THE CONDITION IN THE TRICHOTOMY THEOREM IS OPTIMAL

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ABSTRACT. We show that the assumption $\lambda > \kappa^+$ in the Trichotomy Theorem cannot be relaxed to $\lambda > \kappa$.

1. Introduction

The Trichotomy Theorem specifies three alternatives for the structure of an increasing sequence of ordinal functions modulo an ideal on an infinite cardinal κ — provided the sequence has regular length λ and λ is at least κ^{++}.

The natural context of the Trichotomy Theorem is, of course, pcf theory, where a sequence of ordinal functions on κ usually has length which is larger than $\kappa^{+\kappa}$. However, the trichotomy theorem has already been applied in several proofs to sequences of length κ^{+n}, $(n \geq 2)$ (see [4], [1] and [3]).

Therefore, a natural question to ask is, whether the Trichotomy Theorem is valid also for sequences of length κ^+, namely, whether the lower bound on the length of the sequence can be lowered by one cardinal.

Below we show that the assumption $\lambda \geq \kappa^{++}$ in the Trichotomy Theorem is tight. For every infinite κ, we construct an ideal I over κ and $<_I$-increasing sequence $\overline{f} \subseteq \text{On}^\kappa$ so that all three alternatives in the Trichotomy theorem are violated by $\overline{f}$.

2. The Counter-example

Let κ be an infinite cardinal. Denote by On^κ the class of all functions from κ to the ordinal numbers.

Let I be an ideal over κ. We write $f <_I g$, for $f, g \in \text{On}^k$, if $\{i < \kappa : f(i) \geq g(i)\} \in I$ and we write $f \leq_I g$ if $\{i < \kappa : f(i) > g(i)\} \in I$. A sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \text{On}^\kappa$ is $<_I$-*increasing* if $\alpha < \beta < \lambda$ implies that $f_\alpha <_I f_\beta$ and is $<_I$-*decreasing* if $\alpha < \beta < \lambda$ implies that $f_\beta <_I f_\alpha$.

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A function $f \in \mathrm{On}^k$ is a least upper bound mod I of a sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle \subseteq \mathrm{On}^\kappa$ if $f_\alpha <_I f$ for all $\alpha < \lambda$ and whenever $f_\alpha <_I g$ for all α then $f \le_I g$. A function $f \in \mathrm{On}^\kappa$ is an *exact upper bound* of $\overline{f}$ if $f_\alpha \le f$ for all $\alpha < \lambda$, and whenever $g <_I f$, there exists $\alpha < \lambda$ such that $g <_I f_\alpha$. For subsets t, s of κ, write $t \subseteq_I s$ if $s - t \in I$.

The dual filter I^* of an ideal I over κ is the set of all complements of members of I. The relations $\le_I$, $<_I$ and $\subseteq_I$ will also be written as $\le_{I^*}$, $<_{I^*}$ and $\subseteq_{I^*}$.

Let us quote the theorem under discussion:

Theorem 1. *(The Trichotomy Theorem)*

Suppose $\lambda \ge \kappa^{++}$ is regular, I is an ideal over k and $\overline{f} = \langle f_{\underline{\alpha}} : \alpha < \lambda\rangle$ is a $<_I$-increasing sequence of ordinal functions on k. Then $\overline{f}$ satisfies one of the following conditions:

(1) *$\overline{f}$ has an exact upper bound f with $\mathrm{cf}\, f(i) > \kappa$ for all $i < \kappa$;*
(2) *there are sets $S(i)$ for $i < \kappa$ satisfying $|S(i)| \le \kappa$ and an ultrafilter U over k extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod_{i<\kappa} S(i)$ and $\beta < \lambda$ such that $f_\alpha <_U h_\alpha <_U f_\beta$.*
(3) *there is a function $g : \kappa \to On$ such that the sequence $\overline{t} = \langle t_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I, where $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$.*

Proofs of the Trichotomy Theorem are found in [5], II,1.2, in [3] or in the future version of [2].

Theorem 2. *For every infinite κ there exists an ultrafilter U over κ and a $<_U$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \kappa^+\rangle \subseteq On^\kappa$ such conditions 1, 2 and 3 in the Trichotomy Theorem fail for $\overline{f}$.*

Proof. Let $\lambda = \kappa^+$.

Let us establish some notation.

We recall that every ordinal has an expansion in base λ, namely can be written as a unique finite sum $\sum_{k\le l} \lambda^{\beta_l}\alpha_l$ so that $\beta_{k+1} < \beta_k$ and $\alpha_k < \lambda$. We limit ourselves from now on to ordinalz $\zeta < \lambda^\omega$. For such ordinals, the expansion in base κ contains only finite powers of λ (that is, every β_k is a natural number).

We agree to write an ordinal $\zeta = \lambda^l\alpha_l + \lambda^{l-1}\alpha_{l-1} + \cdots + \alpha_0$ simply as a finite sequence $\alpha_l\alpha_{l-1}\ldots\alpha_0$. We identify an expansion with λ digits with one with $n > l$ digits by adding zeroes on the left. If $\zeta = \alpha_l\alpha_{l-1}\ldots\alpha_0$, we call α_k, for $k \le l$, the *k-th digit* in the expansion of ζ.

For $\alpha < \lambda$ and an integer l, define:

$$A^l_\alpha = \{\alpha_l\alpha_{l-1}\dots\alpha_0 : \alpha_k < \alpha \text{ for all } k \le l\} \tag{1}$$

A^l_α is the set of all ordinals below λ^ω whose expansion in base λ contains $l+1$ or fewer digits from α.

Fact 3. *For all $\alpha < \lambda$ and $l < \omega$,*

(1) $\bigcup_{\alpha<\lambda} A^l_\alpha = \lambda^{l+1}$
(2) *The ordinal* $\sum_{k=0}^{l} \lambda^k = \overbrace{\alpha\alpha\dots\alpha}^{l+1}$ *is the maximal element in* $A^l_{\alpha+1}$
(3) *if* $\zeta = \alpha_l\alpha_{l-1}\dots\alpha_0 \in A^l_\zeta$ *is not maximal in* $A^l_{\alpha+1}$*, then the immediate successor of* ζ *in* $A^n_{\alpha+1}$ *is obtained from* ζ *as follows: let* k *be the first* $k \le l$ *for which* $\alpha_k < \alpha$*. Replace* α_k *by* $\alpha_k + 1$ *and replace* α_m *by* 0 *for all* $m < k$

Fix a partition $\{X_n : n < \omega\}$ of κ with $|X_n| = \kappa$ for all n. Let $n(i)$, for $i < \kappa$, be the unique n for which $i \in X_n$.

By induction on $\alpha < \kappa^+$, define $f_\alpha : \kappa \to \text{On}$ so that:

(1) $f_\alpha(i) \in A^{n(i)}_{\alpha+1} - A^{n(i)}_\alpha$
(2) For all $n, l < \omega$ and finite, strictly increasing, sequences $\langle \alpha_k : k \le l\rangle \subseteq \lambda$ it holds that for every sequence $\langle \zeta_k : k \le l\rangle$ which satisfies $\zeta_k \in A^n_{\alpha_k+1} - A_{\alpha_k}$, there are κ many $i \in X_n$ for which $\bigwedge_{k\le l} f_{\alpha_k}(i) = \zeta_k$.

The first item above says that $f_\alpha(i)$ is an ordinal below λ^ω whose expansion in base λ has $\le n(i)$ digits, at least one of which is α. The second item says that every possible finite sequence of values $\langle \zeta_k : k < l\rangle$ is realized κ many times as $\langle f_{\alpha_\kappa}(i) : k < \lambda\rangle$ for an arbitrary increasing sequence $\langle \alpha_\kappa : k < l\rangle$.

The induction required to define the sequence is straightforward.

Define now, for every $\alpha < \kappa^+$, a function $g_\alpha : \kappa \to \text{On}$ as follows:

$$g_\alpha(i) = \min[(A^n_{\alpha+1}(i) \cup \{\lambda +^{n(i)+1}\}) - f_\alpha(i)] \tag{2}$$

Since $f_\alpha(i) < \lambda^{n(i)+1}$ for $i \in X_n$, the definition is good. If $f_\alpha(i)$ is not maximal in $A^{n(i)}_{\alpha+1}$, then $g_\alpha(i)$ is the *immediate successor of* $f_\alpha(i)$ *in* $A^{n(i)}_{\alpha+1}$. Let us make a note of that:

Fact 4. *There are no members of* $A^{n(i)}_{\alpha+1}$ *between* $f_\alpha(i)$ *and* $g_\alpha(i)$

We have defined two sequences:

$$\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$$
$$\overline{g} = \langle g_\alpha : \alpha < \lambda \rangle$$

Next we wish to find an ideal modulo which $\overline{f}$ is $<_I$-increasing and $\overline{g}$ is a $<_I$-decreasing sequence of upper bounds of $\overline{f}$.

Claim 5. *For every finite increasing sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_l < \lambda$ there exists $i < \kappa$ such that for all $k < l$*

$$f_{\alpha_k}(i) < f_{\alpha_{k+1}}(i) < g_{\alpha_{k+1}}(i) < g_{\alpha_k}(i) \tag{3}$$

Proof. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_l < \lambda$ is given and choose $n > l$. Let $\zeta_0 = \overbrace{\alpha_0\alpha_0 \ldots \alpha_0}^{l+1} \in A^n_{\alpha_0}$. Let ζ_{k+1} be obtained from ζ_k by replacing the first $l+1-k$ digits of ζ_κ by α_{k+1}:

$$\begin{gathered}
\alpha_0\alpha_1 \ldots \alpha_k\alpha_{k+1} \ldots \alpha_l = \zeta_l \\
\vdots \\
\alpha_0\alpha_1 \ldots \alpha_{k-1}\alpha_k \ldots \alpha_k = \zeta_k \\
\vdots \\
\alpha_0\alpha_1 \;\ldots\ldots\; \alpha_1 \ldots \alpha_1 = \zeta_1 \\
\alpha_0\alpha_0 \;\ldots\ldots\; \alpha_0 \ldots \alpha_0 = \zeta_0
\end{gathered}$$

Thus $\zeta_0 < \zeta_1 < \ldots < \zeta_l$ and $\zeta_k \in A^l_{\alpha_k} \subseteq A^n_{\alpha_k}$ is not maximal in $A^n_{\alpha_k}$ (because $n > l$). Let ξ_k be the immediate successor of ζ_k in $A^n_{\beta_k}$.

By Fact 3 above, we have

$$\begin{gathered}
1\overbrace{0 \;\ldots\quad \ldots\; 0 \;\ldots\quad \ldots\; 0}^{l+1} = \xi_0 \\
(\alpha_0 + 1)\, 0 \;\ldots\ldots\ldots\ldots 0 = \xi_1 \\
\vdots \\
\alpha_0\alpha_1 \ldots (\alpha_{k-1} + 1) 0 \ldots 0 = \xi_k \\
\vdots \\
\alpha_0\alpha_1 \;\ldots\ldots\; (\alpha_{l-1} + 1) 0 = \xi_l
\end{gathered}$$

Therefore $\zeta_0 < \zeta_1 < \ldots \zeta_l < \xi_l < \xi_{l-1} < \ldots < \xi_0$. To complete the proof it remains to find some $i \in X_n$ for which $f_{\alpha_k}(i) = \zeta_\kappa$ for

$k \leq l$, and, consequently, by the definition (2) above, $g_{\alpha_k}(i) = \xi_k$. The existence of such $i \in X_n$ is guaranteed by the second condition in the definition of $\overline{f}$. □

For $\alpha < \beta < \lambda$, let

$$C_{\alpha,\beta} = \{i < \kappa : f_\alpha(i) < f_\beta(i) < g_\beta(i) < g_\alpha(i)\} \tag{4}$$

Claim 6. *$\{C_{\alpha,\beta} : \alpha < \beta < \kappa^+\}$ has the finite intersection property*

Proof. Suppose that $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_l, \beta_l$ are given and $\alpha_k < \beta_k < \lambda$ for $k \leq l$. Let $\langle \gamma_m : m < m(*) \rangle$ be the increasing enumeration of $\bigcup_{k \leq l}\{\alpha_k, \beta_k\}$. To show that $\bigcap_{k \leq l} C_{\alpha_k,\beta_k}$ is not empty, it suffices to find some $i < \kappa$ for which the sequence $g_{\gamma_m}(i)$ is decreasing in m and $f_{\gamma_m}(i)$ is increasing in m. The existence of such an $i < \kappa$ follows from Claim 5. □

Let U be any ultrafilter extending $\{C_{\alpha,\beta} : \alpha < \beta < \lambda\}$. Since for every $\alpha < \beta$ it holds that $f_\alpha <_U f_\beta <_U g_\beta <_U g_\alpha$, we conclude that $\overline{f}$ is $<_U$-increasing, that $\overline{g}$ is $<_U$-decreasing and that every g_α is an upper bound of $\overline{f}$ mod U.

Claim 7. *There is no exact upper bound of $\overline{f}$ mod U.*

Proof. It suffices to check that there is no $h \in \mathrm{On}^\kappa$ that satisfies $f_\alpha <_U h <_U g_\alpha$ for all $\alpha < \kappa^+$. Suppose, then, that $h \in \mathrm{On}^k$ satisfies this. Since $h <_U g_0$, we may assume that $g(i) < \lambda^{n(i)+1}$ for all $i < \kappa$ (by changing h on a set outside of U).

Let $i < \kappa$ be arbitrary. Since $\bigcup_{\alpha<\lambda} A^{n(i)}_\alpha = \lambda^{n(i)+1}$, there is some $\alpha(i)$ so that $h(i) \in A^n_\alpha(i)$. By regularity of λ it follows that there is some $\alpha(*) < \lambda$ such that $h(i) \in A^{n(i)}_{\alpha(*)}$ for all $i < \kappa$. By our assumption about h, $f_{\alpha(*)} <_U h <_U g_{\alpha(*)}$. Thus, there is some $i < \kappa$ for which $f_{\beta(*)}(i) < h(i) < g_{\beta(*)}(i)$. However, all three values belong to $A^{n(i)}_{\alpha(*)+1}$, while by Fact 4 there are no members of $A^{n(i)}_{\alpha(*)+1}$ between $f_{\beta(*)(i)}$ and $g_{\alpha(*)}(i)$ — a contradiction. □

Claim 8. *there are no sets $S(i) \subseteq On$ for $i < \kappa$ which satisfy condition 2 in the trichotomy for $\overline{f}$ and U.*

Proof. Suppose that $S(i)$, for $i < \kappa$, and $h_\alpha \in \prod_{i<\kappa} S(i)$ satisfy 2. in the Trichotomy Theorem. Find $\alpha < \lambda$ such that $S(i) \subseteq A^{n(i)}_\alpha$ for all i. Thus $f_\alpha <_U h_\alpha <_U g_\alpha$ — contradiction to 4. □

Claim 9. *there is no $g : \kappa \to On$ such that $g, \overline{f}$ and the dual of U satisfy condition 3. in the Trichotomy Theorem.*

Proof. Let $g : \kappa \to On$ be arbitrary, and let $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$. As $\overline{f}$ is $<_U$-increasing, for every $\alpha < \beta < \lambda$ necessarily $t_\alpha \subseteq_U t_\beta$. Since U is an ultrafilter, every $\subseteq_U$-increasing sequence of sets stabilizes. □

□

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