UNIVERSALITY AMONG GRAPHS OMITTING A COMPLETE BIPARTITE GRAPH
SH706

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ABSTRACT. For cardinals λ, κ, θ we consider the class of graphs of cardinality λ which has no subgraph which is (κ, θ)-complete bipartite graph. The question is whether in such a class there is a universal one under (weak) embedding. We solve this problem completely under GCH. Under various assumptions mostly related to cardinal arithmetic we prove non-existence of universals for this problem. We also look at combinatorial properties useful for those problems concerning κ-dense families.

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Anotated Content

0. Introduction

The problem of "among graphs with λ nodes and no complete subgraph with κ nodes, is there a universal one" (i.e. under weak embedding) is to a large extent solved in Komjath-Shelah [KS95], see more there. E.g. give a complete solution under the assumption of GCH.

Now there are some variants, mainly for graph theorists embedding, i.e. a one to one function mapping an edge to an edge, called here weak or we-embedding; for model theorists an embedding also maps a non-edge to a non-edge, called here strong or ste-embedding. We have the corresponding we-universal and ste-universal. We deal here with the problem "among the graphs with λ nodes and no complete (θ,κ)-bipartite sub-graph in the weak sense, is there a universal one?", see below on earlier results. We call the family of such graphs $\mathfrak{H}_{\lambda,\theta,\kappa}$, and consider both the weak embedding (as most graph theorists use) and the strong embedding. Our neatest result appears in section 3 (see 3.5, 3.15).

Theorem 0.1. *Assume $\lambda\geq\theta\geq\kappa\geq\aleph_0$.*
1) If λ is strong limit then: there is a member of $\mathfrak{H}_{\lambda,\theta,\kappa}$ which is we-universal (= universal under weak embedding) <u>iff</u> there is a member of $\mathfrak{H}_{\lambda,\theta,\kappa}$ which is ste-universal (= universal under strong embeddings) <u>iff</u> $\mathrm{cf}(\lambda)\leq\mathrm{cf}(\kappa)$ *and* $(\kappa<\theta \vee \mathrm{cf}(\lambda)<\mathrm{cf}(\theta))$.
2) If $\lambda=2^\mu=\mu^+$ and $\mu=2^{<\mu}$, <u>then</u>

- (a) *there is no ste-universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$*
- (b) *there is we-universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$ iff $\mu=\mu^\kappa$ and $\theta=\lambda$.*

We give many sufficient conditions for the non-existence of universals (mainly we-universal) and some for the existence, for this dealing with some set-theoretic properties. Mostly when we get "no $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ is we/ste-universal" we, moreover, get "no $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ is we/ste-universal for the bipartite ones". Hence we get also results on families of bi-partite graphs. We do not look at the case $\kappa < \aleph_0$ here.

Rado has proved that: if λ is regular $> \aleph_0$ and $2^{<\lambda} = \lambda$, then $\mathfrak{H}_{\lambda,\lambda,1}$ has a ste-universal member (a sufficient condition for G^* being ste-universal for $\mathfrak{H}_{\lambda,\lambda,1}$ is: for any connected graph G with $< \lambda$ nodes, λ of the components of G are isomorphic to G). Note that $G \in \mathfrak{H}_{\lambda,\lambda,1}$ iff G has λ nodes and the valency of every node is $< \lambda$. Erdös and Rado (see [EH74], in Problem 74) ask what occurs, under GCH to say $\aleph_\omega$. By [She73, 3.1] if λ is strong limit singular then there is a ste-universal graph in $\mathfrak{H}_{\lambda,\lambda,1}$.

Komjath and Pach [KP84] prove that $\diamondsuit_{\omega_1} \Rightarrow$ no universal in $\mathfrak{H}_{\aleph_1,\aleph_1,\aleph_0}$, this holds also for $\mathfrak{H}_{\kappa^+,\kappa^+,\kappa}$ when $\diamondsuit_{S^{\kappa^+}_\kappa}$ holds; subsequently the author showed that $2^\kappa = \kappa^+$ suffice (Theorem 1 there). Then Shafir (see [Sha01, Th.1]) presents this and proves the following:

([Sha01, Th.2]): if $\kappa = \mathrm{cf}(\kappa), \clubsuit_{S^{\kappa^+}_\kappa}$ and there is a MAD family on $[\kappa]^\kappa$ of cardinality κ, then $\mathfrak{H}_{\kappa^+,\kappa^+,\kappa}$ has no we-universal.
([Sha01, Th.3]): if $\kappa \le \theta \le 2^\kappa$ and there is $\mathcal{A} \subseteq [\kappa]^\kappa$ of cardinality κ^+ such that no $B \in [\kappa]^\kappa$ is included in θ of them, then $\mathfrak{H}_{\theta^\kappa,\kappa,\theta}$ has no ste-universal member.
([Sha01, Th.4]): if $\kappa \le \theta \le 2^\kappa$ and $\clubsuit_{S^\lambda_\kappa}$ then $\mathfrak{H}_{\lambda,\theta,\kappa}$ has no ste-universal members.

Here we characterize "$\mathfrak{H}_{\lambda,\theta,\kappa}$ has universal" under GCH (for weak and for strong embeddings). We also in 1.1 prove $\clubsuit_{S^{\kappa^+}_\kappa} \Rightarrow$ no we-universal in $\mathfrak{H}_{\kappa^+,\kappa^+,\kappa}$ (compared to [Sha01, Th.2], we omit his additional assumption "no MAD $\mathcal{A} \subseteq [\kappa]^\kappa, |\mathcal{A}| = \kappa^+$"); in 1.2 we prove more. Also (1.5) $\lambda = \lambda^\kappa \ge 2^\kappa \ge \theta \ge \kappa \Rightarrow$ no universal under strong embedding in $\mathfrak{H}_{\lambda,\kappa,\theta}$ (compared to [Sha01, Th.3] we omit an assumption).

Lately some results for which we originally used [She00] now instead use [She06], [She10] which gives stronger results.

* * *

Notation 0.2.

- We use $\lambda, \mu, \kappa, \chi, \theta$ for cardinals (infinite if not said otherwise)
- We use $\alpha, \beta, \gamma, \varepsilon, \zeta, \xi, i, j$ for ordinals, δ for limit ordinals
- For $\kappa = \mathrm{cf}(\kappa) < \lambda, S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$
- By $X \backslash Y \backslash Z$ we mean $(X \backslash Y) \backslash Z$
- $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$. · We use G for graphs and for bipartite graphs; see below Definition 0.3(1), 0.4(1), it will always be clear from the context which case we intend.

Definition 0.3. 1) A graph G is a pair $(V, R) = (V^G, R^G), V$ a non-empty set, R a symmetric irreflexive 2-place relation on it. We call V the set of nodes of G and $|V|$ is the cardinality of G, denoted by $\|G\|$, and may write $\alpha \in G$ instead of $\alpha \in V^G$.

Let $E^G = \{\{\alpha, \beta\} : \alpha R^G \beta\}$, so we may consider G as (V^G, E^G).

2) We say f is a strong embedding of G_1 into G_2 (graphs) <u>if</u>:

- (a) f is a one-to-one function from G_1 into G_2; pedantically from V^{G_1} into V^{G_2}
- $(b)_{\rm st}$ for $\alpha, \beta \in G_1$ we have
  $\alpha R^{G_1} \beta \Leftrightarrow f(\alpha) R^{G_2} f(\beta)$.

3) we say f is a weak embedding of G_1 into G_2 if

- (a) above and
- $(b)_{\rm we}$ for $\alpha, \beta \in G_1$ we have
  $\alpha R^{G_1} \beta \Rightarrow f(\alpha) R^{G_2} f(\beta)$.

4) The λ-complete graph K_λ is the graph (λ, R) were $\alpha R \beta \Leftrightarrow \alpha \neq \beta$ or any graph isomorphic to it.

Definition 0.4. 1) G is a bipartite graph means $G = (U, V, R) = (U^G, V^G, R^G)$ where U, V are disjoint non-empty sets, $R \subseteq U \times V$. For a bipartite graph G, we would like sometimes to treat as a usual graph (not bipartite), so let G as a graph, $G^{[\rm gr]}$, be $(U^G \cup V^G, \{(\alpha, \beta) : \alpha, \beta \in V \cup U$ and $\alpha R^G \beta \vee \beta R^G \alpha\})$. The cardinality of G is $(|U^G|, |V^G|)$ or $|U^G| + |V^G|$.
2) We say f is a strong embedding of the bipartite graph G_1 into the bipartite graph G_2 <u>if</u>:

- (a) f is a one-to-one function from $U^{G_1} \cup V^{G_1}$ into $U^{G_2} \cup V^{G_2}$ mapping U^{G_1} into U^{G_2} and mapping V^{G_1} into V^{G_2}
- (b) for $(\alpha, \beta) \in U^{G_1} \times V^{G_1}$ we have
  $\alpha R^{G_1} \beta \Leftrightarrow f(\alpha) R^{G_2} f(\beta)$.

3) We say f is a weak embedding of the bipartite graph G_1 into the bipartite graph G_2 if

- (a) f is a one-to-one function from $U^{G_1} \cup V^{G_1}$ into $U^{G_2} \cup V^{G_2}$, f mapping U^{G_1} into U^{G_2} and mapping V^{G_1} into V^{G_2}
- (b) for $(\alpha, \beta) \in U^{G_1} \times V^{G_1}$ we have
  $\alpha R^{G_1} \beta \Rightarrow f(\alpha) R^{G_2} f(\beta)$.

4) In parts (2), (3) above, if G_1 is a bipartite graph and G_2 is a graph then we mean $G_1^{[\rm gr]}, G_2$.
5) The (κ, θ)-complete bipartite graph $K_{\kappa,\theta}$ is (U, V, R) with $U = \{i : i < \kappa\}, V = \{\kappa + i : i < \theta\}, R = \{(i, \kappa + j) : i < \kappa, j < \theta\}$, or any graph isomorphic to it.

Definition 0.5. 1) For a family $\mathfrak{H}$ of graphs (or of bipartite graphs) we say G is ste-universal [or we-universal] for $\mathfrak{H}$ <u>iff</u> every $G' \in \mathfrak{H}$ can be strongly embedded [or weakly embedded] into G.
2) We say $\mathfrak{H}$ has a ste-universal (or we-universal) <u>if</u> some $G \in \mathfrak{H}$ is ste-universal (or we-universal) for $\mathfrak{H}$.

Definition 0.6. 1) Let $\mathfrak{H}_{\lambda,\theta,\kappa} = \mathfrak{H}^{\rm gr}_{\lambda,\theta,\kappa}$ be the family of graphs G of cardinality λ (i.e. with λ nodes) such that the complete (θ, κ)-bipartite graph cannot be weakly embedded into it; gr stands for graph.

2) Let $\mathfrak{H}^{\rm bp}_{\bar\lambda,\theta,\kappa}$ be the family of bipartite graphs G of cardinality $\bar\lambda$ such that the complete (θ,κ)-bipartite graph cannot be weakly embedded into it. If $\bar\lambda = (\lambda,\lambda)$ we may write λ (similarly in (3)); bp stands for bipartite.
3) Let $\mathfrak{H}^{\rm sbp}_{\bar\lambda,\{\theta,\kappa\}} = \mathfrak{H}^{\rm sbp}_{\bar\lambda,\theta,\kappa}$ be the family of bipartite graphs G of cardinality $\bar\lambda$ such that $K_{\theta,\kappa}$ (the (θ,κ)-complete bipartite graph) and $K_{\kappa,\theta}$ (the (κ,θ)-complete bipartite graph) cannot be weakly embedded into it; sbp stands for symmetrically bipartite.
4) $\mathfrak{H}_\lambda = \mathfrak{H}^{\rm gr}_\lambda$ is the family of graphs of cardinality λ and $\mathfrak{H}^{\rm bp}_{\bar\lambda}$ is the family of bipartite graphs of cardinality $\bar\lambda$.

Observation 0.7. 1) The following are equivalent:

- (a) in $\mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}$ there is a we-universal
- (b) in $\{G^{[\rm gr]} : G \in \mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}\}$ there is a we-universal.

2) Similarly for ste-universal.
3) If G is ste-universal for $\mathfrak{H}$ then it is we-universal for $\mathfrak{H}$ (in all versions).
4) Assume that for every $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ there is a bipartite graph from $\mathfrak{H}_{\lambda,\theta,\kappa}$ not x-embeddable into it, then in $\mathfrak{H}^{\rm bp}_{\lambda,\kappa,\theta}$ and in $\mathfrak{H}^{\rm bp}_{\lambda,\theta,\kappa}$ and in $\mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}$ there is no x-universal member; for $x \in \{\rm we, ste\}$.

Proof. (1) $(a) \Rightarrow (b)$: Trivially.

$(b) \Rightarrow (a)$: Assume G is we-universal in $\{G^{[\rm gr]} : G \in \mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}\}$ and let $\langle A_i : i < i^*\rangle$ be its connectivity components. Let A_i be the disjoint union of $A_{i,0}, A_{i,1}$ with no G-edge inside $A_{i,0}$ and no G-edge inside $A_{i,1}$ (exists as $G = G^{[\rm gr]}_*$ for some $G_* \in \mathfrak{H}^{\rm sbp}_{\lambda,\kappa,\theta}$, note that $\{A_{i,0}, A_{i,1}\}$ is unique as $G \restriction A_i$ is connected). Let $A^{m,\alpha}_{i,\ell}$ for $i < i^*, \ell < 2, m < 2, \alpha < \lambda$ be pairwise disjoint sets with $|A^{m,\alpha}_{i,\ell}| = |A_{i,k}|$ when $m = k$. Let G' be the following member of $\mathfrak{H}^{\rm bp}_{\lambda,\theta,\kappa}$: let $U^{G'}$ be the disjoint union of $A^{0,\alpha}_{i,\ell}$ for $i < i^*, \alpha < \lambda$ and $V^{G'}$ be the disjoint union of $A^{1,\alpha}_{i,\ell}$ for $i < i^*, \alpha < \lambda$ and $R^{G'} = \cup\{R^\alpha_{i,\ell} : i < i^*, \ell < 2\}$ where $R^\alpha_{i,\ell}$ are chosen such that $(A^{0,\alpha}_{i,0}, A^{1,\alpha}_{i,1}, R^\alpha_{i,0}) \cong (A_{i,0}, A_{i,1}, R^G \restriction A_{i,0} \times A_{i,1}) \cong (A^{0,\alpha}_{i,1}, A^{1,\alpha}_{i,0}, R^\alpha_{i,1})$. Easily $G' \in \mathfrak{H}^{\rm bp}_{\lambda,\theta,\kappa}$ is we-universal.
2) The same proof.
3), 4) Easy. $\square_{0.7}$

1. Some no we-universal

We show that if $\lambda = \lambda^\kappa \wedge 2^\kappa \geq \theta \geq \kappa$ then in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no ste-universal graph (in 1.5); for we-universal there is a similar theorem if $\theta = \kappa^+$, $\Pr(\lambda,\kappa)$, see 1.4 + Definition 1.2, (this holds when $\lambda = \lambda^\kappa = \mathrm{cf}(\lambda)$ and $\clubsuit_{S^\lambda_\kappa}$).

Claim 1.1. *Assume κ is regular and $\clubsuit_{S^{\kappa^+}_\kappa}$ (see Definition 1.2 below). Then there is no we-universal in $\mathfrak{H}_{\kappa^+,\kappa^+,\kappa}$.*

Definition 1.2. 1) For regular $\kappa < \lambda$ let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$.
2) For regular λ and stationary subset S of λ let $\clubsuit_S$ means that for some $\bar{A} = \langle A_\delta : \delta \in S, \delta \text{ limit}\rangle$ we have:

- (a) A_δ is an unbounded subset of δ
- (b) if A is an unbounded subset of λ then for some (equivalently stationarily many) $\delta \in S$ we have $A_\delta \subseteq A$.

2A) For $\kappa < \lambda$ let $⫯_{\lambda,\kappa}$ mean that for some family $\mathcal{A} \subseteq [\lambda]^\kappa$ of cardinality λ we have $(\forall B \in [\lambda]^\lambda)(\exists A \in \mathcal{A})(A \subseteq B)$.
3) $\Pr(\lambda,\kappa)$ for cardinals $\lambda > \kappa$ means that some $\mathcal{F}$ exemplifies it, which means:

- (a) $\mathcal{F}$ is a family of $\leq \lambda$ functions
- (b) every $f \in \mathcal{F}$ is a partial function from λ to λ
- (c) if $f \in \mathcal{F}$ then $\kappa = \mathrm{otp}(\mathrm{Dom}(f))$ and f strictly increasing
- (d) $f \neq g \in \mathcal{F} \Rightarrow \kappa > |\mathrm{Dom}(f) \cap \mathrm{Dom}(g)|$
- (e) if g is a partial (strictly) increasing function from λ to λ such that $\mathrm{Dom}(g)$ has cardinality λ, then g extends some $f \in \mathcal{F}$.

4) $\Pr'(\lambda,\delta)$ is defined similarly for δ a limit ordinal but clauses (c) + (d) are replaced by:

- $(c)'$ if $f \in \mathcal{F}$ then $\delta = \mathrm{otp}(\mathrm{Dom}(f))$ and f is one to one
- $(d)'$ if $f \neq g \in \mathcal{F}$ then $\mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ is a bounded subset of $\mathrm{Dom}(f)$ and of $\mathrm{Dom}(g)$
- $(e)'$ like (e) but g is just one to one.

Observation 1.3. 1) We have

- (i) $\clubsuit_{S^\lambda_\kappa} \Rightarrow \Pr(\lambda,\kappa)$
- (ii) $\Pr'(\lambda,\kappa) \Rightarrow \Pr(\lambda,\kappa) \Rightarrow ⫯_{\lambda,\kappa}$
- (iii) for any regular cardinal κ we have $\Pr(\lambda,\kappa) \Leftrightarrow \Pr'(\lambda,\kappa)$.

2) If we weaken clause (c) of 1.2(3) to

$(c)^-$ $f \in \mathcal{F} \Rightarrow |\mathrm{Rang}(f)| = \kappa = |\mathrm{Dom}(f)|$

we get equivalent statement (can combine with 1.3(3)).
3) The "one to one" in Definition 1.2(4), clauses $(c)' + (e)$ are not a serious demand, that is, omitting it we get an equivalent definition (in particular, in 1.2(3)).

Proof. Easy.
1) E.g., clause (iii) holds because for any one to one $f : \kappa \to$ Ord, for some $A \in [\kappa]^\kappa$ the function $f \restriction A$ is strictly increasing.
2) Left to the reader.
3) Why? Let pr:$\lambda \times \lambda \to \lambda$ be 1-to-1 onto and $\mathrm{pr}_1,\mathrm{pr}_2 : \lambda \to \lambda$ be such that $\alpha = \mathrm{pr}(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha))$ for every $\alpha < \lambda$.
Let $\mathcal{F}$ be as in the Definition 1.2(4) old version. Then $\{\mathrm{pr}_1 \circ f : f \in \mathcal{F}\}$ will exemplify the new version, i.e. without the 1-to-1.

For the other direction, just use $\{f \in \mathcal{F} : f$ is one to one$\}$. $\square_{1.3}$

Proof. Proof of 1.1 It follows from 1.4 proved below as $\clubsuit_{S^{\kappa^+}_\kappa}$ easily implies $\Pr(\kappa^+, \kappa)$ for regular κ (see 1.3(1)). $\square_{1.1}$

Claim 1.4. *If* $\Pr(\lambda, \kappa)$*, so* $\lambda > \kappa$ *then in* $\mathfrak{H}_{\lambda,\kappa^+,\kappa}$ *there is no we-universal. Moreover, for every* $G^* \in \mathfrak{H}_{\lambda,\kappa^+,\kappa}$ *there is a bipartite* $G \in \mathfrak{H}_{\lambda,\kappa^+,\kappa}$ *of cardinality* λ *not we-embeddable into it.*

Proof. Let G^* be a given graph from $\mathfrak{H}_{\lambda,\kappa^+,\kappa}$; without loss of generality $V^{G^*} = \lambda$.

For any $A \subseteq \lambda$ let

$(*)_0$ (a) $Y^0_A =: \{\beta < \lambda : \beta$ is G^*-connected with every $\gamma \in A\}$
(b) $Y^2_A =: \{\beta < \lambda : \beta$ is G^*-connected with κ members of $A\}$
(c) $Y^1_A =: \{\beta < \lambda : \beta$ is G^*-connected with every $\gamma \in A$ except possibly $< \kappa$ of them$\}$.
Clearly
(d) $A \subseteq B \subseteq \lambda \Rightarrow Y^0_A \supseteq Y^0_B$ and $Y^2_A \subseteq Y^2_B$ and $Y^1_A \supseteq Y^1_B$
(e) $|A| \geq \kappa \Rightarrow Y^0_A \subseteq Y^1_A \subseteq Y^2_A$.

We now note

$(*)_1$ if $A \in [\lambda]^\kappa$ then $|Y^0_A| \leq \kappa$.

[Why? Otherwise we can find a weak embedding of the (κ, κ^+)-complete bipartite graph into G^*]

$(*)_2$ if $A \in [\lambda]^\kappa$ then $|Y^1_A| \leq \kappa$.

[Why? If not choose pairwise disjoint subsets A_i of A for $i < \kappa$ each of cardinality κ, now easily $\gamma \in Y^1_A \Rightarrow |\{i < \kappa : \gamma \notin Y^0_{A_i}\}| < \kappa$ so $Y^1_A \subseteq \bigcup_{i<\kappa} Y^0_{A_i}$ hence if $|Y^1_A| > \kappa$ then for some $i < \kappa, Y^0_{A_i}$ has cardinality $> \kappa$, contradiction by $(*)_1$.]

Let $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$ exemplify $\Pr(\lambda, \kappa)$. Now we start to choose the bipartite graph G:

$\boxtimes_0$ $U^G = \lambda, V^G = \lambda \times \lambda, R^G = \bigcup_{\alpha<\lambda} R^G_\alpha$ and for $\alpha < \lambda$ we have $R^G_\alpha \subseteq \{(\beta, (\alpha, \gamma)) : \alpha < \lambda$ and $\beta \in \mathrm{Dom}(f_\alpha)$ and $\gamma < \lambda\} \subseteq U^G \times V^G$ where R^G_α is chosen below; we let $G_\alpha = (U^G, V^G, R^G_\alpha)$.

Now

$\boxtimes_1$ G is a bipartite graph of cardinality λ
$\boxtimes_2$ the (κ^+, κ)-complete bipartite graph, (i.e. $K_{\kappa^+,\kappa}$, see Definition 0.4(5)) cannot be weakly embedded into G.

[Why? As for any $(\alpha,\gamma) \in V^G$ the set $\{\beta < \lambda : \beta R^G(\alpha,\gamma)\}$ is equal to $\mathrm{Dom}(f_\alpha)$ which has cardinality κ which is $< \kappa^+$; note that we are speaking of weak embedding as bipartite graph, "side preserving"]

$\boxtimes_3$ the (κ,κ^+)-complete bipartite graph K_{κ,κ^+} cannot be weakly embedded into G provided that for each $\alpha < \lambda$,

$\oplus^3_\alpha$ K_{κ,κ^+} cannot be weakly embedded into (U^G, V^G, R^G_α).

[Why? Let $U_1 \subseteq U^G, V_1 \subseteq V^G$ have cardinality κ,κ^+ respectively and let $V_1' = \{\alpha : (\alpha,\gamma) \in V_1$ for some $\gamma\}$. If $|V_1'| \geq 2$ choose $(\alpha_1,\gamma_1),(\alpha_2,\gamma_2) \in V_1$ such that $\alpha_1 \neq \alpha_2$, so $\{\beta < \lambda : (\beta,(\alpha_\ell,\gamma_\ell)) \in R^G$ for $\ell = 1,2\}$ include U_1 hence by the definition of R^G we have $|\mathrm{Dom}(f_{\alpha_1}) \cap \mathrm{Dom}(f_{\alpha_2})| \geq |U_1| = \kappa$, but $\alpha_1 \neq \alpha_2 \Rightarrow |\mathrm{Dom}(f_{\alpha_1}) \cap \mathrm{Dom}(f_{\alpha_2})| < \kappa$ by clause (d) of Definition 1.2(3). So necessarily $|V_1'| \leq 1$. But $|V_1| = \kappa^+$ hence $V_1' \neq \emptyset$, so V_1' is a singleton, say $\{\alpha\}$, and we get a contradiction to $\oplus^3_\alpha$.]

$\boxtimes_4$ G cannot be weakly embedded into G^* provided that for each $\alpha < \lambda$:

$\oplus^4_\alpha$ there is no weak embedding f of (U^G, V^G, R^G_α) into G^* extending f_α.

[Why? Assume toward contradiction that f is a one-to-one mapping from $U^G \cup V^G$ into $V^{G^*} = \lambda$ mapping edges of G to edges of G^*. So $f \restriction U^G$ is a one-to-one mapping from λ to λ hence by the choice of $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$ to witness $\Pr(\lambda,\kappa)$ see clause (e) of Definition 1.2(3) there is α such that $f_\alpha \subseteq f \restriction U^G$. So clearly $\beta R^G(\alpha,\gamma)$ and $\beta \in \mathrm{Dom}(f_\alpha)$ implies $\{f(\beta), f((\alpha,\gamma))\} = \{f_\alpha(\beta), f((\alpha,\gamma))\} \in E^{G^*}$, hence $(\beta,(\alpha,\gamma)) \in R^G_\alpha \Rightarrow \{f(\beta), f((\alpha,\gamma))\} \in E^{G^*}$. This clearly contradicts $\otimes^4_\alpha$ which we are assuming.]

So we are left with, for each $\alpha < \lambda$, choosing $R_\alpha \subseteq \{(\beta,(\alpha,\gamma)) : \beta \in \mathrm{Dom}(f_\alpha), \gamma < \lambda\}$ to satisfy $\oplus^3_\alpha + \oplus^4_\alpha$. The proof splits to cases, fixing α.

Let us denote $B_\alpha = \mathrm{Rang}(f_\alpha), A_\alpha = \mathrm{Dom}(f_\alpha)$ for $\ell = 0,1$ we let $A^\ell_\alpha =: \{\gamma \in A_\alpha : \mathrm{otp}(A_\alpha \cap \gamma) = \ell \bmod 2\}$ and $B^\ell_\alpha =: \{f_\alpha(\gamma) : \gamma \in A^\ell_\alpha\}$.

<u>Case 1</u>: Y^2_B has cardinality $\leq \kappa$ for some $B \in [B_\alpha]^\kappa$.

Choose such $B = B'_\alpha$ and let $A'_\alpha = \{\beta \in A_\alpha : f_\alpha(\beta) \in B'_\alpha\}$. There is a sequence $\bar{C} = \langle C_\zeta : \zeta < \kappa^+\rangle, C_\zeta \in [\kappa]^\kappa$ such that $\xi < \zeta \Rightarrow |C_\xi \cap C_\zeta| < \kappa$.

Let $R_\alpha = \{(\beta,(\alpha,\gamma)) : \gamma < \kappa^+, \beta \in A'_\alpha$ and $\mathrm{otp}(\beta \cap A'_\alpha) \in C_\gamma\}$. Now $\oplus^4_\alpha$ holds because if f is a counter-example, <u>then</u> necessarily by the pigeon-hole principle for some $\gamma < \kappa^+$ we have $f((\alpha,\gamma)) \notin Y^2_{B'_\alpha}$, but clearly (α,γ) is G_α-connected to κ members of A'_α hence $f((\alpha,\gamma))$ is G^*-connected to κ members of B'_α hence $f((\alpha,\gamma)) \in Y^2_{B'_\alpha}$ and we get a contradiction. Also $\oplus^3_\alpha$ holds as $\xi < \zeta < \kappa^+ \Rightarrow |C_\xi \cap C_\zeta| < \kappa$.

So we may assume, for the rest of the proof, that

$\boxtimes_5$ $|Y^2_B| > \kappa$ for every $B \in [B_\alpha]^\kappa$.

<u>Case 2</u>: For some $\ell < 2, |Y^2_{B^\ell_\alpha}| > \kappa$ and for some Z:

(i) $Z \subseteq Y^2_{B^\ell_\alpha} \backslash Y^1_{B^\ell_\alpha}$

(ii) $|Z| \leq \kappa$

(iii) for every $\gamma_0 \in (Y^2_{B^\ell_\alpha} \backslash Y^1_{B^\ell_\alpha}) \backslash Z$ there is $\gamma_1 \in Z$ such that $\kappa > |\{\beta \in B^\ell_\alpha :$ β is G^*-connected to γ_0 but is not G^*-connected to $\gamma_1\}|$.

So we choose such $\ell = \ell(\alpha) < 2, Z = Z_\alpha$ and then we choose a sequence $\langle B_{\alpha,\gamma} : \gamma \in Z_\alpha\rangle$ such that:

$\boxtimes_6$ (a) $B_{\alpha,\gamma}$ is a subset of B^ℓ_α
(b) $|B_{\alpha,\gamma}| = \kappa$
(c) $\gamma_1 \neq \gamma_2 \in Z \Rightarrow B_{\alpha,\gamma_1} \cap B_{\alpha,\gamma_2} = \emptyset$
(d) γ is not G^*-connected to any $\varepsilon \in B_{\alpha,\gamma}$

(this is possible as t_α has $\leq \kappa$ members and $\gamma \in Z_\alpha \Rightarrow \gamma \notin Y^1_{B^\ell_\alpha}$).

Now we can find a sequence $\langle C_{\alpha,\zeta} : \zeta < \kappa^+\rangle$ satisfying

$\boxtimes_7$ (α) $C_{\alpha,\zeta} \subseteq B^\ell_\alpha$
(β) $|C_{\alpha,\zeta}| = \kappa$ moreover $\beta \in Z_\alpha \Rightarrow |C_{\alpha,\zeta} \cap B_{\alpha,\beta}| = \kappa$
(γ) for $\xi < \zeta$ we have $|C_{\alpha,\xi} \cap C_{\alpha,\zeta}| < \kappa$

(e.g. if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ by renaming $B^\ell_\alpha = \kappa$, each $B_{\alpha,\varepsilon}$ is stationary, choose nonstationary $C_{\alpha,\varepsilon} \subseteq \kappa$ inductively on ε; if $\kappa > \mathrm{cf}(\kappa)$ reduce it to construction on regulars, if $\kappa = \aleph_0$ imitate the case $\kappa = \mathrm{cf}(\kappa) > \aleph_0$).

Lastly we choose $R_\alpha = \{(\beta, (\alpha,\gamma)) : \beta \in A_\alpha, \gamma < \kappa^+ \text{ and } f_\alpha(\beta) \in C_{\alpha,\gamma}\}$.

Now $\oplus^3_\alpha$ is proved as in the first case, as for $\oplus^4_\alpha$, if f is a counter-example <u>then</u> clearly for $\gamma < \kappa^+, f((\alpha,\gamma)) \in Y^2_{B_\alpha}$, so as $|Y^2_{B_\alpha}| > \kappa$ by $\boxtimes_5$ and $|Z_\alpha| \leq \kappa$ and $|Y^1_{B^\ell_\alpha}| \leq \kappa$ (by $(*)_2$) necessarily for some $\zeta < \kappa^+, \gamma_0 =: f((\alpha,\zeta)) \in Y^2_{B^\ell_\alpha} \backslash Y^1_{B^\ell_\alpha} \backslash Z_\alpha$. Let $\gamma_1 \in Z_\alpha$ be as guaranteed in clause (iii) in the present case. Now γ_0 is G^*-connected to every member of $C_{\alpha,\zeta}$ as $\gamma_0 = f((\alpha,\zeta))$. Hence γ_0 is G^*-connected to κ members of B_{α,γ_1} (see clause (β) of $\boxtimes_7$ above and the choice of R_α); but γ_1 is not G^*-connected to any member of B_{α,γ_1} (see clause (d) of $\boxtimes_6$ above). Reading clause (iii), we get contradiction.

<u>Case 3</u>: <u>Neither Case 1 nor Case 2</u>.

Recall that α is fixed by induction on $\zeta < \kappa^+$. For $\ell \in \{0,1\}$ we choose $Z^\ell_{\alpha,\zeta}$ such that

$\boxtimes_8$ (a) $Z^\ell_{\alpha,\zeta}$ a subset of $Y^2_{B^\ell_\alpha}$ of cardinality κ
(b) $Z^\ell_{\alpha,\zeta}$ is increasing continuous with ζ
(c) $Y^1_{B^\ell_\alpha} \subseteq Z^\ell_{\alpha,0}$
(d) if $\zeta = \xi + 1$ <u>then</u> there is $\gamma^\ell_{\alpha,\xi} \in Z^\ell_{\alpha,\zeta} \backslash Z^\ell_{\alpha,\xi} \backslash Y^1_{B^\ell_\alpha}$ such that for every $\gamma' \in Z^\ell_{\alpha,\xi} \backslash Y^1_{B^\ell_\alpha}$ we have $\kappa = |\{\beta \in B^\ell_\alpha : \beta \text{ is } G^*\text{-connected to } \gamma^\ell_{\alpha,\xi} \text{ but not to } \gamma'\}|$
(e) if $\zeta = \xi + 1$ and $\gamma \in Z^\ell_{\alpha,\zeta}$ hence $\kappa = |\{\beta \in B^\ell_\alpha : \beta \text{ is connected to } \gamma\}|$ (e.g. $\gamma = \gamma^\ell_{\alpha,\xi}$), <u>then</u> $Y^1_{\{\beta \in B^\ell_\alpha : \beta \text{ is } G^*\text{-connected to } \gamma\}}$ is included in $Z^\ell_{\alpha,\zeta}$
(f) $Z^0_{\alpha,\zeta} \cap (Y^2_{B^0_\alpha} \cap Y^2_{B^1_\alpha}) = Z^1_{\alpha,\zeta} \cap (Y^2_{B^0_\alpha} \cap Y^2_{B^1_\alpha})$.

Why possible? For clause (c) we have $|Y^1_{B^\ell_\alpha}| \leq \kappa$ by $(*)_2$, for clause (d) note that "not Case 2" trying $Z^\ell_{\alpha,\xi} \backslash Y^1_{B^\ell_\alpha}$ as Z, and for clause (e) note again $|Y^1_{\{\beta \in B^\ell_\alpha : \beta \text{ is } G^*\text{-connected to } \gamma^\ell_{\alpha,\varepsilon}\}}| \leq \kappa$ by $(*)_2$.

Having chosen $\langle Z^\ell_{\alpha,\zeta} : \zeta < \kappa^+, \ell < 2\rangle$, we let

$$R_\alpha = \{(\beta,(\alpha,\zeta)) : \quad \text{for some } \ell < 2 \text{ we have: } \beta \in A^\ell_\alpha \text{ and } f_\alpha(\beta) \text{ is } G^*\text{-connected to } \gamma = \gamma^\ell_{\alpha,2\zeta+\ell} \text{ so } \zeta < \kappa^+\}.$$

Now why $\oplus^3_\alpha$ holds? Otherwise, we can find $A \subseteq A_\alpha, |A| = \kappa$ and $B \subseteq \kappa^+, |B| = \kappa^+$ such that $\beta \in A \wedge \xi \in B \Rightarrow \beta R_\alpha(\alpha, \gamma^{\ell(\xi)}_{\alpha,\xi})$ where $\xi = \ell(\xi) \bmod 2$, so for some $\ell < 2$ we have $|A \cap A^\ell_\alpha| = \kappa$, and let

$$A' = \{f_\alpha(\beta) : \beta \in A \cap A^\ell_\alpha\}.$$

Easily $|B| = \kappa^+$ and $|A'| = \kappa$ and $\beta \in A' \wedge \xi \in B \Rightarrow \beta R^{G^*}\gamma^\ell_{\alpha,\xi}$, contradiction to "$K_{\kappa,\kappa^+}$ is not weakly embeddable into G^*".

Lastly, why $\oplus^4_\alpha$ holds? Otherwise, letting f be a counterexample, let $\zeta < \kappa^+$ and $\ell < 2$. Clearly $f((\alpha,\zeta))$ is G^*-connected to every $\beta' \in B^\ell_\alpha$ which is G^*-connected to $\gamma^\ell_{\alpha,2\zeta+\ell}$ hence $f((\alpha,\zeta))$ cannot belong to $Z^\ell_{\alpha,2\zeta+\ell} \backslash Y^1_{B^\ell_\alpha}$ (by the demand in clause (d) of $\boxtimes_8$), but it has to belong to $Z^\ell_{\alpha,2\zeta+\ell+1}$ (by clause (e) of $\boxtimes_8$), so $f((\alpha,\zeta)) \in (Z^\ell_{\alpha,2\zeta+\ell+1} \backslash Z^\ell_{\alpha,2\zeta+\ell}) \cup Y^1_{B^\ell_\alpha}$. Putting together $\ell = 0,1$ we get $f((\alpha,\zeta)) \in ((Z^0_{\alpha,2\zeta+1} \backslash Z^0_{\alpha,2\zeta}) \cup Y^1_{B^0_\alpha}) \cap ((Z^1_{\alpha,2\zeta+2} \backslash Z^1_{\alpha,2\zeta+1}) \cup Y^1_{B^1_\alpha})$ hence $f((\alpha,\zeta)) \in Y^1_{B^0_\alpha} \cup Y^1_{B^1_\alpha}$, but by $(*)_2$ we have $|Y^1_{B^\ell_\alpha}| < \kappa^+$; as this holds for every $\zeta < \kappa^+$ this is a contradiction to "f is one to one". $\square_{1.4}$

Claim 1.5. *1) Assume $\lambda \geq 2^\kappa \geq \theta \geq \kappa$ and $\lambda = \lambda^\kappa$ (e.g. $\lambda = 2^\kappa$).*

Then in $\mathfrak{H}_{\lambda,\kappa,\theta}$ there is no ste*-universal (moreover, the counterexamples are bipartite).*

2) Assume $\Pr(\lambda,\kappa), \lambda \geq \theta \geq \kappa, 2^\kappa \geq \theta$*. Then the conclusion of (1) holds.*

Proof. 1) By the simple black box ([She87, Ch.III,§4]) or [Shear, Ch.VI,§1], i.e. [Sheb])

$\boxtimes$ there is $\bar f = \langle f_\eta : \eta \in {}^\kappa\lambda\rangle, f_\eta$ a function from $\{\eta \restriction i : i < \kappa\}$ into λ such that for every $f : {}^{\kappa>}\lambda \to \lambda$ for some $\eta \in {}^\kappa\lambda$ we have $f_\eta \subseteq f$.

Let $G^* \in \mathfrak{H}_{\lambda,\kappa,\theta}$ and we shall show that it is not ste-universal in $\mathfrak{H}_{\lambda,\kappa,\theta}$, without loss of generality $V^{G^*} = \lambda$. For this we define the following bipartite graph G:

$\boxplus_1$ (i) $U^G = {}^{\kappa>}\lambda$ and $V^G = {}^\kappa\lambda$

(ii) $R^G = \cup\{R^G_\eta : \eta \in {}^\kappa\lambda$ and f_η is a one-to-one function$\}$ where $R^G_\eta \subseteq \{(\eta \restriction i, \eta) : i \in u_\eta\}$ where $u_\eta \subseteq \kappa$ is chosen as follows:

$\boxplus_2$ for $\eta \in {}^\kappa\lambda$ we choose $u_\eta \subseteq \kappa$ such that if possible

$(*)_{\eta,u_\eta}$ for no $\gamma < \lambda$ do we have $(\forall i < \kappa)[f_\eta(\eta \restriction i) R^{G^*}\gamma \equiv i \in u_\eta]$.

If for every $\eta \in {}^\kappa\lambda$ for which f_η is one to one for some $u \subseteq \kappa$ we have $(*)_{\eta,u}$ holds, then clearly by $\boxtimes$ we are done.

Otherwise, for this $\eta \in {}^\kappa\lambda, f_\eta$ is one to one and: there is $\gamma_u < \lambda$ satisfying $(\forall i < \kappa)(f_\eta(\eta \restriction i) R^{G^*}\gamma_u \Leftrightarrow i \in u)$ for every $u \subseteq \kappa$. But then $A' =: \{f_\eta(\eta \restriction 2i) : i < \kappa\}$ and $B' =: \{\gamma_u : u \subseteq \kappa$ and $(\forall i < \kappa) 2i \in u\}$ form a complete $(\kappa, 2^\kappa)$-bipartite subgraph of G^*, contradiction.

2) The same proof. $\square_{1.5}$

2. No we-universal by $\Pr(\lambda, \kappa)$ and its relatives

We define here some relatives of Pr. Here Ps is like Pr but we are approximating $f : \lambda \to \lambda$, and $\Pr_3(\chi, \lambda, \mu, \alpha)$ is a weak version of $(\lambda + \mu)^{|\alpha|} \leq \chi$ (Definition 2.5); we give sufficient conditions by cardinal arithmetic (Claim 2.6, 2.8). We prove more cases of no we-universal: the case θ limit (and $\Pr(\lambda, \kappa)$) in 2.2, a case of $\Pr'(\lambda, \theta^+ \times \kappa)$ in 2.4. We also note that we can replace Pr by Ps in 2.9, and λ strong limit singular of cofinality $>$ cf(κ) in 2.10.

Convention 2.1. $\lambda \geq \theta \geq \kappa \geq \aleph_0$.

Claim 2.2. *If θ is a limit cardinal and* $\Pr(\lambda, \kappa)$, *then there is no* we-*universal graph in $\mathfrak{H}_{\lambda,\theta,\kappa}$ even for the class of bipartite members.*

Proof. Like the proof of 1.4, except that we replace cases 1-3 by:

for every $\alpha < \lambda$ we let $R_\alpha = \{(\beta, (\alpha, \gamma)) : \beta \in \text{Dom}(f_\alpha)$ and $\gamma < |Y^0_{\text{Dom}(f_\alpha)}|^+\}$. Now $\oplus^3_\alpha$ holds as $|Y^0_{\text{Dom}(f_\alpha)}| < \theta$ (by $(*)_1$ there) hence $|Y^0_{\text{Dom}(f_\alpha)}|^+ < \theta$ as θ is a limit cardinal. Lastly $\oplus^4_\alpha$ holds as for some α we have $f_\alpha \subseteq f$ hence the function f maps $\{(\alpha, \gamma) : \gamma < |Y^0_{\text{Dom}(f_\alpha)}|^+\}$ into $Y^0_{\text{Dom}(f_\alpha)}$ but f is a one to one mapping, contradiction. $\square_{2.2}$

Recall

Definition 2.3. For a cardinal λ and a limit ordinal δ, $\Pr'(\lambda, \delta)$ holds <u>when</u> for some $\mathcal{F}$:

- (a) $\mathcal{F}$ a family of $\leq \lambda$ functions
- (b) every $f \in \mathcal{F}$ is a partial function from λ to λ
- (c) $f \in \mathcal{F} \Rightarrow$ otp(Dom $f) = \delta$, and f is one to one
- (d) $f, g \in \mathcal{F}, f \neq g \Rightarrow$ (Dom $f) \cap$ (Dom g) is a bounded subset of Dom(f) and of Dom(g)
- (e) if $g : \lambda \to \lambda$ is a partial function, one to one, and $|\text{Dom } g| = \lambda$, <u>then</u> g extends some $f \in \mathcal{F}$.

Claim 2.4. *Assume*

- (a) $\Pr'(\lambda, \delta^*), \delta^* = \sigma \times \kappa$, *ordinal product* [1]
- (b) $\sigma = \theta^+$ *and* $\lambda \geq \sigma$.

<u>Then</u> there is no we-*universal in $\mathfrak{H}^{\text{gr}}_{\lambda,\theta,\kappa}$ even for the class of bipartite members.*

Proof. Let $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ and we shall prove it is not we-universal; without loss of generality $V^{G^*} = \lambda$.

Let $\mathcal{F}$ be a family exemplifying $\Pr'(\lambda, \delta^*)$, let $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$ let $A_\alpha =$ Dom(f_α) and let it be $\{\beta_{\alpha,\varepsilon,i} : i < \sigma, \varepsilon < \kappa\}$ such that $[\beta_{\alpha,\varepsilon(1),i(1)} < \beta_{\alpha,\varepsilon(2),i(2)} \Leftrightarrow \varepsilon(1) < \varepsilon(2) \vee (\varepsilon(1) = \varepsilon(2)$ and $i(1) < i(2))]$ and for $i < \sigma$ let $A_{\alpha,i} = \{\beta_{\alpha,\varepsilon,i} : \varepsilon < \kappa\}$, so clearly

$(*)_1$ $A_{\alpha,i} \in [\lambda]^\kappa$ and $(\alpha_1, i_1) \neq (\alpha_2, i_2) \Rightarrow |A_{\alpha_1,i_1} \cap A_{\alpha_2,i_2}| < \kappa$.

---

[1] this is preserved by decreasing σ

[Why the second assertion? As $\{\beta_{\alpha,\varepsilon,i} : \varepsilon < \kappa\}$ is an unbounded subset of A_α (of order type κ).]

For $(\alpha, i) \in \lambda \times \sigma$ let $f_{\alpha,i} = f_\alpha \restriction A_{\alpha,i}$ let $B_{\alpha,i} = \mathrm{Rang}(f_{\alpha,i})$ so $|A_{\alpha,i}| = |B_{\alpha,i}| = \kappa$ and let $Y^0_{\alpha,i} = \{\gamma < \lambda : \gamma$ is G^*-connected to every member of $B_{\alpha,i}\}$, so as $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ clearly $|Y^0_{\alpha,i}| < \theta$. As $\sigma = \theta^+ > \theta \geq \kappa$, clearly for each $\alpha < \lambda$ for some $\mu_\alpha < \theta$ we have [2]: $X_\alpha := \{i < \sigma : |Y^0_{\alpha,i}| \leq \mu_\alpha\}$ has cardinality σ. As $\mu_\alpha < \theta$ also $\chi_\alpha := \mu^+_\alpha$ is $< \theta^+ = \sigma$, so $\chi^+_\alpha \leq \sigma = |X_\alpha|$. We choose by induction on $\varepsilon < \chi^+_\alpha$ an ordinal $i^*_{\alpha,\varepsilon} \in X_\alpha$ such that:

$(*)_2$ $i^*_{\alpha,\varepsilon} \notin \{i^*_{\alpha,\zeta} : \zeta < \varepsilon\}$.

Recall that χ_α is a successor cardinal, hence a regular cardinal. So if $\varepsilon \in S^{\chi^+_\alpha}_{\chi_\alpha} = \{\varepsilon < \chi^+_\alpha : \mathrm{cf}(\varepsilon) = \chi_\alpha\}$ recalling $\chi_\alpha = \mu^+_\alpha > |Y^0_{\alpha,i^*_{\alpha,\varepsilon}}|$ there is $\zeta < \varepsilon$ such that $(Y^0_{\alpha,i^*_{\alpha,\varepsilon}} \cap \cup\{Y^0_{\alpha,\xi} : \xi < \varepsilon\} \subseteq \cup\{Y^0_{\alpha,\xi} : \xi < \zeta\})$. Let $g(\varepsilon)$ be the first ordinal ζ having this property, so g is a well defined function with domain $S^{\chi^+}_{\chi_\alpha}$. Clearly, g is a regressive function.

By Fodor's lemma for some stationary $S_\alpha \subseteq S^{\chi^+_\alpha}_{\chi_\alpha}$, and for some B^*_α of cardinality $\leq \chi_\alpha$ we have $\varepsilon \in S_\alpha \Rightarrow B^*_\alpha \supseteq Y^0_{\alpha,i^*_{\alpha,\varepsilon}} \cap \cup\{Y^0_{\alpha,j} : j < i^*_{\alpha,\varepsilon}\}$, in fact: $B^*_\alpha = \cup\{Y^0_{\alpha,i^*_{\alpha,\varepsilon}} : \varepsilon < \varepsilon^*\}$ where $g \restriction S_\alpha$ is constantly ε^* is O.K., we can decrease B^*_α but immaterial here

Now

$(*)_3$ for $\xi \neq \zeta$ from S_α there is no $\beta < \lambda$ such that:
- β is G^*-connected to every $\gamma \in B_{\alpha,i^*_{\alpha,\xi}}$
- β is G^*-connected to every $\gamma \in B_{\alpha,i^*_{\alpha,\zeta}}$
- β is not in B^*_α.

Let $\langle \zeta(\alpha, j) : j < \chi^+_\alpha\rangle$ list S_α in an increasing order. Let G be the following bipartite graph

$(*)_4$
- (a) $U^G = \lambda$
- (b) $V^G = \lambda \times \sigma$
- (c) $R^G = \big\{(\beta, (\alpha, \gamma)) : \alpha < \lambda$ and $\gamma < \chi^+_\alpha$ and $\beta \in A_{\alpha,i^*_{\alpha,\zeta(\alpha,2\gamma)}} \cup A_{\alpha,i^*_{\alpha,\zeta(\alpha,2\gamma+1)}}\big\}$.

Now clearly

$\boxtimes_1$ G is a bipartite graph of cardinality λ.

Also

$\boxtimes_2$ the (θ, κ)-complete bipartite graph $(\in \mathfrak{H}^{\mathrm{bp}}_{(\theta,\kappa)})$ cannot be weakly embedded into G.

[Why $\boxtimes_2$ holds? So let $(\alpha(1), \gamma(1)) \neq (\alpha(2), \gamma(2))$ belong to V^G, the set $\{\beta \in U^G : \beta$ connected to $(\alpha(1), \gamma(1))$ and to $(\alpha(2), \gamma(2))\}$ is included in $\bigcup\limits_{\iota(1),\iota(2)\in\{0,1\}} (A_{\alpha(1),i^*_{\zeta(\alpha(1),2\gamma(1)+\iota(1)}} \cap A_{\alpha(2),i^*_{\zeta(\alpha(2),2\gamma(2)+\iota(2))}})$ which is the union of four sets each of cardinality $< \kappa$ (by $(*)_1$) hence has cardinality $< \kappa \leq \theta$.]

[2] in fact by 2.2 without loss of generality θ is a successor cardinal, so without loss of generality $\mu^+_\alpha = \theta$

$\boxtimes_3$ the (κ,θ)-complete bipartite graph cannot be weakly embedded into G.

[Why? Toward contradiction assume $U_1 \subseteq U^G, V_1 \subseteq V^G$ have cardinality κ,θ respectively and $\beta \in U_1 \wedge (\alpha,\gamma) \in V_1 \Rightarrow \beta R(\alpha,\gamma)$.

Let $(\alpha,\gamma) \in V_1$, clearly $\beta \in U_1 \Rightarrow \beta R^G(\alpha,\gamma) \Rightarrow \beta \in A_{\alpha,i^*_{\alpha,\zeta(\alpha,2\gamma)}} \cup A_{\alpha,i^*_{\alpha,\zeta(\alpha,2\gamma+1)}}$. So $U_1 \subseteq A_{i^*_{\alpha,\zeta(\alpha,2\gamma)}} \cup A_{i^*_{\alpha,\zeta(\alpha,2\gamma+1)}}$.

Now if $(\alpha_1,\gamma_1),(\alpha_2,\gamma_2) \in V_1$ and $\alpha_1 \neq \alpha_2$ then $i,j < \sigma \Rightarrow |A_{\alpha_1,i} \cap A_{\alpha_2,j}| < \kappa$ by the choice of $\mathcal{F}$, so necessarily for some $\alpha_* < \lambda$ we have $V_1 \subseteq \{\alpha_*\} \times \sigma$. But if $(\alpha,\gamma_1) \neq (\alpha,\gamma_2) \in V_1$ then $(\alpha,\gamma_1),(\alpha,\gamma_2)$ has no common neighbour, contradiction.]

$\boxtimes_4$ there is no weak embedding f of G into G^*.

[Why? Toward contradiction assume that f is such a weak embedding. By the choice of $\mathcal{F}$ and $\langle f_\alpha : \alpha < \lambda\rangle$ we can choose $\alpha < \lambda$ such that $f_\alpha \subseteq f{\restriction}U^G$. As f is a weak embedding $\beta < \lambda \wedge \gamma < \lambda \wedge \beta R^G(\alpha,\gamma) \Rightarrow f(\beta) R^{G^*} f(\alpha,\gamma)$, hence $\beta \in \bigcup\limits_i A_{\alpha,i} = \mathrm{Dom}(f_\alpha) \wedge \gamma < \lambda \wedge \beta R^G(\alpha,\gamma) \Rightarrow f_\alpha(\beta) R^{G^*} f(\alpha,\gamma)$. Hence if $\gamma < \sigma$ then $f(\alpha,\gamma)$ is G^*-connected to every $\beta \in B_{\alpha,\zeta(\alpha,2\gamma)} \cup B_{\alpha,\zeta(\alpha,2\gamma)} \cup \{\beta_{\alpha,\gamma}\}$ hence $f(\alpha,\gamma) \in Y^0_{\alpha,\zeta(\alpha,2\gamma)} \cap Y^0_{\alpha,\zeta(\alpha,2\gamma+1)}$ which implies that $f(\alpha,\gamma) \in B^*_\alpha$. So the function f maps the set $\{(\alpha,\gamma) : \gamma < \sigma\}$ into B^*_α. But f is a one-to-one function and B^*_α has cardinality $< \sigma$, contradiction.]

Together we are done. $\square_{2.4}$

Definition 2.5. 1) For $\kappa < \lambda$ and $\delta < \lambda$ we define $\mathrm{Ps}(\lambda,\kappa)$ and $\mathrm{Ps}'(\lambda,\delta)$ similarly to the definition of $\mathrm{Pr}(\lambda,\kappa)$, $\mathrm{Pr}'(\lambda,\delta)$ in Definition 1.2(3),(4) except that we replace clause (e) by

$(e)^-$ if g is a one to one function from λ to λ, <u>then</u> g extends some $f \in \mathcal{F}$

(so the difference is that $\mathrm{Dom}(g)$ is required to be λ).
2) Let $\mathrm{Pr}_3(\chi,\lambda,\mu,\alpha)$ mean that for some $\mathcal{F}$:

- (a) $\mathcal{F}$ a family of partial functions from μ to λ
- (b) $|\mathcal{F}| \leq \chi$
- (c) $f \in \mathcal{F} \Rightarrow \mathrm{otp}(\mathrm{Dom}(f)) = \alpha$
- (d) if $g \in {}^\mu\lambda \Rightarrow (\exists f \in \mathcal{F})(f \subseteq g)$.

Claim 2.6. *1) Assume λ is strong limit, $\lambda > \kappa$ and $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$.*
<u>Then</u> $\mathrm{Pr}'(\lambda,\delta^*)$ holds if $\delta^* < \lambda$ has cofinality $\mathrm{cf}(\kappa)$.
2) If $\lambda = \mu^+ = 2^\mu$, $\mathrm{cf}(\delta^*) \neq \mathrm{cf}(\mu), \delta^* < \lambda$ <u>then</u> $\mathrm{Pr}'(\lambda,\delta^*)$ holds.
3) If $\delta < \lambda$ is a limit ordinal and $\lambda = \lambda^{|\delta|}$ <u>then</u> $\mathrm{Ps}'(\lambda,\delta)$.
4) If $\kappa = \mathrm{cf}(\delta), \kappa < \delta < \lambda, \lambda = \lambda^\kappa$ and $\mathrm{Pr}_3(\lambda,\lambda,\lambda,\alpha)$ for every $\alpha < \delta$, <u>then</u> $\mathrm{Ps}'(\lambda,\delta)$.
5) $\mathrm{Pr}(\lambda,\kappa) \Rightarrow \mathrm{Ps}(\lambda,\kappa)$, $\mathrm{Pr}'(\lambda,\kappa) \Rightarrow \mathrm{Ps}'(\lambda,\kappa)$ and similarly with δ instead of κ.
6) If $\lambda \geq 2^\kappa$ <u>then</u> $\lambda = \mathbf{U}_\kappa(\lambda) \Rightarrow \mathrm{pr}_3(\lambda,\lambda,\lambda,\kappa)$.
7) If $\mathrm{Ps}'(\lambda,\kappa)$ then $\mathrm{Ps}(\lambda,\kappa)$.

Remark 2.7. Recall $\mathbf{U}_\kappa(\lambda) = \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\lambda)$, and for an ideal J on $\kappa, \mathbf{U}_J(\lambda) = \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\kappa$ is such that for every $f \in {}^\kappa\lambda$ for some $A \in \mathcal{P}$ we have $\{i < \kappa : f(i) \in A\} \neq \emptyset \bmod J\}$.

Proof. 1) Let $\langle \lambda_i : i < \text{cf}(\lambda)\rangle$ be increasing continuous with limit λ such that $\delta^* < \lambda_0$ and $2^{\lambda_i} < \lambda_{i+1}$, hence for limit δ, λ_δ is strong limit cardinal of cofinality $\text{cf}(\delta)$. For $\delta \in S^{\text{cf}(\lambda)}_{\text{cf}(\kappa)} = \{\delta < \text{cf}(\lambda) : \text{cf}(\delta) = \text{cf}(\kappa)\}$, let $\langle f_{\delta,\alpha} : \alpha < 2^{\lambda_\delta}\rangle$ list the partial one-to-one functions from λ_δ to λ_δ with domain of cardinality λ_δ. We choose by induction on $\alpha < 2^{\lambda_\delta}$ a subset $A_{\delta,\alpha}$ of $\text{Dom}(f_{\delta,\alpha})$ of order type δ^* unbounded in λ_δ such that $\beta < \alpha \Rightarrow \sup(A_{\delta,\alpha} \cap A_{\delta,\beta}) < \lambda_\delta$; possible as we have a tree with $\text{cf}(\delta)$ levels and 2^{λ_δ} $\text{cf}(\delta)$-branches, each giving a possible $A_{\delta,\alpha} \subseteq \text{Dom}(f_{\delta,\alpha})$ and each $A_{\delta,\beta}(\beta < \alpha)$ disqualifies $\leq \lambda_\delta + |\alpha|$ of them.

Now $\mathcal{F} = \{f_{\delta,\alpha}{\restriction}A_{\delta,\alpha} : \delta \in S^{\text{cf}(\lambda)}_{\text{cf}(\kappa)}$ and $\alpha < 2^{\lambda_\delta}\}$ is as required because if f is a partial function from λ to λ such that $|\text{Dom}(f)| = \lambda$ and f is one to one then $\{\delta < \text{cf}(\lambda) : (\exists^{\lambda_\delta} i < \lambda_\delta)(i \in \text{Dom}(f) \wedge f(i) < \lambda_\delta)\}$ contains a club of $\text{cf}(\lambda)$.
2) This holds as $\diamondsuit_S$ for every stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) \neq \text{cf}(\mu)\}$, see [She79], and without any extra assumption by [She10].
3) By the simple black box (see proof of 1.5, well it was phrased for κ but the same proof, and we have to rename $\lambda, \lambda^{<\kappa}$ as λ; see [Shear, Ch.IV], i.e. [Sheb]).
4) We combine the proof of the simple black box and the definition of Pr_3. Let $\langle \gamma^*_i : i \leq \kappa\rangle$ be increasing continuous, $\gamma_0 = 0, \gamma_\kappa = \delta$. By $\text{Pr}_3(\lambda,\lambda,\lambda,\alpha)$ with $\alpha = \gamma^*_{i+1} - \gamma^*_i$, for each $i < \kappa$, we can find $\mathcal{F}_i$ such that

- $(*)_1$ $\mathcal{F}_i \subseteq \{g : g$ a partial function from λ to λ, $\text{otp}(\text{Dom}(g)) = \gamma^*_{i+1} - \gamma^*_i\}$
- $(*)_2$ $|\mathcal{F}_i| \leq \lambda$
- $(*)_3$ for every $g^* \in {}^\lambda\lambda$ there is $g \subseteq g^*$ from $\mathcal{F}_i$

By easy manipulation

- $(*)^+_3$ if $g^* \in {}^\lambda\lambda$ and $\beta < \lambda$ <u>then</u> there is $g \in \mathcal{F}_i$ such that $g \subseteq g^*$ and $\text{Dom}(g) \subseteq [\beta,\lambda)$.

Clearly $\mathcal{F}_i$ exists by the assumption "$\text{Pr}_3(\lambda,\lambda,\lambda,\alpha)$ for $\alpha < \delta$" so let $\mathcal{F}_i = \{g_{i,\varepsilon} : \varepsilon < \lambda\}$. Now for every $\eta \in {}^\kappa\lambda$ let f^0_η be the following partial function from $({}^{\kappa>}\lambda) \times \lambda$ to λ:

- $(*)_4$ if $i < \kappa, \varepsilon < \lambda$ and $\alpha \in \text{Dom}(g_{i,\eta(i)})$ <u>then</u> $f^0_\eta((\eta \restriction i, \alpha)) = g_{i,\eta(i)}(\alpha)$.

Let h be a one to one function from $({}^{\kappa>}\lambda) \times \lambda$ onto λ such that (if $\text{cf}(\lambda) \geq \delta$ then also in $(*)_5(b)$ we can replace $\neq$ by $<$)

- $(*)_5$ (a) $\eta \in {}^{\kappa>}\lambda \wedge \alpha < \beta < \lambda \Rightarrow h((\eta,\alpha)) < h(\eta,\beta)$
  (b) $\eta \triangleleft \nu \in {}^{\kappa>}\lambda \wedge \alpha < \lambda \wedge \beta < \lambda \wedge \alpha \in \text{Dom}(g_{\ell g(\eta),\nu(\ell g(\eta))}) \Rightarrow h((\eta,\alpha)) < h((\nu,\beta))$.

Let f_η be the following partial function from λ to λ satisfying $f_\eta(\alpha) = f^0_\eta(h^{-1}(\alpha))$ so it suffices to prove that $\mathcal{F} = \{f_\eta : \eta \in {}^\kappa\lambda$ and f_η is one-to-one$\}$ exemplifies $\text{Ps}'(\lambda,\delta)$.

First, clearly each f_η is a partial function from λ to λ. Also for each $i < \kappa$ and $\varepsilon < \lambda$ the function $g_{i,\varepsilon}$ has domain of order type $\gamma^*_{i+1} - \gamma^*_i$, hence by $(*)_5(a)$ also $\eta \in {}^\kappa\lambda \wedge i < \kappa \Rightarrow \text{Dom}(f_\eta{\restriction}\{h(\eta{\restriction}i,\varepsilon) : \varepsilon < \lambda\}$ has order type $\gamma^*_{i+1} - \gamma^*_i$. By $(*)_5(b)$ also $\text{Dom}(f_\eta)$ has order type $\delta = \sum\limits_i (\gamma^*_{i+1} \backslash \gamma^*_i)$.

Now if $f \in \mathcal{F}$ then f_η is one-to-one by the choice of $\mathcal{F}$. Second, let $f : \lambda \to \lambda$ be a one-to-one function and we shall prove that for some $\eta, f_\eta \in \mathcal{F} \wedge f_\eta \subseteq f$.

We choose $\nu_i \in {}^i\lambda$ by induction on $i \le \kappa$ such that $j < i \Rightarrow \nu_j = \nu_i \restriction j$ and $\nu_i \triangleleft \eta \in {}^\kappa\lambda \Rightarrow f_\eta \restriction \{h(\nu_j, \alpha) : j < i, \alpha \in \mathrm{Dom}(g_{j,\nu_i(j)})\} \subseteq f$.

For $i = 0$ and i limit this is obvious and for $i = j+1$ use $(*)^+_3$. So $\eta_\kappa \in {}^\kappa\lambda$ and $f_{\eta_\kappa} \subseteq f$ hence is one-to-one hence $f_{\eta_\kappa} \in \mathcal{F}$ so we are done.
5) Easy (recalling 1.3(3)).
6) Easy.
7) Easy because if $f : \kappa \to \lambda$ is one-to-one then for some $u \subseteq \kappa$ of order type κ, $f \restriction u$ is increasing (trivial if κ is regular, easy if κ is singular). $\square_{2.6}$

Claim 2.8. *1) Each of the following is a sufficient condition to* $\mathrm{Pr}_3(\chi, \lambda, \mu, \alpha)$, *recalling Definition 2.5(2):*

- (a) $\lambda^{|\alpha|} = \lambda = \chi \ge \mu > \alpha$
- (b) $\chi = \lambda \ge \mu > |\alpha|$ *and* $(\forall \lambda_1 < \lambda)(\lambda_1^{|\alpha|} < \lambda)$
- (c) $\chi = \lambda \ge \mu \ge \beth_\omega(|\alpha|)$.

2) If $\chi_1 \le \chi_2, \lambda_1 \ge \lambda_2, \mu_1 \ge \mu_2, \alpha_1 \ge \alpha_2$ *then* $\mathrm{Pr}_3(\chi_1, \lambda_1, \mu_1, \alpha_1)$ *implies* $\mathrm{Pr}_3(\chi_2, \lambda_2, \mu_2, \alpha_2)$.

Proof. 1) If clause (a) holds, this is trivial, just use $\mathcal{F} = \{f : f$ a partial function from μ to λ with $\alpha = \mathrm{otp}(\mathrm{Dom}(f))\}$. If clause (b) holds, note that for every $f \in {}^\mu\lambda$, for some $i_1, i_2 < \lambda$ we have $\alpha \le \mathrm{otp}(\{j < i_1 : j < \mu$ and $f(j) < i_2\})$ and let $\mathcal{F} = \{f : f$ a partial function from μ to λ with bounded range and bounded domain if $\mu = \lambda$ such that $\alpha = \mathrm{otp}(\mathrm{Dom}(f))\}$. If clause (c) holds, use [She00].
2) Trivial. $\square_{2.8}$

Claim 2.9. *1) In 1.4, 1.5(2) and in 2.2 we can weaken the assumption* $\mathrm{Pr}(\lambda, \kappa)$ *to* $\mathrm{Ps}(\lambda, \kappa)$.
2) In 2.4 we can weaken the assumption $\mathrm{Pr}'(\lambda, \delta^*)$ *to* $\mathrm{Ps}'(\lambda, \delta^*)$.

Proof. The same proofs.

We can get another answer on the existence of universals. $\square_{2.9}$

Claim 2.10. *If λ is strong limit,* $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$ *and* $\lambda > \theta (\ge \kappa)$, *then in* $\mathfrak{H}_{\lambda,\theta,\kappa}$ *there is no we-universal member even for the class of bipartite members.*

Proof. Let $\delta := \theta^+ \times \kappa$ (recalling λ is a limit cardinal) by 2.6(1) we have $\mathrm{Pr}'(\lambda, \delta)$ hence by 2.4 we are done. $\square_{2.10}$

Note that Ps may fail.

Claim 2.11. *Assume* $\delta < \lambda$, $\mathrm{cf}(\lambda) \le \mathrm{cf}(\delta)$ *and* $\alpha < \lambda \Rightarrow |\alpha|^{\mathrm{cf}(\delta)} < \lambda$.

Then $\mathrm{Ps}(\lambda, \delta)$ *fails (hence also* $\mathrm{Pr}(\lambda, \delta)$, $\mathrm{Pr}'(\lambda, \delta)$, $\mathrm{Ps}'(\lambda, \delta)$*).*

Proof. Toward contradiction let $\mathcal{F}$ witness $\mathrm{Ps}(\lambda, \delta)$ so $\mathcal{F}$ is of cardinality λ. Let $\langle f_\varepsilon : \varepsilon < \lambda \rangle$ list $\mathcal{F}$; choose an increasing sequence $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ such that $\lambda_i = (\lambda_i)^{\mathrm{cf}(\delta)}$ and $\lambda = \Sigma\{\lambda_i : i < \mathrm{cf}(\lambda)\}$. We choose $\mathcal{U}_i$ by induction on i such that:

$(*)^1_i$
- (a) $\mathcal{U}_i \subseteq \lambda$
- (b) $\lambda_i \subseteq \mathcal{U}_i$
- (c) $|\mathcal{U}_i| = \lambda_i$
- (d) if $f \in \mathcal{F}$ and $\mathrm{Dom}(f) \cap \mathcal{U}_i$ is unbounded in $\mathrm{Dom}(f)$, then $\mathrm{Dom}(f) \cup \mathrm{Rang}(f) \subseteq \mathcal{U}_i$.

For clause (d) note that if $\mathrm{Dom}(f) \cap \mathcal{U}_i$ is unbounded in $\mathrm{Dom}(f)$ then there is $u \subseteq \mathrm{Dom}(f) \cap \mathcal{U}_i$ unbounded in $\mathrm{Dom}(f)$ of order-type $\mathrm{cf}(\delta)$ and such u determines f in $\mathcal{F}$ uniquely.

Now choose $f^* : \lambda \to \lambda$ such that f^* maps $\mathcal{U}_i \setminus \cup \{\mathcal{U}_j : j < i\}$ into $\lambda_{i+1} \setminus \cup \{\mathcal{U}_j : j \leq i\}$ when $i < \mathrm{cf}(\lambda)$ and is increasing, f^* contradict the choice of $\mathcal{F}$. $\square_{2.11}$

3. Complete characterization under GCH

We first resolve the case λ is strong limit and get a complete answer in 3.5 by dividing to cases (in 3.1, 3.2, 3.4 and 2.10), in 3.4 we deal also with other cardinals. This includes cases in which there are universals (3.1, 3.2) and the existence of we-universal and of ste-universal are equivalent. In fact in 3.4 we deal also (in part (2)) with another case: $\kappa = \theta \leq \operatorname{cf}(\lambda), \lambda = \sum\limits_{\alpha<\lambda,\beta<\theta} |\alpha|^{|\beta|}$ (and then there is no we-universal).

Next we prepare the ground for resolving the successor case under GCH (or weaker conditions using also 3.4(2)). If $\lambda = \mu^+ = \theta, \mu = \mu^\kappa$ then there is a we-universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$ (3.7), if $\lambda = \mu^+ = 2^\mu, \kappa < \mu$ (in 3.8, 3.12) we give a sufficient condition for existence. In 3.15 we sum up. We end with stating the conclusion for the classes of bipartite graphs (3.16, 3.17).

Claim 3.1. *Assume λ is strong limit, $\operatorname{cf}(\lambda) \leq \operatorname{cf}(\kappa), \kappa \leq \theta < \lambda$ and $\kappa < \theta \vee \operatorname{cf}(\lambda) < \operatorname{cf}(\kappa)$; hence $\operatorname{cf}(\lambda) < \theta < \lambda$ so λ is singular.*
<u>Then</u> in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a ste-universal member.

Proof. Denote $\sigma = \operatorname{cf}(\lambda)$ and let $\langle \lambda_i : i < \sigma\rangle$ be increasing continuous with limit λ such that $\lambda_0 > \theta$ and $(\lambda_{i+1})^{\lambda_i} = \lambda_{i+1}$. For any graph $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ we can find $\langle V^G_i : i < \sigma\rangle$ such that: $V^G_i \subseteq V^G, \langle V^G_i : i < \sigma\rangle$ is increasing continuous with i with union V^G such that $|V^G_i| = \lambda_i$ and

$(*)_1$ if $x \in V^G \backslash V^G_{i+1}$ then $|\{y \in V^G_{i+1} : y$ is G-connected to $x\}| < \kappa$.

As $\sigma = \operatorname{cf}(\lambda) \leq \operatorname{cf}(\kappa)$ it follows that:

$(*)_2$ if $i < \sigma$ is a limit ordinal and $x \in V^G\backslash V^G_i$ <u>then</u> ($\operatorname{cf}(i) < \sigma \leq \operatorname{cf}(\kappa)$ hence) $|\{y \in V^G_i : y$ is G-connected to $x\}| < \kappa$.

For $i \leq \sigma$ let $T_i = \prod\limits_{j<i} 2^{\lambda_{j+1}}, T^s = \cup\{T_{i+1} : i < \sigma\}, T = \bigcup_{i<\sigma} T_i$.

Let $\bar{A} = \langle A_\eta : \eta \in T^s\rangle$ be a sequence of pairwise disjoint sets such that $\eta \in T_{i+1} \Rightarrow |A_\eta| = \lambda_i$. For $\eta \in T \cup T_\sigma$ let $B_\eta = \cup\{A_{\eta\restriction j} : j < \ell g(\eta), j$ a successor ordinal$\}$. Now we choose by induction on $i \leq \sigma$, for each $\eta \in T_i$ a graph G_η such that:

$\boxplus$ (a) $V^{G_\eta} = B_\eta$ (so for $\eta = \langle\rangle$ this is the graph with the empty set of nodes) and so $|V^{G_\eta}| = \Sigma\{\lambda_j : j < \ell g(\eta)$ successor$\}$

(b) if $\nu \triangleleft \eta$ then G_ν is an induced subgraph of G_η, moreover $(\forall x \in V^{G_\eta}\backslash V^{G_\nu})(x$ is G_η-connected to $< \kappa$ nodes in $V^{G_\nu})$

(c) if $i < \sigma, \eta \in T_i, G$ a graph such that $|V^G| = \lambda_{i+1}$ and $G \in \mathfrak{H}_{\lambda_{i+1},\theta,\kappa}$ and G_η is an induced subgraph of G and $(\forall x \in V^G\backslash V^{G_\eta})(x$ is G-connected to $< \kappa$ members of $V^{G_\eta})$, <u>then</u> for some $\alpha < 2^{\lambda_{i+1}}$ there is an isomorphism from G onto $G_{\eta\hat{}\langle\alpha\rangle}$ which is the identity on B_η

(d) $G_\eta \in \mathfrak{H}_{|G_\eta|,\theta,\kappa}$.

[Can we carry the induction? For $i = 0$ this is trivial. For $i = j + 1$ this is easy, the demand in clause (c) poses no threat to the others. For i limit for $\eta \in T_i$, the graph G_η is well defined satisfying clauses (a), (b) (and (c) is irrelevant), but why

$G_\eta \in \mathfrak{H}_{|G_\eta|,\theta,\kappa}$? Toward contradiction assume $A_0, A_1 \subseteq B_\eta, A_0 \times A_1 \subseteq R^{G_\eta}$ and $\{|A_0|, |A_1|\} = \{\kappa, \theta\}$. If $\ell < 2$ and $\mathrm{cf}(|A_\ell|) \neq \mathrm{cf}(i)$ then for some $j < i, |A_\ell \cap B_{\eta \restriction j}| = |A_\ell|$ so without loss of generality $A_\ell \subseteq B_{\eta \restriction j}$, but then by clause (b) no $x \in B_\eta \backslash B_{\eta \restriction j}$ is G_η-connected to $\geq \kappa$ members of $B_{\eta \restriction j}$ and $|A_\ell| \geq \mathrm{Min}\{\kappa, \theta\} = \kappa$, hence $A_{1-\ell} \subseteq B_{\eta \restriction j}$, so we get contradiction to the induction hypothesis. So the remaining case is $\mathrm{cf}(|A_0|) = \mathrm{cf}(i) = \mathrm{cf}(|A_1|)$ hence $\mathrm{cf}(\theta) = \mathrm{cf}(\kappa) = \mathrm{cf}(i)$; so as we are assuming $\mathrm{cf}(\kappa) \geq \mathrm{cf}(\lambda) \geq \mathrm{cf}(i)$, we necessarily get $\mathrm{cf}(i) = \mathrm{cf}(\lambda) = \mathrm{cf}(\kappa) = \mathrm{cf}(\theta)$. By the last assumption of the claim (i.e. $\kappa < \theta \vee \mathrm{cf}(\kappa) > \mathrm{cf}(\lambda)$) we get that $\kappa < \theta$ and without loss of generality $|A_0| = \kappa, |A_1| = \theta$, so for some $j < \mathrm{cf}(\lambda)$ we have $j < i$ and $|A_1 \cap B_{\eta \restriction j}| \geq \kappa$, so as above $A_0 \subseteq B_{\eta \restriction j}$, hence again as above $A_1 \subseteq B_{\eta \restriction j}$ and we are done.]

We let $G^* = \cup\{G_\eta : \eta \in T\}$. Now we shall check that G^* is as required. First assume toward contradiction that $A, B \subseteq V^{G^*}$ and $A \times B \subseteq R^{G^*}$ and $\{|A|, |B|\} = \{\kappa, \theta\}$. A set $C \subseteq V^{G^*}$ will be called $\bar{A}$-flat if it is included in some $B_\eta, \eta \in T \cup T_\sigma$. Easily above if B is not $\bar{A}$-flat then A is $\bar{A}$-flat. So without loss of generality for some $\eta \in T_\sigma$ we have $A \subseteq B_\eta$ but then $x \in G^* \backslash B_\eta \Rightarrow$ (for some $i(x) < \sigma$ and $\nu \in T_{i(x)}$ we have $x \in B_\nu \backslash B_{\eta \cap \nu}$) $\Rightarrow |\{y \in B_\eta : x$ is G^*-connected to $y\}| < \kappa$, so as $\kappa \leq \theta$ we get $B \subseteq B_\eta$, hence $A, B \subseteq V^{G_\eta}$ and we get contradiction to clause (d) of $\boxplus$.

So $K_{\kappa,\theta}$ does not weakly embed into G^*; also $|V^{G^*}| = \lambda$ so $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$. Lastly, the ste-universality follows from the choice of $\langle V_i^G : i < \sigma\rangle$ for any $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$. That is, we can choose by induction on $i, \eta_i \in T_i$ and an isomorphism f_i from $G \restriction V_i^G$ onto G_{η_i} if $i > 0$, with f_i increasing continuous (and η_i increasing continuous) using for successor i clause (c) of $\boxplus$. $\square_{3.1}$

In the previous claim we dealt with the case of $\kappa, \theta < \lambda$. In the following claim we cover the case of $\theta = \lambda$:

Claim 3.2. *Assume λ is strong limit, $\mathrm{cf}(\lambda) \leq \mathrm{cf}(\kappa) \leq \kappa < \theta = \lambda$; hence λ is singular.*
<u>Then</u> in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a ste-*universal member.*

Proof. Similar to the previous proof and [She73, Th.3.2, p.268]. Let $\sigma = \mathrm{cf}(\lambda)$ and $\bar{\lambda} = \langle \lambda_i : i < \sigma\rangle, \langle T_i : i \leq \sigma\rangle, T^s, T$ be as in the proof of 3.1. For any graph $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ let $h^G : {}^\kappa(V^G) \to \sigma$ be defined by: if $|\{x_\varepsilon : \varepsilon < \kappa\}| = \kappa$ then $h^G(\bar{x})$ is the first $i < \sigma$ such that $\lambda_i \geq |\{y \in V^G : y$ is G-connected to every $x_i, i < \kappa = \ell g(\bar{x})\}|$, otherwise $h^G(\bar{x})$ is not defined. Now we choose $\langle V_i^G : i < \sigma\rangle$ as an increasing continuous sequence of subsets of V^G with union V^G such that if $i < \sigma$ then $|V_i^G| \leq \lambda_i$ and $\bar{x} \in {}^\kappa(V_{i+1}^G) \wedge |\mathrm{Rang}(\bar{x})| = \kappa \wedge h^G(\bar{x}) \leq i+1 \Rightarrow (\forall y \in V^G)$("$y$ is G-connected to x_i for every $i < \kappa$" $\Rightarrow y \in V_{i+1}^G$).

Then when (as in the proof of 3.1) we construct $\langle G_\eta : \eta \in T^s\rangle$ we also construct $\langle h_\eta : \eta \in T^s\rangle, h_\eta : {}^\kappa(B_\eta) \to \sigma$ with the natural demands. That is, we choose by induction on $i \leq \sigma$ for each $\eta \in T_i$ a graph G_η and a function $h_\eta : {}^\kappa(B_\eta) \to \sigma$ with $\boxplus(a), (b), (d)$ and (c') if $i < \sigma, \eta \in T_i, G$ is a graph such that $|V^G| = \lambda_{i+1}, G \in \mathfrak{H}_{\lambda_{i+1},\theta,\kappa}, G_\eta$ an induced subgraph of $G, h : {}^\kappa G \to \sigma$ extends h_η and if $\bar{x} \in {}^\kappa(B_\eta), \kappa = |\{x_\varepsilon : \varepsilon < \kappa\}|$ and $h_\eta(\bar{x}) \leq i$ then no $y \in G \backslash G_\eta$ is G-connected to x_ε for every $\varepsilon < \kappa$ <u>then</u> for some $\alpha < 2^{\lambda_{i+1}}$ there is an isomorphism g from G onto $G_{\eta\hat{}\langle\alpha\rangle}$ which is the identity on B_η an $h_\eta(\langle g(x_\varepsilon) : \varepsilon < \kappa\rangle) = h(\langle x_\varepsilon : \varepsilon < \kappa\rangle)$ when $x_\varepsilon \in G$ for $\varepsilon < \kappa$. In the end we have to check that "$K_{\kappa,\theta}$ is not weakly embeddable into G^*"; if $\mathrm{cf}(\kappa) = \sigma$ we need to look at slightly more (as in the end of the proof of 3.1). $\square_{3.2}$

Remark 3.3. More generally see [She06].

Claim 3.4. *1) Assume λ is strong limit, $\lambda \geq \theta = \kappa$, $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$, then in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no* we-*universal graph, even for the members from $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$.*
2) Assume

- (a) $\kappa = \theta \leq \lambda$
- (b) $(\forall \alpha < \lambda)(\forall \beta < \theta)[|\alpha|^{|\beta|} \leq \lambda]$, *(recall $\kappa = \theta$)*
- (c) $\mathrm{cf}(\lambda) \geq \mathrm{cf}(\kappa)$.

Then in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no we-universal graph, even for the $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$.

Proof. 1) By part (2).
2) Let $\sigma = \mathrm{cf}(\kappa)$. Let $\langle \gamma_i : i < \sigma \rangle$ be (strictly) increasing with limit $\theta = \kappa$.
Without loss of generality

$\boxtimes$ $|\gamma_{i+1} - \gamma_i|$ is (an even ordinal $> \alpha$, for simplicity), finite or a cardinal ($< \kappa$) with cofinality $\neq \mathrm{cf}(\lambda)$.

[Why? If κ is a limit cardinal, trivial as if a cardinal $< \kappa$ fails its successor is O.K.; if κ is a successor cardinal, $\gamma_i = i$ is O.K. and also $\gamma_i = \omega i$ or $\gamma_i = \omega_1 i$ is O.K.]

Given a graph G^* in $\mathfrak{H}_{\lambda,\theta,\kappa}$ without loss of generality $V^{G^*} = \lambda$.

For $i < \sigma$ let $T_i = \{f : f$ is a partial one-to-one mapping from γ_i into $\lambda = V^{G^*}$ with bounded range such that $2\alpha, 2\beta + 1 \in \mathrm{Dom}(f) \Rightarrow f(2\alpha) R^{G^*} f(2\beta + 1)\}$. Let $T = \bigcup_{i<\sigma} T_i$, so T (with the partial order $\triangleleft$) is a tree with $\leq \lambda$ nodes and σ levels; for $\eta \in T$ let $\mathbf{i}(\eta) < \sigma$ be the unique $i < \sigma$ such that $\eta \in T_i$. Let $T^+ = \{\eta :$ for some $\zeta, \ell g(\eta) = \zeta + 1, \eta \restriction \zeta \in T$ is $\triangleleft$-maximal in T and $\eta(\zeta) = 0\}$ and $T^s = \{\eta \in T : \mathbf{i}(\eta)$ is successor$\}$.

Note

$(*)_1$ if $i < \sigma$ is a limit ordinal, $\langle f_j : j < i \rangle$ is $\subseteq$-increasing, $f_j \in T_j$ then $\bigcup_{j<i} f_j \in T_i$ [in other words if f is a function from γ_i to λ such that $j < i \Rightarrow f \restriction \gamma_j \in T_j$ then $f \in T_i$].

[Why? The least obvious demand is $\sup \mathrm{Rang}(f) < \lambda$ which holds as $\mathrm{cf}(i) \leq i < \sigma = \mathrm{cf}(\kappa) \leq \mathrm{cf}(\lambda)$.]

$(*)_2$ there is no $\bar{f} = \langle f_i : i < \sigma \rangle$ increasing such that $i < \sigma \Rightarrow f_i \in T_i$.

[Why? As then $\bigcup_{i<\sigma} f_i$ weakly embed a complete (κ, θ)-bipartite graph into G^*.]

We define a bipartite graph G

$(*)_3$
- (a) $U^G = \{(\eta, \varepsilon) : \eta \in T^+ \cup T^s$ and $\varepsilon \in [\gamma_{\mathbf{i}(\eta)}, \gamma_{\mathbf{i}(\eta)+1})$ is even$\}$
- (b) $V^G = \{(\eta, \varepsilon) : \eta \in T^+$ and $\varepsilon \in [\gamma_{\mathbf{i}(\eta)}, \gamma_{\mathbf{i}(\eta)+1})$ is odd$\}$
- (c) $R^G = \{(\eta_1, \varepsilon_1), (\eta_2, \varepsilon_2) : (\eta_1, \varepsilon_1) \in U^G, (\eta_2, \varepsilon_2) \in V^G$ and $(\eta_1 \trianglelefteq \eta_2) \vee (\eta_2 \trianglelefteq \eta_1)\}$.

Now

(a) $|T| \le \lambda$.

[Why? By clause (b) of the assumption which is a weak form of "λ is strong limit".]

(b) $|T^+| \le \lambda$.

If $(\mathcal{U}_1 \cap (T^+ \times \lambda)) \ne \emptyset \ne (V_1 \cap (T^+ \times \lambda))$ then we can choose $(\eta_1, \varepsilon_1) \in \mathcal{U}, (\eta_2, \varepsilon_2) \in V_1$. [Why? As $\eta \in T^+ \Rightarrow \sup \mathrm{Rang}(\eta) < \lambda$, see $(*)_2$ recalling $i = \ell g(\eta) < \sigma$, $\mathrm{Rang}(\eta) \subseteq \gamma_i < \lambda$.]

(c) $|U^G| \ge |T_1| = \lambda$ and $|V^G| \ge |T_2| = \lambda$

hence

(d) $|U^G| = \lambda = |V^G|$

(e) $G \in \mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$.

[Why? Being bipartite of cardinality λ is obvious; so toward contradiction assume $(U_1 \subseteq U^G \wedge V_1 \subseteq V^G)$ have cardinality κ (recall that $\kappa = \theta$) and $U_1 \times V_1 \subseteq R^G$. Now if $(\eta_\ell, \varepsilon_\ell) \in V_1$ for $\ell = 1,2$ and η_1, η_2 are $\triangleleft$-incomparable, then $(\nu, \varepsilon) \in U_1 \Rightarrow (\nu, \varepsilon)$ is G-connected to (η_1, ε_1) and to $(\eta_2, \varepsilon_2) \Rightarrow \nu \trianglelefteq \eta_1 \wedge \nu \trianglelefteq \eta_2$ hence $|U_1| < \kappa$, a contradiction. So $\eta_* = \{\eta : (\eta, \zeta) \in V_1 \text{ for some } \zeta\}$ is a well defined sequence and belongs to $T \cup T^+$. As T has no σ-branch we get $|V_1| < \kappa$, a contradiction.]

(f) G is not weakly embeddable into G^*.

[Why? If f is such an embedding, we try to choose by induction on $i < \sigma$, a member η_i of T_i, increasing continuous with i such that $(\forall \varepsilon \in \mathrm{Dom}(\eta_i))(\forall j < i)[\gamma_j \le \varepsilon < \gamma_{j+1} \Rightarrow \eta_i(\varepsilon) = f((\eta_i \restriction \gamma_j, \varepsilon))]$. If we succeed we get a contradiction to $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ by $(*)_2$, so we cannot carry the induction for every $i < \sigma$. For $i = 0$ and i limit there are no problems (see $(*)_1$), so for some $i = j+1 < \sigma$, f_j is well defined but we cannot choose f_i. But if $j < \sigma$ consider $f_i = f_j \cup \{(\varepsilon, f((\eta_j, \varepsilon)) : \varepsilon \in [\gamma_j, \gamma_i)\}$. This gives a contradiction except possibly when $\lambda = \sup \mathrm{Rang}(\eta_i)$, but then necessarily by $\boxtimes$, $|\gamma_{i+1} - \gamma_i|$ has cofinality $\ne \mathrm{cf}(\lambda)$, so for $\ell < 2$, for some $\iota_\ell < \sigma$ the set $\{\varepsilon : \gamma_j \le \varepsilon < \gamma_{j+1} \text{ and } f(\eta_j, \varepsilon) < \lambda_{\iota_\ell} \text{ and } \varepsilon = \ell \bmod 2\}$ has cardinality $|\gamma_i - \gamma_j|/2$ which has cofinality $\ne \mathrm{cf}(\lambda)$, and then $f_i := f_j \cup \{(\varepsilon, f(\eta_j, h(\varepsilon))) : \gamma_j \le \varepsilon < \gamma_i$ and $f(\eta_j, \varepsilon) < \lambda_{\max\{\iota_0, \iota_1\}}\}$ is O.K. for any one-to-one function h, from $[\gamma_j, \gamma_i)$ onto $C_1 \cup C_2$, contradiction.] $\square_{3.4}$

Theorem 3.5. *Assume $\lambda \ge \theta \ge \kappa \ge \aleph_0$ and λ is a strong limit cardinal. There is a we-universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$ iff $\mathrm{cf}(\lambda) \le \mathrm{cf}(\kappa)$ and $(\kappa < \theta \vee \mathrm{cf}(\lambda) < \mathrm{cf}(\theta))$ iff there is a ste-universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$.*

Remark 3.6. Similarly for the universal for $\{G^{[\mathrm{gr}]} : g \in \mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}\}$ and for $\mathfrak{H}^{\mathrm{bp}}_{\lambda,\theta,\kappa}, \mathfrak{H}^{\mathrm{bp}}_{\lambda,\{\theta,\kappa\}}$.

Proof. We use freely 0.7(3) and below in each case the middle condition in 3.5 clearly holds or clearly fails and the other conditions hold or fail by the claim quoted in the case.

If $\theta < \lambda$ and $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$ by 2.10 the family $\mathfrak{H}_{\lambda,\theta,\kappa}$ has no we-universal.

If $\theta = \lambda$ and $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$ then $\mathrm{Pr}'(\lambda, \kappa)$ holds by 2.6(1) (and recall that $\mathrm{Pr}(\lambda, \kappa)$ is equivalent here to $\mathrm{Pr}'(\lambda, \kappa)$, since κ is a cardinal, see 1.3(1)(ii)) hence by 2.2 the family $\mathfrak{H}_{\lambda,\theta,\kappa}$ has no we-universal member.

If $\mathrm{cf}(\lambda) < \mathrm{cf}(\kappa)$ and $\theta < \lambda$ by 3.1 the family $\mathfrak{H}_{\lambda,\theta,\kappa}$ has a ste-universal member; the second statement in Theorem 3.5 holds as: if $\kappa < \theta$ easy, if $\kappa \geq \theta$ then $\kappa = \theta$ hence $\mathrm{cf}(\lambda) < \mathrm{cf}(\kappa) = \mathrm{cf}(\theta)$.

If $\mathrm{cf}(\lambda) < \mathrm{cf}(\kappa)$ (hence $\kappa \neq \lambda$ so $\kappa < \lambda$) and $\theta = \lambda$ (so $\kappa < \theta$) by 3.2 the family $\mathfrak{H}_{\lambda,\theta,\kappa}$ has a ste-universal member.

So the remaining case is $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$. If $\kappa < \theta < \lambda$ by 3.1 in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a ste-universal; if $\kappa = \theta \leq \lambda$ by 3.4(1) in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no we-universal member. If $\kappa < \theta = \lambda$ by 3.2 in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a ste-universal member. We have checked all possibilities hence we are done. $\square_{3.5}$

We turn to λ successor cardinal. In the following case, possibly the existence of we-universal and ste-universal are not equivalent, see 3.12(2) + 3.12(4) and Theorem 3.15.

Claim 3.7. *Assume ($\lambda \geq \theta \geq \kappa \geq \aleph_0$ and)*

- *(a)* $\lambda = \mu^+$
- *(b)* $\kappa < \mu$ *and* $\theta = \lambda$
- *(c)* $\mu = \mu^\kappa$.

Then in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a we-universal member.

Proof. If $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ (and without loss of generality $V^G = \lambda$) and $\alpha < \lambda$, then

$(*)_\alpha$ $\{\beta < \lambda : \beta$ is G-connected to $\geq \kappa$ elements $\gamma < \alpha\}$ is bounded in λ say by $\beta_\alpha < \lambda$.

Hence there is a club $C = C_G$ of λ such that:

- *(i)* $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\kappa), \alpha \in C, \beta \in [\alpha, \lambda) \Rightarrow \kappa > \mathrm{otp}\{\gamma < \alpha : \gamma$ is G-connected to $\beta\}$
- *(ii)* $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa), \alpha \in C, \beta \in [\alpha, \lambda) \Rightarrow \kappa \geq \mathrm{otp}\{\gamma < \alpha : \gamma$ is G-connected to $\beta\}$
- *(iii)* if $\alpha \in C$ then $\mu | \alpha$ and $\alpha > \sup(C \cap \alpha) \Rightarrow \mathrm{cf}(\alpha) \neq \mathrm{cf}(\kappa)$.

We shall define G^* with $V^{G^*} = \lambda$ below. For each $\delta < \lambda$ divisible by μ let $\langle a^\delta_i : i < \mu\rangle$ list $\mathcal{P}_\delta = \{a : a \subseteq \delta$, and $|a| < \kappa$ or $\mathrm{otp}(a) = \kappa$ and $\delta = \sup(a)\}$, each appearing μ times, possible as $|\delta| = \mu = \mu^\kappa$, and let

$$\begin{aligned} R^{G^*}_\delta = \{ \quad & \{\beta, \delta + i\} : \delta < \lambda \text{ is divisible by } \mu, \beta < \delta, i < \mu, \beta \in a^\delta_i\} \\ & \cup \big\{\{\delta + i, \delta + j\} : i \neq j < \mu \text{ and } \delta < \lambda \text{ is divisible by } \mu\big\}. \end{aligned}$$

Now clearly we have $\alpha + \mu \leq \beta < \lambda \Rightarrow \kappa > |\{\gamma < \alpha : \gamma$ is G^*-connected to $\beta\}|$ hence $K_{\kappa,\lambda}$ (which is $K_{\kappa,\theta}$ by the assumptions) cannot be weakly embedded into G^*. On the other hand if $G \in \mathfrak{H}_{\lambda,\theta,\kappa}$ without loss of generality $V^G = \lambda$ and let C_G be as above, and let $\langle \alpha_\zeta : \zeta < \lambda\rangle$ list in increasing order $C_G \cup \{0\}$, and we can choose by induction on ζ, a weak embedding f_ζ of $G \restriction \alpha_\zeta$ into $G^* \restriction (\mu \times \zeta)$. So G^* is as required. $\square_{3.7}$

Claim 3.8. *Assume ($\lambda \geq \theta \geq \kappa \geq \aleph_0$ and)*

- *(a)* $\lambda = 2^\mu = \mu^+, \mu$ *is a singular cardinal* (*b*) *$\kappa < \mu$ and* [3] *$\kappa < \theta \leq \lambda$*

(*c*) *for every $\mathcal{P} \subseteq [\mu]^\mu$ of cardinality λ for some* [4] *$B \in [\mu]^\kappa$, for λ sets $A \in \mathcal{P}$ we have $B \subseteq A$*

(*d*) $\mathrm{cf}(\kappa) = \mathrm{cf}(\mu)$.

Then in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no we-*universal member (even for the family of bipartite graphs).*

Proof. Let $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$, without loss of generality $V^{G^*} = \lambda$ and we shall construct a $G \in \mathfrak{H}^{\mathrm{sbp}}_{\langle\mu,\lambda\rangle,\theta,\kappa}$ not weakly embeddable into it. Now we choose $U^G = \mu, V^G = \lambda \backslash \mu$.

Notice that $\lambda^\mu = \lambda$ (by (a)), so let $\langle (f_\alpha, B_\alpha) : \mu \leq \alpha < \lambda \rangle$ list the pairs (f, B) such that $f : \mu \to \lambda$ is one to one, $B \in [\mu]^\mu$ and $f \restriction B$ is increasing such that each pair appears λ times. Let $\beta_B = \sup\{\beta + 1 : \beta < \lambda$ is G^*-connected to μ members of $B\}$ for $B \in [\lambda]^\mu$ (and we shall use it for $B \in [\lambda\backslash\mu]^\mu$, i.e. for subsets of V^G). Now β_B is $< \lambda$ by clause (c) of the assumption. We shall now choose inductively C_α for $\alpha \in [\mu, \lambda)$ such that:

⊛ (*i*) $C_\alpha \subseteq B_\alpha$ is unbounded of order type κ

(*ii*) no $\zeta \in \lambda\backslash\beta_{\mathrm{Rang}(f_\alpha)}$ is G^*-connected to every $f_\alpha(\gamma), \gamma \in C_\alpha$

(*iii*) $\mu \leq \beta < \alpha \Rightarrow |C_\beta \cap C_\alpha| < \kappa$.

In stage α choose $B'_\alpha \subseteq B_\alpha$ of order type μ such that $(\forall\beta)[\mu \leq \beta < \alpha \Rightarrow \sup(B'_\alpha \cap C_\beta) < \mu)]$, that is $B'_\alpha \cap C_\beta$ is bounded in μ equivalently in C_β for $\beta < \alpha$; this is possible by diagonalization, just remember $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$ and $\mu > \kappa$ and clause ⊛(*i*).

Now there is C satisfying

$(*)^\alpha_C$ $C \subseteq B'_\alpha$ is unbounded of order type κ such that no $\zeta \in \lambda\backslash\beta_{\mathrm{Rang}(f_\alpha)}$ is G^*-connected to every $f_\alpha(\gamma)$ for $\gamma \in C$.

[Why? Otherwise for every such C there is a counterexample γ_C and we can easily choose $C_{\alpha,i}$ by induction on $i < \lambda$ such that:

⊠(*i*) $C_{\alpha,i} \subseteq B'_\alpha$

(*ii*) $\sup(C_{\alpha,i}) = \sup(B'_\alpha) = \mu$

(*iii*) $\mathrm{otp}(C_{\alpha,i}) = \kappa$

(*iv*) $(\forall j < i)[\kappa > |C_{\alpha,i} \cap C_{\alpha,j}|]$

$(iv)^+$ moreover, if $j < i$ then $\kappa > |C_{\alpha,i} \cap \cup\{\zeta < \mu : f_\alpha(\zeta)$ is G^*-connected to $\gamma_{C_{\alpha,0} \cup C_{\alpha,j}}\}|$.

This is easy: for clause $(iv)^+$ note that for $C = C_{\alpha,j} \cup C_{\alpha,0}$ by the choice of γ_C we have $\gamma_C \geq \beta_{\mathrm{Rang}(f_\alpha)}$ hence by the choice of $\beta_{\mathrm{Rang}(f_\alpha)}$ clearly $D_C =: \{i < \mu : f_\alpha(i)$ is well defined and G^*-connected to $\gamma_C\}$ has cardinality $< \mu$, so we can really carry the induction on $i < \lambda$, that is any $C \subseteq B'_\alpha$ unbounded in μ of order type κ such that $j < i \Rightarrow |C \cap D_{C_{\alpha,j} \cup C_{\alpha,0}}| < \kappa$ will do.

---

[3] in fact, $\kappa = \theta$ is O.K., but already covered by 3.4(2)

[4] note that if $\beth_\omega(\kappa) \leq \mu$ this clause always holds; and if $2^\kappa \leq \mu$ it is hard to fail it, not clear if its negation is consistent

Let $A_0 = C_{\alpha,0}, A_1 = \{\gamma_{C_{\alpha,0} \cup C_{\alpha,1+i}} : i < \lambda\}$ they form a complete bipartite subgraph of G^* by the definition of $\gamma_{C_{\alpha,0} \cup C_{\alpha,i}}$ and $|C_{\alpha,0}| = \kappa = |A_0|$ (by (iii) of $\boxtimes$) and $|A_1| = \lambda$ (the last: by $(iv)^+$), contradiction. So there is C such that $(*)^\alpha_C$.]

Choose C_α as any such C such that $(*)^\alpha_C$. Lastly define G

$$U^G = \mu$$

$$V^G = \lambda \backslash \mu$$

$$R^G = \{(\beta, \alpha) : \alpha \in V^G, \beta \in C_\alpha\}.$$

Clearly $G \in \mathfrak{H}^{\text{sbp}}_{\lambda,\theta,\kappa}$ recalling $\kappa < \theta$ and $\alpha_1 \neq \alpha_2 \Rightarrow |C_{\alpha_1} \cap C_{\alpha_2}| < \kappa$. Suppose toward contradiction that $f : \lambda \to \lambda$ is a weak embedding of G into G^*, hence the set $Y = \{\alpha < \lambda : \alpha \geq \mu \text{ and } f_\alpha = f \restriction \mu\}$ is unbounded in λ and without loss of generality $\alpha \in Y \Rightarrow \beta_{\text{Rang}(f_\alpha)} = \beta^*$, i.e. is constant. As f is one to one for every $\alpha \in Y$ large enough, $f(\alpha) \in (\beta^*, \lambda)$ and we get easy contradiction to clause $\circledast(ii)$ for α and we are done. (Note that we can add λ nodes to U^G). $\square_{3.8}$

For the next claim, we need another pair of definitions:

Definition 3.9. 1) $\mathfrak{H}^*_\lambda$ is the class of $G = (V^G, R^G, P^G_i)_{i<\lambda}$ where (V^G, R^G) is a graph, $|V^G| = \lambda$ and $\langle P^G_i : i < \lambda\rangle$ is a partition of V^G.
2) We say f is a strong embedding of $G_1 \in \mathfrak{H}^*_\lambda$ into $G_2 \in \mathfrak{H}^*_\lambda$ <u>when</u> it strongly embeds (V^{G_1}, R^{G_1}) into (V^{G_2}, R^{G_2}) mapping $P^{G_1}_i$ into $P^{G_2}_i$ for $i < \lambda$.
3) $G \in \mathfrak{H}^*_\lambda$ is ste-universal is defined naturally.

Definition 3.10. For $\kappa \leq \mu, \mathbf{U}_\kappa(\mu) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\mu]^\kappa \text{ and } (\forall A \in [\mu]^\kappa)(\exists B \in \mathcal{P})(|A \cap B| = \kappa)\}$.

Remark 3.11. If μ is a strong limit cardinal and $\text{cf}(\mu) < \text{cf}(\kappa) \leq \kappa < \mu$, then $\mathbf{U}_\kappa(\mu) = \mu$.

Claim 3.12. *1) Assume* $(\lambda \geq \theta \geq \kappa \geq \aleph_0$ *and)*

- (a) $\lambda = \mu^+ = 2^\mu$
- (b) $\kappa < \mu$ *and* $\text{cf}(\kappa) \neq \text{cf}(\mu)$
- (c) $2^\kappa \leq \mu$ *and* $\mathbf{U}_\kappa(\mu) = \mu$
- (d) (i) $\kappa > \text{cf}(\mu)$ <u>*or*</u>
  - (ii) $\theta < \lambda$ <u>*or*</u>
  - (iii) $\kappa < \text{cf}(\mu)$ *and there are* $C^*_\alpha \subseteq \mu$ *of order type* κ *for* $\alpha < \lambda$ *such that* $u \in [\lambda]^\lambda \Rightarrow \text{otp}[\cup\{C^*_\alpha : \alpha \in u\}] > \kappa$.

<u>*Then*</u> *in* $\mathfrak{H}_{\lambda,\theta,\kappa}$ *there is no we-universal even for the bipartite graphs in* $\mathfrak{H}_{\lambda,\theta,\kappa}$.
2) In part (1) we can replace clause (d) from the assumption by $(d)_1$ *or* $(d)_2$ *or* $(d)_3$ *where*

- $(d)_1$ $\mu^\kappa \geq \lambda$
- $(d)_2$ $\theta = \lambda$ *and among the graphs of cardinality* μ *there is no ste-universal*
- $(d)_3$ $\theta = \lambda$ *and in* $\mathfrak{H}^*_\mu$ *there is no ste-universal,* <u>*then*</u> *still there is no* ste-universal, *even for the bipartite graphs in* $\mathfrak{H}_{\lambda,\theta,\kappa}$.

3) If (a) + (b) of part (1) and $2^{<\mu} = \mu = \mu^\kappa \wedge \theta = \lambda$ <u>then</u> there is a ste-*universal.*
4) If (a),(b) of part (1) and (c),(d) below, <u>then</u> there is a ste-*universal in $\mathfrak{H}_{\lambda,\theta,\kappa}$*

- *(c)* $\mu = \mu^\kappa$
- *(d) in $\mathfrak{H}^*_\mu$ there is a universal, see 3.9(1).*

Remark 3.13. Note that part (2) is not empty: if μ is strong limit singular, $2^\mu = \lambda = \mu^+, \chi = \chi^{<\chi} < \mu$ and $\mathbb{P}$ is the forcing of adding μ Cohen subsets to χ, then in $\mathbf{V}^{\mathbb{P}}$ clause $(d)_2$ holds.

Proof. 1) Let $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ and without loss of generality $V^{G^*} = \lambda$. As in the proof of 3.7 using assumption (c) there is a club C of λ such that

- (i) $\delta \in C, \delta \le \beta < \lambda \Rightarrow \kappa \ge \operatorname{otp}\{\gamma < \delta : \gamma$ is G^*-connected to $\beta\}$
- (ii) $\delta \in C$, $\operatorname{cf}(\delta) \ne \operatorname{cf}(\kappa), \delta \le \beta < \lambda \Rightarrow \kappa > \operatorname{otp}\{\gamma < \beta : \gamma$ is G^*-connected to $\beta\}$
- (iii) μ^2 divides δ for every $\delta \in C$.

Let $S =: \{\delta \in C : \operatorname{cf}(\delta) = \operatorname{cf}(\kappa)\}$; as we have $\diamondsuit_S$, see [She10], so let $\bar f = \langle f_\delta : \delta \in S\rangle, f_\delta \in {}^\delta\delta$ be a one-to-one function such that $(\forall f \in {}^\lambda\lambda)(\exists^{\rm stat}\delta \in S)[f$ is one-to-one $\Rightarrow f_\delta = f \restriction \delta]$. For $\delta \in S$ let $\beta_\delta = \operatorname{Min}(C\backslash(\delta+1))$, and for $i \in [\delta,\beta_\delta)$ we let $a_{\delta,i} = \{\gamma < \delta : f_\delta(\gamma)$ is G^*-connected to $i\}$, so $\operatorname{otp}(a_{\delta,i}) \le \kappa$ by the choice of C, and let $B_\delta = \{i : \delta \le i < \beta_\delta$ and $|a_{\delta,i}| \ge \kappa\}$.

Now for $\delta \in S$ we choose $a^*_\delta \subseteq \delta$ unbounded of order type κ such that $(\forall i \in B_\delta)(a^*_\delta \nsubseteq a_{\delta,i})$.

[Why? First assume $(d)(i)$, i.e. $\kappa > \operatorname{cf}(\mu)$ let $[\delta,\beta_\delta) = \bigcup\limits_{\xi < \operatorname{cf}(\mu)} A_{\delta,\xi}, |A_{\delta,\xi}| < \mu, A_{\delta,\xi}$ is $\subseteq$-increasing continuous with ξ and let $\langle\gamma_{\delta,\varepsilon} : \varepsilon < \kappa\rangle$ be increasing continuous with limit δ satisfying $\mu|\gamma_{\delta,\varepsilon}$ (remember $\delta \in S \Rightarrow \mu^2|\delta$). Now choose $\gamma^*_{\delta,\varepsilon} \in [\gamma_{\delta,\varepsilon},\gamma_{\delta,\varepsilon+1})\backslash \cup\{a_{\delta,i} :$ for some $\zeta < \operatorname{cf}(\mu), \varepsilon = \zeta \bmod \operatorname{cf}(\mu)$ and $i \in A_{\delta,\varepsilon}\}$ for $\varepsilon < \kappa$ and let $a^*_\delta = \{\gamma^*_{\delta,\varepsilon} : \varepsilon < \kappa\}$, it is as required.

Second assume case (ii) of clause (d) of the assumption, so $\theta < \lambda$ hence $\theta \le \mu$. For $\delta \in S$ we choose a sequence $\bar C^\delta = \langle C_{\delta,i} : i < \mu\rangle$ of pairwise disjoint sets, $C_{\delta,i}$ an unbounded subset of $\delta\backslash S$ of order type κ, always exist as $\mu^2|\delta$ (we could have asked moreover that $f_\delta \restriction C_{\delta,i}$ is increasing with limit δ. Now if $f : \lambda \to \lambda$ is one-to-one then $\{\delta \in S : f_\delta = f \restriction \delta$ and for f_δ we can choose $\bar C^\delta\}$ is a stationary subset of λ so this is O.K. but not necessary). If for some $i < \mu$ the set $C_{\delta,0} \cup C_{\delta,1+i}$ is as required on a^*_δ, fine, otherwise for every $i < \mu$ there is $\gamma_i < \lambda$ which is G^*-connected to every $y \in \operatorname{Rang}(f \restriction (C_{\delta,0} \cup C_{\delta,1+i}))$. As any γ_i is G^*-connected to $\le \kappa$ ordinals $< \delta$ and $\langle C_{\delta,i} : i < \mu\rangle$ are pairwise disjoint, clearly $|\{j : \gamma_j = \gamma_i\}| \le \kappa$ hence we can find $Y \subseteq \mu$ such that $\langle\gamma_i : i \in Y\rangle$ is with no repetitions and $|Y| = \theta$. So $A_0 = \operatorname{Rang}(f \restriction C_{\delta,0}), A_1 = \{\gamma_i : i \in Y\}$ exemplify that a complete (κ,θ)-bipartite graph can be weakly embedded into G^*, contradiction.

Lastly, the case clause (iii) of clause (d) holds. let $\langle\gamma_{\delta,\varepsilon} : \varepsilon < \kappa\rangle$ be an increasing sequence with limit δ such that $\mu|\gamma_{\delta,\varepsilon}$; let $\langle C^*_i : i < \lambda\rangle$ be as in clause (iii) of (d) of the assumption and let $C_{\delta,i} := \{\beta+1$: for some $\varepsilon < \kappa$ we have $\gamma_{\delta,\varepsilon} \le \beta < \gamma_{\delta,\varepsilon} + \mu$ and $\beta - \gamma_{\delta,\varepsilon} \in C^*_i$ and $\operatorname{otp}(C^*_i \cap (\beta - \gamma_{\delta,\varepsilon})) < \varepsilon\}$.

Lastly, define the bipartite graph G by $V^G = \lambda, R^G = \{(\gamma,\delta) : \delta \in S, \gamma \in a^*_\delta\}$. Easily $G \in \mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}$ and is not weakly embeddable into G^* by the choice of $\bar f$.

2) Let $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}, V^{G^*} = \lambda$. We choose the club C, the set S and the sequence $\bar f = \langle f_\delta : \delta \in S\rangle$ as in the proof of part (1). We shall choose $\langle a_{\delta,i} : \delta \in S, i < \mu\rangle$ and define G by $V^G = \lambda, E^G = \bigcup\limits_{\delta \in S} E_\delta, E_\delta = \{\{\gamma,\delta+i\} : \gamma \in a^\delta_i, i < i_\mu\} \cup \{\{\delta+i,\delta+j\} : (i,j) \in R_\delta \subseteq \mu \times \mu\}$. Naturally a^δ_i is an unbounded subset of δ of order-type κ.

Now it is sufficient to find for $\delta \in S$ an unbounded subset $C = C_\delta$ of δ of order type κ such that for no $\gamma = \gamma_C < \lambda$ do we have $(\forall \beta < \delta)(\beta \in C \Leftrightarrow f_\delta(\beta) R^{G^*} \gamma)$, in this case $i_\delta = 1$. If this fails then such γ_C is well defined for any unbounded $C \subseteq \delta$ of order type κ; $C_{\delta,i} \subseteq \delta$ unbounded of order type κ, pairwise distinct for $i < \lambda$ and $C_{\delta,1+i} \cap C_{\delta,0} = \emptyset$; then $A_0 =: \{f_\delta(\beta) : \beta \in C_{\delta,0}\}, A_1 =: \{\gamma_{C_{\delta,0} \cup C_{\delta,1+i}} : i < \lambda\}$ exemplifies that the complete (κ,θ)-bipartite graph can be weakly embedded into G^*, contradiction.

Clearly $(d)_2 \Rightarrow (d)_3$ so without loss of generality $(d)_3$ holds. For $\delta \in S$ let $\langle C_{\delta,j} : j \le \mu\rangle$ be a sequence of distinct subsets of μ which include $\{\{\alpha < \delta : f_\delta(\alpha) R^{G^*}(\delta+i)\} : i < i_\delta\}$ and let M_δ be the model expanding $G^* {\restriction} [\delta, \delta+i_\delta]$ by $P^M_j = \{\delta + i : i < i_\delta$ and $(\forall \alpha < \delta)[f_\delta(\alpha) R^{G^*}(\delta+i) \equiv \alpha \in C_{\delta,j}]\}$. As M_δ is not universal in $\mathfrak{H}^*_\mu$, so let $N_\delta \in \mathfrak{H}^*_\mu$ witness this; without loss of generality the universe of N in $[\delta,\delta+\mu)$, and let $a_{\delta,j} = \{f_\delta(\alpha) : \alpha \in C_{\delta,j}\}$ and $R_\delta = R^{N_\delta}$.

Now check.

3) It suffices to prove that the assumptions of part (4) holds, the non-trivial part is clause (d) there, i.e. $\mathfrak{H}^*_\lambda$ has a universal member. But $2^{\le\mu} = \mu$ so either μ is regular so $\mu = \mu^{<\mu}$ or μ is strong limit singular and in both cases this holds by Jonsson or see [Shea].

4) We choose G_α for $\alpha \le \lambda$ by induction on α such that

$\boxplus$ (a) G_α is a graph with set of nodes $(1+\alpha)\mu$

(b) if $\beta < \alpha$ then G_β is an induced subgraph of G

(c) if $\alpha = \beta + 1$ and G is a graph with μ nodes and id_{G_β} is a strong embedding of G_β into G such that $x \in V^G \backslash V^{G_\beta} \Rightarrow$ (x is connected to $\le \kappa$ nodes of G_β) <u>then</u> there is a strong embedding of G into G_α which extends id_{G_β}.

The construction is possible by clause (d) of the assumption. Now as in the proof of 3.8 $G_\lambda \in \mathfrak{H}_{\lambda,\theta,\kappa}$ is ste-universal. $\square_{3.12}$

Remark 3.14. 1) In the choice of $\bar f$ (in the proof of 3.12) we can require that for every $f \in {}^\lambda\lambda$ the set $\{\delta \in S : f_\delta = f \restriction \delta$ and $\delta \cap \mathrm{Rang}(f) = \mathrm{Rang}(f_\delta)\}$ is stationary and so deal with copies of the complete (κ,θ)-bipartite graph with the θ part after the κ part.

2) Probably we can somewhat weaken assumption (c).

Theorem 3.15. *Assume $\lambda \ge \theta \ge \kappa \ge \aleph_0$ and $\lambda = 2^\mu = \mu^+$ and $2^{<\mu} = \mu$.*
1) In $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is a we-universal member <u>iff</u> $\mu^\kappa = \mu \wedge \theta = \lambda$ <u>iff</u> there is no $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ we-universal for $\{G^{[\mathrm{gr}]} : G \in \mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}\}$.
2) In $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is ste-universal, <u>iff</u> $\mu^\kappa = \mu \wedge \theta = \lambda$ <u>iff</u> there is no $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ ste-universal for $\{G^{[\mathrm{gr}]} : G \in \mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}\}$.

Proof. 1) The second iff we ignore as in each case the same claims cited give it too <u>or</u> use 3.17 below. We first prove that there is no we-universal except possibly when $\mu^\kappa = \mu \wedge \theta = \lambda$.

Proving this claim, whenever we point out a case is resolved we assume that it does not occur. We avoid using $2^{<\mu} = \mu$ when we can.

If $\lambda = \lambda^{\theta^+}$ then by 2.6(3) we have $\mathrm{Ps}'(\lambda, \theta^+ \times \kappa)$ so by 2.4 + 2.9(2) there is no we-universal; hence we can assume that $\lambda < \lambda^{\theta^+}$ so (as $\lambda = \lambda^{<\lambda}$) clearly $\lambda \leq \theta^+$ hence $\lambda = \theta \vee \lambda = \theta^+$ that is $\theta = \lambda \vee \theta = \mu$.

If $\kappa = \theta$ then by 3.4(2) there is no we-universal, so we can assume that $\kappa \neq \theta$ hence $\kappa < \theta \leq \lambda$, hence $\lambda = \lambda^\kappa$ so by 2.6(3) we have $\mathrm{Ps}'(\lambda, \kappa)$ hence by 2.6(7) we have $\mathrm{Ps}(\lambda, \kappa)$. So if $\theta = \kappa^+$ then by 1.4 more exactly, 2.9(1) there is no we-universal so without loss of generality $\kappa^+ < \theta$ hence $\kappa < \mu$. If $\mathrm{cf}(\kappa) = \mathrm{cf}(\mu)$ then by 3.8 we are done except if

$(*)_1$ clause (c) of 3.8 fails, (and $\mathrm{cf}(\kappa) = \mathrm{cf}(\mu), \kappa < \mu$).

But (c) of 3.8 holds, as $2^{<\mu} = \mu$, so we can assume

$(*)_2$ $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\mu)$,

so as $2^{<\mu} = \mu, \kappa < \mu$ we get

$(*)_3$ $\mathbf{U}_\kappa(\mu) = \mu$ and $2^\kappa \leq \mu$.

Now we try to apply 3.12(1), so we can assume that we cannot; but clauses (a)-(c) there hold hence clause (d) there fails. So $\kappa \leq \mathrm{cf}(\mu) \wedge \theta \geq \lambda$ (recalling sub-clauses (i),(ii) of 3.12(1)(d)) as $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\mu)$ by $(*)_2$ and $\theta \leq \lambda$ we have $\kappa < \mathrm{cf}(\mu)$ and $\theta = \lambda$. As $2^{<\mu} = \mu$ this implies $\mu^\kappa = \mu$ and $\theta = \lambda = \mu^+$ as promised. All this gives the implication $\Rightarrow$; the other direction by 3.7 gives there is a we-universal.
2) By part (1) and 0.7(3), the only open case is $\mu^\kappa = \mu$ and $\theta = \lambda = \mu^+$ then Claim 3.12(3),(4) applies (clause (c) there follows from $\mu = \mu^\kappa$). $\square_{3.15}$

Recalling Definition 0.6

Claim 3.16. *The results in 3.15 hold for* $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$ *and for* $\mathfrak{H}^{\mathrm{bp}}_{\lambda,\theta,\kappa}$.

Proof. The "no universal" clearly holds by 3.15, so we need the "positive results", and we are done by 3.17 below. $\square_{3.16}$

Claim 3.17. *The results of 3.1, 3.2, 3.7 and 3.12(3),(4) hold for* $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$ *and* $\mathfrak{H}^{\mathrm{bp}}_{\lambda,\theta,\kappa}$.

Proof. In all the cases the isomorphism and embeddings preserve "$x \in U^G$", "$y \in V^G$".

For $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$, in 3.7 we redefine G^* as a bipartite graph (recalling $\langle a^\delta_i : i < \mu\rangle$ lists $\{a \subseteq \delta :\ \mathrm{otp}(a) \leq \kappa$ and if equality holds then $\delta = \sup(a)\}$ for $\delta < \lambda$ divisible by μ)

$$U^{G^*} = \{2\alpha : \alpha < \lambda\}$$

$$V^{G^*} = \{2\alpha + 1 : \alpha < \lambda\}$$

$$\begin{aligned} R^{G^*} = \quad & \{(2\alpha, 2\beta+1) : \text{for some } \delta < \lambda \text{ divisible by } \mu \text{ we have } 2\alpha, 2\beta+1 \in [\delta, \delta+\mu]\} \\ & \cup\{(2\alpha, \delta+2i+1) : \delta < \lambda \text{ divisible by } \mu, i < \mu, 2\alpha < \delta, \alpha \in a^\delta_i\} \\ & \cup\{(\delta+2i, 2\beta+1) : \delta < \lambda \text{ divisible by } \mu, 2\beta+1 < \delta, i < \mu, \beta \in a^\delta_i\} \end{aligned}$$

The proof is similar. For $\mathfrak{H}^{\mathrm{bp}}_{\lambda,\theta,\kappa}$, 3.7 we redefine G^*

$$U^{G^*} = \{2\alpha : \alpha < \lambda\}$$

$$V^{G^*} = \{2\alpha + 1 : \alpha < \lambda\}$$

$$\begin{aligned} R^{G^*} = \quad & \{(2\alpha, 2\beta+1) : \text{for some } \delta < \lambda \text{ divisible by } \mu, \{2\alpha, 2\beta+1\} \subseteq [\delta, \delta+\mu]\} \\ & \cup\{(2\alpha, \delta+2i+1) : \delta < \lambda \text{ divisible by } \mu, i < \mu, 2\alpha < \delta \text{ and } \alpha \in a_i^\delta\} \\ & \cup\{(2\alpha+1, 2\beta) : 2\alpha+1 < 2\beta\}. \end{aligned}$$

The proof of 3.1, 3.2 for $\mathfrak{H}^{\rm sbp}_{\lambda,\theta,\kappa}$ is similar to that of 3.1, 3.2. The G_η is from $\mathfrak{H}^{\rm sbp}_{\lambda,\ell g(\eta)}$ so the isomorphism preserve the $x \in U^G, y \in V^G$. For $\mathfrak{H}^{\rm bp}_{\lambda,\theta,\kappa}$ without loss of generality $\kappa \neq \theta$ hence $\kappa < \theta$ (otherwise this falls under the previous case). We repeat the proof of the previous case carefully; making the following changes, say for 3.1, $\langle V^G_i : i < \sigma\rangle$ is increasing continuous with union $V^G, \langle U^G_i : i < \sigma\rangle$ increasing continuous with union U^G.

$(*)'_1$ if $x \in V^G \backslash V^G_{i+1}$ then $\kappa > |\{y \in U^G_i : y \text{ is } G\text{-connected to } x\}|$.

We leave 3.12(3),(4) to the reader. $\square_{3.17}$

4. More accurate properties

Definition 4.1. Let $\mathrm{Q}(\lambda, \mu, \sigma, \kappa)$ mean: there are $A_i \in [\mu]^\sigma$ for $i < \lambda$ such that for every $B \in [\mu]^\kappa$ there are $< \lambda$ ordinals i such that $B \subseteq A_i$.

Definition 4.2. 1) For $\mu \geq \kappa$ let $\mathrm{set}(\mu, \kappa) = \{A : A$ is a subset of $\mu \times \kappa$ of cardinality κ such that $i < \kappa \Rightarrow \kappa > |\{A \cap (\mu \times i)\}|\}$ and let $\mathrm{set}(\mu, \kappa) = [\kappa]^\kappa$ for $\mu < \kappa$.
2) Assume $\lambda \geq \theta \geq \kappa, \lambda \geq \mu$. Let $\mathrm{Qr_w}(\lambda, \mu, \theta, \kappa)$ mean that some $\bar{A}$ exemplifies it, which means

- (a) $\bar{A} = \langle A_i : i < \alpha \rangle$
- (b) $A_i \in \mathrm{set}(\mu, \kappa)$ for $i < \alpha$
- (c) $\bar{A}$ is (κ, θ)-free which means $(\forall A \in \mathrm{set}(\mu, \kappa))(\exists^{<\theta} i < \alpha)(A \subseteq A_i)$
- (d) $\alpha \leq \lambda$
- (e) if $\bar{A}' = \langle A'_i : i < \alpha' \rangle$ satisfies clauses (a),(b),(c),(d), then for some one to one function π from $\bigcup\limits_{i<\alpha'} A'_i$ into $\bigcup\limits_{i<\alpha} A_i$ and one-to-one function $\varkappa$ from α' to α (or κ to κ) we have $i < \alpha' \Rightarrow \pi(A'_i) \subseteq A_{\varkappa(i)}$.

3) $\mathrm{Qr_{st}}(\lambda, \mu, \theta, \kappa)$ is defined similarly except that we change clause (e) to (e)$^+$ demanding $\pi(A'_i) = A_{\varkappa(i)}$. Let $\mathrm{Qr_{pr}}(\lambda, \mu, \theta, \kappa)$ be defined similarly omitting clause (e).
4) Assume $\lambda \geq \theta \geq \kappa, \lambda \geq \mu$ and $x \in \{w, st\}$. Let $\mathrm{NQr}_x(\lambda, \mu, \theta, \kappa)$ mean that $\mathrm{Qr}_x(\lambda, \mu, \theta, \kappa)$ fails.

Claim 4.3. *1) Assume* $\mathrm{NQr}_w(\lambda, \mu, \theta, \kappa)$ *and* $\lambda = \lambda^{\mu+\kappa}$ *and* $(\lambda > \mu^\kappa) \vee (\mu < \kappa)$. *Then*

- (a) *in* $\mathfrak{H}_{\lambda,\theta,\kappa}$ *there is no* we-*universal member*
- (b) *moreover, for every* $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ *there is a member of* $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$ *not weakly embeddable into it.*

2) Assume $\mathrm{NQr_{st}}(\lambda, \mu, \theta, \kappa)$ *and* $\lambda = \lambda^{\mu+\kappa}$ *and* $(\lambda > \mu^\kappa) \vee (\mu \leq \kappa)$. *Then*

- (a) *in* $\mathfrak{H}_{\lambda,\theta,\kappa}$ *there is no* ste-*universal member*
- (b) *moreover, for every* $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ *there is a member of* $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$ *not strongly embeddable into it.*

3) In parts (1), (2) we can weaken the assumption $\lambda = \lambda^{\mu+\kappa}$ *to*

$\otimes$ $\lambda = \lambda^\kappa > \mu$ *and there is* $\mathcal{F} \subseteq \{f : f$ *a partial one to one function from* λ *to* $\lambda, |\mathrm{Dom}(f)| = \mu\}$ *of cardinality* λ *such that for every* $f^* \in {}^\lambda\lambda$ *there* [5] *is* $f \in \mathcal{F}, f \subseteq f^*$.

Proof. 1), 2) Let $x = w$ for part (1) and $x = \mathrm{st}$ for part (2).

Now suppose that $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ and without loss of generality $V^{G^*} = \lambda$ and we shall construct $G \in \mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$ not x-embeddable into G^* (so part (b) will be proved, and part (a) follows).

Case 1: $\mu < \kappa$ so $\mathrm{set}(\mu, \kappa) = [\kappa]^\kappa$.

[5]we can add "there are λ functions $f \in \mathcal{F}, f \subseteq f^*$, with pairwise disjoint domains", and possibly increasing $\mathcal{F}$ we get it

Similar to the proof of 1.5. Let $\bar{f} = \langle f_\eta : \eta \in {}^\kappa\lambda\rangle$ be a simple black box for one to one functions. It means that each f_η is a one to one function from $\{\eta{\restriction}i : i < \kappa\}$ into λ, such that for every $f : {}^{\kappa>}\lambda \to \lambda$ for some $\eta \in {}^\kappa\lambda$ we have $f_\eta \subseteq f$. We define the bipartite graph G as follows:

$(*)$ (a) $U^G = {}^{\kappa>}\lambda$ and $V^G = ({}^\kappa\lambda) \times \mu$

(b) $R^G = \cup\{R^G_\eta : \eta \in {}^\kappa\lambda\}$ where $R^G_\eta \subseteq \{(\eta \restriction \varepsilon, (\eta, i)) : \varepsilon < \kappa, i < \mu\}$.

Now for each $\eta \in {}^\kappa\lambda$, we choose $\langle \beta_{\eta,i} : i < \alpha_\eta\rangle$ listing without repetitions the set $\{\beta < \lambda : \beta$ is G^*-connected to κ members of $\mathrm{Rang}(f_\eta)\}$ and without loss of generality $\alpha_\eta = |\alpha_\eta|$ and $A_{\eta,i} = \{\varepsilon < \kappa : \beta_{\eta,i}, f_\eta(\eta \restriction \varepsilon)$ are G^*-connected$\}$.

As $G^* \in \mathfrak{H}_{\lambda,\theta,\kappa}$ clearly $A \in [\kappa]^\kappa \Rightarrow |\{i < \alpha_\eta : A \subseteq A_{\eta,i}\}| < \theta$ hence $\alpha_\eta = |\alpha_\eta| \leq 2^\kappa + \theta$ but (see Definition 4.2(2)) we have assumed $\theta \leq \lambda$ and (in 4.3) $\lambda = \lambda^\kappa$ so $2^\kappa \leq \lambda$ hence $\alpha_\eta \leq \lambda$; next let $\bar{A}_\eta = \langle A_{\eta,i} : i < \alpha_\eta\rangle$ and as $\bar{A}_\eta$ cannot be a witness for $\mathrm{Qr}_x(\lambda, \mu, \theta, \kappa)$ but clauses (a), (b), (c), (d) of Definition 4.2(1) hold, hence clause (e) fails so there is $\bar{A}'_\eta = \langle A'_{\eta,i} : i < \alpha'_\eta\rangle$ exemplifies the failure of clause (e) of Definition 4.2 with $\bar{A}_\eta, \bar{A}'_\eta$ here standing for $\bar{A}, \bar{A}'$ there.

Let

$$R^G_\eta = \{(\eta \restriction \varepsilon, (\eta, i)) : \varepsilon < \kappa, i < \alpha'_\eta \text{ and } \varepsilon \in A'_{\eta,i}\}.$$

The proof that G cannot be x-embedded into G^* is as in the proof of 1.5.

<u>Case 2</u>: $\mu^\kappa < \lambda$ (and $\lambda = \lambda^\mu, \kappa \leq \mu < \lambda$).

First note that by the assumptions of the case

$\boxplus_1$ there is $\bar{f} = \langle f_\eta : \eta \in {}^\kappa\lambda\rangle$ such that

(a) f_η is a function from $\bigcup\limits_{\varepsilon<\kappa} (\{\eta \restriction \varepsilon\} \times \mu)$ into λ

(b) if f is a function from $({}^{\kappa>}\lambda) \times \mu$ to λ then we can find $\langle \nu_\rho : \rho \in {}^{\kappa\geq}\lambda\rangle$ such that

(i) $\nu_\rho \in {}^{\ell g(\rho)}\lambda$

(ii) $\rho_1 \triangleleft \rho_2 \Rightarrow \nu_{\rho_1} \triangleleft \nu_{\rho_2}$

(iii) if $\alpha < \beta < \lambda$ and $\rho \in {}^{\kappa>}\lambda$ then $\nu_{\rho\hat{}\langle\alpha\rangle} \neq \nu_{\rho\hat{}\langle\beta\rangle}$

(iv) $f_{\nu_\rho} \subseteq f$ for $\rho \in {}^\kappa\lambda$.

We commit ourselves to

$\boxplus_2$ (a) $U^G = ({}^{\kappa>}\lambda) \times \mu$ and $V^G = \{(\eta, i) : \eta \in {}^\kappa\lambda, i < \lambda\}$

(b) $R^G = \cup\{R^G_\eta : \eta \in {}^\kappa\lambda\}$ where

(c) $R^G_\eta \subseteq \{((\eta \restriction \varepsilon, j), (\eta, i)) : j < \mu, i < \lambda, \varepsilon < \kappa\}$.

We say $\eta \in {}^\kappa\lambda$ is G^*-reasonable if f_η is one to one and for every $\zeta < \kappa$ and $y \in V^{G^*}$ the set $\{(\eta \restriction \varepsilon, j) : \varepsilon < \zeta, j < \mu$ and $f_\eta((\eta \restriction \varepsilon, j))$ is G^*-connected to $y\}$ has cardinality $< \kappa$. We decide

$\boxplus_3$ (a) if η is not G^*-reasonable then $R^G_\eta = \emptyset$

(b) if η is G^*-reasonable let $\langle \beta_{\eta,i} : i < \alpha_\eta \rangle$ list without repetitions the set $\{\beta < \lambda : \beta$ is G^*-connected to at least κ members of $\mathrm{Rang}(f_\eta)\}$; and let $A_{\eta,i} = \{(\varepsilon, j) : \varepsilon < \kappa, j < \mu$ and $f_\eta((\eta \restriction \varepsilon, j))$ is G^*-connected to $\beta_{\eta,i}\}$; clearly $A_{\eta,i} \in$ set(μ, κ) and let $\bar{A}_\eta = \langle A_{\eta,i} : i < \alpha_\eta \rangle$

(c) as $\bar{A}_\eta$ cannot guarantee $\mathrm{Qr}_x(\lambda, \mu, \theta, \kappa)$ necessarily there is $\bar{A}'_\eta = \langle A'_{\eta,i} : i < \alpha'_\eta \rangle$ exemplifying this so $\alpha'_\eta \le \lambda$ and let $R^G_\eta = \{((\eta \restriction \varepsilon, j), (\eta, i)) : i < \alpha'_\eta$, and $(\varepsilon, j) \in A'_{\eta,i}\}$.

The rest should be clear; for every $f : {}^{\kappa >}\lambda \to \lambda$ letting $\langle \nu_\rho : \rho \in {}^{\kappa \ge}\lambda \rangle$ be as in $\boxplus_1$ above, for some $\rho \in {}^\kappa\lambda, \nu_\rho$ is G^*-reasonable.

<u>Case 3</u>: $\mu = \kappa$.

Left to the reader (as after Case 1,2 it should be clear).

3) As in the proof of 2.6(4), it follows that there is $\bar{f}$ as needed. $\square_{4.3}$

Claim 4.4. *1)* $\mathrm{NQr}_{\mathrm{st}}(2^\kappa, \kappa, 2^\kappa, \kappa)$.
2) If $\lambda = \theta = 2^\kappa$ <u>then</u> in $\mathfrak{H}_{\lambda,\theta,\kappa}$ there is no ste-universal even for members of $\mathfrak{H}^{\mathrm{sbp}}_{\lambda,\theta,\kappa}$.

Proof. 1) Think.
2) By part (1) and 4.3. $\square_{4.4}$

Claim 4.5. *1) Assume $\kappa < \lambda$ and $\mathrm{Qr}_x(\lambda, 1, \lambda, \kappa)$ and $\mathfrak{H}^{\mathrm{sbp}}_{(\lambda,\kappa),\lambda,\kappa} \ne \emptyset$, <u>then</u> $\mathfrak{H}^{\mathrm{sbp}}_{(\lambda,\kappa),\lambda,\kappa} = \mathfrak{H}^{\mathrm{bp}}_{(\lambda,\kappa),\lambda,\kappa}$ has a x-universal member.*

Proof. Read the definitions. $\square_{4.5}$

5. Independence results on existence of large almost disjoint families

This deals with a question of Shafir

Definition 5.1. 1) Let $\mathrm{Pr}_2(\mu, \kappa, \theta, \sigma)$ mean: there is $\mathcal{A} \subseteq [\kappa]^\kappa$ such that

- (a) $|\mathcal{A}| = \mu$
- (b) if $A_i \in \mathcal{A}$ for $i < \theta$ and $i \neq j \Rightarrow A_i \neq A_j$ then $|\bigcap\limits_{i<\theta} A_i| < \sigma$.

If we omit σ we mean κ, if we omit μ we mean 2^κ.

Claim 5.2. $\mathrm{Pr}_2(-,-,-,-)$ *has obvious monotonicity properties.*

Claim 5.3. *Assume*

- $(*)$ $\sigma = \sigma^{<\sigma} < \kappa = \kappa^\sigma = \mathrm{cf}(\kappa)$ *and* $(\forall \alpha < \kappa)(|\alpha|^{<\sigma} < \kappa)$ *and* $2^\kappa = \kappa^+ < \chi$ *(so* $\kappa^{++} \leq \chi$*).*

<u>Then</u> *for some forcing notion* $\mathbb{P}$

- (a) $|\mathbb{P}| = \chi^\kappa$
- (b) $\mathbb{P}$ *satisfies the* κ^{++}*-c.c.*
- (c) $\mathbb{P}$ *is* σ*-complete*
- (d) $\mathbb{P}$ *neither collapses cardinals nor changes cofinalities*
- (e) *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $2^\sigma = \chi^\sigma, 2^\kappa = \chi^\kappa$
- (f) *in* $\mathbf{V}^{\mathbb{P}}$ *we have* $\mathrm{Pr}_2(\chi, \kappa, \sigma^+, \sigma)$ *but* $\neg\mathrm{Pr}_2(\kappa^{++}, \kappa, \kappa, \theta)$ *for* $\theta < \sigma$ *recalling 5.2.*

Proof. The forcing is as in a special case of the $\mathbb{Q}$ one in [She12, §2], see history there. Let E be the following equivalence relation on χ

$$\alpha E \beta \Rightarrow \alpha + \kappa = \beta + \kappa.$$

We define the partial order $\mathbb{P} = (P, \leq)$ by

$$\begin{aligned} P = \{f : \quad & f \text{ is a partial function from } \chi \text{ to } \{0,1\} \\ & \text{with domain of cardinality } \leq \kappa \text{ such that} \\ & (\forall \alpha < \chi)(|\mathrm{Dom}(f) \cap (\alpha/E)| < \sigma)\} \end{aligned}$$

$$\begin{aligned} f_1 \leq f_2 \ \underline{\text{iff}} \quad & f_1, f_2 \in P, f_1 \subseteq f_2 \text{ and} \\ & \sigma > |\{\alpha \in \mathrm{Dom}(f_1) : f_1 \restriction (\alpha/E) \neq f_2 \restriction (\alpha/E)\}|. \end{aligned}$$

We define two additional partial orders on P:

$$\begin{aligned} f_1 \leq_{\mathrm{pr}} f_2 \ \underline{\text{iff}} \quad & f_1 \subseteq f_2 \text{ and } f_1, f_2 \in \mathbb{P} \text{ and} \\ & (\forall \alpha \in \mathrm{Dom}(f_1))[f_1 \restriction (\alpha/E) = f_2 \restriction (\alpha/E)]. \end{aligned}$$

$$f_1 \leq_{\mathrm{apr}} f_2 \text{ iff } f_1, f_2 \in \mathbb{P}, f_1 \leq f_2 \text{ and } \mathrm{Dom}(f_2) \subseteq \cup\{\alpha/E : \alpha \in \mathrm{Dom}(f_1)\}.$$

We know (see there)

$(*)_0$ $\mathbb{P}$ is κ^{++}-c.c., $|\mathbb{P}| = \chi^\kappa$

$(*)_1$ $\mathbb{P}$ is σ-complete and $(P, \leq_{\rm pr})$ is κ^+-complete, in both cases the union of an increasing sequence forms an upper bound

$(*)_2$ for each $p, \mathbb{P} \restriction \{q : p \leq_{\rm apr} q\}$ is σ^+-c.c. of cardinality $\kappa^\sigma = \kappa$ and σ-complete

$(*)_3$ if $p \leq r$ then for some q, q' we have $p \leq_{\rm apr} q \leq_{\rm pr} r$ and $p \leq_{\rm pr} q' \leq_{\rm apr} r$; moreover q is unique we denote it by $\mathrm{inter}(p, r)$

$(*)_4$ if $p \Vdash$ "$\underset{\sim}{\tau} \in {}^\kappa\mathrm{Ord}$" then for some q we have
 - (a) $p \leq_{\rm pr} q$
 - (b) if $\alpha < \kappa$ and $q \leq r$ and $r \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\tau}(\alpha) = \beta$" then $\mathrm{inter}(q, r) \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\tau}(\alpha) = \beta$".

This gives that clauses (a),(b),(c),(d) of the conclusion hold. As for clause (e), $2^\kappa \leq \chi^\kappa$ follows from $(*)_5 + |\mathbb{P}| = \chi^\kappa$ and $2^\sigma \leq \chi^\sigma$, too.

Define:

$(*)_5$ (a) $\underset{\sim}{f} := \cup\{p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$
 - (b) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} = \cup \underset{\sim}{G}_{\mathbb{P}}$ is a function from χ to $\{0,1\}$"
 - (c) for $\alpha < \chi$ let $\underset{\sim}{A}_\alpha = \{\gamma < \kappa : \underset{\sim}{f}(\kappa\alpha + \gamma) = 1\}$.

Also easily

$(*)_6$ if $p \Vdash$ "$\underset{\sim}{\tau} \subseteq \sigma$" then for some $u \in [\chi]^{\leq\sigma}$ and σ-Borel function $\mathbf{B} : {}^u2 \to {}^\sigma 2$ and q we have
 $p \leq_{\rm pr} q$
 $q \Vdash$ "$\underset{\sim}{\tau} = \mathbf{B}(\underset{\sim}{f} \restriction u)$"

$(*)_7$ $\Vdash$ "$\underset{\sim}{A}_\alpha \subseteq \kappa$ moreover $\gamma < \kappa \Rightarrow [\gamma, \gamma+\sigma) \cap \underset{\sim}{A}_\alpha \neq \emptyset$ and $[\gamma, \gamma+\sigma) \nsubseteq \underset{\sim}{A}_\alpha$ and $\alpha \neq \beta \Rightarrow \underset{\sim}{A}_\alpha \neq \underset{\sim}{A}_\beta$".

[Why? By density argument.]

$(*)_8$ $\mathcal{A} := \{\underset{\sim}{A}_\alpha : \alpha < \chi\}$ exemplifies $\mathrm{Pr}_2(\chi, \kappa, \sigma^+, \sigma)$.

Why? By $(*)_7 + (*)_5(c)$, $\mathcal{A} \subseteq [\kappa]^\kappa, |\mathcal{A}| = \chi$ so we are left with proving clause (b) of 5.1; its proof will take awhile. So toward contradiction assume that for some $p \in \mathbb{P}$ and $\langle \underset{\sim}{\beta}_\zeta : \zeta < \sigma^+\rangle$ we have

$\boxplus_1$ $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\beta}_\zeta < \chi, \underset{\sim}{\beta}_\zeta \neq \underset{\sim}{\beta}_\xi$ for $\zeta < \xi < \kappa^+$ and $|\bigcap_{\zeta<\sigma^+} \underset{\sim}{A}_{\beta_\zeta}| \geq \sigma$".

By induction on $\zeta \leq \sigma^+$ we choose $p_\zeta \in \mathbb{P}$ such that:

$\boxplus_2$ (α) $p_0 = p$
 - (β) p_ζ is $\leq_{\rm pr}$-increasing continuous
 - (γ) there is r'_ζ such that $p_{\zeta+1} \leq_{\rm apr} r'_\zeta$ and $r'_\zeta \Vdash \underset{\sim}{\beta}_\zeta = \beta^*_\zeta$
 - (δ) $\mathrm{Dom}(p_{\zeta+2}) \cap (\beta^*_\zeta/E) \neq \emptyset$.

No problem because $(P, \leq_{\rm pr})$ is κ^+-complete and $\sigma^+ < \kappa^+$ and $(*)_3$.

Let[6] $q = p_{\sigma^+}$.

We can find r_ζ, β^*_ζ for $\zeta < \sigma^+$ such that $q \leq_{\rm apr} r_\zeta$ and $r_\zeta \Vdash \underset{\sim}{\beta}_\zeta = \beta^*_\zeta$. Let $u_\zeta = \mathrm{Dom}(r_\zeta) \setminus \mathrm{Dom}(q)$, so $|u_\zeta| < \sigma$ by the definition of $\leq_{\rm apr}$. By the Δ-system

[6]if we define $\mathbb{P}$ such that it is only κ-complete, we first choose $y_2, u_3, u^*, v_\zeta, v^*$ and then $q = p_{\zeta(*)}$ for $\zeta(*) = \min\{\zeta < \sigma^+ : |\zeta \cap y| = \sigma\}$

lemma, recalling $\sigma = \sigma^{<\sigma}$ there is $Y \subseteq \sigma^+, |Y| = \sigma^+$ and u^* such that for $\zeta < \xi$ from Y we have $u_\zeta \cap u_\xi = u^*$. Without loss of generality $\zeta \in Y \Rightarrow r_\zeta \restriction u^* = r^*$. Let $v_\zeta = \{\gamma < \kappa : \kappa\beta^*_\zeta + \gamma \in \mathrm{Dom}(r_{\zeta+2})\}$ so $v_\zeta \in [\kappa]^{<\sigma}$.

Possibly further shrinking Y without loss of generality

$$\zeta \neq \xi \text{ from } Y \Rightarrow v_\zeta \cap v_\xi = v^*.$$

So $v^* \in [\kappa]^{<\sigma}$ (in fact follows).

Let $\zeta(*) = \mathrm{Min}(Y)$.

We claim

$$r_{\zeta(*)} \Vdash \text{“} \bigcap_{\zeta<\sigma^+} \underset{\sim}{A}_{\underset{\sim}{\beta}_\zeta} \subseteq v^* \text{”}.$$

As $p \le r_{\zeta(*)}$ this suffices.

Toward contradiction assume that r, α are such that

$\boxplus_3$ $r_{\zeta(*)} \le r \in \mathbb{P}$ and $r \Vdash$ “$\alpha \in \bigcap_{\zeta<\sigma^+} \underset{\sim}{A}_{\underset{\sim}{\beta}_\zeta}$”.

Recall clause (δ) of $\boxplus_2$ (and $\zeta < \sigma^+ \Rightarrow p_\zeta \le_{\mathrm{pr}} p_{\sigma^+} = q$), we know $\zeta < \sigma^+ \Rightarrow \mathrm{Dom}(q) \cap (\beta^*_\zeta/E) \neq \emptyset$ and, of course, $q \le r_{\zeta(*)} \le r$ so by the definition of $\le$ in $\mathbb{P}$, for every $\xi < \sigma^+$ large enough

$\boxplus_4$ (a) $(\beta^*_\xi/E) \cap \mathrm{Dom}(r_\xi)\backslash\mathrm{Dom}(r_{\zeta(*)}) = \emptyset$ hence
(b) r, r_ξ are compatible functions (hence conditions)
(c) $\alpha \notin v_\zeta$.

Let $r^+ = r \cup r_\zeta \cup \{\langle \kappa\beta^*_\zeta + \alpha, 0\rangle\}$.

So easily

$\boxplus_5$ (a) $r \le r^+ \in \mathbb{P}$
(b) $r_\zeta \le r^+$ hence $r^+ \Vdash$ ”$\underset{\sim}{\beta}_\zeta = \beta^*_\zeta$”
(c) $r^+ \Vdash$ “$\alpha \notin \underset{\sim}{A}_{\underset{\sim}{\beta}^*_\zeta}$”.

So we have gotten a contradiction thus proving $(*)_8$.

$(*)_9$ $\Vdash$ “$\neg\mathrm{Pr}_2(\kappa^{++}, \kappa, \theta, \theta)$ if $\theta < \sigma$”.

Why? So toward contradiction suppose $p^* \Vdash$ “$\underset{\sim}{\mathcal{A}} = \{\underset{\sim}{B}_\alpha : \alpha < \kappa^{++}\} \subseteq [\kappa]^\kappa$ exemplifies $\mathrm{Pr}_2(\kappa^{++}, \kappa, \theta, \theta)$”.

For each $\alpha < \kappa^{++}$ we can find p_α such that $\bar{p}, \bar{r}$ and then S, q:

$\circledast_1$ (a) $\bar{p} = \langle p_{\alpha,i} : i \le \kappa\rangle$ is $\le_{\mathrm{pr}}$-increasing continuous (in $\mathbb{P}$)
(b) $p_0 = p^*$
(c) $p_{\alpha,i} \le r_{\alpha,i}$ and $r_{\alpha,i} \Vdash_{\mathbb{P}}$ “$\gamma_{\alpha,i} \in \underset{\sim}{B}_\alpha\backslash i$”
(d) $p_{\alpha,i} \le_{\mathrm{pr}} p_{\alpha,i+1} \le_{\mathrm{ap}} r_{\alpha,i}$
(e) $S_\alpha \subseteq \kappa$ is stationary
(f) $\langle v_{\alpha,i} := \mathrm{Dom}(r_{\alpha,i})\backslash \mathrm{Dom}(p_{\alpha,i+1}) : i \in S\rangle$ is a Δ-system with heart v_α, so $|v_\alpha| < \sigma$
(g) $\langle r_{\alpha,i}{\restriction}v_\alpha : i \in s\rangle$ is constantly r_α
(h) $p_\alpha = p_{\alpha,\kappa} \cup r_\alpha \in \mathbb{P}$ so $p_\kappa \le q$.

Let $u_\alpha = \cup\{\beta/E : \beta \in \mathrm{Dom}(p_\alpha)\}$ so $u_\alpha \in [\chi]^{\le\kappa}$ and $\mathrm{Dom}(r_{\alpha,i}) \subseteq u_\alpha$ for $i < \kappa$. For some $Y \in [\kappa^{++}]^{\kappa^{++}}$ and $\Upsilon^* < \kappa^+$ and stationary $S \subseteq \kappa$ and $\bar\gamma = \langle \gamma_i : i \in S\rangle$ we have

$\circledast_2$ if $\alpha \in Y$ then $\mathrm{otp}(u_\alpha) = \Upsilon^*$ and $S_\alpha = S$ and $\langle \gamma_{\alpha,i} : i \in S\rangle = \bar\gamma$.

Let $g_{\alpha,\beta}$ be the order preserving function from u_β onto u_α.

Again as $2^\kappa = \kappa^+$ without loss of generality

$\circledast_3$ For $\alpha, \beta \in Y$
- (a) $p_\beta = p_\alpha \circ g_{\alpha,\beta}$ and $p_{\beta,i} = p_{\alpha,i} \circ g_{\alpha,\beta}$ and $r_{\beta,i} = r_{\alpha,i} \circ g_{\alpha,\beta}$ for $i < \kappa$
- (b) $u_\alpha \cap u_\beta = u_*$ for $\alpha < \beta < \kappa^{++}$
- (c) $g_{\alpha,\beta}$ is the identity on u_*

and

$\circledast_4$ if $p_\alpha \le_{\rm apr} r_\alpha, p_\beta \le_{\rm apr} r_\beta, r_\beta = r_\alpha \circ g_{\alpha,\beta}$ and $\gamma < \kappa$ then
- (a) $r_\alpha \Vdash$ "$\gamma \in \underset{\sim}{B}_\alpha$" $\Leftrightarrow r_\beta \Vdash$ "$\gamma \in \underset{\sim}{B}_\gamma$"
- (b) $r_\alpha \Vdash$ "$\gamma \notin \underset{\sim}{B}_\alpha$" $\Leftrightarrow r_\beta \Vdash$ "$\gamma \notin \underset{\sim}{B}_\gamma$".

Choose $\langle \beta_\varepsilon : \varepsilon < \kappa^{++}\rangle$ an increasing sequence of ordinals from Y.

Let $p^* = \bigcup\limits_{\varepsilon<\theta} p_{\beta_\varepsilon}$ and $\zeta(*) = \beta_\theta$.

So:

$\circledast_5$ (a) $\mathbb{P} \models$ "$p^* \le p_{\beta_\varepsilon} < q^*$" for $\varepsilon < \theta$
- (b) if $p^* \le q$ then for each $\varepsilon < \theta$ for every large enough $i \in S$, the conditions $q, r_{\beta_\varepsilon,i}$ are compatible hence
- (c) if $q^* \le q$ then for every large enough $i \in S$ the universal $q^+ = q \cup \{r_{\beta_\varepsilon,i} : \varepsilon < \theta\}$ is a well defined function, belongs to $\mathbb{P}$ and is common upper bound of $\{r_{\beta_\varepsilon,i} : \varepsilon < \theta\} \cup q$ hence force $\gamma_i \in B_{\beta_\varepsilon}$ for $\varepsilon < \theta$.

As $\gamma_i \ge i$ clearly

$\circledast_6$ $q^* \Vdash$ "$\bigcap\limits_{\varepsilon<\theta} B_{\beta_\varepsilon}$ is unbounded in κ hence has cardinality κ".

$\square_{5.3}$

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