Large localizations of finite simple groups

Rüdiger Göbel, José L. Rodríguez and Saharon Shelah *

Abstract

A group homomorphism $\eta : H \to G$ is called a *localization* of H if every homomorphism $\varphi : H \to G$ can be 'extended uniquely' to a homomorphism $\Phi : G \to G$ in the sense that $\Phi\eta = \varphi$.

Libman showed that a localization of a finite group need not be finite. This is exemplified by a well-known representation $A_n \to SO_{n-1}(\mathbb{R})$ of the alternating group A_n, which turns out to be a localization for n even and $n \geq 10$. Emmanuel Farjoun asked if there is any upper bound in cardinality for localizations of A_n. In this paper we answer this question and prove, under the generalized continuum hypothesis, that every non abelian finite simple group H, has arbitrarily large localizations. This shows that there is a proper class of distinct homotopy types which are localizations of a given Eilenberg–Mac Lane space $K(H, 1)$ for any non abelian finite simple group H.

0 Introduction

One of the current problems in localization of groups is to decide what algebraic properties of H can be transferred to G by a localization $\eta : H \to G$. Recall that $\eta : H \to G$ is a *localization* if every homomorphism $\varphi : H \to G$ in the diagram

$$\begin{array}{ccc} H & \xrightarrow{\eta} & G \\ \varphi\downarrow & \swarrow_{\Phi} & \\ G & & \end{array} \tag{0.1}$$

*The authors are supported by the project No. G 0545-173, 06/97 of the German-Israeli Foundation for Scientific Research & Development.

The second author is also supported by DGES grant PB97-0202 and DAAD.

GbSh 701 in Shelah's list of publications.

can be extended to a *unique* homomorphism $\Phi : G \to G$ such that $\Phi\eta = \varphi$. This in other words says that $G \cong L_\eta H$ where L_η is the localization functor with respect to η; see e.g. [2, 13, 4, 3]. For example, the properties of being an abelian group or a commutative ring with 1 are preserved. Casacuberta, Rodríguez and Tai [4] have found consequences of these facts for homotopical localizations of abelian Eilenberg–Mac Lane spaces, see also Casacuberta [3].

A further step leads to nilpotent groups. Dwyer and Farjoun showed that every localization of a nilpotent group of class 2 is nilpotent of class 2 or less. A proof is given by Libman in [13] (see also [3]). However, it is unknown if nilpotent groups are preserved under localizations in general.

Another interesting problem is to find an upper bound for the cardinalities of the localizations of a fixed group H. It is easy to see that, if H is finite abelian, then every localization $\eta : H \to G$ is an epimorphism, hence $|G| \leq |H|$. More generally, Libman has shown in [13] that if H is torsion abelian then $|G| \leq |H|^{\aleph_0}$. However, if H is not torsion, this can fail. Indeed, the localizations of $\mathbb{Z}$ are precisely the E-rings, and we know by Dugas, Mader and Vinsonhaler [9] that there exist E-rings of arbitrarily large cardinality. Strüngmann sharpened this result in [22] for almost free E-rings.

The first example observing that a localization of a finite group need not be finite is due to Libman [14]. He showed that the alternating group A_n has a $(n-1)$-dimensional irreducible representation $\eta : A_n \to SO_{n-1}(\mathbb{R})$ which is a localization for any even natural number $n \geq 10$. In the proof he uses that $O_{n-1}(\mathbb{R})$ is complete, $SO_{n-1}(\mathbb{R})$ is simple, and the fact that all automorphisms of $SO_{n-1}(\mathbb{R})$ are conjugation by some element in $O_{n-1}(\mathbb{R})$. This also motivates our Definition 1. Thus, Emmanuel Farjoun asked about the existence of an upper bound for the cardinality of localizations of A_n. We give an answer to this question in Corollary 3, which is a direct consequence of our Main Theorem and Proposition 2.

In fact our result also holds for many other finite groups which we will call *suitable* groups, see Definition 1. We shall assume the generalized continuum hypothesis. GCH will be needed to apply a new combinatorial principle which is similar to 'Shelah's black box' or the diamond principle $\diamondsuit$. But rather than applying some game with a winning strategy for some player we will apply the outcome directly as stated in Proposition 5.3. The proof of this combinatorial result will appear in Chapter 8 of the book by Shelah [21]. The proof can also be recovered from [11], the result is stated for cardinality $\aleph_1$ in

[20] and applied to boolean algebras, moreover see [19]. Accordingly, the group G being constructed will have cardinality $|G| = \lambda^+$, the successor of a regular cardinal λ.

The group theoretical techniques derive from combinatorial group theory and can be found in the book by Lyndon and Schupp [15]. Some aspects are inspired from the solution of Kurosh's problem about Jonsson groups [18]. But unlike there, in this updated version we will not apply cancellation theory which simplifies proofs about certain centralizers of subgroups.

Main Theorem (ZFC + GCH) *Let λ be the successor of an uncountable, regular cardinal and let λ^+ be its successor cardinal. Then any suitable group H is a subgroup of a group $\mathcal{G}$ of size λ^+ with the following properties:*

(a) *Any monomorphism $\phi : H \to \mathcal{G}$ is the restriction of an inner automorphism of $\mathcal{G}$.*

(b) *H has trivial centralizer in $\mathcal{G}$: if $[H, x] = 1$ for some $x \in \mathcal{G}$ then $x = 1$.*

(c) *Any monomorphism $\mathcal{G} \to \mathcal{G}$ is an inner automorphism of $\mathcal{G}$.*

(d) *$\mathcal{G}$ is simple.*

It is interesting to note that we can also require that $\mathcal{G} = \mathcal{G}[H]$, this is to say that the following holds.

(e) *The group $\mathcal{G}$ is generated by copies of H.*

This strong demand can be established if we add $G[H] = G$ to the Definition 3.2 of $\mathcal{K}^*$. It is then easy to see from Lemma 3.11 that the new class $\mathcal{K}^*$ is still large enough to provide $\mathcal{G}$ as in the Main Theorem. We see that the group $\mathcal{G}$ in the Main Theorem is complete, i.e. has trivial center $Z(\mathcal{G})$ and $\mathrm{Aut}(\mathcal{G}) = \mathrm{Inn}\,(\mathcal{G})$, where $\mathrm{Aut}(\mathcal{G})$ denotes the automorphism group of $\mathcal{G}$ and $\mathrm{Inn}\,(\mathcal{G})$ is the normal subgroup $\mathrm{Inn}\,(\mathcal{G}) = \{g^* : g \in \mathcal{G}\}$ which consists of all conjugations

$$g^* : \mathcal{G} \longrightarrow \mathcal{G} \quad (x \longrightarrow g^{-1}xg).$$

It is also obvious that $\mathcal{G}$ from the Main Theorem is co-hopfian in the sense that any monomorphism is an automorphism of $\mathcal{G}$.

Definition 1 *Let H be any group with trivial center and view $H \leq \mathrm{Aut}(H)$ as inner automorphisms of H. Then H is called* suitable *if the following conditions hold:*

1. *H is finite and* $\mathrm{Aut}(H)$ *is complete.*

2. *If $H_1 \leq \mathrm{Aut}(H)$ and $H_1 \cong H$ then $H_1 = H$.*

Note that $\mathrm{Aut}(H)$ has trivial center because H has trivial center. Hence the first condition only requires that $\mathrm{Aut}(H) = \mathrm{Inn}\,(H)$, so H has no outer automorphisms. We want to show that all non abelian finite simple groups are suitable. As a consequence of the classification of finite simple groups the Schreier conjecture holds for all finite groups H, hence the outer automorphism group $\mathrm{Out}(H) \cong \mathrm{Aut}(H)/\mathrm{Inn}\,(H)$ is always soluble. As H is also simple and non abelian then H identified with $\mathrm{Inn}\,(H)$ is a characteristic subgroup of $\mathrm{Aut}(H)$. Hence, looking at the solvable group $\mathrm{Aut}(H)/H$ any copy of H in $\mathrm{Aut}(H)$ must be H. Moreover $\mathrm{Aut}(H)$ is complete by Burnside, see [16, p. 399]. This shows part (a) of the following

Proposition 2 (a) *All non abelian finite simple groups are suitable.*
(b) *The non abelian, finite, simple and complete groups are precisely the following sporadic groups: M_{11}, M_{23}, M_{24}, Co_3, Co_2, Co_1, Fi_{23}, Th, B, M, J_1, Ly, Ru, J_4 and the following Chevalley groups for primes p and natural numbers $n \geq 3$*

$$S_{2n}(2),\ G_2(p)\ (p \neq 2,3),\ F_4(p)\ (p \neq 2),\ E_7(2),\ E_8(p).$$

Part (b) follows by inspection of the list of finite simple groups. It is interesting to know when $H = \mathrm{Aut}(H)$ because in this case our proof becomes visibly simpler.

We finally obtain an answer to Emmanuel Farjoun's question concerning alternating groups A_n for all finite simple non abelian groups.

Corollary 3 *(Assume* ZFC + GCH.*) Any finite simple non abelian group has localizations of arbitrarily large cardinality.*

The localization $A_n \to SO_{n-1}(\mathbb{R})$ for any even $n \geq 10$ induces a map between Eilenberg–Mac Lane spaces $K(A_n, 1) \to K(SO_{n-1}(\mathbb{R}), 1)$ which turns out to be a localization in the homotopy category. This is the first example of a space with a finite fundamental group which admits localizations with an infinite fundamental group. Corollary 3 yields then the following extension.

Corollary 4 *(Assume* ZFC + GCH.*) Let H be a finite simple non abelian group. Then $K(H,1)$ has localizations with arbitrarily large fundamental group.*

Constructions in homotopy theory based on large-cardinal principles were used in Casacuberta, Scevenels and Smith [5].

Paper Sh:701, version 2000-05-04_11. See https://shelah.logic.at/papers/701/ for possible updates.

Acknowledgements. We would like to thank Emmanuel Farjoun for explaining the above problem to us and pointing out further interesting connections between group theory and homotopy theory.

1 Proof of corollaries

Assuming GCH, Proposition 5.3 applies for any cardinal λ^+ with λ uncountable and regular, hence $\mathcal{G}$ in the Main Theorem can have cardinality λ^+. In order to prove Corollary 3, we next show that the inclusion $\eta : H \hookrightarrow \mathcal{G}$ in the Main Theorem is a localization. Suppose that $\varphi : H \to \mathcal{G}$ is a homomorphism. We have to show that there is a unique homomorphism $\Phi : \mathcal{G} \to \mathcal{G}$ such that $\Phi\eta = \varphi$. If $\varphi = 0$ then $\Phi = 0$ makes the diagram (0.1) commutative. To see that it is unique, we note that H is in the kernel K of Φ and $K = \mathcal{G}$ by simplicity, hence $\Phi = 0$. Now suppose that $\varphi \neq 0$. Since H is simple we have that φ is a monomorphism thus by (a) of the Main Theorem there is an element $y \in \mathcal{G}$ such that $\varphi = y^* \restriction H$ where $y^* \restriction H$ denotes the restriction of the map y^* on H. Hence $\Phi = y^*$ satisfies $\Phi\eta = \varphi$. Suppose that $\Phi' : \mathcal{G} \to \mathcal{G}$ is another homomorphism such that $\Phi'\eta = \varphi$. Then $\Phi' \neq 0$ and since $\mathcal{G}$ is simple by (d), the map Φ' is a monomorphism of $\mathcal{G}$, hence an inner automorphism by (c) . Now $(g^{-1}y)^* : \mathcal{G} \to \mathcal{G}$ fixes all elements of H. From (b), we obtain $g^{-1}y = 1$ and thus $\Phi = \Phi'$ as desired.

Recall from [7] or [4] that a map $f : X \to Y$ between two connected spaces is a *homotopical localization* if Y is f-local, i.e. if the map of pointed function spaces

$$\operatorname{map}_*(Y, Y) \to \operatorname{map}_*(X, Y)$$

induced by composition by f is a weak homotopy equivalence. As in the case of groups this says that $Y \simeq L_f X$, where L_f is the localization functor with respect to f.

It turns out that the homotopical localizations of the circle $S^1 = K(\mathbb{Z}, 1)$ are precisely Eilenberg–Mac Lane spaces $K(A, 1)$ where A ranges over the class of all E-rings [4, Theorem 5.11]. And therefore this is proper class (not a set) in view of the result in [9]. Recall that an E-ring is a commutative ring A with identity which is canonically isomorphic to its own ring of additive endomorphisms. Corollary 4 claims that a similar situation holds for $K(H, 1)$ if H is a finite, simple non abelian group. However, in this case, other localizations of $K(H, 1)$ which are not of the form $K(G, 1)$ may exist, see [7, Section 1.E.].

Corollary 4 follows from the fact that every localization of groups $H \to \mathcal{G}$ gives rise to a localization of spaces $K(H,1) \to K(\mathcal{G},1)$. This holds because, for arbitrary groups A and B, the space $\mathrm{map}_*(K(A,1), K(B,1))$ is homotopically discrete and equivalent to the set $\mathrm{Hom}(A,B)$.

Basic facts on homotopy theory can be seen in [23], the monograph by Aubry [1] (lectures of a DMV seminar by Baues, Halperin and Lemaire) as well as in Farjoun's exposition [7] for homotopical localizations.

2 On free products with amalgamation and HNN extensions

If G is a group and $a, b \in G$ then $[a,b] = a^{-1}b^{-1}ab$ denotes the commutator of a and b, and this naturally extends to subset $[A,B]$. Compare Theorem 2.7, p. 187, Theorem 6.6, p. 212 and Theorem 2.4, p. 185 in [15] for the notion of free products with amalgamation and HNN extension. We will need a lemma describing finite subgroups of free products with amalgamations and HNN extensions.

Lemma 2.1 *Let $G^* = G_1 *_{G_0} G_2$ be the free product of G_1 and G_2 amalgamating a common subgroup $G_0 = G_1 \cap G_2$ and let H be a finite subgroup of G^*. Then there exist $i \in \{1,2\}$ and $y \in G^*$ such that $(H')^y \leq G_i$.*

PROOF. Let

$$H^{G^*} = \{H^x : x \in G^*\}$$

be the conjugacy class of H in G^*. If $g \in G^* = G_1 *_{G_0} G_2$ then $|g|$ denotes the length of g, which is an invariant of g, see [15]. We now choose $H' \in H^{G^*}$ subject to the following two conditions.

$$|H' \cap (G_1 \cup G_2)| \text{ is maximal (see also (2.5))} \tag{2.1}$$

and among those let

$$\min\{|h| : h \in H' \setminus G_1 \setminus G_2\} \text{ be minimal, also say } \min\emptyset = 0. \tag{2.2}$$

So there is such an H' which we rename H. If $h \in H$, then h is torsion and by the Torsion Theorem for free products with amalgamation there are $g \in G^*$ and $i \in \{0,1\}$

such that $h^g = y \in G_i$, see [15] or [16]. Hence any $h \in H$ has the form

$$h = gyg^{-1} = g_1 \cdots g_n y g_n^{-1} \cdots g_1^{-1}. \tag{2.3}$$

and we may assume that n minimal with (2.3). If $y \in G_0$, then we can replace y by $g_n y g_n^{-1}$, which is in $G_1 \cup G_2$ and $g_1 \cdots g_{n-1}$ has shorter length, contradicting minimality of n. Hence $y \in G_1 \cup G_2 \setminus G_0$ and say $y \in G_1$ without any restriction. By the same argument we can not have that $g_n \in G_1$, hence $g_n \in G_2$ and $h = g_1 \cdots g_n y g_n^{-1} \cdots g_1^{-1}$ is in normal form and its length is $|h| = 2n + 1$ is odd. We have shown that

$$|h| \text{ is odd for all elements } h \in H. \tag{2.4}$$

If $h_i \in H \cap G_i \setminus G_0$ for $i = 1, 2$ then visibly $h = h_1 h_2$ has length $|h| = 2$ which contradicts (2.4). We conclude

$$H \cap (G_1 \cup G_2) = H \cap G_i \text{ for some } i \in \{1, 2\}, \tag{2.5}$$

hence in (2.1) either $|H \cap G_1|$ or $|H \cap G_2|$ is the maximal integer. By symmetry we may assume that

$$H \cap (G_1 \cup G_2) = H \cap G_1 \text{ and hence } |H \cap G_2| \leq |H \cap G_1|. \tag{2.6}$$

In order to say more about $H \cap G_1$ we fix a left coset representation of $G_0 \leq G_i$ and let $1 \in Z_i \leq G_i$ be a fixed left transversal of G_0 in G_i, i.e. G_i is the disjoint union of $\{zG_0 : z \in Z_i\}$ and $Z = Z_1 \cup Z_2$ is a transversal of G^* over G_0. Following standard notation if $g \in zG_0$ we also write $\bar{g} = z$ for the representative of the coset. We will use the following well-known fact about normal forms with respect to transversals; see [16, pp. 179–181] or [15, pp. 205–206]. Any

$$g \in G^* \textit{ can uniquely} \text{ be expressed as a reduced word } g = \bar{g}_1 \cdots \bar{g}_n g_0 \tag{2.7}$$

with $g_0 \in G_0$ and $g_k \in G_1 \cup G_2 \setminus G_0$ alternating in G_1 and G_2 respectively, e.g. by using [16, p. 179]. If we apply this to (2.3) then (after renaming y as $g_0 y g_0 - 1$) then (2.3) becomes an expression with unique $\bar{g}_i \in Z$

$$h = \bar{g}_1 \cdots \bar{g}_n y \bar{g}_n^{-1} \cdots \bar{g}_1^{-1} \text{ with } \bar{g}_i \in Z. \tag{2.8}$$

Next we claim that

$$H \cap G_1 \not\subseteq G_0. \tag{2.9}$$

Assume $H \cap G_1 \subseteq G_0$ and also assume that the lemma does not hold. Hence there is $h \in H \setminus G_1 \setminus G_2$, which can be expressed as in (2.8). Now we claim that the subgroup

$$H' = H^{\bar{g}_1} \in H^{G^*}$$

violates the maximality (2.1) for H. We may assume by symmetry that $\bar{g}_1 \in G_1 \cup G_2$ belongs to G_1. Hence

$$G_0^{\bar{g}_1} \leq G_1,\ (G_0 \cap H)^{\bar{g}_1} \leq G_1 \cap H' \text{ and } |G_0 \cap H| \leq |G_1 \cap H'|.$$

However $H \cap G_0 = H \cap G_1$ by assumption on H, hence $|H \cap G_1| \leq |H' \cap G_1|$. By maximality (2.1) with (2.6) follows

$$|H \cap G_0| = |H \cap G_1| = |H' \cap G_1|. \tag{2.10}$$

We now consider $h^{\bar{g}_1} \in H'$ with $h \in H$ subject to (2.2). Such an $h \in H$ exists as we assume that the lemma does not hold. Obviously

$$h^{\bar{g}_1} = \bar{g}_2 \cdots \bar{g}_n y \bar{g}_n^{-1} \cdots \bar{g}_2^{-1}$$

from (2.8). We get that $|h^{\bar{g}_1}| = |h| - 2 < |h|$ has shorter length. Hence $h^{\bar{g}_1} \in H' \setminus G_1 \setminus G_2$ by (2.2) is impossible, so necessarily $h^{\bar{g}_1} \in G_1 \cup G_2$. On the other hand $h \notin H \cap G_0$ by (2.2), hence $h^{\bar{g}_1} \in H' \cap (G_1 \cup G_2)$ is a 'new' element when compared with $(H \cap G_0)^{\bar{g}_1} \subseteq H' \cap (G_1 \cup G_2)$, so

$$|H \cap G_0| = |H \cap (G_1 \cup G_2)| < |H' \cap (G_1 \cup G_2)|$$

which contradicts (2.10), and (2.9) follows.

We continue assuming that the lemma does not hold. Now we want to exploit the fact (2.9) that $H \cap G_1$ is relatively large. Let $h \in H \setminus G_1 \setminus G_2$ still be expressed as in (2.8) in normal form. If $x \in H \cap G_1$ then also $hx \in H$ can be represented like h as

$$hx = \bar{g}_{1x} \cdots \bar{g}_{mx} y_x \bar{g}_{mx}^{-1} \cdots \bar{g}_{1x}^{-1} = g_x y_x g_x^{-1} \text{ with } \bar{g}_{ix} \in Z. \tag{2.11}$$

for some $y_x \in G_1 \cup G_2$ and $g_x = \bar{g}_{1x} \cdots \bar{g}_{mx}$ with factors which are representatives alternating from G_1 and G_2, respectively.

Hence

$$\bar{g}_1 \cdots \bar{g}_n y \bar{g}_n^{-1} \cdots \bar{g}_1^{-1} x = \bar{g}_{1x} \cdots \bar{g}_{mx} y_x \bar{g}_{mx}^{-1} \cdots \bar{g}_{1x}^{-1}$$

and by uniqueness of factors from the left of the reduced normal forms for transversals follows element-wise $\bar{g}_1 = \bar{g}_{1x}, \ldots, \bar{g}_n = \bar{g}_{mx}$, hence $m = n$ and $g = g_x$ for all $x \in H \cap G_1$. By (2.11) this is to say that all elements in the coset $h(H \cap G_1)$ of $H \cap G_1$ are conjugate by the same element g. Accordingly, if

$$X = H \cap (G_1 \cup G_2)^{g^{-1}}$$

then $h(H \cap G_1) \subseteq X \subseteq H$ and hence $|H \cap G_1| \leq |X|$.

We now consider $H' = H^g$ and note that $X^g \subseteq H' \cap (G_1 \cup G_2)$. Cosets have the same size, hence using (2.6) and (2.1) for H we get

$$|H \cap G_1| = |h(H \cap G_1)| \leq |X| = |X^g| \leq |H' \cap (G_1 \cup G_2)| \leq |H \cap (\cap G_1 \cup G_2)| = |H \cap G_1|$$

and equality holds. Hence

$$h(H \cap G_1) = H' \cap G_1 \text{ or } h(H \cap G_1) = H' \cap G_2,$$

the coset is a subgroup which is only possible if $h \in H \cap G_1$ or $h \in H \cap G_2$, which however contradicts our choice of h. The lemma holds. $\square$

Note that HNN extensions are obtained by particular successive free products with amalgamation, see [16, p. 182] or [15]. From Lemma 2.1 we have the immediate

Corollary 2.2 *Let G be any group, and $\phi : G_0 \to G_1$ be an isomorphism between two subgroups of G. Consider the HNN extension $G^* = \langle G, t : t^{-1}ht = \phi(h), h \in G_0 \rangle$. If H is a finite subgroup of G^*, then there exists a $y \in G^*$ such that H^y is contained in G.*

The following lemma describes centralizers of finite subgroups in free products with amalgamation.

Lemma 2.3 *Let $G^* = G_1 *_{G_0} G_2$ be the free product of G_1 and G_2 amalgamating a common subgroup G_0. Let $H \leq G_1$ be a non trivial finite subgroup and let $x \in G^*$ be an element which commutes with all elements of H. Then either $x \in G_1$ or $H^g \leq G_0$ for some $g \in G^*$.*

Proof. Suppose $[x, H] = 1$ and $x \notin G_1$. Express x in a reduced normal form

$$x = g_1 g_1' \cdots g_n g_n',$$

that is, $g_i \in G_1 \setminus G_0$, $(1 < i \leq n)$ and $g'_i \in G_2 \setminus G_0$, $(1 \leq i < n)$. The relation $h^{-1}x^{-1}hx = 1$ yields the following

$$h^{-1}{g'}_n^{-1} g_n^{-1} \cdots {g'}_1^{-1}(g_1^{-1}hg_1)g'_1 \cdots g_n g'_n = 1.$$

By the normal form theorem for free products with amalgamation [15, Theorem 2.6 p. 187], this is only possible if $g_1^{-1}hg_1 \in G_0$ for all $h \in H'$. This concludes the proof. $\square$

By similar arguments we have

Lemma 2.4 *Let G be any group, and $\phi : G_0 \to G_1$ be an isomorphism between two subgroups of G. Consider the HNN extension $G^* = \langle G, t : t^{-1}ht = \phi(h), h \in G_0 \rangle$. If H is a non trivial finite subgroup of G^* and $x \in G^*$ such that $[x, H] = 1$, then $x \in G$.*

3 Group theoretic approximations of $\mathcal{G}$

We fix a suitable group H and write $\widehat{H} = \mathrm{Aut}(H)$. Moreover, view $H \leq \widehat{H}$ as subgroup. We also fix an uncountable regular cardinal λ. As usual $C_{G'}(G)$ and $N_{G'}(G)$ denote, respectively, the centralizer and the normalizer of a subgroup G in a group G'.

Let $\mathrm{pInn}\,(G)$ denote the set of *partial inner automorphisms*, which are the isomorphisms $\phi : G_1 \to G_2$ where $G_1, G_2 \leq G$ such that ϕ can be extended to an inner automorphism of G. Hence $\mathrm{pInn}\,(G)$ are all restrictions of conjugations to subgroups of G. In addition we will use the following

Definition 3.1 *Let $M \leq N$ be groups, then $x \in N$ is* nice *over M in N if for any $s, t \in M$ with $x = sxt$ follows $s = t^{-1}$.*

Obviously nice elements over M in N as in the Definition 3.1 are also nice over M in G if $M \leq N \leq G$. This will be used very often in Section 4 and 5. We next consider two particular families $\mathcal{K}^* \subseteq \mathcal{K}$ of groups which will be used to approximate our group $\mathcal{G}$ group theoretically. An ordering will follow in the next section. The class of groups $\mathcal{K}^*$ will be *dense* in $\mathcal{K}$ in the sense that for any group $G \in \mathcal{K}$ is the subgroup of some group $G' \in \mathcal{K}^*$. Moreover we will show that $|G| \cdot \aleph_0 = |G'|$.

Definition 3.2
1. $\mathcal{K}$ consists of all groups G with $|G| < \lambda$ such that $H \leq \widehat{H} \leq G$, and any isomorphic copy of H in G has trivial centralizer in G. That is,

$$\mathcal{K} = \{G : \widehat{H} \leq G, |G| < \lambda, \text{ if } H \cong H' \leq G, x \in G \text{ with } [H', x] = 1, \text{ then } x = 1\}.$$

2. $\mathcal{K}^*$ is the class of all groups G in $\mathcal{K}$ such that any isomorphism between H and a subgroup of G is induced by conjugation with an element in G.

As H is suitable, we have $\widehat{H} \in \mathcal{K}^*$.

Lemma 3.3 *If G and G' are in $\mathcal{K}$ then $G * G' \in \mathcal{K}$. Moreover, all elements in G are nice over G' in $G * G'$.*

Proof. Suppose that $H' \leq G*G'$ with $H' \cong H$ and $x \in G*G'$ such that $[H', x] = 1$. By Lemma 2.1, we can suppose that $(H')^y \leq G$. Hence $[H', x] = 1$ implies $[(H')^y, x^y] = 1$ and we may assume that $H' \leq G$ and $[x, H'] = 1$. If $x \notin G$, then express x in reduced form and consider the commutator $h^{-1}x^{-1}hx = 1$ for any $h \in H$. Replace a first choice $h \neq 1$ by a different one if the first G-factor of $(x)^{-1}$ is cancelled by h^{-1}. Hence the normal form of the commutator shows non-trivial factors and the commutator can not be 1 by the normal form theorem for free products [15, p. 175]. This is impossible, hence $x \in G \in \mathcal{K}$ implies $x = 1$.

The second statement of the lemma follows by an immediate length argument. $\square$

If G is any group in $\mathcal{K}$ and $\phi : A \to B$ is an isomorphism between two subgroups of G isomorphic to H, we want that ϕ is an partially inner automorphism in some extension $G \leq G_2 \in \mathcal{K}$. This follows by using HNN extensions as we next show.

Lemma 3.4 *Let $G \in \mathcal{K}$ and $B \leq G$ be a subgroup isomorphic to H. Then there is $G \leq G_1 \in \mathcal{K}$ such that* $\mathrm{Aut}(B) \leq G_1$.

Proof. Let $\widehat{B} = \mathrm{Aut}(B)$ and $N = N_G(B)$. If $\widehat{B} \leq G$ then let $G_1 = G$. Suppose that $\widehat{B} \not\leq G$. Note that $N = G \cap \widehat{B}$, so we can consider the free product with amalgamation $G_1 := G *_N \widehat{B}$. We shall show that $G_1 \in \mathcal{K}$. Let $H' \leq G_1$ be a subgroup isomorphic to H and $1 \neq x \in G_1$ such that $[H', x] = 1$. By Lemma 2.1 we can suppose that $H' \leq G$ or $H' \leq \widehat{B}$. Suppose that $H' \leq G$, the other case is easier. Let $x = g_1 g_2 \cdots g_n$ be written in a reduced normal form. If $x = g_1 \in G$ then $x = 1$ since $G \in \mathcal{K}$, and this is a contradiction. Hence $x = g_1 \in \widehat{B} \setminus N$. As in Lemma 2.3 we deduce that $H' = (H')^{g_1} \leq N$, thus $H' = B$ since B is suitable. Hence $g_1 \in N$ is a contradiction. If $n = 2$, then we obtain $(H')^{g_1} = (H')^{g_2} = B = H'$. So both g_1 and g_2 are in N, which is a contradiction. Similarly, if $n \geq 3$ we have that g_2 and g_3 are in N. This is again impossible. This concludes the proof. $\square$

By the previous lemma we can suppose that if $B \leq G \in \mathcal{K}$, and if $B \cong H$, then $\widehat{B} \leq G$ as well. If $A, B \leq G$, $A \cong B \cong H$ and $\widehat{A}$, $\widehat{B}$ are conjugate in G then A and B are also conjugate. Indeed, if $g \in G$ such that $g^* : \widehat{A} \longrightarrow \widehat{B}$, then $A^g \leq \widehat{B}$ is a subgroup isomorphic to B, hence $A^g = B$ by Definition 1.

Lemma 3.5 *Let $G \in \mathcal{K}$ and $B \leq \widehat{B} \leq G$. Suppose that H and B are not conjugate in G. Let $\phi : \widehat{H} \to \widehat{B}$ be any isomorphism. Then the HNN extension*

$$G_1 = \langle G, t \,:\, t^{-1}ht = \phi(h) \ \ \textit{for all } h \in \widehat{H} \rangle$$

is also in $\mathcal{K}$.

Proof. Let $H' \leq G_1$ be a subgroup isomorphic to H and $1 \neq x \in G_1$ such that $[H', x] = 1$. By Corollary 2.2 we can suppose that $H' \leq G$ already. Let

$$x = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} g_n$$

be written in a reduced form in G_1, where $g_i \in G$ and there is no subword $t^{-1} g_i t$ with $g_i \in \widehat{H}$ or $t g_i t^{-1}$ with $g_i \in \widehat{B}$ (see [15, p. 181]).

If $x = g_0 \in G$ then $x = 1$, since $G \in \mathcal{K}$. This yields a contradiction. Thus $n \geq 1$. We have $[h, x] = h^{-1}x^{-1}hx = 1$ for every $h \in H'$. In other words, for a fixed $h \neq 1$, the following holds

$$h^{-1} g_n^{-1} t^{-\varepsilon_n} \cdots t^{-\varepsilon_1} (g_0^{-1} h g_0) t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n = 1. \tag{3.1}$$

By the normal form theorem for HNN extensions ([15, p. 182]), either $\varepsilon_1 = 1$ and $g_0^{-1} h g_0 \in \widehat{H}$, or $\varepsilon_1 = -1$ and $g_0^{-1} h g_0 \in \widehat{B}$. Suppose that $\varepsilon_1 = 1$, the other case is analogous. Then $(H')^{g_0} \leq \widehat{H}$, thus $(H')^{g_0} = H$ from 'suitable', and we can replace in (3.1) the subword $t^{-1}(g_0^{-1} h g_0)t$ by $\phi(g_0^{-1} h g_0) \in B$. Repeating the same argument we obtain that $\varepsilon_i = 1$ or -1. Hence one of the two possibilities holds depending on $\varepsilon_2 = 1$ or $\varepsilon_2 = -1$. We have either

$$\mathrm{id} : H_1 \xrightarrow{g_0^*} \widehat{H} \xrightarrow{\phi} \widehat{B} \xrightarrow{g_1^*} \widehat{H} \xrightarrow{\phi} \cdots \xrightarrow{g_n^*} H_1$$

or

$$\mathrm{id} : H_1 \xrightarrow{g_0^*} \widehat{H} \xrightarrow{\phi} \widehat{B} \xrightarrow{g_1^*} \widehat{B} \xrightarrow{\phi^{-1}} \cdots \xrightarrow{g_n^*} H_1.$$

In the first case we have an isomorphism $g_1^*\phi : \widehat{H} \xrightarrow{\phi} \widehat{B} \xrightarrow{g_1^*} \widehat{H}$. Since $\widehat{H}$ is complete there is $g \in \widehat{H}$ such that $g_1^*\phi = g^*$. This yields $\phi = (g_1^{-1}g)^*$, i.e. $\widehat{H}$ and $\widehat{B}$ are conjugate, and thus H and B are conjugate, but this is impossible by hypothesis. In the second case we have $g_1 \in \widehat{B}$ by completeness. But, on the other hand $g_1 \notin \widehat{B}$ since x is written in a reduced form. We conclude that G_1 is in $\mathcal{K}$. $\square$

Lemma 3.6 *Let $A \leq \widehat{A} \leq G \in \mathcal{K}$ and $B \leq \widehat{B} \leq G$. If $\phi : A \longrightarrow B$ is any isomorphism, then there is $G \leq G_2 \in \mathcal{K}$ such that $\phi \in \mathrm{pInn}\,(G_2)$. Moreover, G_2 can be obtained from G by at most two successive HNN extensions.*

Proof. If $\phi \in \mathrm{pInn}\,(G)$ we take $G_2 = G$. Suppose that $\phi \notin \mathrm{pInn}\,(G)$. If $\widehat{H}$ and $\widehat{A}$ are conjugate, we take $G_1 = G$. Otherwise, we consider the HNN extension

$$G_1 = \langle G, t_1 : t_1^{-1} h t_1 = \phi_1(h) \;\text{ for all } h \in \widehat{H}\rangle$$

where ϕ_1 is any isomorphism between $\widehat{H}$ and $\widehat{A}$. By Lemma 3.5 we know that $G_1 \in \mathcal{K}$. Now, if H and B are conjugate in G_1, we take $G_2 = G_1$. It follows automatically that $\phi \in \mathrm{pInn}\,(G_1)$, since $\widehat{H}$ is complete. If $\widehat{H}$ and $\widehat{B}$ are not conjugate in G_1, we consider a new HNN extension

$$G_2 = \langle G_1, t_2 : t_2^{-1} h t_2 = \phi_2(h) \;\text{ for all } h \in \widehat{H}\rangle$$

where ϕ_2 is any isomorphism between $\widehat{H}$ and $\widehat{B}$. Again $G_2 \in \mathcal{K}$ by Lemma 3.5. In that case we have an isomorphism $(t_2^{-1})^* \phi {t_1}^* : \widehat{H} \to \widehat{H}$, which equals g^* for some $g \in \widehat{H}$ by completeness. Thus $\phi = (t_2 g t_1^{-1})^* \restriction A$. This shows that $\phi \in \mathrm{pInn}\,(G_2)$. $\square$

Lemma 3.7 *Let $G \in \mathcal{K}$ and suppose that $G' \in \mathcal{K}$ or G' does not contain any subgroup isomorphic to H. Let $g \in G$ and $g' \in G'$ with $o(g) = o(g')$. Then $(G * G')/N \in \mathcal{K}$ where N is the normal subgroup of $G * G'$ generated by $g^{-1}g' \in G * G'$.*

Proof. The group $\overline{G} = (G * G')/N$ is a free product with amalgamation, hence G and G' can be seen as subgroups of $\overline{G}$ respectively. Suppose that we have a subgroup $H' \leq \overline{G}$ isomorphic to H and $x \in \overline{G}$ such that $[H', x] = 1$. By Lemma 2.1 we can assume that H' is already contained in G. Suppose that $x \neq 1$. By Lemma 2.3 it follows that either $x \in G$ or a conjugate of H' is contained in $\langle g\rangle$. In the first case $x = 1$ from $G \in \mathcal{K}$ is a contradiction. The second case is obviously impossible. Thus $\overline{G} \in \mathcal{K}$. $\square$

The proof of the next lemma is obvious.

Lemma 3.8 *(a) Let $\gamma < \lambda$ and $\{G_i : i < \gamma\}$ be an ascending continuous chain of groups in $\mathcal{K}$. Then the union $G = \bigcup_{i<\gamma} G_i$ also belongs to $\mathcal{K}$.*

(b) Let $\gamma < \lambda$ and let $\{G_i : i < \gamma\}$ be an ascending continuous chain of groups in $\mathcal{K}^*$. Then the union $G = \bigcup_{i<\gamma} G_i$ also belongs to $\mathcal{K}^*$.

(c) $\mathcal{K}^*$ is dense in $\mathcal{K}$.

Using Lemma 3.3 and Lemma 3.8 we obtain that, for every cardinal $\kappa < \lambda$, the free product $*_{\alpha\in\kappa}\widehat{H}_\alpha$ belongs to $\mathcal{K}$, where $\widehat{H}_\alpha$ is an isomorphic copy of $\widehat{H}$ for every $\alpha \in \kappa$. Now we can apply the following lemma to obtain a group $G' \in \mathcal{K}^*$, such that $*_{\alpha\in\kappa}\widehat{H}_\alpha \leq G'$.

Lemma 3.9 *Let $G^* = G *_{G_0} G'$ be the free product of G and G' amalgamating a common subgroup G_0. If G, G' and G_0 are in $\mathcal{K}^*$, then $G^* \in \mathcal{K}$.*

PROOF. Let $H' \leq G^*$ be a subgroup isomorphic to H, and $1 \neq x \in G^*$ such that $[H', x] = 1$. By Lemma 2.1 we can assume that $H' \leq G_0$ and $x = g_1 g_2 \cdots g_n$, is written in a reduced form of length bigger than two. Then we have $g_1^* : H' \longrightarrow (H')^{g_1}$ both of them inside G_0. Since $G_0 \in \mathcal{K}^*$ there exists $g \in G_0$ such that $g^* : H' \longrightarrow (H')^{g_1}$. We can also suppose that the automorphism group $\widehat{H}'$ is already in G_0 by Lemma 3.4. Hence the composition $(g_1^{-1}g)^* : H' \to H'$ is an automorphism, which is inner by completeness. Thus, $g_1^{-1}g \in G_0$ and $g_1 \in G_0$. This is a contradiction, since x was written in a reduced form. $\square$

In order to show that our final group $\mathcal{G}$ is simple we only must consider normal subgroups N of $\mathcal{G}$ which are cyclically generated, i.e. there is an $1 \neq x \in \mathcal{G}$ with $N = \langle x^{\mathcal{G}} \rangle$. We need that $N = \mathcal{G}$. There are two natural cases depending on the order of x. The case that x has infinite order is taken care by the next Proposition 3.10. Hence assuming that all elements of infinite order are conjugate, a consequence of Proposition 3.10, we only need to note that any element g of finite order can be written as a product of two elements of infinite order, just take y from a different factor then $g = (gy)y^{-1}$. Hence $\mathcal{G} = N$. If x has finite order, then there is a conjugate y of x such that xy has infinite order. Hence $xy \in N$ and the first case applies.

Proposition 3.10 *Let G be a group in $\mathcal{K}$. Let $g, f \in G$, where $o(f) = o(g) = \infty$ and g does not belong to the normal subgroup generated by f. Then there is a group $\overline{G} \in \mathcal{K}$ such that $G \leq \overline{G}$ and g is conjugate to f in $\overline{G}$.*

Proof. Let $\alpha : \langle f \rangle \to \langle g \rangle$ be the isomorphism mapping f to g. By hypothesis $\alpha \notin \mathrm{pInn}\, G$. As in Lemma 3.6 consider the HNN extension $\overline{G} = \langle G, t : t^{-1}ft = g\rangle$. We must show that $\overline{G} \in \mathcal{K}$. Clearly $|\overline{G}| < \lambda$ and consider any H' with $H \cong H' \leq G$ and any $x \in \overline{G}$ with $[H', x] = 1$. As above we may assume that $H' \leq G$ and $x \in \overline{G}$ with $[H', x] = 1$. Now we apply Lemma 2.3. $\square$

The last lemma of this section is not needed for proving the Main Theorem but it is used to show the additional property (e) of the group $\mathcal{G}$ in Section 1 mentioned after Main Theorem.

Lemma 3.11 *If $g \in G \in \mathcal{K}$, then there is a group $\overline{G} \in \mathcal{K}$, such that $G \leq \overline{G}$, with $|\overline{G}| = |G| \cdot \aleph_0$ and $g \in H(\overline{G})$.*

Proof. Suppose that $o(g) = \infty$ and that $g \notin H(G)$. Let H_1 and H_2 be two isomorphic copies of H. Choose a non trivial element $h \in H$ and let h_1 and h_2 be its copies in H_1 and H_2 respectively. Now define

$$\overline{G} = (G * H_1 * H_2)/N$$

where N is the normal subgroup generated by $g^{-1}h_1h_2$. Then $\overline{G} \in \mathcal{K}$ by Lemma 3.7 and moreover $g \in H(\overline{G})$.

If $o(g) = n < \infty$ we first embed $G \leq (G * K)/N$ where $K = \langle x_1, x_2 : (x_1x_2)^n = 1\rangle$ and N is the normal closure of $g^{-1}x_1x_2$. Then by the Lemma 3.7 $(G * K)/N \in \mathcal{K}$. Now, since $o(x_1) = o(x_2) = \infty$, we can apply the first case. $\square$

4 Approximations of $\mathcal{G}$ as a λ^+-uniform poset

We recall some basic notions of set theory from [12]. In particular $\mathrm{cf}\,(\alpha)$ denotes the cofinality of an ordinal infinite cardinal α and a cardinal λ is *regular* if $\mathrm{cf}\,(\lambda) = \lambda$. Throughout let λ be a fixed uncountable regular cardinal and λ^+ it successor. We will write $\mathcal{P}_{<\lambda}(\lambda)$ for the set of all subsets of λ of cardinality $< \lambda$. From GCH we have $|\mathcal{P}_{<\lambda}(\lambda)| = \lambda^{<\lambda} = \lambda$.

In this section we have to fit the group extensions of the last sections into a poset $\mathbb{P}$ defined in the appendix A of the paper. Let λ and H be as in Main Theorem, and $H \leq \widehat{H}$ as before. Write

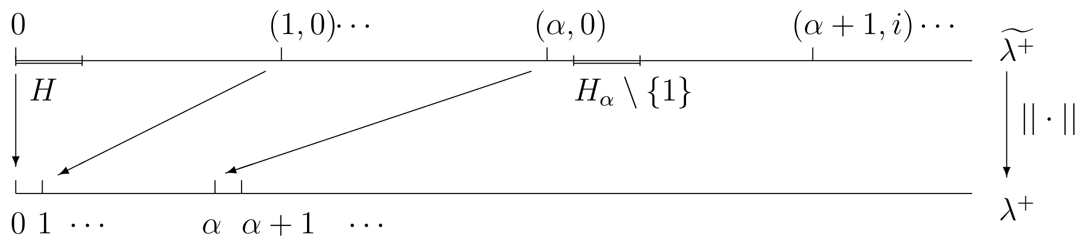
$$\widetilde{\lambda^+} := \{(\alpha, i) : \alpha \in \lambda^+,\ i < \lambda\}. \tag{4.1}$$

with the lexicographical ordering. Hence $\widetilde{\lambda^+}$ is a well-ordered set of cardinality λ^+, an ordinal $< \lambda^{++}$. Hence

$$\pi : \lambda^+ \longrightarrow \widetilde{\lambda^+} \ (\alpha \longrightarrow (\alpha, 0)) \text{ is a canonical embedding.}$$

If $x = (\alpha, i) \in \widetilde{\lambda^+}$ with $i < \lambda$, we write $||x|| = \alpha$ and call α the *norm* of x. We define the *domain* of a subset X of $\widetilde{\lambda^+}$ as the set $\operatorname{dom} X = \{||x|| : x \in X\}$.

The following picture illustrates how we can embed for instance $H * H_\alpha$ in $\widetilde{\lambda^+}$, with $\operatorname{dom}(H * H_\alpha) = \{0, \alpha\}$.



Definition 4.1 *Let $u \subset \lambda^+$ be a subset of cardinality $< \lambda$ with $0 \in u$. Then a group G of size $|G| < \lambda$ is called an u-group if the following holds:*

(a) $\operatorname{Dom} G \subset \widetilde{\lambda^+}$ *and* $\operatorname{dom} G = u$, *where* $\operatorname{Dom} G$ *denotes the underlying set of elements of the group G. We will identify* $\operatorname{Dom} G = G$.

(b) For every $0 \neq \delta \in \lambda^+$ the subset $G \cap \pi(\delta)$ is a subgroup of G. Moreover $G \cap \pi(\delta)$ belongs to $\mathcal{K}^*$ given in Section 3.

We will rewrite the elements $p = (\alpha, u) \in \mathbb{P}$ (see the Appendix A) in the form $p = (G^p, u^p)$ or simply $p = (G, u)$ where G is a u-group. If u is fixed, then G runs through all u-groups of cardinality $< \lambda$. By GCH this is a set of cardinality λ, hence this modification of $\mathbb{P}$ agrees with the requirement that (only) $\alpha < \lambda$ (codes these algebraic structures). Next we define an ordering on $\mathbb{P}$ which will use 'nice' elements from Definition 3.1. Now we say that
$p \leq q$ in $\mathbb{P}$, which is the case if and only if the following two conditions hold:

1. $G^p \leq G^q$

2. If $\delta \in \lambda^+$ and $x \in G^p$ is nice over $G^p \cap \pi(\delta)$ in G^p, then $x \in G^q$ is nice over $G^q \cap \pi(\delta)$ in G^q.

Theorem 4.2 $(\mathbb{P}, \leq)$ *is a* λ^+*-uniform partially ordered set.*

PROOF. We must define elements in $\mathbb{P}$ and have to check the conditions listed in Definition A.1:

First we trade $\widehat{H}$ into a 0-set for example, as indicated by the diagram. Hence $(\widehat{H}, \{0\})$ is our first element in $\mathbb{P}$.

Suppose G_i is a u_i-group for each $i = 0, 1, 2$ such that $G_0 = G_1 \cap G_2$. We turn the free product $G^* = G_1 *_{G_0} G_2$ into a 'weak' u-group for $u = u_1 \cup u_2$ which is a u-group except that the subgroups $G^* \cap \pi(\delta)$ $(\delta < \lambda^+)$ may not be in $\mathcal{K}^*$.

If $g \in G^* = G_1 *_{G_0} G_2 \setminus G_1 \setminus G_2$ then choose a transversal of G^* as in (2.7) and write g uniquely as indicated there. Turn g into an unused ordinal $g \in \widetilde{\lambda^+}$ such that its norm $||g||$ is the maximum of the norms of those factors. With HNN extension we can deal similarly, which is left as an exercise. In order to satisfy (b) of the Definition 4.1, we apply Lemma 3.8 extending $G^* \cap \pi(\delta)$ accordingly and identifying with unused ordinals of the intervals related to u. We see that G^* becomes a subgroup of the set $\widetilde{\lambda^+}$ with $\operatorname{dom} G^* = u$ and G^* is a subgroup of some u-group G obtained by iterated applications of free products with amalgamation and unions of such chains. Hence it follows from Definition 3.1 that nice elements in G_0 over $G_0 \cap \pi(\delta)$ remain nice in $G \cap \pi(\delta)$, which we will use silently to check $1, \ldots, 8$ in Definition A.1:

1. Let $p, q \in \mathbb{P}$ such that $p \leq q$ and $p = (G^p, u^p)$ and $q = (G^q, u^q)$. Then $G^p \leq G^q$ and $\operatorname{dom} G^p \subseteq \operatorname{dom} G^q$, or equivalently $\operatorname{dom} p \subseteq \operatorname{dom} q$.

2. Let $p, q, r \in \mathbb{P}$ such that $p, q \leq r$. With the same notation as above we have that G^p, G^q are subgroups of G^r and all of them belong to $\mathcal{K}^*$. Consider $G' \leq G^r$ generated by G^p and G^q. It is clear that $u = \operatorname{dom} G' = \operatorname{dom} p \cup \operatorname{dom} q$. Using the fact that G^r is in $\mathcal{K}^*$, we can add to G' the elements of G^r, with norm in u and obtain a new group $G'' \in \mathcal{K}^*$ such that $G' \leq G'' \leq G^r$. Hence $r' = (G'', u)$ is the required element in $\mathbb{P}$.

Condition 4. can be shown similarly, 3., 5. and 6. are obvious in view of the previous remarks.

7. (Indiscernibility) Suppose that $p = (G^p, u^p) \in \mathbb{P}$ and $\varphi : u^p \to u'$ is an order-isomorphism in λ^+. We define a set

$$G' = \{(\varphi(\alpha), i) \in \widetilde{\lambda^+} : \alpha \in u^p, (\alpha, i) \in G\} \subseteq \widetilde{\lambda^+}$$

and give to G' multiplication canonically induced by G. The map φ induces a map of p to some $\varphi(p) = (G', u') \in \mathbb{P}$ which we also denote by φ; it is order preserving on $u^p \to u'$

and a group isomorphism on $G \to G'$. The fact that $q \leq p$ implies $\varphi(q) \leq \varphi(p)$ is also clear.

8. (Amalgamation property) Using the same notations let be G^p and G^q from p and q respectively and let $r = (G^r, u^r)$ be such that $G^r = G^p \cap G^q$. The free product $G^* = G^p *_{G^r} G^q$ with amalgamated subgroup G^r by Lemma 3.9 belongs to $\mathcal{K}$. By Lemma 3.8 there is a group $G \in \mathcal{K}^*$ such that $G^* \leq G$. Using the remarks at the beginning we can trade G into an (isomorphic) u-group which we also call G with $\operatorname{dom} G = u = \operatorname{dom} G^p \cup \operatorname{dom} G^q$. Hence $s = (G, u)$ is as required for 8. $\square$

It may help to make the following

Definition 4.3 *Let G_i be u_i-groups for $i = 1, 2$. A map $\psi : G_1 \to G_2$ is a strong isomorphism if $\psi : G_1 \to G_2$ is a group isomorphism which preserves the order on $\operatorname{dom} G_i$, this is to say that*

$$\operatorname{dom} G_1 \to \operatorname{dom} G_2 \quad (\, ||x|| \to ||\psi(x)|| \,) \quad (x \in G_1)$$

is an order isomorphism.

The map defined in 7. is such a strong isomorphism.

Below we will introduce certain density systems on $\mathbb{P}$ which will ensure the requirements stated in the Main Theorem.

The group $\mathcal{G}$ at the end will be

$$\mathcal{G} = \bigcup_{\alpha < \lambda^+} \mathcal{G}_\alpha, \tag{4.2}$$

where each $\mathcal{G}_\alpha$ is the union of all groups in the directed system $\mathbb{G}_\alpha$ over $\mathbb{P}_\alpha$ (see Definition A.2), where

$$\mathbb{P}_\alpha = \{p \in \mathbb{P} : \operatorname{dom} p \subseteq \alpha\}.$$

Every $\mathcal{G}_\alpha$ has cardinality $\leq \lambda$, so that (4.2) is a λ^+-filtration of $\mathcal{G}$. For the rest of this section we fix the following notation: Let be $\alpha < \beta < \lambda^+$, $u \subseteq v \subseteq \lambda^+$ with $|v| < \lambda$, and define

$$\mathbb{E} := \{p \in \mathbb{P}/\mathbb{G}_\alpha : v \subseteq \operatorname{dom} p \subseteq v \cup \beta\}.$$

Recall from Definition A.2 that $p \in \mathbb{E}$ if and only if $p \restriction \alpha \in \mathbb{G}_\alpha$, in our setting $p = (G^p, u^p)$ this is to say that $(G^p \cap \pi(\alpha), u^p \cap \alpha) \in \mathbb{G}_\alpha$. Note that this will follow for density systems (below) from condition 2 of the ordering on $\mathbb{P}$. We define

the density system for $|\mathcal{G}| = \lambda^+$ to be the set

$$D_\alpha(u,v) := \{p \in \mathbb{E} : u \cup \{\alpha\} \subseteq \operatorname{dom} p \subseteq v \cup \beta\}. \tag{4.3}$$

Proposition 4.4 *The collection D_α of $D_\alpha(u,v)$ as in (4.3) is a density system over $\mathbb{G}_\alpha$.*

Proof. We will use the notation above and from Definition A.2. It is clear that $D_\alpha(u,v)$ is closed upwards in $\mathbb{E}$. To show that $D_\alpha(u,v)$ is dense in $\mathbb{E}$ it is enough to consider the case that $\alpha \notin u$ and $q = (G,u) \in \mathbb{E}$. As in the proof of Theorem 4.2 we can find a group $G' \in \mathcal{K}^*$, such that $G \leq G'$, which contains a subgroup H' isomorphic to H and $\alpha \in \operatorname{dom} H' \subseteq v \cup \beta$. In particular, we have $\alpha \in \operatorname{dom} G'$. Moreover, the group G' can be constructed in such a way that $\operatorname{dom} G' \subseteq v \cup \beta$. By the remarks above $p := (G', \operatorname{dom} G') \in \mathbb{E}$, nice elements in G are also nice in G', hence $q \leq p \in D_\alpha(u,v)$ shows density. The second condition in Definition A.3 can be verified similarly to the proof of 7. in Theorem 4.2. $\square$

The density systems to make $\mathcal{G}$ simple

If $x, y \in \mathcal{G}_\alpha$ and $o(x) = o(y) = \infty$ then let

$$D_{x,y}(u,v) = \{p \in \mathbb{E} : p = (G^p, u^p)\ , u \subseteq u^p \subseteq v \cup \beta, \quad x \in \langle y^{G^p} \rangle\}, \tag{4.4}$$

where $\langle y^{G^p} \rangle = \langle y^z : z \in G^p \rangle$.

Proposition 4.5 *$D_{x,y}$ as in (4.4) is a density system over $\mathbb{G}_\alpha$.*

Proof. If $q = (G^q, u^q) \in \mathbb{E}$ we may assume that $y, x \in G^q$. If $x \notin \langle y^{G^q} \rangle$ we can construct a v-group G represented by $p \in \mathbb{E}$ extending q such that $x \in \langle y^{G^q} \rangle$: Apply Proposition 3.10 and define $G_1 = \langle G^q, t : x = t^{-1}yt \rangle$, as an HNN extension, where t is a new element in v with $||t|| = ||x||$. Using again the argument from the proof of Theorem 4.2 we can also find $G_1 \leq G$ with $G \in \mathcal{K}^*$ and $\operatorname{dom}(G) \subset v \cap \beta$ which gives $p \leq q = (G,u) \in \mathbb{E}$ and x and y are conjugate in G, hence $q \in D_{x,y}$ showing density. $\square$

The density systems to trade monomorphisms of $\mathcal{G}$ into inner automorphism

For any subgroup $K \leq \mathcal{G}_\alpha$ of cardinality $< \lambda$ and any monomorphism $\psi : \mathcal{G}_\alpha \to \mathcal{G}_\alpha$ define the set $D_{\psi \restriction K}(u,v)$ as

$$\{p = (G^p, u^p) \in \mathbb{E} : K \leq G^p \ \ \exists y \in G^p \text{ with } \psi \restriction K = y^* \restriction K, \operatorname{dom} p \subseteq v \cup \beta\} \tag{4.5}$$

Note that the just defined system (running over all ψ and K) of sets $D_{\psi\restriction K}(u,v)$ has (only) size λ.

We also have a special case of the last density system (4.5) which we state explicitly because it serves for a different purpose.

The density system to conjugate copies of H in $\mathcal{G}$

In this case we choose for each $H' \leq \mathcal{G}_\alpha$ with isomorphism $\psi : H \to H'$ the set

$$D_\psi(u,v) = \{p = (G^p, u^p) \in \mathbb{E} : H' \leq G^p \text{ and } \exists g \in G^p, \psi \restriction H = g^* \restriction H\} \tag{4.6}$$

Proposition 4.6 *The collection of all $D_{\psi\restriction K}(u,v)$ as in (4.5) [respectively $D_\psi(u,v)$ as in (4.6)] is a density system over $\mathbb{G}_\alpha$.*

Proof. Apply Lemma 3.8 and Definition 3.2: If $p \in \mathbb{E}$ then we find a group G such that $\psi \restriction K = y^* \restriction K$ for some $y \in G$ by HNN extension. By Theorem 4.2 we also find $G \leq G' \in \mathcal{K}^*$ and we trade G' into a v-group with $||y|| = \alpha$ and $v = u \cup \{\alpha\}$ as we did before. Hence Proposition 4.6 follows. $\square$

The density for many nice elements

For α as above we also choose

$$D(u,v) = \{p = (G^p, u^p) \in \mathbb{E} : \exists\, q \in \mathbb{E},\ \exists x \in G^p \text{ torsion-free and } \langle x \rangle * G^q = G^p\} \tag{4.7}$$

The density can easily be checked as before.

5 Proof of the Main Theorem

We will fix for the rest of this paper a particular word of a free group (i.e. a term in group theory) $\tau(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) = [\mathbf{x_1}\mathbf{x_2}, \mathbf{x_3}\mathbf{x_4}]$ which is the commutator of products in free variables $\mathbf{x_1}, \dots, \mathbf{x_4}$. We will also use the notion of a group isomorphism which is at the same time 'level preserving', the strong isomorphism from Definition 4.3

The main result for proving the Main Theorem is the following lemma concerning this word.

Main Lemma 5.1 *Let $\mathbb{P}$ be the λ^+-uniform poset defined in the last section. Assume that the following three properties hold in $\mathbb{P}$.*

1. *There are ordinals* $\delta_1 < \delta_2 < \delta_3 < \delta_4$ *and approximations* $p_i \in \mathbb{P}_{\delta_{i+1}}$ $(i = 1, 2, 3)$ *and* $p_4 \in \mathbb{P}$ *with* $p_i \restriction \delta_i = p_0 \in \mathbb{P}$ *for* $i = 1, \dots, 4$.

2. *There is a nice element* $x_1 \in G^{p_1} \setminus G^{p_0}$ *over* G^{p_0} *and an element* $y_1 \in G^{p_1} \setminus x_1^{G^{p_0}}$.

3. *Let* $\varphi_i : G^{p_1} \to G^{p_i}$ *for* $i = 2, 3, 4$ *be a strong isomorphism and* $\varphi_i(x_1) = x_i$, $\varphi_i(y_1) = y_i$ *in* G^{p_i} *respectively.*
   Then we can find in $\mathbb{P}$ *an approximation* $q \geq p_1, \dots, q_4$ *such that in the group* G^q *we have*
   $$\tau(x_1, x_2, x_3, x_4) = 1 \quad \textit{but } \tau(y_1, y_2, y_3, y_4) \neq 1.$$

Proof. We assume the hypothesis of the lemma, in particular we have elements x_i, y_i with $(i = 1, 2, 3, 4)$ in the appropriate groups G^{p_i}. From $\mathbb{P}$ we can choose an approximation $q = (G, v) \geq p_1, p_2$ such that $G = G^{p_1} *_{G^{p_0}} G^{p_2}$, hence $x =: x_1 x_2 \in G$. First we want to show that

1. *For any word* $\sigma(x, \bar{t}) = (x_1x_2)^{n_1} t_1 (x_1x_2)^{n_2} t_2 \cdots = 1$ *for a finite set* $\bar{t} = \{t_1, \dots, t_k\}$ *from* G^{p_0} *follows that the* t_i*'s commute with* x.

2. x *is nice in* G *over* G^{p_0}.

At the end we want to apply the normal form theorem for free products with amalgamation [15] to the equation

$$1 = (x_1x_2)^{n_1} t_1 (x_1x_2)^{n_2} t_2 \cdots \text{ with } t_i \in G^{p_0} \text{ and } n_i \in \mathbb{Z} \setminus 0.$$

Naturally we can distinguish four cases, where we use the following notation: Let $x_{\iota_1}^{\varepsilon_1}$ be the *last* x_i-term which appears in $(x_1x_2)^{n_1}$, similarly let $x_{\iota_2}^{\varepsilon_2}$ be the *first* x_i-term which appears in $(x_1x_2)^{n_2}$.

Obviously we have the following possibilities:

1. $n_1 > 0 \Rightarrow (\iota_1 = 2 \text{ and } \varepsilon_1 = 1)$

2. $n_1 < 0 \Rightarrow (\iota_1 = 1 \text{ and } \varepsilon_1 = -1)$

3. $n_2 > 0 \Rightarrow (\iota_2 = 1 \text{ and } \varepsilon_2 = 1)$

4. $n_2 < 0 \Rightarrow (\iota_2 = 2 \text{ and } \varepsilon_2 = -1)$

Hence the displayed equality can only hold if $\iota_1 = \iota_2 \in \{1,2\}$. To be definite we take $\iota_1 = \iota_2 = 1$ which by the cases implies $\varepsilon_1 = -1, \varepsilon_2 = 1, n_1 < 0$ and $n_2 > 0$. Hence the term in the last displayed equation connecting x^{n_1} and x^{n_2} is $x_1^{-1}t_1x_1$ and the equality for normal forms forces $s =: x_1^{-1}t_1x_1 \in G^{p_0}$. Hence $x_1 = t_1x_1s^{-1}$ which by hypothesis is a nice element, hence $s = t_1$ and $[x_1, t_1] = 1$. Using φ_2 also $[x_2, t_1] = 1$ and induction shows that all the t_i's commute with x_1 and x_2 and in particular any equation of the form $txs = x$ with $s, t \in G^{p_0}$ implies $s = t^{-1}$, hence x is nice.

Next we consider the relationship of the x_i's and the y_i's. We have the following

Claim 5.2 *If there are group terms (words) $y_1 = \sigma_1(x_1, \bar{t}) \in G^{p_1}$ and $y_1y_2 = \sigma_2(x, \bar{t}')$ with $\bar{t}, \bar{t}' \subseteq G^{p_0}$, then there is an $s \in G^{p_0}$ such that $y_1 = \sigma_1(x_1, \bar{t}) = s^{-1}x_1s$.*

Proof. Using the action of φ_2 we have

$$\sigma_1(x_1, \bar{t})\sigma_1(x_2, \bar{t}) = y_1y_2 = \sigma_2(x_1x_2, \bar{t}').$$

By hypothesis $y_i \in G^{p_i} \setminus G^{p_0}$, hence the displayed element has length 2. Let $\sigma_2(x, \bar{t}') = t_1x^{n_1}t_2 \dots t_kx^{n_k}$ be in canonical form as before. Hence the normal form forces $k = 1$ and if $|n_1| > 1$ then $t_1(x_1x_2)^{n_1}t_2$ has length > 2, so also $n_1 \in \{\pm 1\}$. We arrive at two cases

$$\sigma_1(x_1, \bar{t})\sigma_1(x_2, \bar{t}) = t_1x_1x_2t_2 \tag{5.1}$$

or

$$\sigma_1(x_1, \bar{t})\sigma_1(x_2, \bar{t}) = t_1x_2^{-1}x_1^{-1}t_2. \tag{5.2}$$

In the first case normal form forces that there is $s \in G^{p_0}$ such that $\sigma_1(x_1, \bar{t}) = t_1x_1s$ as well as $\sigma_1(x_2, \bar{t}) = s^{-1}x_2t_2$. Application of φ_2 also gives $\sigma_1(x_2, \bar{t}) = t_1x_2s$, hence $t_1x_2s = s^{-1}x_2t_2$. Recall that x_2 (like x_1) is nice, hence $t_1 = s^{-1}$ and $s = t_2$, the claim follows (in this case). In the other case we have an element (5.2) which is written in normal form at the same time as products from exchanged factors G^{p_1} and G^{p_2} which is impossible. $\square$

In order to complete the proof of the Main Lemma 5.1 we define two more extensions of G^q in $\mathbb{P}$. Let $q' \in \mathbb{P}$ be given by $\operatorname{dom} q' = \operatorname{dom} p_3 \cup \operatorname{dom} p_4$ such that as groups we have $G^{q'} = G^{p_3} *_{G^{p_0}} G^{p_4}$. Recall that $G^q = G^{p_1} *_{G^{p_0}} G^{p_2}$. Hence we find a strong isomorphism (which is order preserving on dom ...) which is

$$\varphi : G^q \to G^{q'} \text{ extending } \varphi_3,\ \varphi_4\varphi_0^{-1},$$

hence

$$\varphi(x_1) = x_3,\ \varphi(x_2) = x_4,\ \varphi(x) = \varphi(x_1x_2) = x_3x_4,\ \varphi(y_1) = y_3,\ \varphi(y_2) = y_4$$

and we let

$$x^1 = x = x_1x_2,\ x^2 = x_3x_4,\ y^1 = y_1y_2,\ y^2 = y_3y_4$$

which are in the appropriate factors G^q respectively $G^{q'}$ but not in G^{p_0}, moreover $\varphi(x^1) = x^2, \varphi(y^1) = y^2$. If there is a group term $y^1 = \sigma(x^1, \bar{t})$ with $\bar{t} \subseteq G^{p_0}$ then we can apply the last Claim 5.2 to see that $y^1 = s^{-1}x^1s \in (x^1)^{G^{p_0}}$ which was excluded by hypothesis of the Main lemma. We conclude that

$$y^1 \notin \langle G^{p_0}, x^1\rangle \leq G^q \text{ and similarly } y^2 \notin \langle G^{p_0}, x^2\rangle \leq G^{q'}. \tag{5.3}$$

Now we define a final approximation $r \in \mathbb{P}$ with $\operatorname{dom} r = \operatorname{dom} q' \cup \operatorname{dom} q$. Group theoretically we get G^r in several steps:

To easy notation let

$$K_0 = G^{p_0}, K_1 = G^q, K_2 = G^{q'}, K_1' = \langle K_0, x^1\rangle \leq K_1, K_2' = \langle K_0, x^2\rangle \leq K_2,$$

and define $L' = K_1' *_{K_0} K_2' = \langle K_0, x^1, x^2\rangle$, and if $N = \langle [x^1, x^2]^{L'}\rangle \triangleleft L'$ is the normal subgroup of L' generated by the commutator $[x^1, x^2]$ then let $L = L'/N$. However $K_0 \cap N = 1$ and $K_0 \leq L$ canonically, hence L is the free product of K_0 with the free abelian group $\langle x^1N, x^2N\rangle$ of rank 2. Using only the group operation of Section 4, the group L now obviously can be made into an element in $\mathbb{P}$. Note that also $N \cap K_i' = 1$, we get a canonical embedding $K_i' \leq L$ and can consider $M_i = K_i *_{K_i'} L$ for $i = 1, 2$; finally put $G^r = M_1 *_L M_2$. From the normal subgroup N follows in G^r that $[x^1, x^2] = 1$. On the other hand from (5.3) it follows that $y^i \notin K_i'$, $y_i \in G^{p_i}$ hence $y^1 = y_1y_2 \in G^q \setminus K_1'$ and similarly $y^2 \in G^{q'} \setminus K_2'$. By definition of M_i also $y^i \in M_i \setminus K_i'$ and $[y^1, y^2]$ can not cancel in G^r, this is to say that $[y^1, y^2] \neq 1$. $\square$

Using now directly the Main Theorem 1.11 (which is in terms of model theory) from the forthcoming book Shelah [21] (or slight modifications in [11, 20] or [19] we get the following proposition. Its proof like earlier 'black boxes' (see the appendix of [6] for instance), also this case is based counting arguments but using $\diamondsuit$ on $\{\alpha \in \lambda^+ : \operatorname{cf}(\alpha) = \lambda\}$ and $\{\alpha \in \lambda^+ : \operatorname{cf}(\alpha) = \omega\}$ as well. The latter explains why the generalized continuum hypothesis gets into the proposition.

The statement of the proposition depends on $\mathbb{P}$, the density systems constructed in Section 4 and also (substantially) on the Main Lemma 5.1.

Proposition 5.3 *1. Assuming ZFC + GCH, there is an ascending sequence of ordinals $\zeta_\alpha < \lambda^+$ and a continuous ascending chain of admissible ideals $\mathbb{G}_\alpha \subseteq \mathbb{P}_{\zeta_\alpha}$ which meets all density systems constructed in Section 4.*

2. Let $\mathcal{G}_\alpha = \bigcup \mathbb{G}_\alpha$ and $\mathcal{G} = \bigcup_{\alpha<\lambda^+} \mathcal{G}_\alpha$.

3. If for all $\alpha < \lambda^+$ there is a nice element $x_\alpha \in \mathcal{G} \setminus \mathcal{G}_\alpha$ over $\mathcal{G}_\alpha$ and there is $y_\alpha \in \mathcal{G} \setminus x_\alpha^{\mathcal{G}_\alpha}$, then there are four ordinals $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that $[x_{\alpha_1}x_{\alpha_2}, x_{\alpha_3}x_{\alpha_4}] = 1$ and $[y_{\alpha_1}y_{\alpha_2}, y_{\alpha_3}y_{\alpha_4}] \neq 1$.

For simplicity we will say that $\zeta_\alpha = \alpha$ without less of generality. We will apply this black box for proving the Main Theorem 1.

PROOF. Obviously $H \leq \mathcal{G}$ because any approximation has H as subgroup.

$\mathcal{G}$ has cardinality λ^+: By Proposition 5.3 we have that for every $\alpha < \lambda^+$, there is a group G in $\mathbb{G}$ such that $\alpha \in \operatorname{dom} G$. Hence $\operatorname{dom} \mathcal{G} = \lambda^+$ and $\lambda^+ = |\mathcal{G}|$ follows immediately.

Property (d): Let $x, y \in \mathcal{G}$ be both of infinite order. We can apply Proposition 4.5 and the density condition from Proposition 5.3 to see that x and y are conjugate in $\mathcal{G}$. If x has finite order and y has infinite order we can easily find an element $x' = xz \in \mathcal{G}$ of infinite order such that $z^{-1}y$ has infinite order as well. Hence the first case applies and similarly we work of also y has finite order. Hence in any case $x \in \langle y^G \rangle$.

Property (a) follows from (4.6) and Proposition 4.6.

Property (b) was carried on inductively by our choice of $\mathcal{K}^*$, hence this also holds for $\mathcal{G}$. Finally we have to show that

Property (c) holds, this is to say that

any monomorphism $\psi : \mathcal{G} \to \mathcal{G}$ is an inner automorphism of $\mathcal{G}$.

Suppose $\psi : \mathcal{G} \to \mathcal{G}$ is a counterexample, a monomorphism which is not inner. We first want to show that the following holds.

$$\text{There is } \alpha_* < \lambda^+ \text{ such that } \psi(x) \in x^{\mathcal{G}_\alpha} \text{ for all } x \in \mathcal{G} \setminus \mathcal{G}_{\alpha_*}. \tag{5.4}$$

Otherwise

$$\text{for all } \alpha < \lambda^+ \text{ there is } x_\alpha \in \mathcal{G} \setminus \mathcal{G}_\alpha \text{ with } y_\alpha = \psi(x_\alpha) \notin x_\alpha^{\mathcal{G}_\alpha} \tag{5.5}$$

From (4.7) it is easy to see by a change of elements x_α that

$$\text{we may also assume that each } x_\alpha \text{ is nice over } \mathcal{G}_\alpha \text{ in } \mathcal{G}. \tag{5.6}$$

By Proposition 5.3, based also on the Main Lemma 5.1 together with (5.5) we can find four ordinals $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that $[x_{\alpha_1}x_{\alpha_2}, x_{\alpha_3}x_{\alpha_4}] = 1$ and $[y_{\alpha_1}y_{\alpha_2}, y_{\alpha_3}y_{\alpha_4}] \neq 1$. However $1 = \psi([x_{\alpha_1}x_{\alpha_2}, x_{\alpha_3}x_{\alpha_4}]) = [y_{\alpha_1}y_{\alpha_2}, y_{\alpha_3}y_{\alpha_4}] \neq 1$ is a contradiction. Hence we may assume that (5.4) holds. $\square$

First we claim that

$$\forall X \subseteq \mathcal{G},\ |X| = \lambda^+ \Rightarrow (\exists\ X' \subseteq X, \exists y_* \in \mathcal{G},\ |X'| = \lambda^+, \psi(x) = x^{y_*}\ \forall x \in X'). \tag{5.7}$$

Let $\downarrow z = \min\{\alpha < \lambda^+ : z \in \mathcal{G}_\alpha\}$ for any $z \in \mathcal{G}$. Let S be the set of all limit ordinals $\alpha < \lambda^+$ with $\alpha_* < \alpha$. Hence S is a stationary subset of λ^+ of cardinality λ^+ and from (5.4) we can choose a sequence $x_\alpha \in X \setminus \mathcal{G}_\alpha$ and $y_\alpha = \psi(x_\alpha) \in x_\alpha^{\mathcal{G}_\alpha}$ for all $\alpha \in S$. We may assume that $\{\downarrow y_\alpha : \alpha \in S\}$ is bounded in λ^+. For otherwise the function $f : (S \to \lambda^+)(\alpha \to f(\alpha) = \downarrow y_\alpha)$ satisfies $f(\alpha) = \downarrow y_\alpha < \alpha$ and is regressive. By Fodor's lemma there is a sationary subset S' of S and $\beta < \lambda^+$ such that $f(\alpha) = \beta$ for all $\alpha \in S'$, see Jech [12, p. 59, Theorem 22]. As $|S'| = \lambda^+$ we can replace S by S'. We now continue using S and choose the new $X = \{x_\alpha : \alpha \in S\}$. From $|\mathcal{G}_\beta| = \lambda < |X| = \lambda^+$ we also find an equipotent subset of X, call it X again, and

$$y_* \in \mathcal{G}_\beta \text{ such that } \psi(x) = x^{y_*} \text{ for all } x \in X.$$

The claim (5.7) is shown.

We now choose a set $X = \{x_\alpha^1 \in \mathcal{G} \setminus \mathcal{G}_\alpha : \alpha < \lambda^+\}$ of nice elements over $\mathcal{G}_\alpha$ in $\mathcal{G}$ with $|X| = \lambda^+$ and apply (5.7). Hence we may assume that $\psi \restriction X = y_*^* \restriction X$ for some $y_* \in \mathcal{G}$. Replacing ψ by $\psi(y_*^{-1})^*$ we may assume that $y_* = 1$, hence

$$\psi \restriction X = \mathrm{id}_X.$$

We now assume for contradiction that $\psi \neq \mathrm{id}_{\mathcal{G}}$. There is an $x_* \in \mathcal{G}$ such that $\psi(x_*) \neq x_*$. Next we choose a second sequence of nice elements over $\mathcal{G}_a$ which is $\{x_\alpha^2 \in \mathcal{G} \setminus \mathcal{G}_\alpha : \psi(x_\alpha^2) \neq x_\alpha^2 : \alpha < \lambda^+\}$. If a first choice fails for any subsequence, the we multiply each of these elements by x_*. For $\alpha > \alpha_*$ the new elements are obviously nice and do what we want. We apply once more the claim (5.4) and find some α_* and $t_\alpha \in \mathcal{G}_{\alpha_*}$ such that $\psi(x_\alpha^2) = (x_\alpha^2)^{t_\alpha}$ for all $\alpha_* < \alpha < \lambda^+$. By a pigeon hole argument we may assume that $t = t_\alpha$ for all $\alpha_* < \alpha < \lambda^+$. Hence we found a sequence of pairs of nice elements over $\mathcal{G}_{\alpha_*}$:

$$x_\alpha^1, x_\alpha^2 \in \mathcal{G} \setminus \mathcal{G}_\alpha \text{ with } y_\alpha^1 = \psi(x_\alpha^1) = x_\alpha^1,\ y_\alpha^2 = \psi(x_\alpha^2) = (x_\alpha^2)^t \neq x_\alpha^2.$$

Recall the properties of $\mathbb{P}$: If $\alpha_* < \alpha < \alpha_1 < \alpha_2$ we can choose $p_1 \in \mathbb{P}_{\alpha_1}$ such that $x^i_{\alpha_1}, y^i_{\alpha_1} \in G^{p_1} \setminus \pi(\alpha)$, $i = 1, 2$ and $t \in G^{p_1} \cap \pi(\alpha_*)$. Moreover we find $p_2 \in \mathbb{P}_{\alpha_2}$ such that $G^{p_1} \cap G^{p_2} = G^{p_1} \cap \pi(\alpha)$ and let $p_0 = p_1 \restriction \alpha = (G^{p_1} \cap \pi(\alpha), \operatorname{dom}(G^{p_1} \cap \pi(\alpha)))$. There is a 'level preserving' strong isomorphism

$$\varphi : G^{p_1} \to G^{p_2} \text{ with } \varphi \restriction G^{p_0} = \mathrm{id}_{G^{p_0}},$$

which carries the $x^i_{\alpha_1}, y^i_{\alpha_1}$'s to G^{p_2}. Let $\varphi(x^i_{\alpha_1}) = x^i_{\alpha_2}$ which is another nice element over $\mathcal{G}_{\alpha_*}$ and $\varphi(y^i_{\alpha_1}) = y^i_{\alpha_2}$ for $i = 1, 2$. From $t \in G^{p_0}, \varphi \restriction G^{p_0} = \mathrm{id}_{G^{p_0}}$ and the above equations we have $x^1_{\alpha_2} = y^1_{\alpha_2}$ and $(x^2_{\alpha_2})^t = y^2_{\alpha_2}$. We now choose $p_3 \in \mathbb{P}$ with $p_1, p_2 \leq p_3$, hence $x^1_{\alpha_1}, x^2_{\alpha_2}$ are nice over G^{p_0}. From $t \in G^{p_0}$ and the last equalities follows that

$$y^1_{\alpha_1} y^2_{\alpha_2} = \psi(x^1_{\alpha_1} x^2_{\alpha_2}) = \psi(x^1_{\alpha_1})\psi(x^2_{\alpha_2}) = x^1_{\alpha_1}(x^2_{\alpha_2})^t \notin (x^1_{\alpha_1} x^2_{\alpha_2})^{\mathcal{G}_{\alpha_*}}.$$

On the other hand $x^1_{\alpha_1}, x^2_{\alpha_2} \in \mathcal{G} \setminus \mathcal{G}_{\alpha_*}$, hence

$$y^1_{\alpha_1} y^2_{\alpha_2} = \psi(x^1_{\alpha_1} x^2_{\alpha_2}) \in (x^1_{\alpha_1} x^2_{\alpha_2})^{\mathcal{G}_{\alpha_*}}$$

by (5.4) is a contradiction. Hence $\psi = \mathrm{id}_{\mathcal{G}}$ is an inner automorphism, which contradicts our initial hypothesis and property (c) of the Main Theorem follows.

A The appendix: A standard λ^+-uniform set

Definition A.1 A *standard λ^+-uniform partial order* is a partial order $\leq$ defined on a subset $\mathbb{P}$ of $\lambda \times \mathcal{P}_{<\lambda}(\lambda^+)$. Its elements are pairs $p = (\alpha, u) \in \lambda \times \mathcal{P}_{<\lambda}(\lambda^+)$. We write $u = \operatorname{dom} p$ and call u the *domain* of p. [The ordinal α is a code for some algebraic structures under investigation that u can support, in our case α represents a group on the set u.] These approximations $p \in \mathbb{P}$ satisfy the following conditions:

1. (Compatibility of the orders) If $p \leq q$ then $\operatorname{dom} p \subseteq \operatorname{dom} q$.

2. For all $p, q, r \in \mathbb{P}$ with $p, q \leq r$ there is $r' \in \mathbb{P}$ such that $p, q \leq r' \leq r$ and $\operatorname{dom} r' = \operatorname{dom} p \cup \operatorname{dom} q$.

3. If $\{p_\alpha : \alpha < \delta\}$ is an increasing sequence in $\mathbb{P}$ of length $\delta < \lambda$ then it has a least upper bound $q \in \mathbb{P}$ with $\operatorname{dom} q = \bigcup_{\alpha<\delta} \operatorname{dom} p_\alpha$; we say that $q = \bigcup_{\alpha<\delta} p_\alpha$.

4. If $p \in \mathbb{P}$ and $\alpha < \lambda^+$ then there is $q \in \mathbb{P}$ such that $q \leq p$ and $\operatorname{dom} q = \operatorname{dom} p \cap \alpha$ and there is a unique maximal such q for which we write $q = p \restriction \alpha$.

5. (Continuity I) If δ is a limit then $p \restriction \delta = \bigcup_{\alpha<\delta} p \restriction \alpha$.

6. (Continuity II) If $\{p_\alpha : \alpha < \delta\}$ is an increasing sequence in $\mathbb{P}$ of length $\delta < \alpha$ then
$$(\bigcup_{i<\delta} p_i) \restriction \alpha = \bigcup_{i<\delta} (p_i \restriction \alpha).$$

7. (Indiscernibility) If $p = (\alpha, u) \in \mathbb{P}$ and $\varphi : u \to u'$ is an order-isomorphism in λ^+, then $\varphi(p) := (\alpha, \varphi(u)) \in \mathbb{P}$. Moreover, if $q \leq p$ then $\varphi(q) \leq \varphi(p)$.

8. (Amalgamation property) For every $p, q \in \mathbb{P}$ and $\alpha < \lambda^+$ with $p \restriction \alpha \leq q$ and $\operatorname{dom} p \cap \operatorname{dom} q = \operatorname{dom} p \cap \alpha$ there is an $r \in \mathbb{P}$ such that $p, q \leq r$.

Consider the following filtration of $\mathbb{P}$, where $\alpha < \lambda^+$,
$$\mathbb{P}_\alpha := \{p \in \mathbb{P} : \operatorname{dom} p \subseteq \alpha\}.$$

Definition A.2 Let $(\mathbb{P}, \leq)$ be a λ^+-uniform partially ordered set and $\alpha < \lambda^+$. A subset $\mathbb{G} \subseteq \mathbb{P}_\alpha$ is an *admissible ideal of* $\mathbb{P}_\alpha$ if the following holds:

1. $\mathbb{G}$ is closed downward, i.e. if $p \in \mathbb{G}$ and $q \leq p$ then $q \in \mathbb{G}$.

2. $\mathbb{G}$ is λ-directed, i.e. if $\mathcal{A} \subset \mathbb{G}$ and $|\mathcal{A}| < \lambda$ then $\mathcal{A}$ has an upper bound in $\mathbb{G}$.

3. (Maximality) If $p \in \mathbb{P}$ is compatible with all $q \in \mathbb{G}$ then $p \in \mathbb{G}$. (Recall that p is compatible with q if there is an $r \in \mathbb{P}$ such that p, $q \leq r$.)

For an admissible ideal $\mathbb{G}$ of $\mathbb{P}_\alpha$, we define
$$\mathbb{P}/\mathbb{G} := \{p \in \mathbb{P} : p \restriction \alpha \in \mathbb{G}\}.$$

This consists of all extensions of elements in $\mathbb{G}$. Note that this notion is compatible with taking direct limits in the following sense. Let $\{\zeta_\beta : \beta < \alpha\}$ be an increasing sequence of ordinals in λ^+ converging to ζ. Suppose that for every β we have an admissible ideal $\mathbb{G}_\beta$ of $\mathbb{P}_{\zeta_\beta}$. Then there is a unique minimal admissible ideal containing the set theoretic union of the $\mathbb{G}_\beta$'s. With a slight misuse of notation we write
$$\mathbb{G}_{<\alpha} := \bigcup_{\beta<\alpha} \mathbb{G}_\beta,$$
for this admissible ideal of $\mathbb{P}_\zeta$, see Hart, Laflamme and Shelah [11, p. 173, Lemma 1.3].

Definition A.3 Let $\mathbb{G}$ be an admissible ideal of $\mathbb{P}_\alpha$ and $\alpha < \beta < \lambda^+$. An (α, β)-*density system over* $\mathbb{G}$ is a function

$$D : \{(u, v) : u \subseteq v \in \mathcal{P}_{<\lambda}(\lambda^+)\} \longrightarrow \mathcal{P}(\mathbb{P})$$

such that the following holds:

1. $D(u, v) \subseteq \{p \in \mathbb{P}/\mathbb{G} : \operatorname{dom} p \subseteq v \cup \beta\}$ is a dense and upward-closed subset.
2. If $(u, v), (u', v')$ and $u \cap \beta = u' \cap \beta$, $v \cap \beta = v' \cap \beta$ and there is an order-isomorphism from v onto v' which maps u onto u', then for any ordinal γ we have

$$(\gamma, v) \in D(u, v) \Longleftrightarrow (\gamma, v') \in D(u', v').$$

An admissible ideal $\mathbb{G}'$ of $\mathbb{P}_{\alpha'}$ for some $\alpha' < \lambda^+$ *meets* the (α, β)-density system D (over $\mathbb{G}$) if $\alpha < \alpha'$, $\mathbb{G} \subseteq \mathbb{G}'$ and for each $u \in \mathcal{P}_{<\lambda}(\alpha')$ there is a $v \in \mathcal{P}_{<\lambda}(\alpha')$, with $u \subseteq v$ and such that $D(u, v) \cap \mathbb{G}' \neq \emptyset$.

References

[1] M. Aubry, *Homotopy theory and models, Based on lectures held at a DMV Seminar in Blaubeuren by H. J. Baues, S. Halperin and J.-M. Lemaire,* DMV Seminar, Vol. **24**, Birkhäuser Verlag, Basel, 1995.

[2] C. Casacuberta, *Recent advances in unstable localization*, CRM Proceedings and Lecture Notes **6**, Amer. Math. Soc., Providence, 1994, 1–22.

[3] C. Casacuberta, *On structures preserved by idempotent transformations of groups and homotopy types*, preprint 1999.

[4] C. Casacuberta, J. L. Rodríguez and J.-Y. Tai, *Localizations of abelian Eilenberg–Mac Lane spaces of finite type*, preprint 1998.

[5] C. Casacuberta, D. Scevenels and J. H. Smith, *Implications of large-cardinal principles in homotopical localization*, preprint 1998.

[6] A. L. S. Corner, R. Göbel, Prescribing endomorphism algebras, a unified treatment. Proc. London Math. Soc. **50**, (1985) 447–479.

Paper Sh:701, version 2000-05-04_11. See https://shelah.logic.at/papers/701/ for possible updates.

[7] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. vol. 1622, Springer-Verlag, Berlin–Heidelberg–New York 1996.

[8] M. Dugas, R. Göbel, *On locally finite p-groups and a problem of Philip Hall's*, J. Algebra **159**(1) (1993) 115–138.

[9] M. Dugas, A. Mader and C. Vinsonhaler, *Large E-rings exist*, J. Algebra **108** (1987) 88–101.

[10] P. C. Eklof and A. H. Mekler, *Almost Free Modules. Set-theoretic Methods*, North-Holland, Amsterdam, New York, 1990.

[11] B. Hart, C. Laflamme and S. Shelah, *Models with second order properties V: A general principle*, Ann. Pure Appl. Logic, **64** (1993) 169–194.

[12] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[13] A. Libman, *Cardinality and nilpotency of localizations of groups and G-modules*, Israel J. Math., to appear.

[14] A. Libman, *A note on the localization of finite groups,* J. Pure Appl. Algebra, to appear

[15] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer Ergebinsberichte **89**, Berlin–Heidelberg–New York, 1977.

[16] D. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Math. vol **80**, Berlin–Heidelberg–New York, 1996.

[17] P. E. Schupp, *A characterization of inner automorphisms*, Proc. Amer. Math. Soc. **101** (1987) 226–228.

[18] S. Shelah, *On a problem of Kurosh, Jonsson groups and applications*, S. I. Adian, W. W. Boone, G. Higman, eds., Word Problems II, North-Holland, Amsterdam, 1980, 373–394.

[19] S. Shelah, *Can you take Solovay's inaccessible away?*, Israel J. Math. **48** (1984) 1–47.

Paper Sh:701, version 2000-05-04_11. See https://shelah.logic.at/papers/701/ for possible updates.

[20] S. Shelah, *Uncountable constructions for b. a., e.c. groups and Banach spaces*, Israel J. Math. **51** (1985) 273–297.

[21] S. Shelah, Non Structure Theory, Oxford University Press (2000) in preparation

[22] L. Strüngmann, *Almost-free $E(R)$-algebras over countable domains*, Ph. D. Thesis, University of Essen (1998).

[23] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math. vol **61**, Springer-Verlag, Berlin–Heidelberg–New York, 1978.

Fachbereich 6, Mathematik und Informatik, Universität Essen,
D-45117 Essen, Germany, e-mail: R.Goebel@Uni-Essen.De

Departament de Matemàtiques, Universitat Autònoma de Barcelona,
E–08193 Bellaterra, Spain, e-mail: jlrodri@mat.uab.es

Institute of Mathematics, Hebrew University,
91904 Jerusalem, Israel, e-mail: shelah@math.huji.ac.il