ANTICHAINS IN PRODUCTS OF LINEAR ORDERS

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Abstract. We show that:

(1) For many regular cardinals λ (in particular, for all successors of singular strong limit cardinals, and for all successors of singular ω-limits), for all $n \in \{2, 3, 4, \dots\}$: There is a linear order L such that L^n has no (incomparability-)antichain of cardinality λ, while L^{n+1} has an antichain of cardinality λ.
(2) For any nondecreasing sequence $\langle \lambda_n : n \in \{2, 3, 4, \dots\}\rangle$ of infinite cardinals it is consistent that there is a linear order L such that, for all n: L^n has an antichain of cardinality λ_n, but no antichain of cardinality λ_n^+.

1. Introduction

For any nontrivial linear ordering L and any natural number $n > 1$, the set L^n (ordered by the "pointwise" or "product" order) is partial order which is not linear any more. A natural measure for its non-linearity is obtained by considering the possible sizes (cardinalities) of antichains in L^n (that is, sets of pairwise incomparable elements).

Haviar and Ploščica in [2] asked: Can there be a linear ordering L and an infinite cardinal number λ such that for some natural number $n > 1$, the partial order L^n does not have antichains of size λ, while L^{n+1} has such antichains?

Farley [1] has constructed, for any *singular* cardinal κ, a linear order L of size κ such that L^3 has an antichain of cardinality κ while L^2 does not.

We will be mainly interested in this question for *regular* cardinals.

First we show in ZFC that there are many successor cardinals λ (including $\aleph_{\omega+1}$) with the following property:

> For every $n \geq 2$ there is a linear order J of size λ such that J^n has no antichain of size λ, while J^{n+1}) does.

This proof is given in section 2. It uses a basic fact from pcf theory.

In section 3 we show that partial orders L^n can all be very different as far as the possible sizes of antichains in these orders are concerned. More

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precisely, we show that for any nondecreasing sequence of infinite cardinals $\langle \lambda_n : 2 \leq n < \omega \rangle$ there is a cardinal-preserving forcing extension of the universe in which we can find a linear order L such that for all $n \in \{2, 3, \ldots\}$: in L^n there are antichains of cardinality λ_n, but no larger ones.

For example, it is consistent that there is a linear order L such that L^2 has no uncountable antichain, while L^3 does.

Here we use forcing. The heart of this second proof is the well-known Δ-system lemma.

2. A ZFC proof

Let μ be a regular cardinal. We will write D_μ for the filter of cobounded sets, i.e.,

$$D_\mu = \{A \subseteq \mu : \exists i < \mu \; \mu \setminus i \subseteq A\}$$

For any sequence $\langle \lambda_i : i < \mu \rangle$ of cardinals, we write $\prod_{i<\mu} \lambda_i$ for the the set of all functions f with domain μ satisfying $f(i) < \lambda_i$ for all i. The relation $f \sim_{D_\mu} g \Leftrightarrow \{i < \mu : f(i) = g(i)\} \in D_\mu$ is an equivalence relation. We call the quotient structure $\prod_i \lambda_i / D_\mu$ (and we often do not distinguish between a function f and its equivalence class). $\prod_i \lambda_i / D_\mu$ is partially ordered by the relation

$$f <_{D_\mu} g \quad \text{iff} \quad \{i < \mu : f(i) < g(i)\} \in D_\mu$$

Definition 2.1. For any partial order $(P, \leq)$ and any regular cardinal λ we say $\lambda = \mathrm{tcf}(P)$ ("λ is the true cofinality of P") iff there is an increasing sequence $\langle p_i : i < \lambda \rangle$ such that $\forall p \in P \, \exists i < \lambda : p \leq p_i$.

Remark 2.2.
1. For any partial order P there is at most one regular cardinal λ which can be the true cofinality of P, that is: if $\langle p_i : i < \lambda \rangle$ and $\langle p'_j : j < \lambda' \rangle$ are both as above, then $\lambda = \lambda'$.
2. Every linear order has a true cofinality
3. If P has true cofinality, then P is upward directed
4. There are partial orders which are upward directed but have no true cofinality, for example $\omega \times \omega_1$.

Theorem 2.3. *Assume that*
1. $\langle \lambda_i : i < \mu \rangle$ *is an increasing sequence of regular cardinals*
2. *For each* $j < \mu$, $\left|\prod_{i<j} \lambda_i\right| < \lambda_j$
3. λ *is regular and* $\mathrm{tcf}(\prod_{i<\mu} \lambda_i / D_\mu) = \lambda$,
4. $n \geq 2$.

Then there is a linear order J *of size* λ *such that*
- J^{n+1} *has an antichain of size* λ
- J^n *has no antichain of size* λ

To show that there are indeed many cardinals λ of the form $\prod_{i<\mu} \lambda_i / D_\mu$ as in our theorem, we use theorem II.1.5 from [3, page 50]:

Theorem 2.4. *If κ is singular and $\mathrm{cf}\kappa = \mu$, then there is a strictly increasing sequence $\langle \lambda_i : i < \mu \rangle$ of regular cardinals such that $\kappa = \sum_{i<\mu} \lambda_i$ and $\left(\prod_{i<\mu} \lambda_i, <_{D_\mu}\right)$ has true cofinality κ^+.*

It is clear that in the above theorem we can replace $\langle \lambda_i : i < \mu \rangle$ by any increasing cofinal subsequence $\langle \lambda_{f(j)} : j < \mu \rangle$.

Conclusion 2.5. *Whenever $\lambda = \kappa^+$, where κ is a singular strong limit cardinal, or even only the following holds:*

$$\kappa \text{ is singular, and } \forall \kappa' < \kappa : (\kappa')^{<\mathrm{cf}(\kappa)} < \kappa,$$

then we can find a sequence $\langle \lambda_i : i < \mathrm{cf}(\kappa) \rangle$ as in the assumption of theorem 2.3.

For example, if $\lambda = \aleph_{\omega+1}$, then there is an increasing sequence $\langle n_k : k \in \omega \rangle$ of natural numbers such that $\mathrm{tcf}(\prod_{k\in\omega} \aleph_{n_k}/D_\omega) = \aleph_{\omega+1}$. So theorem 2.3 implies for any $n > 1$ there is a linear order J such that J^n has no antichain of size $\aleph_{\omega+1}$, whereas J^{n+1} has one.

Proof. Let $\mu = \mathrm{cf}(\kappa)$. We can start with a sequence $\langle \lambda_i : i < \mu \rangle$ such that $\prod_{i<\mu} \lambda_i/D_\mu$ has true cofinality $\lambda = \kappa^+$. Choose $f : \mu \to \mu$ increasing cofinal such that $\prod_{j<i} \lambda_{f(j)} < \lambda_{f(i)}$ for all $i < \mu$, then $\langle \lambda_{f(j)} : j < \mu \rangle$ is as required. □

The proof of theorem 2.3 will occupy the rest of this section. The assumption of theorem 2.3 says $\mathrm{tcf} \prod_{i<\mu} \lambda_i/D_\mu) = \lambda$, so we may fix a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod_{i<\mu} \lambda_i$ such that for all $\alpha < \beta < \lambda$ we have $f_\alpha <_{D_\mu} f_\beta$.

We start by writing $\mu = \bigcup_{\ell=0}^n A_\ell$ as a disjoint union of $n+1$ many D_μ-positive (i.e., unbounded) sets. For $\ell = 0, \dots, n$ we define a linear order $<_\ell$ on λ as follows:

Definition 2.6. For any two distinct functions $f, g \in \prod_{i<\mu} \lambda_i$ we define

$$d(f,g) = \sup\{i < \mu : f{\restriction}i = g{\restriction}i\} = \max\{i < \mu : f{\restriction}i = g{\restriction}i\} \tag{1}$$

That is: if $f \neq g$, then $d(f,g) = \min\{j : f(j) \neq g(j)\}$ is the first point where f and g differ.

For $\alpha, \beta \in \lambda$ we define $\alpha <_\ell \beta$ iff:

$$\begin{aligned} &\text{letting } i := d(f_\alpha, f_\beta), \\ &\text{either } i \in A_\ell \text{ and } f_\alpha(i) < f_\beta(i) \\ &\text{or } i \notin A_\ell \text{ and } f_\alpha(i) > f_\beta(i) \end{aligned} \tag{2}$$

We leave it to the reader to check that $<_\ell$ is indeed a linear order on λ.

We now define J to be the "ordinal sum" of all the orders $<_\ell$:

Definition 2.7. Let

$$J = \bigcup_{\ell=0}^n \{\ell\} \times (\lambda, <_\ell)$$

with the "lexicographic" order, i.e., $(\ell_1, \alpha_1) < (\ell_2, \alpha_2)$ iff $\ell_1 < \ell_2$, or $\ell_1 = \ell_2$ and $\alpha_1 <_{\ell_1} \alpha_2$.

Claim 2.8. J^{n+1} has an antichain of size λ.

Proof. Let $\vec{t}_\alpha = \langle (0,\alpha), \ldots, (n,\alpha)\rangle \in J^{n+1}$.

For any $\alpha \neq \beta$ we have to check that $\vec{t}_\alpha$ and $\vec{t}_\beta$ are incomparable. Let $i^* = d(f_\alpha, f_\beta)$, and find ℓ^* such that $i^* \in A_{\ell^*}$. Wlog assume $f_\alpha(i^*) < f_\beta(i^*)$. Then $\alpha <_{\ell^*} \beta$, but $\alpha >_\ell \beta$ for all $\ell \neq \ell^*$, i.e., $(\ell^*, \alpha) <_J (\ell^*, \beta)$, but $(\ell, \alpha) >_J (\ell, \beta)$ for all $\ell \neq \ell^*$.

□

Finishing the proof of 2.3. It remains to show that J^n does not have an antichain of size λ. Towards a contradiction, assume that $\langle \vec{t}_\beta : \beta < \lambda\rangle$ is an antichain in J^m, $m \leq n$, and m as small as possible. Let $\vec{t}_\beta = (t_\beta(1), \ldots, t_\beta(m)) \in J^m$. For $k = 1, \ldots, m$ we can find functions ℓ_k, ξ_k such that

$$\forall \beta < \lambda \; \forall k: \; t_\beta(k) = (\ell_k(\beta), \xi_k(\beta))$$

Thinning out we may assume that the functions $\ell_1, \ldots, \ell_m$ are constant. We will again write $\ell_1, \ldots, \ell_m$ for those constant values.

We may also assume that for each k the function $\beta \mapsto \xi_k(\beta)$ is either constant or strictly increasing. If any of the functions ξ_k is constant we get a contradiction to the minimality of m, so all the ξ_k are strictly increasing. So we may moreover assume that $\beta < \gamma$ implies $\xi_k(\beta) < \xi_{k'}(\gamma)$ for all k, k', and in particular $\beta \leq \xi_k(\beta)$ for all β, k.

Now define $g^+_\beta, g^-_\beta \in \prod_{i<\mu} \lambda_i$ for every $\beta < \lambda$ as follows:

$$\begin{aligned} g^+_\beta(i) &= \max(f_{\xi_1(\beta)}(i), \ldots, f_{\xi_n(\beta)}(i)) \\ g^-_\beta(i) &= \min(f_{\xi_1(\beta)}(i), \ldots, f_{\xi_n(\beta)}(i)) \end{aligned} \tag{3}$$

Claim. The set

$$C := \{i < \mu : \; \forall \beta \; \{g^-_\gamma(i) : \gamma > \beta\} \text{ is unbounded in } \lambda_i\} \tag{4}$$

is in the filter D_μ.

Proof of the Claim. Let $S = \mu \setminus C$, or more explicitly:

$$S := \{i < \mu : \exists \beta < \lambda \, \exists s < \lambda_i \;\; \{g^-_\gamma(i) : \gamma > \beta\} \subseteq s\}$$

We have to show that S is a bounded set.

For each $i \in S$ let $\beta_i < \lambda$ and $h(i) < \lambda_i$ be such that $\{g^-_\gamma(i) : \gamma > \beta_i\} \subseteq h(i)$. Let $\beta^* = \sup\{\beta_i : i \in S\} < \lambda$, and extend h arbitrarily to a total function on μ. Since the sequence $\langle f_\alpha : \alpha < \lambda\rangle$ is cofinal in $\prod_{i<\mu} \lambda_i / D_\mu$, we can find $\gamma > \beta^*$ such that $h <_{D_\mu} f_\gamma$.

We have $\gamma \leq \xi_k(\gamma)$ for all $k \in \{1, \ldots, m\}$, so the sets

$$X_k := \{i < \mu : h(i) < f_{\xi_k(\gamma)}(i)\}$$

are all in D_μ. Now if S were positive mod D_μ (i.e., unbounded), then we could find $i_* \in S \cap X_1 \cap \cdots \cap X_m$. But then $i^* \in X_1 \cap \cdots \cap X_m$ implies

$$h(i_*) < g^-_\gamma(i_*),$$

and $i \in S$ implies

$$g^-_\gamma(i_*) < h(i_*),$$

a contradiction.

This shows that C is indeed a set in the filter D_μ. □

We will now use the fact that $m < n+1$. Let

$$\ell^* \in \{0, \dots, n\} \setminus \{\ell_1, \dots, \ell_m\}.$$

Since A_{ℓ^*} is positive mod D_μ, we can pick

$$i^* \in A_{\ell^*} \cap C \tag{5}$$

Using the fact that $i^* \in C$ and definition (4) we can find a sequence $\langle \beta_\sigma : \sigma < \lambda_{i^*} \rangle$ such that

$$\forall \sigma < \sigma' < \lambda_{i^*}: \; g^+_{\beta_\sigma}(i^*) < g^-_{\beta_{\sigma'}}(i^*), \tag{6}$$

We now restrict our attention from $\langle \vec{t}_\beta : \beta < \lambda \rangle$ to the subsequence $\langle \vec{t}_{\beta_\sigma} : \sigma < \lambda_{i^*} \rangle$; we will show that this sequence cannot be an antichain. For notational simplicity only we will assume $\beta_\sigma = \sigma$ for all $\sigma < \lambda_{i^*}$.

Recall that $\vec{t}_\sigma = \langle (\ell_1, \xi_1(\sigma)), \dots, (\ell_m, \xi_m(\sigma)) \rangle$. For each $\sigma < \lambda_{i^*}$ define

$$\vec{x}_\sigma := \langle f_{\xi_1(\sigma)} \restriction i^*, \dots, f_{\xi_n(\sigma)} \restriction i^* \rangle \in \left(\prod_{j < i^*} \lambda_j \right)^m .$$

Since $\left| \prod_{j<i^*} \lambda_j \right| < \lambda_{i^*}$, there are only $< \lambda_{i^*}$ many possible values for $\vec{x}_\sigma$, so we can find $\sigma_1 < \sigma_2 < \lambda_{i^*}$ such that $\vec{x}_{\sigma_1} = \vec{x}_{\sigma_2}$.

Now note that by (3) and (6) we have

$$f_{\xi_k(\sigma_1)}(i^*) \le g^+_{\sigma_1}(i^*) < g^-_{\sigma_2}(i^*) \le f_{\xi_k(\sigma_2)}(i^*). \tag{7}$$

Hence $d(f_{\xi_k(\sigma_1)}, f_{\xi_k(\sigma_2)}) = i^*$ for $k = 1, \dots, m$.

Since $i^* \in A_{\ell^*}$ we have for all k: $i^* \notin A_{\ell_k}$. From (1), (2), (7) we get

$$\xi_k(\sigma_1) <_{\ell_k} \xi_k(\sigma_2) \qquad \text{for } k = 1, \dots, m.$$

Hence $(\ell_k, \xi_k(\sigma_1)) < (\ell_k, \xi_k(\sigma_2))$ for all k, which means $\vec{t}_{\sigma_1} < \vec{t}_{\sigma_2}$. □

3. Consistency

Theorem 3.1. *Assume $\aleph_0 \le \lambda_2 \le \lambda_3 \le \cdots$, where $\lambda_n^{<\kappa} = \lambda_n$ for all n, and $\kappa^{<\kappa} = \kappa$. Then there is a forcing notion $\mathbb{P}$ which satisfies the κ^+-cc and is κ-complete, and a $\mathbb{P}$-name $\underset{\sim}{I}$ of a subset of 2^κ (where 2^κ is endowed with the lexicographic order, which is inherited by $\underset{\sim}{I}$) such that*

$$\Vdash_{\mathbb{P}} \forall n > 1 : \underset{\sim}{I}^n \text{ has antichains of size } \lambda_n, \text{ but no larger ones.}$$

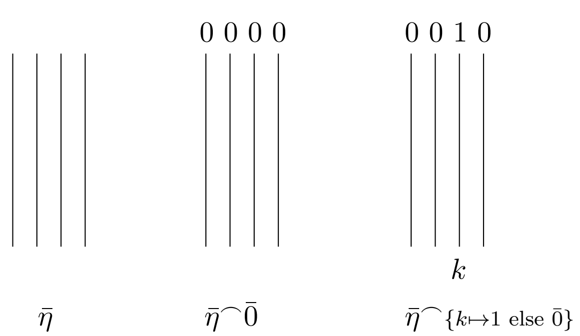
Remark 3.2. At first reading, the reader may want to consider the special case $\kappa = \omega$, $\lambda_{n+2} = \aleph_n$. Note that 2^ω is order isomorphic to the Cantor set, a subset of the real line $\mathbb{R}$, so we obtain as a special case of theorem 3.1:

> Consistently, there is a set $I \subseteq \mathbb{R}$ such that for each n, I^{n+1} admits much larger antichains than I^n.

Notation 3.3.
1. We let $\lambda_1 = 0$, $\lambda_\omega = \sup\{\lambda_n : n < \omega\}$.
2. It is understood that 2^α is linearly ordered lexicographically, and $(2^\alpha)^m$ is partially ordered by the pointwise order.
3. For $\alpha \le \beta \le \kappa$, $\eta \in 2^\alpha$, $\nu \in 2^\beta$, we define
$$\eta \trianglelefteq \nu \quad \text{iff} \quad \nu \text{ extends } \eta, \text{ i.e., } \eta \subseteq \nu$$
4. For $\bar\eta = \langle \eta(0), \ldots, \eta(n-1)\rangle \in (2^\alpha)^n$, $\bar\nu = \langle \nu(0), \ldots, \nu(n-1)\rangle \in (2^\beta)^n$, we let
$$\bar\eta \trianglelefteq \bar\nu \text{ iff } \eta(0) \trianglelefteq \nu(0), \ldots, \eta(n-1) \trianglelefteq \nu(n-1).$$
5. For $\eta \in 2^\alpha$, $i \in \{0,1\}$ we write $\eta^\frown i$ for the element $\nu \in 2^{\alpha+1}$ satisfying $\eta \trianglelefteq \nu$, $\nu(\alpha) = i$.

Definition 3.4. Let $\bar\eta \in (2^\alpha)^m$, $k \in \{0, \ldots, m-1\}$, $m \ge 2$. We define $\bar\eta^\frown\bar 1$, $\bar\eta^\frown\bar 0$, $\bar\eta^\frown\{k\mapsto 1 \text{ else } \bar 0\}$, $\bar\eta^\frown\{k\mapsto 0 \text{ else } \bar 1\}$ in $(2^{\alpha+1})^m$ as follows: All four are $\trianglelefteq$-extensions of $\bar\eta$, and:

- $\bar\eta^\frown\bar 0(n) = \eta(n)^\frown 0$ for all $n < m$.
- $\bar\eta^\frown\bar 1(n) = \eta(n)^\frown 1$ for all $n < m$.
- $\bar\eta^\frown\{k\mapsto 0 \text{ else } \bar 1\}(n) = \eta(n)^\frown 1$ for all $n \ne k$, $\bar\eta^\frown\{k\mapsto 0 \text{ else } \bar 1\}(k) = \eta(n)^\frown 0$.
- $\bar\eta^\frown\{k\mapsto 1 \text{ else } \bar 0\}(n) = \eta(n)^\frown 0$ for all $n \ne k$, $\bar\eta^\frown\{k\mapsto 1 \text{ else } \bar 0\}(k) = \eta(n)^\frown 1$.



Fact 3.5.
1. If $\alpha \le \beta \le \kappa$, $\bar\eta, \bar\eta' \in (2^\alpha)^n$ are incomparable, $\bar\nu, \bar\nu' \in (2^\beta)^n$, $\bar\eta \trianglelefteq \bar\nu$, $\bar\eta' \trianglelefteq \bar\nu'$, then also $\bar\nu$ and $\bar\nu'$ are incomparable.

(2) $\bar\eta^\frown\bar 0 < \bar\eta^\frown\bar 1$.
(3) $\bar\eta^\frown\{k\mapsto 0 \text{ else } \bar 1\}$ and $\bar\eta^\frown\{k\mapsto 1 \text{ else } \bar 0\}$ are incomparable.

Definition 3.6. We let $\mathbb{P}$ be the set of all "conditions"

$$p = (u^p, \alpha^p, \langle \bar\eta^p_\xi : \xi \in u^p\rangle)$$

satisfying the following requirements for all m:

- $u^p \in [\lambda_\omega]^{<\kappa}$
- $\alpha^p < \kappa$
- For all $\xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1})$: $\bar\eta^p_\xi = \langle \eta^p_\xi(0), \dots, \eta^p_\xi(m-1)\rangle \in (2^{\alpha^p})^m$.
- For all $\xi \neq \xi'$ in $u^p \cap (\lambda_m \setminus \lambda_{m-1})$, $\bar\eta^p_\xi$ and $\bar\eta^p_{\xi'}$ are incomparable in $(2^{\alpha^p})^m$.

We define $p \le q$ ("q is stronger than p") iff

- $u^p \subseteq u^q$
- $\alpha^p \le \alpha^q$
- for all $\xi \in u^p$, $\bar\eta^p_\xi \trianglelefteq \bar\eta^q_\xi$

Fact 3.7. (1) For all $\alpha < \kappa$: The set $\{p \in \mathbb{P} : \alpha^p \ge \alpha\}$ is dense in $\mathbb{P}$.
(2) For all $\xi < \lambda_\omega$: The set $\{p \in \mathbb{P} : \xi \in u^p\}$ is dense in $\mathbb{P}$.

Fact and Definition 3.8. We let $\langle \underset{\sim}{\bar\nu}_\xi : \xi < \lambda_\omega\rangle$ be the "generic object", i.e., a name satisfying

$$\forall m \in \omega\ \forall p \in \mathbb{P}\ \forall \xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1}) : p \Vdash_{\mathbb{P}} \underset{\sim}{\bar\nu}_\xi \in (2^\kappa)^m$$
$$\forall p \in \mathbb{P}\ \forall \xi \in u^p : p \Vdash \bar\eta^p_\xi \trianglelefteq \underset{\sim}{\bar\nu}_\xi$$

(This definition makes sense, by fact 3.7.)

Clearly, $\Vdash \xi, \xi' \in \lambda_m \setminus \lambda_{m-1} \Rightarrow \underset{\sim}{\bar\nu}_\xi, \underset{\sim}{\bar\nu}_{\xi'}$ incompatible.

We let $\Vdash \underset{\sim}{I} = \bigcup_{m=2}^\infty \{\nu_\xi(\ell) : \xi \in \lambda_m \setminus \lambda_{m-1}, \ell < m\}$.

Lemma 3.9. *Let $\mathbb{P}$, $\underset{\sim}{I}$ be as in 3.6 and 3.8.*
Then $\Vdash_{\mathbb{P}} I^m$ has antichains of size λ_m, but no larger ones.

It is clear that $\mathbb{P}$ is κ-complete, and κ^+-cc is proved by an argument similar to the Δ-system argument below. So all the λ_m stay cardinals.

We can show by induction that $\Vdash \alpha(\underset{\sim}{I}^m) > \lambda_m$, i.e., $\underset{\sim}{I}^m$ has an antichain of size λ_m: This is clear if $\lambda_m = \lambda_{m-1}$ (and void if $m = 0$); if $\lambda_m > \lambda_{m-1}$ then $\langle \bar\nu_\xi : \xi \in \lambda_m \setminus \lambda_{m-1}\rangle$ will be forced to be antichain.

It remains to show that (for any m) there is no antichain of size λ_m^+ in $\underset{\sim}{I}^m$.

Fix $m^* \in \omega$, and assume wlog that $\lambda_{m^*+1} > \lambda_{m^*}$.

[Why is this no loss of generality? If $\lambda_{m^*} = \lambda_\omega$, then the cardinality of $\underset{\sim}{I}$ is at most λ_{m^*}, and there is nothing to prove. If $\lambda_{m^*} = \lambda_{m^*+1} < \lambda_\omega$, then replace m^* by $\min\{m \ge m^* : \lambda_m < \lambda_{m+1}\}$]

Towards a contradiction, assume that there is a condition p and a sequence of names $\langle \underset{\sim}{\bar\rho}_\beta : \beta < \lambda_{m^*}^+\rangle$ such that

$$p \Vdash \langle \underset{\sim}{\bar\rho}_\beta : \beta < \lambda_{m^*}^+\rangle \text{ is an antichain in } \underset{\sim}{I}^{m^*}$$

Let $\bar{\underset{\sim}{\rho}}_\beta = (\underset{\sim}{\rho}_\beta(n) : n < m^*)$. For each $\beta < \lambda^+_{m^*}$ and each $n < m^*$ we can find a condition $p_\beta \geq p$ and

$$m_n(\beta) \in \omega \qquad \ell_n(\beta) < m_n(\beta) \qquad \xi_n(\beta) \in \lambda_{m_n(\beta)} \setminus \lambda_{m_n(\beta)-1}$$

such that

$$p_\beta \Vdash \rho_\beta(n) = \nu_{\xi_n(\beta)}(\ell_n(\beta))$$

We will now employ a Δ-system argument.

We define a family $\langle \zeta^\beta : \beta < \lambda^+_{m^*} \rangle$ of functions as follows: Let i_β be the order type of u^{p_β}, and let

$$u^\beta = u^{p_\beta} = \{\zeta^\beta(i) : i < i_\beta\} \text{ in increasing enumeration}$$

By 3.7.2 may assume $\xi_n(\beta) \in u^\beta$, say $\xi_n(\beta) = \zeta^\beta(i_n(\beta))$.

By thinning out our alleged antichain $\langle \bar{\underset{\sim}{\rho}}_\beta : \beta < \lambda^+_{m^*} \rangle$ we may assume

- For some $i^* < \kappa$, for all β: $i_\beta = i^*$
- For some $\alpha^* < \kappa$, for all β: $\alpha^{p_\beta} = \alpha^*$
- For each $i < i^*$ there is some $m_{\langle i \rangle}$ such that for all β: $\zeta^\beta(i) \in \lambda_{m_{\langle i \rangle}} \setminus \lambda_{m_{\langle i \rangle}-1}$
- For each $i < i^*$ there is some $\bar{\eta}_{\langle i \rangle} \in (2^{\alpha^*})^{m_{\langle i \rangle}}$ such that for all β: $\bar{\eta}^{p_\beta}_{\zeta^\beta(i)} = \bar{\eta}_{\langle i \rangle}$. (Here we use $\lambda_m^{<\kappa} = \lambda_m$.)
- the family $\langle u^\beta : \beta < \lambda^+_{m^*} \rangle$ is a Δ-system, i.e., there is some set $u^* \in [\lambda_\omega]^{<\kappa}$ such that for all $\beta \neq \gamma$: $u^\beta \cap u^\gamma = u^*$.
- Moreover: there is a set $\Delta \subseteq i^*$ such that for all β: $u^* = \{\zeta^\beta(i) : i \in \Delta\}$. Since ζ^β is increasing, this also implies $\zeta^\beta(i) = \zeta^\gamma(i)$ for $i \in \Delta$.)
- For each $n < m^*$, the functions m_n, ℓ_n and i_n are constant. (Recall that these functions map $\lambda^+_{m^*}$ into ω.) We will again write m_n, ℓ_n, i_n for these constant values.

Note that for $i \in i^* \setminus \Delta$ all the $\zeta^\beta(i)$ are distinct elements of $\lambda_{m_{\langle i \rangle}}$, hence:

$$i \notin \Delta \text{ implies } \lambda^+_{m^*} \leq \lambda_{m_{\langle i \rangle}}, \text{ hence } m_{\langle i \rangle} > m^*.$$

Now pick $k^* \leq m^*$ such that $k^* \notin \{\ell_n : n < m^*\}$. Pick any distinct $\beta, \gamma < \lambda^+_{m^*}$. We will find a condition q extending p_β and p_γ, such that $q \Vdash \bar{\rho}_\beta \leq \bar{\rho}_\gamma$.

We define q as follows:

- $u^q := u_\beta \cup u_\gamma = u^* \mathbin{\dot\cup} \{\zeta^\beta(i) : i \in i^* \setminus \Delta\} \mathbin{\dot\cup} \{\zeta^\gamma(i) : i \in i^* \setminus \Delta\}$.
- $\alpha^q = \alpha^* + 1$.
- For $\xi \in u^*$, say $\xi = \zeta^\beta(i) = \zeta^\gamma(i)$, recall that $\bar{\eta}^{p_\beta}_\xi = \bar{\eta}_{\langle i \rangle} = \bar{\eta}^{p_\gamma}_\xi$. We let $\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown \bar{0}$ (see 3.3).
- For $\xi = \zeta^\beta(i)$, $i \in i^* \setminus \Delta$, we have $\bar{\eta}^{p_\beta}_\xi = \bar{\eta}_{\langle i \rangle} \in (2^{\alpha^*})^{m_{\langle i \rangle}}$, where $m_{\langle i \rangle} > m^*$. So $k^* \leq m^* < m_{\langle i \rangle}$, hence $\bar{\eta}_{\langle i \rangle}{}^\frown \{k^* \mapsto 1 \text{ else } \bar{0}\}$ is well-defined. We let

$$\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown \{k^* \mapsto 1 \text{ else } \bar{0}\}$$

- For $\xi = \zeta^\gamma(i)$, $i \in i^* \setminus \Delta$, we let

$$\bar\eta^q_\xi = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 0 \text{ else } \bar 1\}$$

We claim that q is a condition. The only nontrivial requirement is the incompatibility of all $\bar\eta^q_\xi$: Let $\xi, \xi' \in u^q$, $\xi \neq \xi'$, and assume that $\xi, \xi' \in \lambda_m \setminus \lambda_{m-1}$ for some m.

If $\xi, \xi' \in u^\beta$, then the incompatibility of $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ follows from the incompatibility of $\bar\eta^{p_\beta}_\xi$ and $\bar\eta^{p_\beta}_{\xi'}$. The same argument works for $\xi, \xi' \in u_\gamma$.

So let $\xi \in u_\beta \setminus u^*$, $\xi' \in u_\gamma \setminus u^*$. Say $\xi = \zeta^\beta(i)$, $\xi' = \zeta^\gamma(i')$.

If $i \neq i'$, then $\bar\eta_{\langle i\rangle} = \bar\eta^{p_\beta}_{\zeta^\beta(i)} = \bar\eta^{p_\gamma}_{\zeta^\gamma(i)}$ and $\bar\eta_{\langle i'\rangle} = \bar\eta^{p_\gamma}_{\zeta^\gamma(i')}$ are incompatible (because p_γ is a condition). From $\bar\eta_{\langle i\rangle} \trianglelefteq \bar\eta^q_\xi$ and $\bar\eta_{\langle i'\rangle} \trianglelefteq \bar\eta^q_{\xi'}$ we conclude that also $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ are incompatible.

Finally, we consider the case $i = i'$.

We have

$$\bar\eta^q_\xi = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 0 \text{ else } \bar 1\} \qquad \bar\eta^q_{\xi'} = \bar\eta_{\langle i\rangle}{}^\frown\{k^*\mapsto 1 \text{ else } \bar 0\}$$

so by 3.7.3, $\bar\eta^q_\xi$ and $\bar\eta^q_{\xi'}$ are incompatible.

This concludes the construction of q. We now check that $q \Vdash \bar\rho_\beta \leq \bar\rho_\gamma$, i.e., $q \Vdash \rho_\beta(n) \leq \rho_\gamma(n)$ for all n. Clearly, $q \Vdash \rho_\beta(n) = \nu_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \bar\eta^q_{\zeta^\beta(i_n)} = \eta_{\langle i_n\rangle}{}^\frown\bar 0$. Here we use the fact that $k^* \neq \ell_n$. Similarly, $q \Vdash \rho_\gamma(n) = \nu_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \eta_{\langle i_n\rangle}{}^\frown\bar 1$.

Hence $q \Vdash \bar\rho_\beta \leq \bar\rho_\gamma$.

This concludes the proof of theorem 3.1

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