}, E_{p_i}) \in \mathbb{Q}$ for $i < \kappa^+$. Using the Δ-System Lemma and the assumption that $\kappa^{<\kappa} = \kappa$, after passing to a suitable subsequence if necessary, we can suppose that the following conditions are satisfied.

(i) There exists a fixed ordinal δ such that $\delta_{p_i} = \delta$ for all $i < \kappa^+$.

(ii) There exists a fixed subgroup H such that $H_{p_i} \cap H_{p_j} = H$ for all $i < j < \kappa^+$.

(iii) There exists a fixed function E such that $E_{p_i} \restriction H = E$ for all $i < \kappa^+$.

Now fix any two ordinals $i < j < \kappa^+$. Let $H^+ = \langle H_{p_i}, H_{p_j} \rangle$ be the subgroup generated by $H_{p_i} \cup H_{p_j}$; and let E^+ be any extension of $E_{p_i} \cup E_{p_j}$ which satisfies condition (c). Then $q = (\delta, H^+, E^+)$ is a common lower bound of p_i and p_j. $\square$

Claim 2.10. *For each $\alpha < \kappa$, the set $C_\alpha = \{p \in \mathbb{Q} \mid \delta_p \geq \alpha\}$ is dense in $\mathbb{Q}$.*

Proof of Claim 2.10. Let $p = (\delta_p, H_p, E_p) \in \mathbb{Q}$. Then we can suppose that $\delta_p < \alpha$. We can define an isomorphic embedding $\varphi : H_p \to \mathrm{Sym}(H_p)$ by setting $\varphi(h)(x) = hx$ for all $x \in H_p$. Since $|H_p| \leq |\delta_p|$, it follows that there exists an isomorphic embedding $\varphi_\xi : H_p \to \mathrm{Sym}(\{\xi\} \times \xi)$ for each $\delta_p \leq \xi < \alpha$. Hence there exists a condition $q = (\delta_q, H_q, E_q) \leq p$ such that $H_q = H_p$ and $\delta_q = \alpha$. $\square$

Claim 2.11. *For each $\pi \in W$, the set $D_\pi = \{p \in \mathbb{Q} \mid \pi \in H_p\}$ is dense in $\mathbb{Q}$.*

Proof of Claim 2.11. Let $p = (\delta_p, H_p, E_p) \in \mathbb{Q}$. Let $H^+ = \langle H_p, \pi \rangle$ be the subgroup generated by $H_p \cup \{\pi\}$, and let E^+ be any extension of E_p to H^+ which satisfies condition (c). Then $q = (\delta_q, H_q, E_q) \leq p$. $\square$

Let $F \subseteq \mathbb{Q}$ be a generic filter, and let $V^{\mathbb{Q}} = V[F]$ be the corresponding generic extension. Working within $V^{\mathbb{Q}}$, for each $\pi \in W$, let

$$e(\pi) = \bigcup \{e^p_{\pi,\xi} \mid \text{ There exists } p \in F \text{ such that } \pi \in H_p \text{ and } \xi < \delta_p\}.$$

Then $e(\pi) \in \mathrm{Sym}(\Omega)$. Let $\mathrm{Sym}_\kappa(\Omega) = \{\psi \in \mathrm{Sym}(\Omega) \mid |\mathrm{supp}(\psi)| < \kappa\}$; and define the function

$$f : W \to \mathrm{Sym}(\Omega)/\mathrm{Sym}_\kappa(\Omega)$$

by $f(\pi) = e(\pi)\,\mathrm{Sym}_\kappa(\Omega)$. Then it is enough to show that f is an isomorphic embedding.

Claim 2.12. *If $1 \neq \pi \in W$, then $f(\pi) \neq 1$.*

Proof of Claim 2.12. Choose a condition $p = (\delta_p, H_p, E_p) \in F$ such that $\pi \in H_p$. If ξ is any ordinal such that $\delta_p \leq \xi < \kappa$, then $e(\pi) \restriction \{\xi\} \times \xi \neq id_{\{\xi\} \times \xi}$. Hence $e(\pi) \notin \mathrm{Sym}_\kappa(\Omega)$. $\square$

Claim 2.13. *f is a group homomorphism.*

Proof of Claim 2.13. Let π_1, $\pi_2 \in W$. Let $p = (\delta_p, H_p, E_p) \in F$ be a condition such that π_1, $\pi_2 \in H_p$. Let ξ be any ordinal such that $\delta_p \leq \xi < \kappa$; and let $q \in F$ be a condition such that $q \leq p$ and $\xi < \delta_q$. Then

$$e^q_{\pi_1,\xi} \circ e^q_{\pi_2,\xi} = e^q_{\pi_1 \circ \pi_2,\xi}.$$

It follows that

$$e(\pi_1)\operatorname{Sym}_\kappa(\Omega) \circ e(\pi_2)\operatorname{Sym}_\kappa(\Omega) = e(\pi_1 \circ \pi_2)\operatorname{Sym}_\kappa(\Omega).$$

□

Finally it is easily checked that $|\mathbb{Q}| = \lambda$; and it follows that $V^{\mathbb{Q}} \vDash 2^\kappa = \lambda$. This completes the proof of Lemma 2.7. □

Summing up our work in this section, we obtain the following result.

Theorem 2.14. *Let V be a transitive model of ZFC and let κ, λ be cardinals such that $\omega < \kappa^{<\kappa} = \kappa < \lambda = \lambda^\kappa$. Let α be any ordinal such that $\alpha < \lambda^+$. Then there exists a notion of forcing $\mathbb{Q}$ such that*

(1) *$\mathbb{Q}$ is κ-closed;*
(2) *$\mathbb{Q}$ has the κ^+-c.c.;*

and such that the following statements are true in the generic extension $V^{\mathbb{Q}}$.

(a) $2^\kappa = \lambda$.
(b) *There exist groups $H < G < \operatorname{Sym}(\kappa)$ such that $|H| = \kappa$ and the normaliser tower of H in G terminates in exactly α steps.*

Proof. Let γ be an ordinal such that $\max\{\alpha, \lambda\} \leq \gamma < \lambda^+$. Then $|W_\gamma| = \lambda$. Let $\mathbb{Q}$ be the notion of forcing obtained by applying Lemma 2.7 to $W = W_\gamma$. By Lemma 2.5, $\mathbb{Q}$ satisfies our requirements. □

3. Closed groups of uncountable degree

In this section, we will prove Theorem 1.12. Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$, and let G be a group which satisfies the κ^+-compatibility condition. Let L be a first-order language consisting of κ binary relation symbols. The following notion of forcing $\mathbb{P}$ is designed to adjoin a structure $\mathcal{M}$ of cardinality κ for the language L such that $G \simeq \operatorname{Aut}\mathcal{M}$. This is sufficient; for then we can use one of the standard procedures to code $\mathcal{M}$ into a graph Γ of cardinality κ such that $\operatorname{Aut}\Gamma \simeq \operatorname{Aut}\mathcal{M}$. (Cf. Section 5.5 of Hodges [3].)

Definition 3.1. Suppose that $L_0 \subseteq L$ and that $\mathcal{N}$ is a structure for the language L_0. Then a *restricted atomic type* in the free variable v for the language L_0 using parameters from $\mathcal{N}$ is a set t of formulas of the form $R(v, a)$, where $R \in L_0$ and $a \in \mathcal{N}$. An element $c \in \mathcal{N}$ is said to *realise* t if $\mathcal{N} \models \varphi[c]$ for every formula $\varphi(v) \in t$. If no element of $\mathcal{N}$ realises t, then t is said to be *omitted* in $\mathcal{N}$. Notice that if t is omitted in $\mathcal{N}$, then t is not the *trivial* restricted atomic type $\emptyset$.

Definition 3.2. Let $\mathbb{P}$ be the notion of forcing consisting of the conditions

$$p = (H, \pi, \mathcal{N}, T)$$

such that the following hold.

(a) H is a subgroup of G such that $|H| < \kappa$.
(b) There exists an ordinal $0 < \delta < \kappa$ and a subset $L(\mathcal{N}) \in [L]^{<\kappa}$ such that $\mathcal{N}$ is a structure with universe δ for the language $L(\mathcal{N})$.
(c) $\pi : H \to \operatorname{Aut}\mathcal{N}$ is a group homomorphism.
(d) T is a set of restricted atomic types in the free variable v for the language $L(\mathcal{N})$ using parameters from $\mathcal{N}$. Furthermore, $|T| < \kappa$; and each $t \in T$ is omitted in $\mathcal{N}$.

We set $(H_2, \pi_2, \mathcal{N}_2, T_2) \leq (H_1, \pi_1, \mathcal{N}_1, T_1)$ if and only if

(1) $H_1 \leqslant H_2$.
(2) $\mathcal{N}_1$ is a substructure of $\mathcal{N}_2$.
(3) For all $h \in H_1$ and $\alpha \in \mathcal{N}_1$, $\pi_2(h)(\alpha) = \pi_1(h)(\alpha)$.
(4) $T_1 \subseteq T_2$.

It is clear that the components $(H, \pi, \mathcal{N})$ in each condition $p \in \mathbb{P}$ are designed to generically adjoin a structure $\mathcal{M}$ of cardinality κ for the language L, together with an embedding π^* of G into $\operatorname{Aut}\mathcal{M}$. The set T of restricted atomic types is needed to kill off potential extra automorphisms $g \in \operatorname{Aut}\mathcal{M} \smallsetminus \pi^*[G]$, and thus ensure that π^* is surjective.

Lemma 3.3. *$\mathbb{P}$ is κ-closed.*

Proof. This is clear. □

Lemma 3.4. *$\mathbb{P}$ has the κ^+-c.c.*

Proof. Suppose that $p_i = (H_i, \pi_i, \mathcal{N}_i, T_i) \in \mathbb{P}$ for $i < \kappa^+$. After passing to a suitable subsequence if necessary, we can suppose that the following conditions hold.

(1) There exists a fixed structure $\mathcal{N}$ such that $\mathcal{N}_i = \mathcal{N}$ for all $i < \kappa^+$.
(2) There exists a fixed set of restricted atomic types T such that $T_i = T$ for all $i < \kappa^+$.
(3) There is a fixed group H such that for each $i < \kappa^+$, there exists an isomorphism $f_i : H \simeq H_i$.
(4) For each $i < \kappa^+$, let $\psi_i : H \to \operatorname{Aut}\mathcal{N}$ be the embedding defined by $\psi_i = \pi_i \circ f_i$. Then $\psi_i = \psi_j$ for all $i < j < \kappa^+$.

Since G satisfies the κ^+-compatibility condition, there exist ordinals $i < j < \kappa^+$ and a surjective homomorphism $\varphi : \langle H_i, H_j\rangle \to H_i$ such that

(a) $\varphi \circ f_j = f_i$; and
(b) $\varphi \restriction H_i = id_{H_i}$.

Let $\langle H_i, H_j\rangle \to \operatorname{Aut}\mathcal{N}$ be the homomorphism defined by $\pi = \pi_i \circ \varphi$. Clearly $\pi_i \subseteq \pi$. Note that if $x \in H_j$, then

$$\begin{aligned}\pi_i \circ \varphi(x) &= \pi_i \circ (\varphi \circ f_j) \circ f_j^{-1}(x)\\ &= \pi_i \circ f_i \circ f_j^{-1}(x)\\ &= \pi_j(x).\end{aligned}$$

Thus we also have that $\pi_j \subseteq \pi$. Consequently, we can define a condition $p \le p_i$, p_j by

$$p = (\langle H_i, H_j\rangle, \pi, \mathcal{N}, T)\,.$$

□

Lemma 3.5. *For each $p = (H, \pi, \mathcal{N}, T) \in \mathbb{P}$, there exists $p^+ = (H, \pi^+, \mathcal{N}^+, T) \le p$ such that $\pi^+ : H \to \operatorname{Aut}\mathcal{N}^+$ is an embedding.*

Proof. Let $\mathcal{N}^+$ be the structure for the language $L(\mathcal{N})$ such that

(a) the universe of $\mathcal{N}^+$ is the disjoint union $\mathcal{N} \sqcup H$;
(b) for each relation $R \in L(\mathcal{N})$, $R^{\mathcal{N}^+} = R^{\mathcal{N}}$.

In particular, if $x \in \mathcal{N}^+ \smallsetminus \mathcal{N}$, then x only realises the trivial restricted atomic type $\emptyset$ over $\mathcal{N}$. Hence none of the restricted atomic types in T is realised in $\mathcal{N}^+$. Let $\pi^+ : H \to \operatorname{Aut} \mathcal{N}^+$ be the embedding such that for each $h \in H$,

(i) $\pi^+(h)(x) = \pi(h)(x)$ for all $x \in \mathcal{N}$; and

(ii) $\pi^+(h)(x) = hx$ for all $x \in H$.

Then $p^+ = (H, \pi^+, \mathcal{N}^+, T) \leq p$. □

There is a slight inaccuracy in the proof of Lemma 3.5, as the universe of $\mathcal{N}^+$ should really be an ordinal $\delta < \kappa$. However, the proof can easily be repaired: simply replace $\mathcal{N}^+$ by a suitable isomorphic structure. Similar remarks apply to the proofs of Lemmas 3.6 and 3.8.

Lemma 3.6. *For each* $a \in G$,

$$D_a = \{(H, \pi, \mathcal{N}, T) \mid a \in H\}$$

is a dense subset of $\mathbb{P}$.

Proof. Let $a \in G$ and $p = (H, \pi, \mathcal{N}, T) \in \mathbb{P}$. We can suppose that $a \notin H$. Let $H^+ = \langle H, a \rangle$. Let $C = \{g_i \mid i \in I\}$ be a set of left coset representatives for H in H^+, chosen so that $1 \in C$. Let $\mathcal{N}^+$ be the structure for the language $L(\mathcal{N})$ such that

(a) the universe of $\mathcal{N}^+$ is the cartesian product $C \times \mathcal{N}$; and

(b) for each relation $R \in L(\mathcal{N})$,

$$((g_i, x), (g_j, y)) \in R^{\mathcal{N}^+} \text{ iff } i = j \text{ and } (x, y) \in R^{\mathcal{N}}.$$

By identifying each $x \in \mathcal{N}$ with the element $(1, x) \in \mathcal{N}^+$, we can regard $\mathcal{N}$ as a substructure of $\mathcal{N}^+$. Once again, each element $(g_i, x) \in \mathcal{N}^+ \smallsetminus \mathcal{N}$ only realises the trivial restricted atomic type $\emptyset$ over $\mathcal{N}$; and hence none of the restricted atomic types in T is realised in $\mathcal{N}^+$.

Define an action of H^+ on $\mathcal{N}^+$ as follows. If $g \in H^+$ and $(g_i, x) \in \mathcal{N}^+$, then

$$g(g_i, x) = (g_j, \pi(h)(x)),$$

where $j \in I$ and $h \in H$ are such that $gg_i = g_j h$. It is easily checked that this action yields a homomorphism $\pi^+ : H^+ \to \operatorname{Aut} \mathcal{N}^+$; and that $(H^+, \pi^+, \mathcal{N}^+, T) \leq p$. □

Lemma 3.7. *For each* $\alpha < \kappa$,

$$E_\alpha = \{(H, \pi, \mathcal{N}, T) \mid \alpha \in \mathcal{N}\}$$

is a dense subset of $\mathbb{P}$.

Proof. Left to the reader. □

Let $F \subseteq \mathbb{P}$ be a generic filter, and let $V^{\mathbb{P}} = V[F]$ be the corresponding generic extension. Working within $V^{\mathbb{P}}$, define

$$\mathcal{M} = \bigcup \{\mathcal{N} \mid \text{ There exists } p = (H, \pi, \mathcal{N}, T) \in F\}$$

and

$$\pi^* = \bigcup \{\pi \mid \text{ There exists } p = (H, \pi, \mathcal{N}, T) \in F\}.$$

Then $\mathcal{M}$ is a structure for L of cardinality κ, and π^* is an embedding of G into $\operatorname{Aut} \mathcal{M}$. So the following lemma completes the proof of Theorem 1.12.

Lemma 3.8. $\pi^* : G \to \operatorname{Aut} \mathcal{M}$ *is a surjective homomorphism.*

Proof. Suppose that $g \in \operatorname{Aut} \mathcal{M} \setminus \pi^*[G]$. Let $\widetilde{\mathcal{M}}$, $\widetilde{\pi}$ be the canonical $\mathbb{P}$-names for $\mathcal{M}$ and π^*; and let $\widetilde{g}$ be a $\mathbb{P}$-name for g. Then there exists a condition $p \in F$ such that

$$p \Vdash \widetilde{g} \in \operatorname{Aut} \widetilde{\mathcal{M}} \text{ and } \widetilde{g} \neq \widetilde{\pi}(h) \text{ for all } h \in G.$$

Using the fact that $\mathbb{P}$ is κ-closed, we can inductively construct a descending sequence of conditions $p_m = (H_m, \pi_m, \mathcal{N}_m, T_m)$ for $m \in \omega$ such that the following hold.

(a) $p_0 = p$.

(b) For all $x \in \mathcal{N}_m$, there exists $y \in \mathcal{N}_{m+1}$ such that $p_{m+1} \Vdash \widetilde{g}(x) = y$.

(c) For all $h \in H_m$, there exists $z \in \mathcal{N}_{m+1}$ such that $p_{m+1} \Vdash \widetilde{g}(z) \neq \pi_{m+1}(h)(z)$.

Let $q = (H, \pi, \mathcal{N}, T)$ be the greatest lower bound of $\{p_m \mid m \in \omega\}$ in $\mathbb{P}$. Then $q \leq p$, and there exists $g^* \in \operatorname{Aut} \mathcal{N} \setminus \pi[H]$ such that $q \Vdash \widetilde{g} \restriction \mathcal{N} = g^*$. Let $\mathcal{N}^+$ be the structure defined as follows.

(1) The universe of $\mathcal{N}^+$ is the disjoint union $\mathcal{N} \sqcup H$.

(2) For each relation $R \in L(\mathcal{N})$, $R^{\mathcal{N}^+} = R^{\mathcal{N}}$.

(3) For each $x \in \mathcal{N}$, let $R_x \in L \setminus L(\mathcal{N})$ be a new binary relation symbol. Then we set $(h, y) \in R_x^{\mathcal{N}^+}$ iff $h \in H$, $y \in \mathcal{N}$ and $\pi(h)(x) = y$.

Once again, it is clear that none of the restricted atomic types in T is realised in $\mathcal{N}^+$. Let $\pi^+ : H \to \operatorname{Sym} \mathcal{N}^+$ be the embedding such that

(i) $\pi^+(h)(x) = \pi(h)(x)$ for all $x \in \mathcal{N}$; and
(ii) $\pi^+(h)(x) = hx$ for all $x \in H$.

Then it is easily checked that $\pi^+[H] \leqslant \operatorname{Aut} \mathcal{N}^+$. Finally let t be the restricted atomic type defined by

$$t = \{R_x(v, g^*(x) \mid x \in \mathcal{N}\};$$

and let $T^+ = T \cup \{t\}$.

Claim 3.9. *t is omitted in $\mathcal{N}^+$.*

Proof of Claim 3.9. If $z \in \mathcal{N}^+$ realises t, then $z = h \in H$. And if $x \in \mathcal{N}$, then $\mathcal{N}^+ \models R_x(h, g^*(x))$, and so $\pi(h)(x) = g^*(x)$. But this contradicts the fact that $g^* \in \operatorname{Aut} \mathcal{N} \smallsetminus \pi[H]$. □

Thus $q^+ = (H, \pi^+, \mathcal{N}^+, T^+) \in \mathbb{P}$. To simplify notation, suppose that $q^+ \in F$; so that $g^* \subseteq g$. Then for each $x \in \mathcal{N}$, we have that $\mathcal{M} \models R_x(1, x)$, and hence $\mathcal{M} \models R_x(g(1), g^*(x))$. But this means that $g(1) \in \mathcal{M}$ realises t, which is the final contradiction. □

4. τ_κ IS INCREASING

In this section, we will complete the proof of Theorem 1.4. So let $V \models GCH$ and let κ, $\lambda \in V$ be uncountable cardinals such that $\kappa < \operatorname{cf}(\lambda)$. Let α be any ordinal such that $\alpha < \lambda^+$. It is well-known that there exists a centreless group T of cardinality κ such that $\operatorname{Aut} T = \operatorname{Inn} T$. (For example, we can take $T = PGL(2, K)$, where K is a rigid field of cardinality κ. The existence of such a field follows from Lemma 1.7.) Thus we can assume that $\alpha \geq 1$. First consider the case when κ is a regular cardinal. Let $\mathbb{Q}$ be the notion of forcing which is given by Theorem 2.14. Then the following statements are true in the corresponding generic extension $M = V^{\mathbb{Q}}$.

(a) $\kappa^{<\kappa} = \kappa$.
(b) $2^\kappa = \lambda$.
(c) There exist groups $H < G < \operatorname{Sym}(\kappa)$ such that $|H| = \kappa$ and the normaliser tower of H in G terminates in exactly α steps.

Let $\mathbb{P} \in M$ be the notion of forcing, given by Theorem 1.12, which adjoins a graph Γ of cardinality κ such that $G \simeq \operatorname{Aut}\Gamma$. Then, applying Lemma 1.8, we find that the following statements are true in $M^{\mathbb{P}}$.

(1) $2^\kappa = \lambda$.
(2) There exists a centreless group T of cardinality κ such that $\tau(T) = \alpha$.

Next suppose that κ is a singular cardinal. By the above argument, there is a generic extension $V^{\mathbb{P}*\mathbb{Q}}$ in which the following statements are true.

(i) $2^{\omega_1} = 2^\kappa = \lambda$.
(ii) There exists a centreless group T of cardinality ω_1 such that $\tau(T) = \alpha$.

Let $G = T \times \operatorname{Alt}(\kappa)$. Clearly $|G| = \kappa$; and the following theorem implies that $\tau(G) = \tau(T) = \alpha$. This completes the proof of Theorem 1.4.

Theorem 4.1. *Suppose that $\omega \leq \theta < \kappa$. If H is a centreless group of cardinality θ such that $\tau(H) \geq 1$, then $\tau(H \times \operatorname{Alt}(\kappa)) = \tau(H)$.*

Corollary 4.2. *If $\omega \leq \theta < \kappa$, then $\tau_\theta \leq \tau_\kappa$.*

The remainder of this section will be devoted to the proof of Theorem 4.1. Let $G = H \times \operatorname{Alt}(\kappa)$; and let H_β, G_β be the β^{th} groups in the automorphism towers of H, G respectively. We will eventually prove by induction that $G_\beta = H_\beta \times \operatorname{Sym}(\kappa)$ for all $\beta \geq 1$. To accomplish this, we need to keep track of $\varphi[\operatorname{Alt}(\kappa)]$ for each automorphism φ of G_β. The next lemma shows that for all $\varphi \in \operatorname{Aut} G_\beta$, either $\varphi[\operatorname{Alt}(\kappa)] \leqslant H_\beta$ or $\varphi[\operatorname{Alt}(\kappa)] \leqslant \operatorname{Sym}(\kappa)$. The main point will be to eliminate the possibility that $\varphi[\operatorname{Alt}(\kappa)] \leqslant H_\beta$. This is straightforward when β is a successor ordinal. To deal with the case when β is a limit ordinal, we will make use of the result that $\operatorname{Alt}(\kappa)$ is strictly simple.

Lemma 4.3. *Suppose that A is a simple nonabelian normal subgroup of the direct product $H \times S$. Then either $A \leqslant H$ or $A \leqslant S$.*

Proof. Let $1 \neq g = xy \in A$, where $x \in H$ and $y \in S$. If $y = 1$, then the conjugacy class $g^A = x^A$ is contained in H, and so $A = \langle g^A \rangle \leqslant H$. So suppose that $y \neq 1$. Let $\pi : H \times S \to S$ be the canonical projection map. Then $1 \neq y \in \pi[A] \leqslant S$ and $\pi[A] \simeq A$. Hence there exists an element $z \in \pi[A] \leqslant S$ such that $zyz^{-1} \neq y$. Since

$A \trianglelefteq H \times S$, it follows that

$$1 \neq zyz^{-1}y^{-1} = zxyz^{-1}y^{-1}x^{-1} = zgz^{-1}g^{-1} \in A \cap S.$$

Arguing as above, we now obtain that $A \leqslant S$. □

Definition 4.4. Let H be a subgroup of the group G. Then H is said to be an *ascendant subgroup* of G if there exist an ordinal γ and a set of subgroups $\{H_\beta \mid \beta \leq \gamma\}$ such that the following conditions are satisfied.

(a) $H_0 = H$ and $H_\gamma = G$.

(b) If $\beta < \gamma$, then $H_\beta \trianglelefteq H_{\beta+1}$.

(c) If δ is a limit ordinal such that $\delta \leq \gamma$, then $H_\delta = \bigcup_{\beta<\delta} H_\beta$.

Definition 4.5. A group A is *strictly simple* if it has no nontrivial proper ascendant subgroups.

Theorem 4.6 (Macpherson and Neumann [7])**.** *For each $\kappa \geq \omega$, the alternating group* $\mathrm{Alt}(\kappa)$ *is strictly simple.*

□

Lemma 4.7. *Suppose that H is a centreless group. Let $\tau = \tau(H)$ and let H_τ be the τ^{th} group in the automorphism tower of H. If A is a strictly simple normal subgroup of H_τ, then $A \leqslant H_0$.*

Proof. Let $\beta \leq \tau$ be the least ordinal such that $A \leqslant H_\beta$. First suppose that β is a limit ordinal. Then $A = \bigcup_{\gamma<\beta}(A \cap H_\gamma)$, and $A \cap H_\gamma \trianglelefteq A \cap H_{\gamma+1}$ for all $\gamma < \beta$. Consequently, if $\gamma < \beta$ is the least ordinal such that $A \cap H_\gamma \neq 1$, then $A \cap H_\gamma$ is a nontrivial ascendant subgroup of A. But this contradicts the assumption that A is strictly simple.

Next suppose that $\beta = \gamma + 1$ is a successor ordinal. Since $A \cap H_\gamma$ is a proper normal subgroup of A, it follows that $A \cap H_\gamma = 1$. Now notice that $A \leqslant H_{\gamma+1} \leqslant N_{H_\tau}(H_\gamma)$ and $H_\gamma \leqslant H_\tau = N_{H_\tau}(A)$, This implies that $[A, H_\gamma] \leqslant A \cap H_\gamma = 1$. (For example, see Lemma 1.1.3 of Suzuki [12].) But then $A \leqslant C_{H_{\gamma+1}}(H_\gamma) = 1$, which is a contradiction. The only remaining possibility is that $\beta = 0$. □

We will also make use of the well-known results that $\mathrm{Aut}(\mathrm{Alt}(\kappa)) = \mathrm{Sym}(\kappa)$ and $\mathrm{Aut}(\mathrm{Sym}(\kappa)) = \mathrm{Sym}(\kappa)$. (For example, see Theorem 11.4.8 of Scott [9].)

Proof of Theorem 4.1. Let $\tau = \tau(H)$; and let

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \dots H_\beta \trianglelefteq H_{\beta+1} \trianglelefteq \dots H_\tau = H_{\tau+1} = \dots$$

be the automorphism tower of of H. Let $G = H \times \mathrm{Alt}(\kappa)$; and let

$$G = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots G_\beta \trianglelefteq G_{\beta+1} \trianglelefteq \dots$$

be the automorphism tower of G. We will prove by induction on $\beta \geq 1$ that $G_\beta = H_\beta \times \mathrm{Sym}(\kappa)$.

First consider the case when $\beta = 1$. Let $\varphi \in \operatorname{Aut} G$ be any automorphism. Then $\varphi[\mathrm{Alt}(\kappa)]$ is a simple nonabelian normal subgroup of the direct product $G = H \times \mathrm{Alt}(\kappa)$. By Lemma 4.3, either $\varphi[\mathrm{Alt}(\kappa)] \leqslant H$ or $\varphi[\mathrm{Alt}(\kappa)] \leqslant \mathrm{Alt}(\kappa)$. Since $|\varphi[\mathrm{Alt}(\kappa)]| = \kappa > \theta = |H|$, it follows that $\varphi[\mathrm{Alt}(\kappa)] \leqslant \mathrm{Alt}(\kappa)$. As $\varphi[\mathrm{Alt}(\kappa)]$ is a normal subgroup of G, we must have that $\varphi[\mathrm{Alt}(\kappa)] = \mathrm{Alt}(\kappa)$. Note that $C_G(\mathrm{Alt}(\kappa)) = H$. Hence we must also have that $\varphi[H] = H$. It follows that

$$G_1 = \operatorname{Aut} G = \operatorname{Aut} H \times \operatorname{Aut}(\mathrm{Alt}(\kappa)) = H_1 \times \mathrm{Sym}(\kappa).$$

Next suppose that $\beta = \gamma+1$ and that $G_\gamma = H_\gamma \times \mathrm{Sym}(\kappa)$. Let $\varphi \in \operatorname{Aut} G_\gamma$ be any automorphism. By Lemma 4.3, either $\varphi[\mathrm{Alt}(\kappa)] \leqslant H_\gamma$ or $\varphi[\mathrm{Alt}(\kappa)] \leqslant \mathrm{Sym}(\kappa)$. As $\mathrm{Alt}(\kappa)$ is a strictly simple group, Lemma 4.7 implies that $\varphi[\mathrm{Alt}(\kappa)] \leqslant \mathrm{Sym}(\kappa)$. Since $\varphi[\mathrm{Alt}(\kappa)]$ is a simple normal subgroup of G_γ, it follows that $\varphi[\mathrm{Alt}(\kappa)] = \mathrm{Alt}(\kappa)$. Using the facts that $C_{G_\gamma}(\mathrm{Alt}(\kappa)) = H_\gamma$ and $C_{G_\gamma}(H_\gamma) = \mathrm{Sym}(\kappa)$, we now see that $\varphi[H_\gamma] = H_\gamma$ and $\varphi[\mathrm{Sym}(\kappa)] = \mathrm{Sym}(\kappa)$. Hence

$$G_{\gamma+1} = \operatorname{Aut} G_\gamma = \operatorname{Aut} H_\gamma \times \operatorname{Aut}(\mathrm{Sym}(\kappa)) = H_{\gamma+1} \times \mathrm{Sym}(\kappa).$$

Finally no difficulties arise when β is a limit ordinal. □

References

[1] N. G. De Bruijn, *Embedding theorems for infinite groups*, Indag. Math. **19** (1957), 560–569.

[2] E. Fried and J. Kollár, *Automorphism groups of fields*, in Universal Algebra (E. T. Schmidt et al., eds.), Coloq. Math. Soc. Janos Boyali, vol **24**, 1981, pp. 293–304.

[3] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications **42**, Cambridge University Press, 1993.

[4] J. A. Hulse, *Automorphism towers of polycyclic groups*, J. Algebra **16** (1970), 347–398.

[5] D. W. Kueker, *Definability, automorphisms and infinitary languages*, in The Syntax and Semantics of Infinitary Languages (ed. J. Barwise), Lecture Notes in Math. **72**, Springer, Berlin, 1968, pp. 152–165.

[6] K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1980.

[7] H. D. Macpherson and P. M. Neumann, *Subgroups of infinite symmetric groups*, J. London Math. Soc. (2) **42** (1990), 64–84.

[8] A. Rae and J. E. Roseblade, *Automorphism towers of extremal groups*, Math. Z. **117** (1970), 70–75.

[9] W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[10] S. Shelah, *First order theory of permutation groups*, Israel J. Math. **14** (1973), 149–162.

[11] S. Solecki, *Polish group topologies*, preprint (1997).

[12] M. Suzuki, *Group Theory II*, Grundlehren der mathematischen Wissenschaften **248**, Springer-Verlag 1986.

[13] S. Thomas, *The automorphism tower problem*, Proc. Amer. Math. Soc. **95** (1985), 166–168.

[14] S. Thomas, *The automorphism tower problem II*, to appear in Israel J. Math.

[15] H. Wielandt, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Z. **45** (1939), 209–244.

Department of Mathematics, Ohio University, Athens, Ohio 45701

Mathematics Department, The Hebrew University, Jerusalem, Israel

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019