CONTINUOUS IMAGES OF SETS OF REALS

TOMEK BARTOSZYNSKI AND SAHARON SHELAH

ABSTRACT. We will show that, consistently, every uncountable set can be continuously mapped onto a non measure zero set, while there exists an uncountable set whose all continuous images into a Polish space are meager.

1. INTRODUCTION

Let $\mathcal{J}$ be a σ-ideal of subsets of a Polish space Y. Assume also that $\mathcal{J}$ contains singletons and has Borel basis. Let $\mathsf{non}(\mathcal{J}) = \min\{|X| : X \subseteq Y \ \& \ X \notin \mathcal{J}\}$.

In this paper we are concerned with the family

$$\mathsf{NON}(\mathcal{J}) = \{X \subseteq \mathbb{R} : \text{for every continuous mapping } F : X \longrightarrow Y,\ F``(X) \in \mathcal{J}\}.$$

Note that $\mathsf{NON}(\mathcal{J})$ contains all countable sets. Moreover, $\mathsf{NON}(\mathcal{J})$ is closed under countable unions but need not be downward closed, thus it may not an ideal. However, $\mathsf{NON}(\mathcal{J})$ is contained in the σ-ideal

$$\mathsf{NON}^\star(\mathcal{J}) = \{X \subseteq \mathbb{R} : \text{for every continuous mapping } F : \mathbb{R} \longrightarrow Y,\ F``(X) \in \mathcal{J}\}.$$

It is also not hard to see that $\mathsf{NON}^\star(\mathcal{J})$ consists of those sets whose uniformly continuous images are in $\mathcal{J}$.

Let $\mathcal{N}$ be the ideal of measure zero subsets of 2^ω with respect to the standard product measure μ, and let $\mathcal{M}$ be the ideal of meager subsets of 2^ω (or other Polish space Y).

We will show that

- $\mathsf{ZFC} \vdash \mathsf{NON}(\mathcal{M})$ contains an uncountable set.
- It is consistent that $\mathsf{NON}^\star(\mathcal{N}) = \mathsf{NON}(\mathcal{N}) = [\mathbb{R}]^{\leq\aleph_0}$.
- It is consistent that: $\mathsf{NON}^\star(\mathcal{J}) = [\mathbb{R}]^{\leq\aleph_0} \iff \mathsf{non}(\mathcal{J}) < 2^{\aleph_0}$.

Observe that if $\mathsf{NON}^\star(\mathcal{J}) = [\mathbb{R}]^{\leq\aleph_0}$ then $\mathsf{non}(\mathcal{J}) = \aleph_1$. On the other hand for all σ-ideals $\mathcal{J}$ considered in this paper, continuum hypothesis implies that $\mathsf{NON}(\mathcal{J})$ contains uncountable sets.

Finally notice that one can show in ZFC that there exists an uncountable universal measure zero set (see [7]), i.e. a set whose all homeomorphic (or even Borel isomorphic) images are all of measure zero. Therefore one cannot generalize the consistency results mentioned above by replacing the word "continuous" by "homeomorphic" in the definition of $\mathsf{NON}(\mathcal{N})$.

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2. Category

In this section we show that $\mathsf{NON}(\mathcal{M})$ contains an uncountable set. This was proved in [9], the proof presented here gives a slightly stronger result.

For $f, g \in \omega^\omega$ let $f \leq^\star g$ mean that $f(n) \leq g(n)$ for all but finitely many n. Let

$$\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega \ \& \ \forall h \in \omega^\omega \ \exists f \in F \ f \not\leq^\star h\}.$$

Theorem 1 ([9]). *There exists a set $X \subseteq \mathbb{R}$ of size $\mathfrak{b}$ such that*

(1) *every continuous image of X into ω^ω is bounded,*
(2) *every continuous image of X into a Polish space is meager,*
(3) *if $\mathfrak{b} \leq \mathsf{non}(\mathcal{N})$ then every continuous image of X into $\mathbb{R}$ has measure zero.*

Proof Let $Z \subseteq (\omega+1)^\omega$ consist of functions f such that

(1) $\forall n \ f(n) \leq f(n+1)$,
(2) $\forall n \ \big(f(n) < \omega \rightarrow f(n) < f(n+1)\big)$.

Note that Z is a compact subset of $(\omega+1)^\omega$ thus it is homeomorphic to 2^ω. For an increasing sequence $s \in (\omega+1)^{<\omega}$ let $q_s \in (\omega+1)^\omega$ be defined as

$$q_s(k) = \begin{cases} s(k) & \text{if } k < |s| \\ \omega & \text{otherwise} \end{cases} \quad \text{for } k \in \omega.$$

Note that the set $\mathbb{Q} = \{q_s : s \in \omega^{<\omega}\}$ is dense in Z. Put $X' = \{f_\alpha : \alpha < \mathfrak{b}\}$ such that

(1) $\forall \alpha \ f_\alpha \in Z$,
(2) $f_\alpha \leq^\star f_\beta$ for $\alpha < \beta$,
(3) $\forall f \in \omega^\omega \ \exists \alpha \ f_\alpha \not\leq^\star f$.

Let $X = X' \cup \mathbb{Q}$. We will show that X is the set we are looking for.

(1) Suppose that $F : X \longrightarrow \omega^\omega$ is continuous. We only need to assume that F is continuous on $\mathbb{Q}$. Without loss of generality we can assume for every $x \in X$, $F(x) \in \omega^\omega$ is an increasing function.

Lemma 2. *There exists a function $g \in \omega^\omega$ such that for every $x \in X$ and $n \in \omega$,*

$$F(x)(n) \leq g(n) \ \text{if } x(n) > g(n).$$

Proof Fix $n \in \omega$ and for each $s \in (\omega+1)^n$ let $I_s \subseteq (\omega+1)^\omega$ be a basic open set containing q_s such that for $x \in \mathsf{dom}(F) \cap I_s$, $F(x)(n) = F(q_s)(n)$. For every s the set $I_s{\restriction}n = \{x{\restriction}n : x \in I_s\}$ is open $(\omega+1)^n$ and the family $\{I_s{\restriction}n : s \in (\omega+1)^n\}$ is a cover of $(\omega+1)^n$. By compactness there are sequences $s_1, \dots, s_k$ such that $(\omega+1)^n = I_{s_1}{\restriction}n \cup \dots \cup I_{s_k}{\restriction}n$. Find N so large that if $x(n) > N$ then $x \in I_{s_j}$ for some $j \leq k$. Define

$$g(n) = \max\{N, F(q_{s_1})(n), \dots, F(q_{s_k})(n)\}. \quad \square$$

Let $g \in \omega^\omega$ be the function from the above lemma. Find α_0 such that $f_{\alpha_0} \not\leq^\star g$. Let $\{u_n : n \in \omega\}$ be an increasing enumeration of $\{n : g(n) < f_{\alpha_0}(n)\}$. Put $h(n) = g(u_n)$ for $n \in \omega$ and note that for $\beta > \alpha_0$ and sufficiently large n we have

$$F(f_\beta)(n) \leq F(f_\beta)(u_n) < g(u_n) = h(n).$$

Since the set $\{F(f_\beta) : \beta \leq \alpha_0\} \cup \{F(q_s) : s \in \omega^{<\omega}\}$ has size $< \mathfrak{b}$ we conclude that $F``(X)$ is bounded in ω^ω.

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(2) Suppose that F is a continuous mapping from X into a Polish space Y with metric ρ. Observe that F is not onto and fix a countable dense set $\{q_n : n \in \omega\}$ disjoint with $F"(X)$. For $x \in X$ let $f_x \in \omega^\omega$ be defined as

$$f_x(n) = \min\left\{k : \rho(f(x), d_n) > \frac{1}{k}\right\}.$$

In particular,

$$f(x) \notin B\left(d_n, \frac{1}{f_x(n)}\right) = \left\{z : \rho(d_n, z) < \frac{1}{f_x(n)}\right\}.$$

Note that the mapping $x \mapsto f_x$ is continuous and find a function $h \in \omega^\omega$ such that $f_x \leq^\star h$ for $x \in X$. Put

$$G = \bigcap_m \bigcup_{n>m} B\left(d_n, \frac{1}{h(n)}\right)$$

and note that G is a comeager set disjoint from $F"(X)$.

(3) Let $\mathbb{Q} \subseteq U$ be an open set. Define $g \in \omega^\omega$ as

$$g(0) = \min\{k : \forall x\ x(0) > k \to x \in U\}$$

and for $n > 0$

$$g(n) = \min\left\{k : \forall x\ \Big(\big(\forall j < n\ x(j) < g(j)\ \&\ x(n) > k\big) \to x \in U\Big)\right\}.$$

Let α_0 be such that $f_{\alpha_0} \not\leq^\star g$. It follows that $f_\beta \in U$ for $\beta > \alpha_0$.

Suppose that $F : X \longrightarrow \mathbb{R}$ is continuous (on $\mathbb{Q}$). Let $\{q_n : n \in \omega\}$ be enumeration of $\mathbb{Q}$. Let $I^k_n \ni q_n$ be a basic open set such that $F"(I^k_n)$ has diameter $< 2^{-n-k}$. Put $H = \bigcap_k \bigcup_n I^k_n$. It is clear that $F"(H)$ has measure zero. Fix α_0 such that for all $\beta > \alpha_0$, $f_\beta \in H$. It follows that for $\beta > \alpha_0$, $F(f_\beta)$ belongs to a measure zero set $F"(H)$. By the assumption, the remainder of the set $F"(X)$ has size $< \mathsf{non}(\mathcal{N})$ which finishes the proof. $\square$

The set X constructed above is not hereditary, and for example $X \setminus \mathbb{Q}$ can be continuously mapped onto an unbounded family. A hereditary set having property (1) of theorem 1 cannot be constructed in ZFC. Miller showed in [6] that it is consistent that every uncountable set has a subset that can be mapped onto an unbounded family. This holds in a model where there are no σ-sets, i.e. every uncountable set has a G_δ subset which is not F_σ.

It is open whether a hereditary set having property (2) of theorem 1 can be constructed in ZFC.

3. Making NON($\mathcal{J}$) small.

We will start with the following:

Definition 3. *A set $X \subseteq 2^\omega$ has strong measure zero if for every function $g \in \omega^\omega$ there exists a function $f \in (\omega^{<\omega})^\omega$ such that $f(n) \in 2^{g(n)}$ for every n and*

$$\forall x \in X\ \exists^\infty n\ x{\restriction}g(n) = f(n).$$

Let $\mathcal{SN}$ denote the collection of all strong measure zero sets.

If the above property fails for some g then we say that g witnesses that $X \notin \mathcal{SN}$.

For $g \in \omega^\omega$ we will define a forcing notion $\mathbb{P}_g$ such that:

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(1) $\mathbb{P}_g$ is proper,
(2) there exists a family $\{F_n : n \in \omega\} \in \mathbf{V}^{\mathbb{P}_g}$ such that
   (a) $\forall n \; F_n : 2^\omega \longrightarrow 2^\omega$ is continuous,
   (b) if $X \subseteq 2^\omega$, $X \in \mathbf{V}$ and g witnesses that $X \notin \mathcal{SN}$ then

$$\mathbf{V}^{\mathbb{P}_g} \models \bigcup_n F_n``(X) = 2^\omega.$$

Let $\mathbb{L}$ be the Laver forcing and suppose that g is a Laver real over $\mathbf{V}$. It is well known that:

Lemma 4. [5] *If $X \subseteq 2^\omega$, $X \in \mathbf{V}$ is uncountable then $\mathbf{V}[g] \models$"g witnesses that X does not have strong measure zero."*

Theorem 5. *It is consistent with* ZFC *that for every σ-ideal $\mathcal{J}$*

$$\mathsf{NON}(\mathcal{J}) = [\mathbb{R}]^{\leq \aleph_0} \iff \mathsf{non}(\mathcal{J}) < 2^{\aleph_0}.$$

Proof Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration such that $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbb{L} \star \mathbb{P}_{\dot{g}}$ for $\alpha < \omega_2$. Suppose that $\mathcal{J}$ is a σ-ideal and $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{non}(\mathcal{J}) = \aleph_1$. It follows that for some $\alpha < \omega_2$, $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathbf{V}^{\mathcal{P}_\alpha} \cap 2^\omega \notin \mathcal{J}$.

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_{\omega_2}} \cap 2^\omega$ is uncountable. Let $\beta > \alpha$ be such that $X \cap \mathbf{V}^{\mathcal{P}_\beta}$ is uncountable. In $\mathbf{V}^{\mathcal{P}_\beta \star \mathbb{L}}$ the Laver real witnesses that $X \notin \mathcal{SN}$ and so $\mathbf{V}^{\mathcal{P}_{\beta+1}} \models \bigcup_n F_n``(X) = 2^\omega$. Hence $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \bigcup_n F_n``(X) \notin \mathcal{J}$ which means that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \exists n \in \omega \; F_n``(X) \notin \mathcal{J}$. $\square$

4. Defintion of $\mathbb{P}_g$

Let us fix the following notation. Suppose that $\langle F_n : n \in \omega \rangle$ are nonempty sets. Let $T^{max} = \bigcup_n \prod_{j=0}^{n-1} F_j$. For a tree $T \subseteq T^{max}$ let $T{\restriction}n = T \cap \prod_{j=0}^{n-1} F_j$. For $t \in T{\restriction}n$ let $\mathsf{succ}_T(t) = \{x \in F_n : t^\frown x \in T\}$ be the set of all immediate successors of t in T, and let $T_t = \{s \in T : s \subseteq t \text{ or } t \supseteq s\}$ be the subtree determined by t. Let $\mathsf{stem}(T)$ be the shortest $t \in T$ such that $|\mathsf{succ}_T(t)| > 1$.

Fix a sequence $\langle \varepsilon^k_j : j \leq k \}$ such that

(1) $\forall k \; 0 < \varepsilon^k_0 < \varepsilon^k_1 < \cdots < \varepsilon^k_k < 2^{-k}$.
(2) $\forall k \; \forall j < k \; (\varepsilon^k_{j+1})^3 > \varepsilon^k_j$.
(3) $\forall k \; \forall j < k \; \frac{\varepsilon^k_{j+1}}{2^{k^2}} > \varepsilon^k_j$.
(4) $\forall k \; \forall j < k \; \frac{\varepsilon^k_j}{\varepsilon^k_{j+1}} < \varepsilon^k_k$.

For example $\varepsilon^k_j = 2^{-k^2 4^{k-j}}$ for $j \leq k$ will work.

Suppose that a strictly increasing function $g \in \omega^\omega$ is given. Fix an increasing sequence $\langle n_k : k \in \omega \rangle$ such that $n_0 = 0$ and

$$n_{k+1} \geq g\left((k+1) 2^{n_k} \frac{2^{n_k 2^{n_k}}}{\varepsilon^k_0}\right) \text{ for } k \in \omega.$$

For the choice of ε^k_j above $n_{k+1} = g\left(2^{1999^{n_k}}\right)$ will be large enough.

Let $F_k = \{f : \mathsf{dom}(f) = 2^{[n_k, n_{k+1})}, \mathsf{range}(f) = \{0,1\}\}$. For $A \subseteq F_k$ let

$$\|A\| = \max\left\{\ell : \frac{|A|}{|F_k|} \geq \varepsilon^k_\ell\right\}.$$

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Consider the tree

$$T^{max} = \bigcup_k \prod_{j=0}^{k} F_j.$$

Let $\mathbb{P}_g$ be the forcing notion which consists of perfect subtrees $T \subseteq T^{max}$ such that

$$\lim_{k\to\infty} \min\{\|\mathsf{succ}_T(s)\| : s \in T{\restriction}k\} = \infty.$$

For $T, S \in \mathbb{P}_g$ and $n \in \omega$ define $T \geq S$ if $T \subseteq S$ and $T \geq_n S$ if $T \geq S$ and

$$\forall s \in S \ \big(\|\mathsf{succ}_S(s)\| \leq n \rightarrow \mathsf{succ}_S(s) = \mathsf{succ}_T(s)\big).$$

It is easy to see that $\mathbb{P}_g$ satisfies Axiom A, thus it is proper.

Suppose that $G \subseteq \mathbb{P}_g$ is a generic filter over $\mathbf{V}$. Let $\bigcap G = \langle f_0, f_1, f_2, \ldots\rangle \in \prod_k F_k$. Define $F_G : 2^\omega \longrightarrow 2^\omega$ as

$$F_G(x)(k) = f_k\big(x{\restriction}[n_k, n_{k+1})\big) \text{ for } x \in 2^\omega,\ k \in \omega.$$

First we show that $\mathbb{P}_g$ is ω^ω-bounding. The arguments below are rather standard, we reconstruct them here for completeness but the reader familiar with [8] will see that they are a part of a much more general scheme.

Lemma 6. *Suppose that $I \subseteq \mathbf{V}$ is a countable set, $n \in \omega$ and $T \Vdash_{\mathbb{P}_g} \dot{a} \in I$. There exists $S \geq_n T$ and $k \in \omega$ such that for every $t \in S{\restriction}k$ there exists $a_t \in I$ such that $S_t \Vdash_{\mathbb{P}_g} \dot{a} = a_t$.*

Proof Let $S \subseteq T$ be the set of all $t \in T$ such that T_t satisfies the lemma. In other words

$$S = \{t \in T : \exists k_t \in \omega\ \exists T' \geq_n T_t\ \forall s \in T'{\restriction}k_t\ \exists a_s \in I\ T'_s \Vdash_{\mathbb{P}_g} \dot{a} = a_s\}.$$

We want to show that $\mathsf{stem}(T) \in S$. Notice that if $s \notin S$ then

$$\|\mathsf{succ}_S(s)\| \leq \varepsilon_n^{|s|}.$$

Suppose that $\mathsf{stem}(T) \notin S$ and by induction on levels build a tree $\bar{S} \geq_n T$ such that for $s \in \bar{S}$,

$$\mathsf{succ}_{\bar{S}}(s) = \begin{cases} \mathsf{succ}_T(s) & \text{if } \|\mathsf{succ}_T(s)\| \leq n \\ \mathsf{succ}_T(s) \setminus \mathsf{succ}_S(s) & \text{otherwise} \end{cases}.$$

Clearly $\bar{S} \in \mathbb{P}_g$ since $\|\mathsf{succ}_{\bar{S}}(s)\| \geq \|\mathsf{succ}_{\bar{T}}(s)\| - 1$ for s containing $\mathsf{stem}(T)$. That is a contradiction since $\bar{S} \cap S = \emptyset$ which is impossible. □

In our case we have even stronger fact:

Lemma 7. *Suppose that $T \Vdash_{\mathbb{P}_g} \dot{A} \subseteq 2^{<\omega}$. There exists $S \geq T$ such that for all but finitely many n, for every $t \in S{\restriction}n$ there exists $A_t \subseteq 2^n$ such that $S_t \Vdash_{\mathbb{P}_g} \dot{A}{\restriction}n = A_t$.*

In particular, if $T \Vdash_{\mathbb{P}_g} \dot{x} \in 2^\omega$ then there exists $S \geq T$ such that for every for all but finitely many n, for every $t \in S{\restriction}n$ there exists $s_t \in 2^n$ such that $S_t \Vdash_{\mathbb{P}_g} \dot{x}{\restriction}n = s_t$.

Proof It is enough to prove the first part. By applying lemma 6 we can assume that there exists an increasing sequence $\langle k_n : n \in \omega\rangle$ such that for every $t \in T{\restriction}k_n$ there exists $A_t \subseteq 2^{<n}$ such that $T_t \Vdash_{\mathbb{P}_g} \dot{A}{\restriction}n = A_t$.

Let $n_0 = |\mathsf{stem}(T)|$. Build by induction a family of trees $\{T_{n,l} : n > n_0, n \leq l \leq k_n\}$ such that

$$\forall s \in T_{n,l}{\restriction}l\ \exists A_s \subseteq 2^n\ (T_{n,l})_s \Vdash_{\mathbb{P}_g} \dot{A}{\restriction}n = A_s.$$

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Let $T_{n_0+1,k_{n_0}} = T$ and suppose that $T_{n,l}$ has been constructed. If $l = n$ let $T_{n+1,k_{n+1}} = T_{n,n}$, otherwise construct $T_{n,l-1}$ as follows – by the induction hypothesis for $s \in T_{n,l}\restriction l-1$ and every $f \in \mathsf{succ}_{T_{n,l}}(s)$, there exists $A_{s^\frown f} \subseteq 2^n$ such that

$$(T_{n,l})_{s^\frown f} \Vdash_{\mathbb{P}_g} \dot{A}\restriction n = A_{s^\frown f}.$$

Fix A such that $\{f : A_{s^\frown f} = A\}$ has the largest size and put

$$\mathsf{succ}_{T_{n,l-1}}(s) = \begin{cases} \{f \in \mathsf{succ}_{T_{n,l}}(s) : A_{s^\frown f} = A\} & \text{if } s \in T_{n,l}\restriction l-1 \\ \mathsf{succ}_{T_{n,l}}(s) & \text{otherwise} \end{cases}$$

Finally let $S = \bigcap_n T_{n,n}$. Clearly S has the required property provided that it is a member of $\mathbb{P}_g$. Note that for an element $s \in S\restriction k$,

$$|\mathsf{succ}_S(s)| \geq \frac{|\mathsf{succ}_{T_{k,k}}(s)|}{2^{k^2}}.$$

By the choice of sequence $\langle \varepsilon^k_l : k, l\rangle$, it follows that $\|\mathsf{succ}_S(s)\| \geq \|\mathsf{succ}_T(s)\| - 1$ if $|s| > |\mathsf{stem}(S)|$. Thus $S \in \mathbb{P}_g$ which finishes the proof. $\square$

Next we show that $\mathbb{P}_g$ adds a continuous function which maps sets that do not have strong measure zero onto sets that are not in $\mathcal{J}$.

Let $\mathbb{Q} = \{x \in 2^\omega : \forall^\infty n\ x(n) = 0\}$ be the set of rationals in 2^ω.

Theorem 8. *Suppose that g witnesses that $X \subseteq \mathbf{V} \cap 2^\omega$, $X \in \mathbf{V}$ does not have strong measure zero. Then $\Vdash_{\mathbb{P}_g} F_{\dot{G}}``(X) + \mathbb{Q} = 2^\omega$.*

In particular,

$$V^{\mathbb{P}_g} \models \exists q \in \mathbb{Q}\ F_q``(X) \notin \mathcal{J},$$

where $F_q : 2^\omega \longrightarrow 2^\omega$ is defined as $F_q(x) = F_{\dot{G}}(x) + q$ for $x \in 2^\omega$.

PROOF We start with the following:

Lemma 9. *Suppose that $\frac{1}{2} > \varepsilon > 0$, $I \subseteq \omega$ is finite and $A \subseteq 2^{2^I}$, $\dfrac{|A|}{|2^{2^I}|} \geq \varepsilon$. Let*

$$Z = \left\{ s \in 2^I : \exists i_s \in \{0,1\}\ \frac{|\{f \in A : f(s) = i_s\}|}{|2^{2^I}|} < \varepsilon^3 \right\}.$$

Then $|Z| \leq \dfrac{1}{\varepsilon}$.

PROOF Suppose otherwise. By passing to a subset we can assume that $|Z| = \dfrac{1}{\varepsilon}$. Let

$$A' = \{f \in A : \forall s \in Z\ f(s) = i_s\}.$$

By the assumption

$$\frac{|A'|}{|2^{2^I}|} \geq \varepsilon - \frac{1}{\varepsilon} \cdot \varepsilon^3 = \varepsilon - \varepsilon^2 > \frac{\varepsilon}{2}.$$

On the other hand the sets $I_s = \{f \in 2^{2^I} : f(s) = 0\}$, $s \in 2^I$ are probabilistically independent and have "measure" $\dfrac{1}{2}$. It follows that

$$\frac{|A'|}{|2^{2^I}|} \leq \frac{1}{2^{\frac{1}{\varepsilon}}} < \frac{\varepsilon}{2},$$

which gives a contradiction. $\square$

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Lemma 10. *Suppose that $T \Vdash_{\mathbb{P}_g} \dot{z} \in 2^\omega$. There exists a sequence $\langle J_k : k \in \omega \rangle$ such that for every $k \in \omega$*

(1) $J_k \subseteq 2^{[n_k, n_{k+1})}$,

(2) $|J_k| \leq \dfrac{2^{n_k 2^{n_k}}}{\varepsilon_0^k}$,

and if $x \in \mathbf{V} \cap 2^\omega$ and $x \restriction [n_k, n_{k+1}) \notin J_k$ for all but finitely many k then there exists $S \geq T$ such that

$$S \Vdash_{\mathbb{P}_g} F_{\dot{G}}(x) =^\star \dot{z}.$$

Proof Suppose that $T \Vdash_{\mathbb{P}_g} \dot{z} \in 2^\omega$. Let $k_0 = |\mathsf{stem}(T)|$. By lemma 7, we can assume that

$$\forall k > k_0 \ \forall t \in T \restriction k \ \exists i_t \in \{0,1\} \ T_t \Vdash_{\mathbb{P}_g} \dot{z}(k) = i_t.$$

For $k > k_0$ and $s \in T \restriction k$ let

$$J_k^s = \{x \in 2^{[n_k, n_{k+1})} : \exists i \in \{0,1\} \ \|\{f \in \mathsf{succ}_T(s) : f(x) = i\}\| < \|\mathsf{succ}_T(s)\| - 1\}.$$

By lemma 9, $|J_k^s| \leq \frac{1}{\varepsilon_0^k}$. Put $J_k = \bigcup_{s \in T \restriction k} J_k^s$ and note that

$$|J_k| \leq \frac{1}{\varepsilon_0^k} \prod_{i=0}^{k-1} 2^{2^{n_{i+1} - n_i}} \leq \frac{2^{n_k 2^{n_k}}}{\varepsilon_0^k}.$$

Suppose that $x \restriction [n_k, n_{k+1}) \notin J_k$ for $k \geq k^\star \geq k_0$. Define $S \geq T$

$$\mathsf{succ}_S(t) = \begin{cases} \{f \in \mathsf{succ}_T(t) : f(x \restriction [n_k, n_{k+1}) = i_t\} & \text{if } s \in T \restriction k \text{ and } k > k^\star \\ \mathsf{succ}_T(s) & \text{otherwise} \end{cases}.$$

By the choice of x, $\|\mathsf{succ}_S(s)\| \geq \|\mathsf{succ}_T(s)\| - 1$ for $s \in S$. Thus $S \in \mathbb{P}_g$, and

$$S \Vdash_{\mathcal{P}} \forall k > k^\star \ F_{\dot{G}}(x)(k) = \dot{z}(k). \quad \square$$

Suppose that $T \Vdash_{\mathbb{P}_g} \dot{z} \in 2^\omega$ and let $\{J_k : k \in \omega\}$ be the sequence from lemma 10. Let

$$U = \{s \in 2^{<\omega} : \exists k \ |s| = n_{k+1} \ \& \ s \restriction [n_k, n_{k+1}) \in J_k\}.$$

Let $s_1, s_2, \dots$ be the list of elements of U according to increasing length. Note that by the choice of g, $|s_k| \geq g(k)$ for $k \in \omega$. Since g witnesses that $X \notin \mathcal{SN}$ (and any bigger function witnesses that as well) there is $x \in X$ such that

$$\forall^\infty k \ x \restriction \mathsf{dom}(s_k) \neq s_k.$$

Since initial parts of $s_k' s$ exhaust all possibilities it follows that for sufficiently large $k \in \omega$,

$$x \restriction [n_k, n_{k+1}) \neq s_l \restriction [n_k, n_{k+1}) \text{ for all } l \text{ such that } |s_l| = n_{k+1}.$$

In particular,

$$\forall^\infty k \ x \restriction [n_k, n_{k+1}) \notin J_k.$$

By lemma 10 we conclude that $\Vdash_{\mathbb{P}_g} F_{\dot{G}}(x) =^\star \dot{z}$. Since $\dot{z}$ was arbitrary, it follows that $\Vdash_{\mathbb{P}_g} F_{\dot{G}}"(X) + \mathbb{Q} = 2^\omega$. As $\mathcal{J}$ is a σ-ideal we conclude that

$$\Vdash_{\mathbb{P}_g} \exists q \in \mathbb{Q} \ F_{\dot{G}}"(X) + q \notin \mathcal{J}. \quad \square$$

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5. Measure

Theorem 5 is significant only if in the constructed model there are some interesting σ-ideals $\mathcal{J}$ such that $\mathsf{non}(\mathcal{J}) < 2^{\aleph_0}$. We will show some examples of such ideals, the most important being the ideal of measure zero sets $\mathcal{N}$.

Definition 11. *A family $\mathcal{A} \subseteq [\omega]^\omega$ is called a splitting family if for every infinite set $B \subseteq \omega$ there exists $A \in \mathcal{A}$ such that*

$$|A \cap B| = |(\omega \setminus A) \cap B| = \aleph_0.$$

We say that $\mathcal{A}$ is strongly non-splitting if for every $B \in [\omega]^\omega$ there exists $C \subseteq B$ which witnesses that $\mathcal{A}$ is not splitting.

Let

$$\mathsf{S} = \{X \subseteq [\omega]^\omega : X \text{ is strongly non-splitting}\}.$$

It is easy to see that S is a σ-ideal.

Theorem 12. *It is consistent that for every uncountable set $X \subseteq 2^\omega$ there exists a continuous function $F : 2^\omega \longrightarrow 2^\omega$ such that $F"(X)$ does not have measure zero. In particular, it is consistent that*

$$\mathsf{NON}(\mathcal{N}) = \mathsf{NON}(\mathsf{S}) = \mathcal{SN} = [\mathbb{R}]^{\leq\aleph_0}.$$

Proof Let $\mathbf{V}^{\mathcal{P}_{\omega_2}}$ be the model constructed in the proof of theorem 5. To show the first part it is enough to show that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{non}(\mathcal{N}) = \aleph_1$. Since is well known that $\mathsf{non}(\mathsf{S}) \leq \mathsf{non}(\mathcal{N})$, it follows that $\mathsf{NON}(\mathsf{S}) = [\mathbb{R}]^{\leq\aleph_0}$. This was known to be consistent (see [1]). Finally, it is well known that if $X \in \mathcal{SN}$ and $F : X \longrightarrow 2^\omega$ is uniformly continuous then $F"(X) \in \mathcal{SN} \subseteq \mathcal{N}$. Thus $\mathcal{SN} = [\mathbb{R}]^{\leq\aleph_0}$.

To finish the proof we have to show that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{non}(\mathcal{N}) = \aleph_1$. By theorem 6.3.13 of [2], in order to show that it suffices to show that both $\mathbb{P}_g$ and $\mathbb{L}$ satisfy certain condition (preservation of $\sqsubseteq^{random}$) which is an iterable version of preservation of outer measure. Theorem 7.3.39 of [2] shows that $\mathbb{L}$ satisfies this condition. Exactly the same proof works for $\mathbb{P}_g$ provided that we show:

Theorem 13. *If $X \subseteq 2^\omega$, $X \in \mathbf{V}$ and $\mathbf{V} \models X \notin \mathcal{N}$ then $\mathbf{V}^{\mathbb{P}_g} \models X \notin \mathcal{N}$.*

Proof The sketch of the proof presented here is a special case of a more general theorem (theorem 3.3.5 of [8]).

Fix $1 > \delta > 0$ and a strictly increasing sequence $\langle \delta_n : n \in \omega\rangle$ of real numbers such that

(1) $\sup_n \delta_n = \delta$.
(2) $\forall^\infty n\ \delta_{n+1} - \delta_n > \varepsilon^n_n$.

Suppose that $\Vdash_{\mathbb{P}_g} X \notin \mathcal{N}$. Without loss of generality we can assume that X is forced to have outer measure one. Let $\dot{A}$ be a $\mathbb{P}_g$-name such that $\Vdash_{\mathbb{P}_g} \dot{A} \subseteq 2^{<\omega}$ & $\mu([A]) \geq \delta$ and suppose that $T \Vdash_{\mathbb{P}_g} X \cap [\dot{A}] = \emptyset$. Let $n_0 = |\mathsf{stem}(T)|$. By lemma 7, we can assume that

$$\forall n > n_0\ \forall t \in T{\restriction}n\ \exists A_t \subseteq 2^n\ T_t \Vdash_{\mathbb{P}_g} \dot{A}{\restriction}n = A_t.$$

Fix $n > n_0$ and define by induction sets $\{A^n_t : t \in T{\restriction}m,\ n_0 \leq m \leq n+1\}$ such that

(1) $A^n_t \subseteq 2^{n+1}$ for $t \in T$,
(2) $|A^n_t| \cdot 2^{-n-1} \geq \delta_m$ for $t \in T{\restriction}m$.

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For $t \in T\restriction n+1$ let $A^n_t = A_t$. Suppose that sets A^n_t are defined for $t \in T\restriction m$, $m > n_0$. Let $t \in T\restriction m-1$ and consider the family $\{A^n_{t^\frown f} : f \in \mathsf{succ}_T(t)\}$. By the induction hypothesis, $|A^n_{t^\frown f}| \cdot 2^{-n-1} \geq \delta_m$ Let

$$A^n_t = \{s \in 2^{n+1} : \|\{f : s \in A_{t^\frown f}\| \geq \|\mathsf{succ}_T(t)\| - 1\}.$$

A straightforward computation (recall Fubini theorem) shows that the requirement that we put on the sequence $\langle \delta_n : n \in \omega\rangle$ implies that $|A^n_t| \cdot 2^{-n-1} \geq \delta_{m-1}$. In particular, $A^n_{\mathsf{stem}(T)} \cdot 2^{-n-1} \geq \delta_{n_0}$ for all n. Let $B = \{x \in 2^\omega : \exists^\infty n\ x\restriction n+1 \in A^n_{\mathsf{stem}(T)}\}$. Clearly $\mu(B) \geq \delta_{n_0}$, so $B \cap X \neq \emptyset$. Fix $x \in B \cap X$. We will find $S \geq T$ such that $S \Vdash_{\mathbb{P}_g} x \in [\dot{A}]$, which will give a contradiction.

For each n such that $x \in A^n_{\mathsf{stem}(T)}$ let $S_n \subseteq T\restriction n$ be a finite tree such that

1. $\mathsf{stem}(S_n) = \mathsf{stem}(T)$,
2. for every $t \in S_n$, $n_0 < |t| < n$, $\|\mathsf{succ}_{S_n}(t)\| \geq \|\mathsf{succ}_T(t)\|$,
3. for every $t \in S_n$, $|t| = n$, $x \in A_t$.

The existence of S_n follows from the inductive definition of A^n_t's. By König lemma, there exists $S \subseteq T$ such that for infinitely many n, $S\restriction n = S_n$. It follows that $S \in \mathbb{P}_g$ and $S \Vdash_{\mathbb{P}_g} \exists^\infty n\ x\restriction n \in \dot{A}\restriction n$. Since $\dot{A}$ is a tree we conclude that $s \Vdash_{\mathbb{P}_g} x \in [\dot{A}]$. $\square$

6. More on $\mathsf{NON}(\mathcal{J})$

In this section we will discuss the model obtained by iterating the forcing $\mathbb{P}_g$ alone.

Theorem 14. *It is consistent with* ZFC *that for every σ-ideal $\mathcal{J}$ such that* $\mathsf{non}(\mathcal{J}) < 2^{\aleph_0}$,

$$\mathsf{NON}(\mathcal{J}) \subseteq \mathsf{NON}(\mathcal{SN}) \subseteq [\mathbb{R}]^{<2^{\aleph_0}}.$$

Proof Elements of $\mathsf{NON}(\mathcal{SN})$ are traditionally called C'-sets. As we remarked earlier, $\mathcal{SN} = \mathsf{NON}^\star(\mathcal{SN})$. However, in [3] it is proved that assuming CH, $\mathsf{NON}(\mathcal{SN}) \subsetneq \mathcal{SN}$.

Let $\mathbf{V}$ be a model satisfying CH and let $\{g_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ be a dominating family. Let $\{S_\alpha : \alpha < \omega_1\}$ be such that

1. $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$,
2. $S_\alpha \subseteq \{\xi < \omega_2 : \mathsf{cf}(\xi) = \omega_1\}$,
3. S_α is stationary for all α.

Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\rangle$ be a countable support iteration such that for $\beta \in S_\alpha$, $\Vdash_\beta \dot{\mathcal{Q}}_\beta \simeq \mathbb{P}_{g_\alpha}$. If $\beta \notin \bigcup_\alpha S_\alpha$ let $\dot{\mathcal{Q}}_\beta$ be trivial forcing.

Suppose that $\mathcal{J}$ is a σ-ideal and $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{non}(\mathcal{J}) = \aleph_1$. It follows that for some $\alpha < \omega_2$, $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathbf{V}^{\mathcal{P}_\alpha} \cap 2^\omega \notin \mathcal{J}$.

Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_{\omega_2}} \cap 2^\omega$ is uncountable.

Case 1 $|X| = \aleph_1$ and $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models X \notin \mathsf{NON}(\mathcal{SN})$.

Let $\beta > \alpha$ be such that

1. $X \in \mathbf{V}^{\mathcal{P}_\beta}$,
2. there is a continuous function $H : X \longrightarrow 2^\omega$, $H \in \mathbf{V}^{\mathcal{P}_\beta}$ such that $\mathbf{V}^{\mathcal{P}_\beta} \models H"(X) \notin \mathcal{SN}$,
3. $\beta \in S_\gamma$ and $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models g_\gamma$ witnesses that $H"(X) \notin \mathsf{NON}(\mathcal{SN})$.

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It follows from the properties of $\mathbb{P}_{g_\gamma}$ that $\mathbf{V}^{\mathcal{P}_{\beta+1}} \models \bigcup_n F_n"(H"(X)) = 2^\omega$. Hence $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \bigcup_n F_n"(H"(X)) \notin \mathcal{J}$, which means that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \exists n \in \omega\ F_n"(H"(X)) \notin \mathcal{J}$.

Case 2 $|X| = 2^{\aleph_0} = \aleph_2$.

It is well known (see [4] or theorem 8.2.14 of [2]) in a model obtained by a countable support iteration of ω^ω-bounding forcing notions there are no strong measure zero sets of size $2^{\aleph_0}$. In particular, $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models X \notin \mathcal{SN}$. Let g_β be a witness to that. Let $X_\beta = X \cap \mathbf{V}^{\mathcal{P}_\beta}$. Standard argument shows that

$$C = \{\gamma < \omega_2 : \mathbf{V}^{\mathcal{P}_\gamma} \models X_\gamma \notin \mathcal{SN}\}$$

is a ω_1-club. Fix $\delta \in C \cap S_\beta$ and argue as in the Case 1. $\square$

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Department of Mathematics and Computer Science, Boise State University, Boise, Idaho 83725 U.S.A.
Email address: tomek@math.idbsu.edu, http://math.idbsu.edu/~tomek

Department of Mathematics, Hebrew University, Jerusalem, Israel
Email address: shelah@sunrise.huji.ac.il, http://math.rutgers.edu/~shelah/