THE RELATIVE CONSISTENCY OF $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$

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Abstract. We prove the consistency result from the title. By forcing we construct a model of $\mathfrak{g} = \aleph_1$, $\mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$.

0. Introduction

We recall the definitions of the three cardinal characteristics in the title and the abstract. We write $A \subseteq^* B$ if $A \setminus B$ is finite. We write $f \leq^* g$ if $f, g \in {}^\omega\omega$ and $\{n : f(n) > g(n)\}$ is finite.

Definition 0.1. *(1) A subset $\mathcal{G}$ of $[\omega]^\omega$ is called groupwise dense if*

– for all $B \in \mathcal{G}$, $A \subseteq^* B$ we have that $A \in \mathcal{G}$ and

– for every partition $\{[\pi_i, \pi_{i+1}) : i \in \omega\}$ of ω into finite intervals there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathcal{G}$.

The groupwise density number, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

(2) $\mathrm{Sym}(\omega)$ is the group of all permutations of ω. If $\mathrm{Sym}(\omega) = \bigcup_{i<\kappa} K_i$ and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\langle K_i : i < \kappa\rangle$ is increasing and continuous, K_i is a proper subgroup of $\mathrm{Sym}(\omega)$, we call $\langle K_i : i < \kappa\rangle$ a cofinality witness. We call the minimal such κ the cofinality of the symmetric group, short $\mathrm{cf}(\mathrm{Sym}(\omega))$.

(3) The bounding number $\mathfrak{b}$ is

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \wedge (\forall g \in {}^\omega\omega)(\exists f \in \mathcal{F}) f \not\leq^* g\}.$$

Simon Thomas asked whether $\mathfrak{g} \neq \mathrm{cf}(\mathrm{Sym}(\omega))$ is consistent [9, Question 3.1]. In this work we prove:

Theorem 0.2. $\mathfrak{g} < \mathrm{cf}(\mathrm{Sym}(\omega))$ *is consistent relative to* ZFC.

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1. Forcings destroying many cofinality witnesses

In this section we introduce two families of forcings that will be used in certain steps of our planned iteration of length $\aleph_2$. The plot is: If $\mathfrak{b}$ is large, there is some way to destroy all shorter cofinality witnesses because by Claims 1.6 and 1.5 none of the subgroups in a cofinality witness contains all permutations respecting a given equivalence relation. In our intended construction, we shall extend suitable intermediate models with a forcing built upon such an equivalence relation and thus prevent possible cofinality witnesses to be lifted to the forcing extension and all further extensions (Claim 1.4).

Here we show some details about destroying one cofinality witness that can be put separately before we launch into an iteration. The additional task, to increase the bounding number along the way, will be taken care of only in the next section.

Definition 1.1. *(1) We work with the following set of equivalence relations:*

$$\begin{aligned}\mathcal{E}_{con} = \{E :\, & E \text{ is an equivalence relation of } \omega,\\ & \text{each equivalence class } [n]_E \text{ is a finite interval of even length and}\\ & \omega = \liminf\langle |[n]_E| \,:\, n < \omega\rangle\}.\end{aligned}$$

We say $b \subseteq \omega$ respects $E \in \mathcal{E}_{con}$ if $(nEm \wedge m \in b) \to n \in b$. A partial permutation π of ω respects E if $\mathrm{dom}(\pi)$ respects E and we have that $n \in \mathrm{dom}(\pi) \to nE\pi(n)$.

(2) Let Q be the set of p such that

(a) p is a permutation of some subset $\mathrm{dom}(p)$ of ω,

(b) $\omega \setminus \mathrm{dom}(p)$ is infinite.

We order Q by inclusion.

(3) For $E \in \mathcal{E}_{con}$, Q_E is the set of p satisfying (2)(a) – (b) and additionally

(c) p respects E.

Part (1) of the following claim is important for later use, whereas part (2) will never be used directly.

Claim 1.2. *(1) If $E \in \mathcal{E}_{con}$ and $p \in Q_E$ and $\underset{\sim}{\tau}$ is a Q_E-name of an ordinal and b is a finite subset of $\omega \setminus \mathrm{dom}(p)$ respecting E, then there is some q such that*

(a) $p \leq q$ and $b \subseteq \omega \setminus \mathrm{dom}(q)$,

(b) *if π is a permutation of b and it respects E then $q \cup \pi$ forces a value to $\underset{\sim}{\tau}$.*

(2) *Q_E is proper, ${}^\omega\omega$-bounding, nep (see [6]) and Souslin.*

Proof. (1) Note that there are only finitely many permutations of b (that respect E). So we can treat them consecutively and find stonger and stronger q's.

(2) Let $N \prec H(\chi, \in)$ be such that $Q_E \in N$ and $p \in N$, $\chi \geq (2^\omega)^+$. Let $\underset{\sim}{\tau}_n$, $n \in \omega$, be a list of all Q_E-names for ordinals that are in N. Let b_n, $n \in \omega$, be a list of pairwise disjoint E-classes such that $\bigcup_{n\in\omega} b_n$ is infinite. Now take q_n by induction starting with $q_0 = p$. We let $i(-1) = 0$. If q_n, $i(n-1)$ are chosen, take $i(n) > i(n-1)$ such that $\mathrm{dom}(q_n) \cap \bigcup_{0\leq k\leq n} b_{i(k)} = \emptyset$. Now take q_{n+1} treating q_n, $\underset{\sim}{\tau}_n$ and $\bigcup_{0\leq k\leq n} b_{i(k)}$ as in the proof of part (1). Hence $\bigcup_{0\leq k\leq n} b_{i(k)} \subseteq \omega \setminus \mathrm{dom}(q_m)$ for all $n, m \in \omega$. We have that $q = \bigcup q_n \in Q_E$ and that $q \Vdash_{Q_E} (\forall n \in \omega)\underset{\sim}{\tau}_n \in \check{N}$. By [7, III, Theorem 2.12], Q_E is proper.

Q_E is ${}^\omega\omega$-bounding: Let $\underset{\sim}{f}$ be a name for a function from ω to ω. Again let b_n, $n \in \omega$, be a list of pairwise disjoint E-classes such that $\bigcup_{n\in\omega} b_n$ is infinite. Now take q_n by induction starting with $q_0 = p$. If q_n is chosen, take $i(n)$ such that $\mathrm{dom}(q_n) \cap b_{i(n)} = \emptyset$. Now take q_{n+1} treating q_n, $\underset{\sim}{\tau}_n$ and $b_{i(n)}$ as in part (2) of this claim and look which values for $\underset{\sim}{f}(n)$ the finitely many permutations in (1)(b) force. Take $g(n)$ to be the maximum of them. We have that $q = \bigcup q_n \in Q_E$ and that $q \Vdash_{Q_E} (\forall n)\underset{\sim}{f}(n) \leq g(n)$.

nep (non-elementary properness): We use much less than $N \prec H(\chi, \in)$. We use that $E \in N \subseteq H(\chi, \in)$. See [6].

Souslin: $p \in Q_E$, $q \leq q$ and $p \perp q$ can be expressed in $\Sigma^1_1(E)$-formulas. $\square$

We shall work with the following special subsets of $\mathrm{Sym}(\omega)$.

Definition 1.3. (1) *For $E \in \mathcal{E}_{con}$ and $A \subseteq \omega$ we define:*

$$S_{E,A} := \{\pi \in Q_E \,:\, \pi \restriction (\omega \setminus A) = id\}.$$

(2) *We set $\mathcal{F} := \{f \,:\, f \in {}^\omega\omega, f(n) \geq n, \lim\langle f(n) - n \,:\, n \in \omega\rangle = \infty\}$. For $f \in \mathcal{F}$ we set $S_f := \{\pi \in \mathrm{Sym}(\omega) \,:\, (\forall n)(\pi(n) \leq f(n) \;\wedge\; \pi^{-1}(n) \leq f(n))\}$.*

The following claim describes the basic step in order to increase $\mathrm{cf}(\mathrm{Sym}(\omega))$.

Claim 1.4. *Assume*

(a) *$\langle K_i \,:\, i < \kappa\rangle$ is a cofinality witness, and K_0 contains all permutations that move only finitely many points,*

(b) *$\underset{\sim}{R}$ is a Q_E-name of a forcing notion,*

(c) *$E \in \mathcal{E}_{con}$, and for no $i < \kappa$ and no coinfinite $A \in [\omega]^\omega$ respecting E we have that $K_i \supseteq S_{E,A}$.*

Then in $\mathbf{V}^{Q_E * \underset{\sim}{R}}$ we cannot find a cofinality witness $\langle K'_i : i < \kappa\rangle$ such that $\bigwedge_{i<\kappa} \Big(K'_i \cap \mathrm{Sym}(\omega)^{\mathbf{V}} = K_i\Big)$.

Proof. Let $\underset{\sim}{f} = \bigcup\{p : p \in \underset{\sim}{G}_{Q_E}\}$ be a Q_E-name of a permutation of ω. It suffices that

$$(*) \qquad \Vdash_{Q_E} \text{“for unboundedly many } i < \kappa, \\ \text{for some } g \in K_i \text{ we have } \underset{\sim}{f} \circ g \circ (\underset{\sim}{f})^{-1} \in K_{i+1} \setminus K_i.\text{”}$$

Why does this suffice? Suppose that $(*)$ holds and we had found a cofinality witness $\langle K'_i : i < \kappa\rangle$ in $\mathbf{V}^{Q_E * \underset{\sim}{R}}$ such that $\bigwedge_{i<\kappa} \Big(K'_i \cap \mathrm{Sym}(\omega)^{\mathbf{V}} = K_i\Big)$. Let G be $Q_E * \underset{\sim}{R}$-generic over $\mathbf{V}$. Take $j < \kappa$ such that $\underset{\sim}{f}[G] \in K'_j$. Then we find according to $(*)$ some $i \geq j$ and some $g \in K_i$ such that $\underset{\sim}{f}[G] \circ g \circ (\underset{\sim}{f}[G])^{-1} \in K_{i+1} \setminus K_i \subseteq \mathbf{V}$. But this contradicts the facts that $\underset{\sim}{f}[G] \circ g \circ (\underset{\sim}{f}[G])^{-1} \in K'_i$ (because this is a subgroup) and $K'_i \cap \mathrm{Sym}(\omega)^{\mathbf{V}} = K_i$.

Proof of $(*)$: Let $p \in Q_E$ and $j < \kappa$. Let $\omega \setminus \mathrm{dom}(p)$ be the disjoint union of A_0, A_1, both infinite subsets of ω respecting E.

Let $g_0 \in \mathrm{Sym}(\omega)$ be such that it has order two and $\{n : g_0(n) \neq n\} = A_0$. Take $A'_0 \supseteq A_0$ such that $A'_0 \setminus A_0$ is infinite. Let $g'_0 \in \mathrm{Sym}(\omega)$ be such that it has order two and $\{n : g'_0(n) \neq n\} = A'_0$. Let $g_0, g'_0 \in K_{i(*)}$, $i(*) \geq j$.

Also $S' = \{g \in S_{E,A'_0} : g$ has order two and does not have a fixed point in some coinfinite subset of A'_0 or does not have a fixed point in $A'_0\}$ together with all permutations that move only finitely many points generates S_{E,A'_0}. In order to see this, write each element π of S_{E,A'_0} as a union of disjoint cycles. All cycles are of finite length, because π respects E. Let π_ℓ, $\ell < L$, enumerate all the disjoint cycles in one fixed E-class $[n]_E$, so that $\pi \restriction [n]_E = \prod_{\ell<L} \pi_\ell$. First we write any cycle as $\pi_\ell = (a_0, a_1, \ldots, a_{k-1})$, which means that $\pi_\ell(a_i) = a_{i+1}$ and $\pi_\ell(a_{k-1}) = a_0$.

In the case of even k, we write π_ℓ as a product of two permutations of order two, whose domain is $\{a_0, a_1, \ldots, a_{k-1}\}$ and $\{a_1, \ldots, a_{\frac{k}{2}-1}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\}$ respectively: $\pi_\ell = \pi^1_\ell \circ \pi^0_\ell$, where $\pi^0_\ell = (a_0, a_{k-1})(a_2, a_{k-1}) \ldots (a_{\frac{k}{2}-1}, a_{\frac{k}{2}})$ and $\pi^1_\ell = (a_1, a_{k-1})(a_3, a_{k-2}) \ldots (a_{\frac{k}{2}-1}, a_{\frac{k}{2}+1})$.

For odd k, we have that $(a_0, a_1, \ldots, a_{k-1}) = (a_0, a_{k-1})(a_0, \ldots, a_{k-2})$. We write $\pi^2_\ell = (a_0, a_{k-1})$ and treat $\pi'_\ell = (a_0, \ldots, a_{k-2})$ according to the former case and thus get $\pi_\ell = \pi^2_\ell \circ \pi^1_\ell \circ \pi^0_\ell$. In order to have more uniform notation we choose π^2_ℓ to be the identity on the domain of the cycle in the case of even length.

So we decompose $\pi \restriction [n]_E = \prod_{\ell<L} \pi^0_\ell \circ \prod_{\ell<L} \pi^1_\ell \circ \prod_{\ell<L} \pi^2_\ell$. For $j = 0,1,2$, $\pi^j_{[n]_E} = \prod_{\ell<L} \pi^j_\ell$ is a permutation of order 2, and depending on the the number of cycles of uneven length in π there may be fixed points in A'_0 in $\pi^0_{[n]_E}$. There are fixed points in $\pi^1_{[n]_E}$ and $\pi^2_{[n]_E}$. We set $\pi^j = \bigcup\{\pi^j_{[n]_E} : n \in R\}$ for some set of representatives R for E. Now the set of fixed points of π^j is either $\omega \setminus A'_0$ or $\omega \setminus A$ for some subset A of A'_0. W.l.o.g. we assume that A is infinite.

By assumption S_{E,A'_0} is not included in any K_i, so in particular not included in $K_{i(*)}$. Hence there is $g_1 \in S' \setminus K_{i(*)}$. Take i such that $g_1 \in K_{i+1} \setminus K_i$. Necessarily we have $\kappa > i \geq i(*) \geq j$.

First case: g_1 has finitely many fixed points in A'_0. By changing it slightly we may assume that is has no fixed point in A'_0. Now there is a permutation f of A'_0 respecting E such that f is an isomorphism from (A'_0, g_1) onto (A'_0, g'_0), because any two permutations of order two without fixed points are conjugated. Hence $n \in A'_0 \Rightarrow f(g'_0(n)) = g_1(f(n))$.

Second case: g_1 has infinitely many fixed points in A'_0. Of course g_1 moves infinitely many points in A'_0. Now there is a permutation f of A'_0 respecting E such that f is an isomorphism from (A'_0, g_1) onto (A'_0, g_0), because any two permutations of order two with an infinite and coinfinite set of fixed points are conjugated. Hence $n \in A'_0 \Rightarrow f(g_0(n)) = g_1(f(n))$.

Let $q = p \cup f$. The condition q forces that $\underset{\sim}{f} \circ g_0 \circ (\underset{\sim}{f})^{-1} = g_1$, or $\underset{\sim}{f} \circ g'_0 \circ (\underset{\sim}{f})^{-1} = g_1$, $g_1 \in K_{i+1} \setminus K_i$, and $i \in (j, \kappa)$, $g_0, g'_0 \in K_{i(*)} \subseteq K_i$, so $(*)$ is proved. $\square$

Claim 1.5. *Assume that $\langle K_i : i < \kappa\rangle$ is a cofinality witness. Assume that K_0 contains all permutations that move only finitely many points. Then the following are equivalent:*

- (α) *There is some $E \in \mathcal{E}_{con}$, such that for every $i < \kappa$ and for every E-respecting $A \in [\omega]^{\aleph_0}$ we do have $K_i \not\supseteq S_{E,A}$.*
- (β) *For every $E \in \mathcal{E}_{con}$, for every $i < \kappa$ and for every E-respecting $A \in [\omega]^{\aleph_0}$ we do have $K_i \not\supseteq S_{E,A}$.*
- (γ) *There is some $f \in \mathcal{F}$, such that for every $i < \kappa$ do we have that $S_f \not\subseteq K_i$.*
- (δ) *For every $f \in \mathcal{F}$, for every $i < \kappa$ do we have that $S_f \not\subseteq K_i$.*

Proof. The implications $(\beta) \Rightarrow (\alpha)$ and $(\delta) \Rightarrow (\gamma)$ are trivial. We shall not use $(\beta) \Rightarrow (\alpha)$ but close a circle of implications as follows: $(\beta) \Rightarrow (\delta)$ and $(\alpha) \Rightarrow (\beta)$ and $(\gamma) \Rightarrow (\alpha)$.

Now we prove $\neg(\delta) \Rightarrow \neg(\beta)$. Let f and i^* exemplify the failure of (δ).

By the definition of $\mathcal{F}$ we have that $\lim\langle f(n) - n : n \in \omega\rangle = \infty$. Hence we may choose a strictly increasing sequence $\langle k_i : i \in \omega\rangle$ such that $(\forall i \in \omega)(\forall n \geq$

$k_i)(f(n) \geq i+n)$. Then we take $E = \{[k_i, k_i+i) : i \in \omega\} \cup \{[k_i+i, k_{i+1}) : i \in \omega\}$ and $A = \bigcup_{i \in \omega}[k_i, k_i + i)$. A is infinite and coinfinite. Then we have that $S_{E,A} \subseteq S_f \subseteq K_{i^*}$, so $\neg(\beta)$.

Now we show $\neg(\beta)$ implies $\neg(\alpha)$. This follows from

Subclaim 1: For all $E, E' \in \mathcal{E}_{con}$ and E-respecting $A \in [\omega]^{\aleph_0}$ there are $f_1, f_2 \in \mathrm{Sym}(\omega)$ such that

$$S_{E',\omega} \subseteq ((f_1)^{-1} \circ S_{E,A} \circ f_1) \circ ((f_2)^{-1} \circ S_{E,A} \circ f_2).$$

Proof. Enumerate the E'-classes with order type ω. Let f_1 inject the even-numbered E'-classes into high enough (there are large enough ones by the definition of $\mathcal{E}_{con}$) E classes that lie in A. The E-classes need not be covered, it is enough that $nE'm \to f_1(n)Ef_1(m)$. We fill this function up to a permutation of ω and call it f_1. Let f_2 do the same with the odd-numbered E'-classes. If $g \in S_{E',\omega}$ then $g = g_1 \circ g_2$ where g_1 is the identity on odd-numbered E'-classes and g_2 is the identity on even-numbered E'-classes. We have that $f_i \circ g_i \circ (f_i)^{-1} \in S_{E,A}$ for $i = 1, 2$ and thus Subclaim 1 and ($\neg(\beta)$ implies $\neg(\alpha)$) are proved.

To complete a cycle of implications, we show $\neg(\alpha) \Rightarrow \neg(\gamma)$. First we need a similar claim:

Subclaim 2: For all $E \in \mathcal{E}_{con}$, E-respecting $A \in [\omega]^{\aleph_0}$ there are there is $f \in \mathrm{Sym}(\omega)$ such that

$$S_{E,\omega\setminus A} \subseteq f^{-1} \circ S_{E,A} \circ f.$$

Proof. Enumerate the E-classes which lie in $\omega \setminus A$ with order type less or equal ω. Let f inject them into high enough E classes that lie in A. As above, the E-classes need not be covered, it is enough that $nEm \to f(n)Ef(m)$. We fill this function up to a permutation of ω and call it f. If $g \in S_{E,\omega\setminus A}$ we have that $f \circ g \circ f^{-1} \in S_{E,A}$, and thus Subclaim 2 is proved.

Now suppose $\neg(\alpha)$. To prove $\neg(\gamma)$ let $f \in \mathcal{F}$. We choose by induction on $k \in \omega$, m_i such that $m_0 = 0$, $m_{k+1} > m_k$ and $(\forall n < m_k)(f(n) < m_{k+1})$.

Now we define two equivalence relations.

$$E_0 = \{[m_{2k}, m_{2k+2}) : k \in \omega\},$$
$$E_1 = \{[m_{2k+1}, m_{2k+3}) : k \in \omega\} \cup \{[0, m_1)\}.$$

By our assumption $\neg(\alpha)$ there is some $i < \kappa$ and there are E-respecting $A_0, A_1 \in [\omega]^\omega$ such that $S_{E_\ell,A_\ell} \subseteq K_i$ for $\ell = 0, 1$. Now note that

$(*)_1$ If $\pi \in S_f$ then we can find $\pi_\ell \in S_{E_\ell,\omega}$ for $\ell = 0, 1$ such that $\pi = \pi_1 \circ \pi_0$. Why?

By the definition of S_f and E_ℓ, for any $x \in \omega$, $xE_0\pi(x)$ or $xE_1\pi(x)$. Now we choose $\pi_0(x)$ and $\pi_1(x)$ by cases.

We write π as a (possibly infinite) product of disjoint finite or infinite cycles. It is enough to show how to decompose each cycle. We write it explicitly for a finite cycle $(a_0, a_1, \dots a_{k-1})$. Infinite cycles are not harder to treat. We write aE'_1b for (aE_1b and not aE_0b). Then we have, say, $a_0E_0a_1$, …, $a_{i_1-1}E_0a_{i_1}$, $a_{i_1}E'_1a_{i_1+1}$, $a_{i_1+1}E_0a_{i_1+2}$, …, $a_{i_2-1}E_0a_{i_2}$, $a_{i_2}E'_1a_{i_2+1}$, $a_{i_2+1}E_0a_{i_2+2}$, … ,$a_{i_{r_{max}}}E'_1a_{i_{r_{max}}+1}$, …, $a_{k-1}E_0a_0$ through the whole cycle. We assumed that $i_{r_{max}} < k-1$. The complementary case is treated similarly.

Since $(a_0, a_1, \dots a_{k-1})$ is a cycle, for each n_{2k+1} we have: If it appears for some r as a border in $a_{i_r}E'_1a_{i_r+1}$ in the sense that $a_{i_r} < n_{2k+1} \le a_{i_r+1}$ then there is a matching $i_{r'}$, call it $h(i_r)$ such that $a_{h(i_r)}E'_1a_{h(i_r)+1}$ and $a_{h(i_r)} \ge n_{2k+1} > a_{h(i_r)+1}$. For all involved n_{2k+1}, we choose matching pairs so that $\{i_r : 0 \le r < r_{max}\}$ is partitioned into pairs $\{i_r, h(i_r)\}$. We set $g = h \cup h^{-1}$ and thus get a bijection of $\{i_r : 0 \le r < r_{max}\}$.

Now we set $\pi_0(a_j) = a_{j+1}$ if $j \ne i_r$ for all r. We set $\pi_0(a_{i_r}) = a_{h(i_r)+1}$ for $0 \le r < r_{max}$. So π_0 is a bijection of $\{a_0, \dots a_{k-1}\}$ and it respects E_0.

Now we set $\pi_1(a_j) = a_j$ if $j \ne i_r + 1(\mathrm{mod} k)$ for all r. We set $\pi_1(a_{h(i_r)+1}) = a_{i_r+1}$ for $0 \le r < r_{max}$. So π_1 is a bijection of $\{a_0, \dots a_{k-1}\}$ and it respects E_1. Now it is easy to check that $\pi = \pi_1 \circ \pi_0$.

$(*)_2$ Let for $\ell = 0, 1$ choose $f_\ell \in K_j$ as in Subclaim 2, such that $S_{E_\ell,\omega\setminus A_\ell} \subseteq (f_\ell)^{-1} \circ S_{E_\ell,A_\ell} \circ f_\ell$. W.l.o.g. $j \ge i$. Since $S_{E_\ell,\omega} = S_{E_\ell,A_\ell} \circ S_{E_\ell,\omega\setminus A_\ell}$ we have that $S_{E_\ell,\omega} \subseteq K_j$ for $\ell = 0, 1$ and hence by $(*)_1$ that $S_f \subseteq K_j$, that is $\neg(\gamma)$. $\square$

Claim 1.6. *Assume that $\langle K_i : i < \kappa\rangle$ is a cofinality witness such that K_0 contains all the permutations that move only finitely any points. If $\mathfrak{b} > \kappa$, then clause (γ) of Claim 1.5 holds (and hence all the other clauses hold as well).*

Proof. For each $i < \kappa$ choose $\pi_i \in \mathrm{Sym}(\omega) \setminus K_i$. Since $\mathfrak{b} > \kappa$ there is some $f \in {}^\omega\omega$ such that $(\forall i < \kappa)(\forall^\infty n)(\pi_i(n) < f(n))$ and w.l.o.g. $f \in \mathcal{F}$. if S_f were a subset of K_i, then we had that $\pi_i \in K_i$, which is not the case. So f exemplifies clause (γ) of Claim 1.5. $\square$

Definition 1.7. *(1) Let $E \in \mathcal{E}_{con}$. We set*

$$Q'_E = \{f : f \text{ is a permutation of some coinfinite subset of } \omega \text{ such that}$$
$$(a)\ n \in \mathrm{dom}(f) \Rightarrow nEf(n),$$
$$(b) \text{ for every } k < \omega \text{ for some } n \text{ we have } k \leq |[n]_E) \setminus \mathrm{dom}(f)|\}.$$

The order is by inclusion.

(2) We call $\bar{f} = \langle f_i : i < \alpha \rangle$, Q'_E-o.k. if $\alpha \leq \omega_1$ and for $i \leq j < \alpha$, $f_i \subseteq^* f_j \in Q'_E$ (i.e. $\{n \in \mathrm{dom}(f_i) : n \notin \mathrm{dom}(f_j) \vee f_i(n) \neq f_j(n)\}$ is finite). For $\bar{f}$ being Q'_E-o.k. we set $Q'_E(\bar{f}) = \{g \in Q'_E : g =^* f_i$ for some $i\}$, where $f_i =^* g$ iff $f_i \subseteq^* g$ and $g \subseteq^* f_i$. The order is inherited from Q'_E.

(3) We write $\trianglelefteq$ for the initial segment relation for sequences of ordinal length, i.e., $\langle g_\beta : \beta < \gamma \rangle \trianglelefteq \langle f_\beta : \beta < \alpha \rangle$ iff $\langle g_\beta : \beta < \gamma \rangle = \langle f_\beta : \beta < \gamma \rangle$.

Remarks. 1) Claims 1.4 and 1.5 hold for Q'_E as well with the analogously modified definition of $S'_{E,A}$. This is shown with the same proofs. The domains of the involved partial permutations must be arranged such that they respect 1.7(1)(b), but they need not be unions of equivalence classes. The $q \in Q_E$ fulfil requirement 1.7(1)(b) automatically, because we have that $\lim \langle |[n]_E| : n \in \omega \rangle = \omega$ and that the domain of q needs to be coinfinite and needs to be a union of equivalence classes.

2) Both Q_E and Q'_E can serve for our purpose. Q'_E exhibits the following "independence of E": For $E_0, E_1 \in \mathcal{E}_{con}$ $(\forall p \in Q'_{E_1})$ $(\exists q)$ $(p \leq q \in Q'_{E_1} \wedge (Q'_{E_1})_{\geq p} \cong Q'_{E_0})$.

3) Note that for $\alpha < \omega_1$, if $\bar{f} = \langle f_\beta : \beta \in \alpha \rangle$ Q'_E-o.k., then we have that $Q'_E(\bar{f})$ is Cohen forcing.

Claim 1.8. *Let E be as in Definition 1.7.*

(1) Q'_E is proper, even strongly proper, with the Sacks property (the last is more than Q_E).

(2) If $\bar{f} = \langle f_\beta : \beta < \alpha \rangle$ is as in 1.7(2), and $\alpha < \omega_1$ and $Q'_E(\bar{f}) \subseteq M$, $\omega + 1 \subseteq M \subseteq (H(\chi), \in)$, M a countable model of ZFC^-, then we can find f_α such that

(a) $\bar{f}\hat{\ } f_\alpha$ is Q'_E-o.k.

(b) If $\bar{f}\hat{\ } f_\alpha \trianglelefteq \bar{f}'$ and $\bar{f}'$ is Q'_E-o.k., then f_α is $(M, Q'_E(\bar{f}'))$-generic. The genericity is independent of $\bar{f}'$ in the following sense: For every I there is some finite $J \subseteq I$, $J \in M$ such that for all $\bar{f}'$

the following holds: If $I \subseteq Q'_E(\bar{f}')$ is predense and a member of M, then J is predense above f_α in $Q'_E(\bar{f}')$.

Proof. (1) We prove the Sacks property. Let $\underset{\sim}{f} \in V^{Q'_E} \cap {}^\omega\omega$. We take $b_{i(n)}$ as in the proof of the ${}^\omega\omega$-boundedness for Q_E (which applies also to Q'_E) in Claim 1.2, but we do not require that $b_{i(n)}$ respects E. Additionally we choose $b_{i(n)}$ so small that there are only fewer than n permutations of $\bigcup_{k\leq n} b_{i(k)}$. So often, but not cofinitely often, $b_{i(n)}$ will be empty. Then we take q_n as there and collect into $S(n)$ all the possible values forced by $q_n \cup \pi$ for $\underset{\sim}{f}(n)$, when π ranges over the permutations of b_n.

(2) Let $\langle \bar{f}'^n : n \in \omega \rangle$ enumerate all the β-sequences in M that are Q'_E-o.k. for all $\beta \in [\alpha, \omega_1]$. Let $\underset{\sim}{\tau_n}$, b_n, $n \in \omega$ be as in the proof of 1.2, $\underset{\sim}{\tau_n}$ a $Q'_E(\bar{f}'^n)$-name. We take an enumeration such that each $\underset{\sim}{\tau_n}$ appears infinitely often. First we choose $f^0_\alpha \supseteq^* f_\beta$ for all $\beta < \alpha$. Here we use that $\alpha < \omega_1$. Next we choose f^n_α $\subseteq^*$-increasing with n, and $i(n)$ strictly increasing with n such that $\bigcup_{k\leq n} b_{i(k)} \cap \mathrm{dom}(f^n_\alpha) = \emptyset$ and such that if $\bar{f}\hat{\ } f^n_\alpha \trianglelefteq \bar{f}'^n$ and π is a permutation of $\bigcup_{k\leq n} b_{i(k)}$ then $f^n_\alpha \cup \pi \Vdash_{Q'_E} \underset{\sim}{\tau_n} \in V$. Let J contain one member i of I for each permutation π of $\bigcup_{k\leq n} b_{i(k)}$ that i is compatible with $f^n_\alpha \cup \pi$. Thus J is a finite subset of I. The choice of f^n_α is independent of $\bar{f}'^n$, because ($f^n_\alpha \cup \pi \Vdash_{Q'_E} \underset{\sim}{\tau_n} \in V$ and $\bar{f}\hat{\ } f^n_\alpha \trianglelefteq \bar{f}'$) implies $f^n_\alpha \cup \pi \Vdash_{Q'_E(\bar{f}')} \underset{\sim}{\tau_n} \in \mathbf{V}$, independently of the choice of $\bar{f}'$. We set $f_\alpha = \bigcup_{n\in\omega} f^n_\alpha$, and by one of the equivalent characterizations of $(M, Q'_E(\bar{f}'))$-genericity [7, III, Theorem 2.12] we are done. $\square$

2. Arranging $\mathfrak{g} = \aleph_1, \mathfrak{b} = \mathrm{cf}(\mathrm{Sym}(\omega)) = \aleph_2$

Starting from a ground model with a suitable diamond sequence we find a forcing extension with the constellation from the section headline. The requirements on the ground model can be established by a well-known forcing (see [4, Chapter 7]) starting from any ground model, and are also true in L (see [3]).

Definition 2.1. *(1) We say $\mathcal{A}$ is a $(\kappa, \mathfrak{g})$-witness if $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and*

- (α) *$\mathcal{A} \subseteq [\omega]^{\aleph_0}$,*
- (β) *if $k < \omega$ and $f_\ell \colon \omega \to \omega$ is injective for $\ell < k$ then for some $\mathcal{A}' \subseteq \mathcal{A}$ of cardinality $< \kappa$ we have that for any A that is a finite union of members of $\mathcal{A} \setminus \mathcal{A}'$*

$$\{n : \bigwedge_{\ell<k} f_\ell(n) \notin A\} \text{ is infinite.}$$

(2) We say $\bar{M}$ κ-exemplifies $\mathcal{A}$ if

- *(a) $\mathcal{A}$ is a $(\kappa, \mathfrak{g})$-witness,*

(b) $\bar{M} = \langle M_i : i < \kappa \rangle$ *is* $\prec$*-increasing and continuous, and* $\omega + 1 \subseteq M_0$ *and* $\mathcal{P}(\omega) \subseteq \bigcup_{i<\kappa} M_i$,

(c) $M_i \subseteq (H(\chi), \in)$ *is a model of* ZFC^- *and* $|M_i| < \kappa$ *and* $(M_i \models |X| < \kappa) \Rightarrow X \subseteq M_i$,

(d) $\bar{M} \restriction (i+1) \in M_{i+1}$,

(e) *for* i *non-limit, there is* $\mathcal{A}_i \in M_i$ *such that* $\mathcal{A} \cap M_i = \mathcal{A}_i$,

(f) *if* $i < \kappa$, $k < \omega$ *and* $f_\ell \in M_i$ *is an injective function from* ω *to* ω *for* $\ell < k$, *and* $k' < \omega$, $A_\ell \in \mathcal{A} \setminus M_i$ *for* $\ell < k'$, *then*

$$\{n : \bigwedge_{\ell<k} f_\ell(n) \notin A_0 \cup \cdots \cup A_{k'-1}\} \text{ is infinite.}$$

(3) *We say* $\bar{M}$ *leisurely exemplifies* $\mathcal{A}$ *if (a) to (f) above are fulfilled and additionally;*

(g) $\kappa = \sup\{i : M_{i+1} \models \text{“}\mathcal{A}_{i+1} = \aleph_0\text{”}\}$.

Definition 2.2. *(1) We say* $(P, \underset{\sim}{\mathcal{A}})$ *is a* (μ, κ)*-approximation if*

(α) P *is a c.c.c. forcing notion,* $|P| \le \mu$,

(β) $\underset{\sim}{\mathcal{A}}$ *is a set of* P*-names of members of* $([\omega]^{\aleph_0})^{\mathbf{V}^P}$, *each hereditarily countable, and for simplicity they are forced to be pairwise distinct,*

(γ) $\Vdash_P$ *“*$\underset{\sim}{\mathcal{A}}$ *is a* $(\kappa, \mathfrak{g})$*-witness.”*

(2) *If* $\mu = \kappa$ *we may write just* κ*-approximation. If* $\kappa = \aleph_1$ *we may omit it. We write* $(*, \kappa)$*-approximation if it is a* (μ, κ)*-approximation for some* μ.

(3) $(P_1, \underset{\sim}{\mathcal{A}}_1) \le^\kappa_{app} (P_2, \underset{\sim}{\mathcal{A}}_2)$ *if:*

(a) $(P_\ell, \underset{\sim}{\mathcal{A}}_\ell)$ *is a* $(*, \kappa)$*-approximation.*

(b) $P_1 \lessdot P_2$,

(c) $\underset{\sim}{\mathcal{A}}_1 \subseteq \underset{\sim}{\mathcal{A}}_2$ *(as a set of names, for simplicity),*

(d) *if* $k < \omega$ *and* $\underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k-1} \in \underset{\sim}{\mathcal{A}}_2 \setminus \underset{\sim}{\mathcal{A}}_1$ *then*

$$\Vdash_{P_2} \text{“ if } B \in ([\omega]^{\aleph_0})^{V^{P_1}},$$

$$f_\ell \in ({}^B\omega)^{V^{P_1}} \text{for } \ell < k \text{ are injective, then}$$

$$\left\{ n \in B : \bigwedge_{\ell<k} f_\ell(n) \notin \bigcup_{\ell<k} \underset{\sim}{A}_\ell \right\} \text{ is infinite”.}$$

Remark. We mean $\underset{\sim}{\mathcal{A}}_1 \subseteq \underset{\sim}{\mathcal{A}}_2$ as a set of names. It is no real difference if $\underset{\sim}{\mathcal{A}}$ is a P-name in 2.2(1) and if in (3) we have $\Vdash \underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k-1} \in \underset{\sim}{\mathcal{A}}_2 \setminus \underset{\sim}{\mathcal{A}}_1$.

Claim 2.3. *$\leq^\kappa_{app}$ is a partial order.*

Proof. We check (3) clause (d) of the definition. Let $(P_1, \underset{\sim}{\mathcal{A}}_1) \leq^\kappa_{app} (P_2, \underset{\sim}{\mathcal{A}}_2)$ and $(P_2, \underset{\sim}{\mathcal{A}}_2) \leq^\kappa_{app} (P_3, \underset{\sim}{\mathcal{A}}_3)$. Let $k < \omega$, $\underset{\sim}{f}_\ell$ be P_1-names of injective functions from ω to ω . Let $G \subseteq P_3$ be generic over $\mathbf{V}$. So let $A_\ell \in \underset{\sim}{\mathcal{A}}_3[G]$ for $\ell < m$. We assume that for $\ell < m_0 \leq m$ we have that $\underset{\sim}{A}_\ell \in \underset{\sim}{\mathcal{A}}_2$ and that that $\{\underset{\sim}{A}_\ell \,:\, \ell < m\} \subseteq \underset{\sim}{\mathcal{A}}_3 \setminus \underset{\sim}{\mathcal{A}}_2$. By the assumptions on P_1 we have that $B_1 = \big\{n < \omega \,:\, \bigwedge_{\ell<k} f_\ell(n) \notin \bigcup\{A_\ell \,:\, \ell < m_0\}\big\}$ is infinite. It belongs to $\mathbf{V}[G \cap P_2]$. Since we have that $(P_2, \underset{\sim}{\mathcal{A}}_2) \leq^\kappa_{app} (P_3, \underset{\sim}{\mathcal{A}}_3)$ and $\{\underset{\sim}{A}_\ell \,:\, \ell \in [m_0, m)\} \subseteq \underset{\sim}{\mathcal{A}}_3 \setminus \underset{\sim}{\mathcal{A}}_2$ and $B_1, f_0, \ldots, f_{k-1} \in \mathbf{V}[G \cap P_2]$, by Definition 2.2(3) clause (d) we are done.

Claim 2.4. *If $\langle (P_i, \underset{\sim}{\mathcal{A}}_i) \,:\, i < \delta\rangle$ is a $\leq^\kappa_{app}$-increasing continuous sequence (continuous means that in the limit steps we take unions), then $(P, \underset{\sim}{\mathcal{A}}) = (\bigcup_{i<\delta} P_i, \bigcup_{i<\delta} \underset{\sim}{\mathcal{A}}_i)$ is an $\leq^\kappa_{app}$-upper bound of the sequence, in particular, a $(*, \kappa)$-approximation.*

Proof. The only problem is "$(P, \underset{\sim}{\mathcal{A}})$ is a κ-approximation."

Case 1: $\mathrm{cf}(\delta) > \aleph_0$. Let $k < \omega$, $\underset{\sim}{f}_\ell$ be P-names of injective functions from ω to ω . So for some $i < \delta$ we have that $\langle \underset{\sim}{f}_\ell \,:\, \ell < k\rangle$ is a P_i-name. Let $G \subseteq P$ be generic over $\mathbf{V}$. In $\mathbf{V}[G \cap P_i]$, there is some $\underset{\sim}{\mathcal{A}}' \subseteq \underset{\sim}{\mathcal{A}}$ such that $\underset{\sim}{\mathcal{A}}' \in ([\underset{\sim}{\mathcal{A}}_i[G \cap P_i]]^{<\kappa})^{\mathbf{V}[G\cap P_i]}$ as required in $\mathbf{V}[G \cap P_i]$ for $\langle \underset{\sim}{f}_\ell[G \cap P_i] \,:\, \ell < k\rangle$. We shall show that $\underset{\sim}{\mathcal{A}}'$ is as required in $\mathbf{V}[G]$ for $\langle \underset{\sim}{f}_\ell[G \cap P_i] \,:\, \ell < k\rangle$. So let $A_\ell \in \underset{\sim}{\mathcal{A}}[G] \setminus \underset{\sim}{\mathcal{A}}'[G]$ for $\ell < m$, w.l.o.g. $\underset{\sim}{A}_\ell \in \underset{\sim}{\mathcal{A}}$, $A_\ell = \underset{\sim}{A}_\ell[G]$. We assume that for $\ell < m_0 \leq m$ we have that $\underset{\sim}{A}_\ell \in \underset{\sim}{\mathcal{A}}_i$ and that $j < \delta$ is such that $\{\underset{\sim}{A}_\ell \,:\, \ell < m\} \subseteq \underset{\sim}{\mathcal{A}}_j$. By the assumptions on P_i we have that $B_1 = \big\{n < \omega \,:\, \bigwedge_{\ell<k} f_\ell(n) \notin \bigcup\{A_\ell \,:\, \ell < m_0\}\big\}$ is infinite. It belongs to $\mathbf{V}[G \cap P_i]$. Since we have that $(P, \underset{\sim}{\mathcal{A}}_i) \leq^\kappa_{app} (P_j, \underset{\sim}{\mathcal{A}}_j)$ and $\{\underset{\sim}{A}_\ell \,:\, \ell \in [m_0, m)\} \subseteq \underset{\sim}{\mathcal{A}}_j \setminus \underset{\sim}{\mathcal{A}}_i$ and $B_1, f_0, \ldots, f_{k-1} \in \mathbf{V}[G \cap P_i]$, by Definition 2.2(3) clause (d) we are done.

Case 2: $\mathrm{cf}(\delta) = \aleph_0$. W.l.o.g. $\delta = \omega$. So let $k < \omega$, $p \in P$, $p \Vdash$ " for $\ell < k$, $\underset{\sim}{f}_\ell \in {}^\omega\omega$ is injective." By renaming we may assume w.l.o.g. that $p \in P_0$. For every $m < \omega$ we find $\langle \underset{\sim}{f}^m_\ell \,:\, \ell < k\rangle$ such that

$(*)_1$ $\underset{\sim}{f}^m_\ell$ is a P_m-name for a P/G_m-name for an injective function from ω to ω,

$(*)_2$ if $p \in G_m \subseteq P_m$, G_m generic over $\mathbf{V}$ and $m, n < \omega$, then for densely many $q \in P/G_m$ we have that $p \Vdash_{P_m}$ "$q \Vdash_{P/G_m} \bigwedge_{\ell<k} (\underset{\sim}{f}_\ell) \restriction n = (\underset{\sim}{f}^m_\ell[G_m])) \restriction n$".

We give explicit names in the case that $\underset{\sim}{f}_\ell$ is written in the form $\underset{\sim}{f}_\ell = \{((n, a_{\ell,n}), p) \,:\, p \in A_{\ell,n}, n \in \omega, a_{\ell,n} \in \omega\}$ and $A_{\ell,n}$ are suitable maximal antichains. Then we write $\underset{\sim}{f}^m_\ell = \{(((n, a_{\ell,n}), p[G_m]), p \restriction P_m) \,:\, p \in A_{\ell,n}, n \in$

$\omega, a_{\ell,n} \in \omega\}$. Here the $\restriction$ is a projection function that comes with $P_m \lessdot P$ (is a complete suborder of) as explained in [1].

Let A be the union of all antichains appearing in the names $\underset{\sim}{f}^m_\ell$. By the c.c.c. A is countable. So easily $p \Vdash_{P_m}$ "$\underset{\sim}{f}^m_\ell \in {}^\omega A$ is injective".

By the hypothesis on P_m and $\underset{\sim}{\mathcal{A}}_m$ we have that $p \Vdash_{P_m}$ "there is $\underset{\sim}{\mathcal{A}}_m \in [\underset{\sim}{\mathcal{A}}_m]^{<\kappa}$ as in 2.2(1)". As P_m is c.c.c. and because of the form of $\underset{\sim}{\mathcal{A}}_m$ there is $\underset{\sim}{\mathcal{A}}'_m$ a set of $<\kappa$ names from $\underset{\sim}{\mathcal{A}}_m$ such that

$$\text{if } \underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k-1} \in \underset{\sim}{\mathcal{A}}_m \setminus \underset{\sim}{\mathcal{A}}'_m \text{ then}$$
$$p \Vdash_{P_m} \text{"}\left\{ n \,:\, \bigwedge_{\ell<k} \underset{\sim}{f}^m_\ell(n) \notin \underset{\sim}{A}_0 \cup \dots \cup \underset{\sim}{A}_{k'-1} \right\} \text{ is infinite."}$$

So it is enough to show that $\underset{\sim}{\mathcal{A}}' = \bigcup_{m<\omega} \underset{\sim}{\mathcal{A}}'_m$ is as required. Let $k' < \omega$, $\underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k'-1} \in \underset{\sim}{\mathcal{A}} \setminus \underset{\sim}{\mathcal{A}}'$ and towards a contradiction assume that $q \Vdash$ "$\{n < \omega : \bigwedge_{\ell<k} \underset{\sim}{f}_\ell(n) \notin \underset{\sim}{A}_0 \cup \dots \cup \underset{\sim}{A}_{k'-1}\} \subseteq [0, m^*]$." So for some m we have that $q \in P_m$, $\underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k'-1} \in \underset{\sim}{\mathcal{A}}_m \setminus \underset{\sim}{\mathcal{A}}'_m$. Let $q \in G_m \subseteq P_m$ be P_m generic over $\mathbf{V}$. In $\mathbf{V}[G_m]$ we have that $B' = \{n \in \omega : \bigwedge_{\ell<k} \underset{\sim}{f}^m_\ell[G_m](n) \notin \underset{\sim}{A}_0[G_m] \cup \dots \cup \underset{\sim}{A}_{k'-1}[G_m]\}$ is infinite. So we can find $n \in B'$ such that $n > m^*$. Now there are densely many $q' \in P/G_m$ forcing $\underset{\sim}{f}_\ell(n) = \underset{\sim}{f}^m_\ell(n)$, so w.l.o.g. $q \le q' \in P/G_m$, and we find $p' \in G$ such that $p \le p' \in P$ and $p' \Vdash$ "$\underset{\sim}{f}_\ell(n) = \underset{\sim}{f}^m_\ell(n)$". Contradiction. $\square$

Claim 2.5. *Assume that $(P, \underset{\sim}{\mathcal{A}})$ is a κ-approximation.*

(1) *If $\Vdash$ "$\underset{\sim}{Q}$ is Cohen or just $<\kappa$-centred ", then $(P * \underset{\sim}{Q}, \underset{\sim}{\mathcal{A}})$ is a κ-approximation, and $(P, \underset{\sim}{\mathcal{A}}) \le^\kappa_{app} (P * \underset{\sim}{Q}, \underset{\sim}{\mathcal{A}})$.*

(2) *If in addition $\Vdash_P$ "$\langle w_n : n < \omega\rangle$ is a set of finite non-empty pairwise disjoint subsets of ω", and Q is Cohen forcing, and $\underset{\sim}{\eta}$ is the $P * \underset{\sim}{Q}$-name of the generic, then $(P * \underset{\sim}{Q}, \underset{\sim}{\mathcal{A}} \cup \{\bigcup\{w_n : \underset{\sim}{\eta}(n) = 1\}\})$ is a κ-approximation, and $\le^\kappa_{app}$-above $(P, \underset{\sim}{\mathcal{A}})$.*

Proof. (1) Let $G \subseteq P$ be P-generic over $\mathbf{V}$. We work in $\mathbf{V}[G]$. It is enough to prove that in $(\mathbf{V}[G])^Q$, $\mathcal{A} = \underset{\sim}{\mathcal{A}}[G]$ is a $(\kappa, \mathfrak{g})$-witness. let $Q = \bigcup_{m \in \mu} Q_m$, Q_m directed, $\mu < \kappa$. So let $\Vdash_Q$ "$\underset{\sim}{f}_0, \dots \underset{\sim}{f}_{k-1} \in {}^\omega\omega$ are injective." For each $m < \mu$ we find $\langle f^m_\ell : \ell < k\rangle$ such that

$(*)_1$ f^m_ℓ is a partial function from ω to ω,

$(*)_2$ if $q \in Q_m$, $m < \mu$, $n < \omega$ then $q \nVdash_Q$ "$\bigvee_{\ell<k} (\exists n' < n)(f^m_\ell(n')$ is defined and $\underset{\sim}{f}_\ell(n') \ne f^m_\ell(n'))$".

Just take $f^m_\ell = \underset{\sim}{f}_\ell[Q_m]$. Since Q_m is directed, this is well-defined. If there is some ℓ such that $\mathrm{dom}(f^m_\ell)$ is infinite, then for $\langle f^m_\ell : \ell < k, \mathrm{dom}(f^m_\ell) \text{ infinite}\rangle$ we choose some $\mathcal{A}'_m \in [\mathcal{A}]^{<\kappa}$ as required in Definition 2.1(1). If there is so such

ℓ, then we let $\mathcal{A}'_m = \emptyset$. Let $\mathcal{A}' = \bigcup_{m<\mu} \mathcal{A}'_m$, it is clearly as required. This is shown similarly to 2.4. In the the end of the proof of 2.4 write $\{0\}$ instead of G_m and Q_m instead of P_m.

(2) We prove clause (d) of 2.2(3). Let $G \subseteq P$ be P-generic over $\mathbf{V}$. So let $f_0, \dots, f_{k-1} \in V[G]$, $B \in ([\omega]^\omega)^{\mathbf{V}[G]}$ and we should prove that $\{n \in B : \bigwedge_{\ell<k} f_\ell(n) \notin \bigcup\{w_m : \underset{\sim}{\eta}[G](n) = 1\}\}$ is infinite. As $\underset{\sim}{\eta}$ is Cohen and the w_n are pairwise disjoint and finite and non-empty, this follows from a density argument. $\square$

An ultrafilter D on ω is called Ramsey iff for every function $f\colon \omega \to \omega$ there is some $A \in D$ such that $f \restriction A$ is injective or is constant.

Claim 2.6. *Assume that*

- *(a)* $\mathbf{V} \models \mathrm{CH}$,
- *(b)* $P = \langle (P_i, \underset{\sim}{\mathcal{A}}_i) : i \leq \delta \rangle$ *is* $\leq^{\aleph_1}_{app}$*-increasing and continuous and* $|P_i| \leq \aleph_1$,
- *(c)* $\mathrm{cf}(\delta) = \aleph_1 = |\delta|$,
- *(d)* $\delta = \sup\{i < \delta : P_{i+1} = P_i * \mathrm{Cohen}, \underset{\sim}{\mathcal{A}}_{i+1} = \underset{\sim}{\mathcal{A}}_i\}$,
- *(e)* $G \subseteq P_\delta$ *is* P_δ*-generic over* $\mathbf{V}$*, and in* $\mathbf{V}[G]$ *we have* $\mathcal{A} = \bigcup_{i<\kappa} \underset{\sim}{\mathcal{A}}_i[G]$.

Then

- *(1)* *In* $\mathbf{V}[G]$ *there is* $\bar{M}$ *leisurely exemplifying* $\mathcal{A}$.
- *(2)* *In* $\mathbf{V}[G]$ *there is a Ramsey ultrafilter* D *such that for every* $f \in {}^\omega\omega$ *which is not constant on any set in* D *and for all but countably [*$< \kappa$*] many* $A \in \mathcal{A}$ *we have that* $\{n : f(n) \notin A\} \in D$*. In short we say "*$D$ *is* $\mathcal{A}$*-Ramsey [*$(\kappa, \mathcal{A})$*-Ramsey]".*

Proof. (1) By renaming, w.l.o.g. $\delta = \aleph_1$. Let $\chi \geq (2^{\aleph_0})^+$ and let $\bar{M}^0 = \langle M^0_i : i < \omega_1 \rangle$ be increasing and continuous and $M^0_i \prec (H(\chi), \in, <^*_\chi)$, M^0_i countable and $\underset{\sim}{\mathcal{A}}_i \in M^0_i$ and $\bar{M}^0 \restriction (i+1) \in M^0_{i+1}$ and such that $\mathcal{P}(\omega) \subseteq \bigcup_{i<\omega_1} M^0_i$ and hence $\bigcup_{i<\omega_1} \underset{\sim}{\mathcal{A}}_i \in M^0_i$. Let $M^1_i = M^0_i[G]$, $\mathcal{A}_i = \underset{\sim}{\mathcal{A}}_i[G]$. Since P is proper, we have that M^1_i is countable. For any $i < \omega_1$ we shall find $j(i) \geq i, j(i-1)+1$ and $N_{j(i)}$ such that

$$(**)\quad \begin{array}{l} (\alpha)\ M^1_{j(i)} \subseteq N_{j(i)} \subseteq M^1_{j(i)+1}, \\ (\beta)\ N_{j(i)} \models |\mathcal{A}_{j(i)}| = \aleph_0, \\ (\gamma)\ N_{j(i)} \in M^1_{j(i)+1}, \\ (\delta)\ \mathcal{A}_\delta \cap N_{j(i)} = \mathcal{A}_\delta \cap M^1_{j(i)}, \\ (\varepsilon)\ (\underset{\sim}{f} \in \bigcup\limits_{i<\omega_1} M^0_i \ \wedge\ \underset{\sim}{f}[G] \in M^1_i \cap {}^\omega\omega) \rightarrow \underset{\sim}{f} \text{ is a } P_{j(i)}\text{-name}, \\ (\zeta)\ M^1_i \models |X| < \aleph_1 \Rightarrow X \subseteq M^1_{j(i)}. \end{array}$$

In M^1_i, choose $j = j(i)$ according to the premise (d) such that $\sup(M^1_i \cap \omega_1) < j < \omega_1$ and $P_{j+1} = P_j * \text{Cohen}$, $\underset{\sim}{\mathcal{A}}_{j+1} = \underset{\sim}{\mathcal{A}}_j$ and such that (ε) and (ζ) are true. In M^0_{j+1} we define the forcing notion $R_j = \{g : g$ is a function from some $n < \omega$ into $\underset{\sim}{\mathcal{A}}_{j+1} \cap M^0_{j+1}\}$. This is a variant of Cohen forcing, and hence we can interpret R_j as the Cohen forcing in P_{j+1}. We let $\hat{g}$ be generic and set $N_j = M^1_j[\hat{g}]$. Now we take a club C in ω_1 such that $(\forall \alpha \in C)(\forall \beta < \alpha)(j(\beta) < \alpha)$. We let $\langle c(i) : i < \omega_1\rangle$ be an increasing enumeration of C. Finally we let for $i < \omega_1$, $M_i = M^1_{c(i)}$ for limit i.

We have to show that in $\mathbf{V}[G]$, $\bar{M}$ κ-exemplifies $\mathcal{A}$. That is, according to 2.1(2):

(a) $\mathcal{A}$ is an $(\aleph_1, \mathfrak{g})$-witness,

(b) $\bar{M} = \langle M_i : i < \aleph_1\rangle$ is $\prec$-increasing and continuous, and $\omega + 1 \subseteq M_0$ and $\mathcal{P}(\omega) \subseteq \bigcup_{i<\kappa} M_i$,

(c) $M_i \subseteq (H(\chi), \in)$ is a model of ZFC$^-$ and $|M_i| < \aleph_1$ and $(M_i \models |X| < \aleph_1) \Rightarrow X \subseteq M_i$,

(d) $\bar{M} \restriction (i+1) \in M_{i+1}$,

(e) for non-limit i there is $\mathcal{A}_i \in M_i$ such that $\mathcal{A} \cap M_i = \mathcal{A}_i$,

(f) if $i < \aleph_1$, $k < \omega$ and $f_\ell \in M_i$ is an injective function from ω to ω for $\ell < k$, and $k' < \omega$, $A_\ell \in \mathcal{A} \setminus M_i$ for $\ell < k'$, then
$$\Big\{n : \bigwedge_{\ell<k} f_\ell(n) \notin A_0 \cup \dots \cup A_{k'-1}\Big\} \text{ is infinite.}$$

Item (a) follows from 2.4. The items (b) and (c) follow from $M^0_i \prec (H(\chi), \in, <^*_\chi)$, M^0_i countable and $\bar{M}^0 \restriction (i+1) \in M^0_{i+1}$ and such that $\mathcal{P}(\omega) \subseteq \bigcup_{i<\omega_1} M^0_i$.

The item (d) is clear by our choice of M_i.

The item (e) follows from (δ) in $(**)$ and the fact that the $c(i)$'s are limits and $\bigcup_{\beta<c(i)} N_{j(\beta)} = \bigcup_{\beta<c(i)} M_{j(\beta)+1}$.

To show item (f), suppose that $i < \omega_1$ and $f_\ell \in M_i$ for $\ell < k$ and $A_\ell \in \mathcal{A} \setminus M_i$. Then we have that $\underset{\sim}{f_\ell} \in V^{P_i}$ and $A_\ell \in \mathcal{A} \setminus \mathcal{A}_i$ (the latter holds by (e)) and $\mathcal{A}_i =$

$\underset{\sim}{\mathcal{A}}_i[G] = \underset{\sim}{\mathcal{A}}_i[G_i]$ by our choice of C. Hence we may use $(P_i, \underset{\sim}{\mathcal{A}}_i) \leq^{\aleph_1}_{app} (P_{\omega_1}, \underset{\sim}{\mathcal{A}})$ and get from 2.2(3)(d) if $k < \omega$ and $\underset{\sim}{A}_0, \dots, \underset{\sim}{A}_{k-1} \in \underset{\sim}{\mathcal{A}} \setminus \underset{\sim}{\mathcal{A}}_i$ then

$$\begin{aligned}\Vdash_{P_{\omega_1}} \text{“ if } & B \in ([\omega]^{\aleph_0})^{V^{P_i}}, \\ & \underset{\sim}{f}_\ell \in ({}^B\omega)^{V^{P_i}} \text{for } \ell < k, \text{ then} \\ & \left\{ n \in B : \bigwedge_{\ell<k} \underset{\sim}{f}_\ell(n) \notin \bigcup_{\ell<k} \underset{\sim}{A}_\ell \right\} \text{ is infinite”,}\end{aligned}$$

so we get the desired property in $\mathbf{V}[G]$.

(2) We work in $\mathbf{V}[G]$. We take $\langle M_i : i < \omega_1 \rangle$ as in (1), and choose by induction on $i < \omega_1$ sets B_i such that

(α) $B_i \in M_{i+1}$,

(β) $j < i \Rightarrow B_i \subseteq^* B_j$,

(γ) if $i = j+1$ and $f \in M_j \cap {}^\omega\omega$ is injective and $A \in \mathcal{A} \cap (M_i \setminus M_j)$, then $B_i \subseteq^* \{n : f(n) \notin A\}$,

(δ) if i is limit and $f \in M_i \cap \omega^\omega$ then for some n^* we have that $f \restriction (B_i \setminus n^*)$ is constant or $f \restriction (B_i \setminus n^*)$ is injective.

(ε) B_i is $<^*_\chi$-first of the sets fulfilling (α) – (δ).

Now it is easy to carry out the induction and to show that D, the filter generated by $\{B_i : i < \omega_1\}$ is as required. We use property (f) of $\bar{M}$ in order to show that requirement (γ) is no problem. $\square$

Claim 2.7. *Assume that in* $\mathbf{V}$

(a) $\mathcal{A}$ *is a* $(\kappa, \mathfrak{g})$*-witness,*

(b) D *is a* $(\kappa, \mathcal{A})$*-Ramsey,*

(c) $Q_D = \{(w, A) : w \in [\omega]^{<\omega}, A \in D\}$, $(w, A) \leq (w', A')$ *iff* $w \subseteq w' \subseteq w \cup A$ *and* $A' \subseteq A$.

Then $\Vdash_{Q_D}$ *“*$\mathcal{A}$ *is a* $(\kappa, \mathfrak{g})$*-witness.”.*

Proof. For $u \in [\omega]^{<\aleph_0}$ let $Q_u = \{(u, A) : A \in D\}$. This is a directed subset and we have that $Q_D = \bigcup \{Q_u : u \in [\omega]^{<\aleph_0}\}$. So assume that $p = (w, A) \in Q_D$ and

$$p \Vdash_{Q_D} \text{“}\underset{\sim}{f}_\ell \in {}^\omega\omega \text{ is injective for } \ell < k\text{”.}$$

For $q \in Q_D$ we write $\mathrm{pos}(q) = \{s \in [\omega]^{<\aleph_0} : \exists B\ (s, B) \geq q\}$, the set of possible finite extensions. For $s \in \mathrm{pos}(u, B)$ we set $q^{[s]} = (s, B \setminus \max(s))$. As usual we write $p || \varphi$ if $p \Vdash \varphi$ or $p \Vdash \neg\varphi$ and $q \geq_{tr} p$ iff $q \geq p$ and $q = (w^q, A^q)$, $p = (w^p, A^p)$ and $w^q = w^p$.

For every $u \in \mathrm{pos}(p)$ we define $f^u_\ell \in {}^\omega(\omega+1)$ as follows:

$$(\otimes) \qquad \begin{aligned} &f^u_\ell(n) = m \text{ if } (\exists p \in Q_u)(p \Vdash \underset{\sim}{f}_\ell(n) = m),\\ &f^u_\ell(n) = \omega \text{ if } (\forall m)\neg(\exists p \in Q_u)(p \Vdash \underset{\sim}{f}_\ell(n) = m). \end{aligned}$$

Since D is Ramsey [5] (without Ramsey but using memory [8]) we have that Q_D has the pure decision property:

$$(\otimes\otimes) \qquad \begin{aligned} &\forall p \in Q_D \exists q \geq_{tr} p \, \forall \ell < k \, \forall m \in \omega \, \forall n \in (\omega+1)\\ &\Big(((\exists q' \geq q)(q' || \underset{\sim}{f}_\ell(n) = m)) \rightarrow (\exists s \in \mathrm{pos}(q'))(q^{[s]} || \underset{\sim}{f}_\ell(n) = m\Big). \end{aligned}$$

We apply this to our initial p and get some q as in $(\otimes\otimes)$, which we fix. For every $u \in \mathrm{pos}(q)$ and $\ell < k$ we can find $g^u_\ell \in {}^\omega\omega$ injective, such that if $\{n : f^u_\ell(n) < \omega\} \in D$ then $\{n : f^u_\ell(n) = g^u_\ell(n)\} \in D$.

We call u (v,n)-critical if

$$(*)^u_{v,n} \qquad \begin{aligned} &(\alpha)\ u \in [\omega]^{<\omega},\\ &(\beta)\ \emptyset \neq v \subseteq \{0,\dots,k-1\},\\ &(\gamma)\ \ell \in v \Rightarrow f^u_\ell(n) = \omega,\\ &(\delta)\ \{m : (\forall \ell \in v) f^{u\cup\{m\}}_\ell(n) < \omega\} \in D,\\ &(\varepsilon)\ \ell < k \wedge \ell \notin v \rightarrow \{m : f^{u\cup\{m\}}_\ell(n) = f^u_\ell(n)\} \in D. \end{aligned}$$

For u (v,n)-critical and $\ell \in v$ note that $\lim_D \langle f^{u\cup\{m\}}_\ell(n) : m < \omega\rangle = \infty$. Proof: If for some $k < \omega$, $\{m : f^{u\cup\{m\}}_\ell(n) < k\} \in D$, then there is some $k' < k$ such that $X = \{m : f^{u\cup\{m\}}_\ell(n) = k'\} \in D$. For $m \in X$, we choose a witness $p_m \in Q_{u\cup\{m\}}$, $p_m \Vdash \underset{\sim}{f}_\ell(n) = k'$. Since D is Ramsey, we may glue all the witnesses together (find a common second component), and thus get a condition in Q_u that shows that $f^u_\ell(n) < \omega$, in contrast to condition (γ).

As D is Ramsey for some $A = A_{u,v,n} \in D$ we have if $\ell \in v$ then $\langle f^{u\cup\{m\}}_\ell(n) : m \in A\rangle$ is without repetition.

So we can find for $\ell \in v$ injective functions $h^{u,v,n}_\ell \in {}^\omega\omega$ such that $\{m : f^{u\cup\{m\}}_\ell(n) = h^{u,v,n}_\ell(m)\} \in D$.

For each injective function $h \in {}^\omega\omega$ we have that $\mathcal{A}_h = \{A \in \mathcal{A} : \{n : h(n) \in A\} \in D\}$ is empty or at least of cardinality strictly less than κ. Let $\mathcal{A}' = \bigcup\{\mathcal{A}_h : h = g^u_\ell$ for some $\ell < h$, $u \in [\omega]^{<\aleph_0}$ or $h = h^{u,v,n}_\ell$ where u is (v,n)-critical and $\ell \in v$ and $\emptyset \neq v \subseteq k$ $\}$. So $\mathcal{A}' \subseteq \mathcal{A}$ is of cardinality strictly less than κ and it is enough to prove that if $A_0, \dots A_{k'-1} \in \mathcal{A} \setminus \mathcal{A}'$ then $\Vdash_Q$ "$\{n : \bigwedge_{\ell<k} \underset{\sim}{f}_\ell(n) \notin A_0 \cup \dots \cup A_{k'-1}\}$ is infinite".

Let $A_0, \ldots, A_{k'-1}$ be given. Set $B^* = A_0 \cup \cdots \cup A_{k'-1}$. Towards a contradiction we assume that $p^* \in Q_D$, $p^* \geq q$ and $n^* < \omega$ and

$$p^* \Vdash \text{“}(\forall n)\left(n^* < n < \omega \to \bigvee_{\ell<k} \underset{\sim}{f}_\ell(n) \in B^*\right)\text{”}.$$

Let $M \prec (H(\chi), \in)$ be countable such that the following are elements of M: p^*, D, $\underset{\sim}{f}_\ell$ for $\ell < k$, A_ℓ for $\ell < k'$, $\mathcal{A}'$, $\langle g^u_\ell : u \in [\omega]^{<\aleph_0}, \ell < k\rangle$, $\langle h^{u,v,n}_\ell : u \in [\omega]^{<\aleph_0}, \ell \in v, \emptyset \neq v \subseteq k\rangle$.

Let $p^* = (u^*, A^*)$. Let $A^\odot \in d$ and $A^\odot \subseteq A^*$ be such that $(\forall Y \in D \cap M)(A^\odot \subseteq^* Y)$ and $\min(A^\odot) \geq \sup(u^*)$. It is obvious that $u^* \cup A^\odot$ is generic real for Q_D over M, i.e.: $\{(u', A') \in Q_D \cap M : u' \subseteq u^* \cup A^\odot \subseteq u' \cup A'\}$ is a subset of a $(Q_D)^M$-generic over M.

As $A_0, \ldots, A_{k'-1} \in \mathcal{A} \setminus \mathcal{A}' \subseteq \mathcal{A} \setminus \bigcup_{\ell<k} \mathcal{A}_{g^{u^*}_\ell}$ there is $n^\odot \in [n^*, \omega)$ such that $\ell < k \Rightarrow g^{u^*}_\ell(n^\odot) \notin B^*$ and $g^{u^*}_\ell(n^\odot) = f^{u^*}_\ell(n^\odot)$. Let

$$\mathcal{U} = \{u : u^* \subseteq u \subseteq u^* \cup A^\odot, u \text{ finite, } (\forall \ell < k)(f^u_\ell(n^\odot) < \omega \to f^u_\ell(n^\odot) \notin B^*\}.$$

Now clearly $u^* \in \mathcal{U}$. Choose $u^\odot \in \mathcal{U}$ such that $|\{\ell : f^{u^\odot}_\ell(n^\odot) = \omega\}|$ is minimal. If it is zero, we are done. So assume that is is not zero.

We choose by induction on $i < \omega$ n_i such that

$$(\diamondsuit) \qquad \begin{aligned} &n_i \in A^\odot, \\ &n_i < n_{i+1}, \\ &\sup(u^\odot) < n_i. \\ &\ell < k \to f^{u^\odot}_\ell(n^\odot) = f^{u^\odot \cup \{n_j : j<i\}}_\ell(n^\odot). \end{aligned}$$

By the pure decision property there is some $s \in \mathrm{pos}(u^\odot, A^\odot)$ such that $(s, A^\odot \setminus \max(s))$ decides $\underset{\sim}{f}_\ell(n^\odot)$. So for some i and $\{n_j : 0 \leq j < i\}$ we cannot choose n_i. Let $u^\triangle = u^\odot \cup \{n_j : j < i\}$. Let $v = \{\ell < k : \{m : f^{u^\triangle \cup \{m\}}_\ell(n^\odot) \neq f^{u^\triangle}_\ell(n^\odot)\} \in D\} \subseteq \{0, \ldots, k-1\}$. Let $C = \{m : (\ell \in v \to f^{u^\triangle \cup \{m\}}_\ell(n^\odot) \neq f^{u^\triangle}_\ell(n^\odot))$ and $(\ell \notin v \to f^{u^\triangle \cup \{m\}}_\ell(n^\odot) = f^{u^\triangle}_\ell(n^\odot))\}$. So $C \in D$ and necessarily $\ell \in v \wedge m \in C \Rightarrow f^{u^\triangle \cup \{m\}}_\ell(n^\odot) < f^{u^\triangle}_\ell(n^\odot) = \omega$. So $u^\triangle$ is $(v, n^\odot)$-critical. Hence $C_1 = \{m : \bigwedge_{\ell \in v} h^{u^\triangle, v, n^\odot}_\ell(m) \notin B^*\} \in D$. Choose $n_i \in C_1 \cap C \cap A^\odot$ large enough. If $v = \emptyset$, it can serve as n_i and we have a contradiction. Recall that $h^{u^\triangle, v, n^\odot}_\ell(n_i) = f^{u^\triangle \cup \{n_i\}}_\ell(n^\odot) < \infty$. If $v \neq \emptyset$, then $u^\triangle \cup \{n_i\}$ contradicts the choice of $u^\odot$, because we had required that $|\{\ell : f^{u^\odot}_\ell(n^\odot) = \omega\}|$ is minimal.

$\square$

Later we shall use Claim 1.6 in order to fulfil premise (2) of the following Claim 2.8, which is together with 2.4, 2.5, 2.6, 2.7 the justification of the single steps of our final construction of length $\aleph_2$. Claim 2.8 serves to show that certain (and in the end we want to have: all) cofinality witnesses in intermediate ZFC models are not cofinality witnesses any more in any forcing extension.

Claim 2.8. *Assume that* $\mathbf{V}$, $\mathrm{cf}(\delta) = \omega_1$, $\langle (P_i, \underset{\sim}{\mathcal{A}}_i) : i \leq \delta \rangle$ *are as in 2.6, and*

(1) $\Vdash_{P_\delta}$ "$\langle \underset{\sim}{K}_i : i < \omega_1 \rangle$ *is a cofinality witness and* $\{f \in \mathrm{Sym}(\omega) : (\forall^\infty n) f(n) = n\} \subseteq \underset{\sim}{K}_0$".

(2) *Let* $E_0 = \{(n_1, n_2) : (\exists n)(n_1, n_2 \in [n^2, (n+1)^2)\}$, $A = \bigcup\{[(2n)^2, (2n+1)^2) : n \in \omega\}$. *(Any* $E \in \mathcal{E}_{con}$ *and* $A \in [\omega]^{\aleph_0}$ *could have served as well.) Assume that in* $\mathbf{V}^{P_\delta}$, $S_{E_0,A}$ *is not included in any* K_i.

(3) $\delta = \sup\{\alpha : \underset{\sim}{Q}_\alpha$ *is Cohen,* $\underset{\sim}{\mathcal{A}}_\alpha = \underset{\sim}{\mathcal{A}}_{\alpha+1}\}$.

Then *there is a* P_δ*-name* $\underset{\sim}{Q}$ *such that*

(α) $(P_\delta, \underset{\sim}{\mathcal{A}}_\delta) \leq^\kappa_{app} (P_\delta * \underset{\sim}{Q}, \underset{\sim}{\mathcal{A}}_\delta)$,

(β) $\Vdash_{P_\delta}$ "$\underset{\sim}{Q} \subseteq \underset{\sim}{Q}'_{E_0}$ *(where* Q'_{E_0} *is from 1.7).*

(γ) $\Vdash_{P_\delta * \underset{\sim}{Q}}$ "$\underset{\sim}{g} = \bigcup\{\underset{\sim}{f} : (p, \underset{\sim}{f}) \in \underset{\sim}{G}(P_\delta * \underset{\sim}{Q})\}$ *is a permutation of* ω *and for arbitrarily large* $i < \omega_1$, $\langle \underset{\sim}{g}, \underset{\sim}{K}_i \rangle_{\mathrm{Sym}(\omega)} \cap \mathrm{Sym}(\omega)^{\mathbf{V}[P_\delta]} \neq \underset{\sim}{K}_i$".

Proof. As in 2.6, we assume w.l.o.g. $\delta = \omega_1$. We can find in $\mathbf{V}$, $\bar{\underset{\sim}{g}}^* = \langle \underset{\sim}{g}^*_i : i < \omega_1 \rangle$ such that $\Vdash_{P_{\omega_1}}$ "$\underset{\sim}{g}^*_i \in \mathrm{Sym}(\omega) \setminus \underset{\sim}{K}_i$, $\underset{\sim}{g}^*_i \in S_{E_0,A}$, and $\underset{\sim}{g}^*_0 \in M_0 \prec (\underset{\sim}{H}(\chi, \in)$, M_0 countable". In $\mathbf{V}$ we now choose by induction on $i < \omega_1$ $\underset{\sim}{M}_i, \underset{\sim}{N}_i, \underset{\sim}{p}_i, \alpha_i$ such that

(a) $\langle \underset{\sim}{M}_j : j \leq i \rangle$ is a sequence of $\mathbf{V}^{P_\delta}$-names as in 2.6,

(b) $\Vdash_{P_\delta} \bar{Q}, \underset{\sim}{\mathcal{A}}, \bar{\underset{\sim}{g}}^*, \langle \underset{\sim}{K}_i : i < \omega_1 \rangle \in \underset{\sim}{M}_0$,

(c) $\underset{\sim}{N}_i = \{\underset{\sim}{\tau}_{1,n} : n \in \omega\}$ is a countable P_{α_i}-name such that $\Vdash_{P_{\alpha_i}}$ "$M_i[\underset{\sim}{G}_{P_{\alpha_i}}] \subseteq N_i \subseteq (H(\chi)^{\mathbf{V}[P_{\alpha_i}]}, \in), ||N_i|| = \aleph_0, N_i \models \mathrm{ZFC}^-$",

(d) $\underset{\sim}{p}_i \in Q'_{E_0}$ is hereditarily countable and a P_{α_i}-name of a member Q'_{E_0}, $\Vdash_{P_{\alpha_i}} \langle \underset{\sim}{p}_j : j \leq i \rangle$ is $\subseteq^*$-increasing and $\in \underset{\sim}{N}_i, \underset{\sim}{p}_i \in \underset{\sim}{N}_i$,

(e) in $\mathbf{V}^{P_\delta}$ we have $\underset{\sim}{M}_i[G_\delta] = M_i$ and $\langle \underset{\sim}{N}_j : j \leq i \rangle \in M_{i+1}$, $\sup(M_i \cap \omega_1) \leq \alpha_i \in M_{i+1}$, $\underset{\sim}{Q}_{\alpha_i}$ is Cohen and $\underset{\sim}{\mathcal{A}}_{\alpha_i} = \underset{\sim}{\mathcal{A}}_{\alpha_i+1}$,

(f) if $\underset{\sim}{I} \in N_i$ is a P_{α_i}-name of a predense subset of $Q'_{E_0}(\langle \underset{\sim}{p}_j : j < i \rangle) = \underset{\sim}{Q}_{\alpha_i}$, then some finite $J(\underset{\sim}{I}) \subseteq \underset{\sim}{I}$, $J(\underset{\sim}{I}) \in N_i$, is predense above $\underset{\sim}{p}_i$ in $Q'_{E_0}(\langle \underset{\sim}{p}_j : j \leq i \rangle)$ in the universe $\mathbf{V}^{P_{\alpha_i+1}}$.

At limit stages i we take for M_i the union of the former M_j. Otherwise choose M_i as required. Next we choose α_i such that $\sup(M_i \cap \omega_1) \leq \alpha_i < \omega_1$ and $\underset{\sim}{Q}_{\alpha_i}$ is Cohen and $\underset{\sim}{\mathcal{A}}_{\alpha_i} = \underset{\sim}{\mathcal{A}}_{\alpha_{i+1}}$. We work in $\mathbf{V}[P_{\alpha_i}]$. We set $N^0_i = M_i[\underset{\sim}{G}_{P_{\alpha_i}}]$. We now interpret the Cohen forcing as $R_0 \times R_1 \times R_2$ where

$$R_0 = \{h : (\exists n < \omega) h \colon n \to \mathcal{P}(\omega)^{M_i}\}$$

ordered by inclusion. In $N_i^1 = N_i^0[G_{R_0}] = M_i[G_{P_{\alpha_i}}][G_{R_0}]$ we let

$$R_1 = \{(n,q) : n < \omega, q \in Q'_{E_0}(\langle p_j : j < i\rangle)\},$$

ordered by $(n_1,q_1) \le (n_2,q_2) \Leftrightarrow n_1 \le n_2 \wedge q_1 \restriction n = q_2 \restriction n \wedge q_1 \le q_2$. Since $(Q'_{E_0})^{N_i^1}$ is countable we have that R_1 is Cohen forcing. Let $N_i^2 = N_i^1[G_{R_0}, G_{R_1}] = M_i[G_{P_{\alpha_i}}][G_{R_0}][G_{R_1}]$, $q_i' = \bigcup\{q : (n,q) \in G_{R_1}\}$.

Now we choose $q_i \supseteq^* q_i'$ such that q_i has the properties of f_α in 1.8(2)(b) for the sequence $\bar f = \langle p_j : j < i\rangle$. So clearly $q_i \in (Q'_{E_0})^{\mathbf{V}[P_{\alpha_i+1}]}$, $\bigwedge_{j<i} p_j \subseteq^* q_i$.

We can find in N_i^2 a sequence $\langle w_k^i : k < \omega\rangle$ and h_i^* such that

$$(\diamondsuit\diamondsuit)\begin{cases} k_1 \ne k_2 \Rightarrow w_{k_1}^i \cap w_{k_2}^i = \emptyset, \\ w_k^i \text{ is included in some } E_0\text{-equivalence class,} \\ w_k^i \subseteq \omega \setminus \mathrm{dom}(q_i), \\ \forall n \exists m \left(\left| [m]_E \setminus \mathrm{dom}(q_i) \setminus \bigcup_{k\in\omega} w_k^i \right| > n \right), \\ h_i^* \in \mathrm{Sym}(\omega), \\ h_i^* \text{ maps } \{[n]_{E_0} : n \in A\} \text{ onto } \{w_k^i : k < \omega\} \\ \text{more precise, } \hat{h_i^*} \text{ does this, where for } b \subseteq \omega,\ \hat{h_i^*}(b) = \mathrm{range}(h_i^* \restriction b). \end{cases}$$

Let

$$R_2 = \left\{ f : (\exists m < \omega) \left(f \text{ is a permutation of } \bigcup_{k<m} w_k^i \text{ mapping } w_k^i \text{ into itself} \right) \right\},$$

ordered by inclusion. In $N_i^3 = N_i^2[G_{R_2}]$ let $f_i^{\odot} = \bigcup G_{R_2}$ so $N_i^3 = N_i^2[f_i^{\odot}]$.

So $N_i^3 \in \mathbf{V}^{P_{\alpha_i}+1}$, and hence is a P_{α_i+1}-name. As P_{α_i+1} has the c.c.c., we can assume that this name is hereditarily countable. Now $N_i^3 \cap \omega_1 = N_i^0 \cap \omega_1 = M_i[G_{\alpha_i}] \cap \omega_1 = \delta_i < \omega_1$, hence $N_i^3 \cap \mathrm{Sym}(\omega)^{\mathbf{V}[P_\delta]} \subseteq K_{\delta_i}$. Let

$$f_i^{\boxdot} = (h_i^* \circ g_{\delta_i}^* \circ (h_i^*)^{-1} \restriction \bigcup_{k<\omega} w_k^i) \circ f_i^{\odot}.$$

It is still generic for R_2 over $\mathbf{V}^{P_{\alpha_i}}[G_{R_0}, G_{R_1}]$. We set $N_i^4 = N_i^3[f_i^{\boxdot}]$, $q_i^4 = q_i \cup f_i^{\boxdot}$. Now (N_i^4, q_i^4) are as required. We choose $(\underset{\sim}{N}_i, \underset{\sim}{q}_i^4)$ by taking P_{ω_1}-names $(\underset{\sim}{N}_i, \underset{\sim}{q}_i^4)$ in $\mathbf{V}$ for them. Finally we choose by 1.8(2) some $\underset{\sim}{p}_i$ such that $\underset{\sim}{p}_i \ge \underset{\sim}{q}_i^4$ and $\underset{\sim}{p}_i$ is $(\underset{\sim}{N}_i, \underset{\sim}{Q}_{\alpha_i})$-generic over $\underset{\sim}{N}_i$ and as in (f).

Item (α) of the conclusion is seen as follows: We have for $i < \omega_1$ that $\mathbf{V}^{P_{\omega_1}} \models$ "$Q'_{E_0}(\langle \underset{\sim}{p}_j : j < i\rangle)$ is c.c.c.". Hence we have by 2.5 that $(P_\delta, \underset{\sim}{\mathcal{A}}_\delta) \le_{app}^\kappa (P_\delta * Q'_{E_0}(\langle \underset{\sim}{p}_j : j < i\rangle), \underset{\sim}{\mathcal{A}}_\delta)$, and $(P_\delta * Q'_{E_0}(\langle \underset{\sim}{p}_j : j < i\rangle), \underset{\sim}{\mathcal{A}}_\delta) \le_{app}^\kappa (P_\delta * Q'_{E_0}(\langle \underset{\sim}{p}_j : j < k\rangle), \underset{\sim}{\mathcal{A}}_\delta)$ for $i < k \in \omega_1$. Since $\underset{\sim}{Q} = \underset{\sim}{Q}'_{E_0}(\langle \underset{\sim}{p}_j : j < \omega_1\rangle) = \bigcup_{i<\omega_1} \underset{\sim}{Q}'_{E_0}(\langle \underset{\sim}{p}_j : j < i\rangle)$ we can apply 2.4.

Item (β) of the conclusion follows from the choice of $\underset{\sim}{Q}$.

For item (γ): Fix i. Note that $\delta_i \geq i$. We have in $\mathbf{V}^{P_{\omega_1}}$ that $f_i^{\boxdot} \in K_{\delta_i} = \underset{\sim}{K}_{\delta_i}[G_{\omega_1}]$. We have that $p_i \in (Q'_{E_0})^{\mathbf{V}^{P_{\alpha_i}}}$ and

$$p_i \Vdash_{P_{\omega_1} * \underset{\sim}{Q}} \underset{\sim}{g} \restriction \bigcup_{k \in \omega} w_k^i = \underset{\sim}{f}_i^{\boxdot} \restriction \bigcup_{k \in \omega} w_k^i$$

and hence

$$(\odot) \qquad p_i \Vdash_{P_{\omega_1} * \underset{\sim}{Q}} \underset{\sim}{g}_{\delta_i}^* \restriction A = (h_i^*)^{-1} \circ \underset{\sim}{g} \circ (f_i^{\odot})^{-1} \circ (h_i^*) \restriction A,$$

and thus, since $g_{\delta_i} \restriction A$ contains the same information as g_{δ_i} since the latter is in $S_{E_0,A}$, the equation $\odot$ gives a witness in $\langle \underset{\sim}{g}, K_{\delta_i} \rangle_{\mathrm{Sym}(\omega)} \cap \mathrm{Sym}(\omega)^{\mathbf{V}[P_{\omega_1}]} \setminus \underset{\sim}{K}_{\delta_i}$ and hence shows the inequality claimed in (γ). $\square$

In order to organize the bookkeeping in our final construction of length $\aleph_2$ we use $\diamondsuit(S^2_1)$ in order to guess the names $\langle \underset{\sim}{K}_i : i < \omega_1 \rangle$ of objects that we do not want to have as cofinality witnesses. We recall $S^2_1 = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \aleph_1\}$.

For $E \subseteq \omega_2$ being stationary in ω_2 we have the combinatorial principle $\diamondsuit(E)$: There is a sequence $\langle X_\delta : \delta \in E \rangle$ such that for every $X \subseteq \omega_2$ the set $\{\delta \in E : X_\delta = X \cap \delta\}$ is stationary in ω_2.

For more information about this and related principles and their relative consistency we refer the reader to [3, 2].

Conclusion 2.9. *Assume that* $\mathbf{V}$ *fulfils* $2^{\aleph_0} = \aleph_1$ *and* $\diamondsuit_{S^2_1}$. *Then for some forcing notion* $P \in \mathbf{V}$ *of cardinality* $\aleph_2$ *in* $\mathbf{V}^P$ *we have that* $\mathfrak{g} = \aleph_1$ *and* $\mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{b} = \aleph_2$.

Proof. Let $H(\aleph_2) = \bigcup_{i < \aleph_2} B_i$, B_i increasing and continuous, $B_{i+1} \supseteq [B_i]^{\leq \aleph_0}$ and $\langle X_i \subseteq B_i : i \in S^2_1 \rangle$ is a $\diamondsuit_{S^2_1}$-sequence. We choose by induction on $i < \aleph_2$ $(P_i, \underset{\sim}{\mathcal{A}}_i, d_i)$ such that

- (α) $(P_i, \underset{\sim}{\mathcal{A}}_i)$ is an $\aleph_1$-approximation, $|P_i| \leq \aleph_1$,
- (β) $(P_i, \underset{\sim}{\mathcal{A}}_i)$ is $\leq^\kappa_{app}$-increasing and continuous,
- (γ) d_i is a function from $\underset{\sim}{\mathcal{A}}_i$ to ω_1,
- (δ) if $i < \aleph_2$ and $\langle \underset{\sim}{w}_k : k < \omega \rangle$ is a P_i-name and $\Vdash_{P_i} \langle \underset{\sim}{w}_k : k < \omega \rangle$ are non-empty pairwise distinct and $\gamma < \omega_1$ then for some $j \in (i, \omega_2)$ we have that $\Vdash_{P_{j+1}}$ for some infinite $u \subseteq \omega$ and some $\underset{\sim}{A} \in \underset{\sim}{\mathcal{A}}_{j+1}$ we have that $\bigcup_{k \in u} w_k \subseteq \underset{\sim}{A} \in \underset{\sim}{\mathcal{A}}_{j+1} \wedge d_{j+1}(\underset{\sim}{A}) = \gamma$,
- (ε) for arbitrarily large $i < \omega_2$ we have that $\Vdash_{P_i}$ "$Q_i = Q_{D_i}$ and D_i is a Ramsey ultrafilter",
- (ζ) if $i \in S^2_1$ and $P_i \subseteq B_i$, X_i code of the P_i-name $\langle \underset{\sim}{K}_j : j < \omega_1 \rangle$ and $\Vdash_{P_i}$ "$\langle \underset{\sim}{K}_j : j \in \omega_1 \rangle$ is a cofinality witness of $\mathrm{Sym}(\omega)^{\mathbf{V}[P_i]}$ and $\{f \in$

$\mathrm{Sym}(\omega)^{\mathbf{V}[P_i]}$ respects E_0 and $\supset id_{\omega\setminus A_0}\}$ is not included in any $\underset{\sim}{K}_j$", then $\Vdash_{P_{i+1}}$ " for some $f \in \mathrm{Sym}(\omega)$ for arbitrarily large $j < \omega_1$ we have $\langle \underset{\sim}{K}_j, f\rangle_{\mathrm{Sym}(\omega)} \cap (\underset{\sim}{K}_{j+1})^{\mathbf{V}_i} \neq (\underset{\sim}{K}_j)^{\mathbf{V}_i}$".

Can we carry out such an iteration? We freely use the existence of limits from Claim 2.4 and that $\leq^*_{app}$ is a partial order 2.3. The step $i = 0$ is trivial. So we have to take care of successor steps.

If $i = j + 1$ and $j \notin S^2_1$ then we can use 2.5 to define $(P_\alpha, \underset{\sim}{\mathcal{A}}_\alpha)$, and taking care of clause (δ) by bookkeeping.

If $i = j + 1$ and $j \in S^2_1$ and the assumption of clause (ζ) holds, we apply 2.8 to satisfy clause (ζ), using $\underset{\sim}{Q}'_\zeta = \underset{\sim}{Q}$ from 2.8(β).

If $i = j+1$ and $j \in S^2_1$ but the assumption of clause (ζ) fails (which necessarily occurs stationarily often), we apply 2.6 and 2.7.

Having carried out the induction we let $P = \bigcup_{\alpha<\omega_2} P_\alpha$, $\underset{\sim}{\mathcal{A}} = \bigcup_{\alpha<\omega_2} \underset{\sim}{\mathcal{A}}_\alpha$, $d = \bigcup_{\alpha<\omega_2} d_\alpha$. So $(P, \underset{\sim}{\mathcal{A}})$ is an $(\aleph_2, \aleph_1)$-approximation. For $\gamma \in \omega_1$ we set $\underset{\sim}{\mathcal{A}}^{\langle\gamma\rangle} = \{\underset{\sim}{A} \in \underset{\sim}{\mathcal{A}} : d(\underset{\sim}{A}) = \gamma\}$. Now clearly $\mathbf{V}^{P_{\aleph_2}} \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Let $G \subseteq P$ be generic.

We show: $\Vdash_P \mathfrak{g} = \aleph_1$. For $\delta < \aleph_1$ we have that $\underset{\sim}{\mathcal{A}}^{\langle\delta\rangle}[G]$ is groupwise dense by clause (δ), and always $\mathfrak{g} \geq \aleph_1$. So it is enough to show that the intersection of the $\underset{\sim}{\mathcal{A}}^{\langle\delta\rangle}[G]$ is empty. Suppose that it is not, i.e. that there is some $B \in [\omega]^\omega$ such that for $\delta < \omega_1$ there is some $A_\delta \in \underset{\sim}{\mathcal{A}}^{\langle\delta\rangle}[G]$ such that for all δ, $B \subseteq^* A_\delta$. Now let $h \colon \omega \to B$ be an injective function. But now we have a contradiction to $(P, \underset{\sim}{\mathcal{A}})$ being $(\aleph_2, \aleph_1)$-approximation (see 2.2(3)) and to property (2.1(β)) of $\underset{\sim}{\mathcal{A}}$ is a $(\aleph_1, \mathfrak{g})$-witness.

We show that $\Vdash_P \mathfrak{b} = \aleph_2$. This follows from clause (ε).

Finally we show that $\Vdash \mathrm{cf}(\mathrm{Sym}(\omega)) > \aleph_1$. Suppose that $\langle \underset{\sim}{K}_j[G_{\omega_2}] : j < \omega_1\rangle$ is a cofinality witness in $\mathbf{V}[G_{\omega_2}]$. Then there is a club subset $\underset{\sim}{C}$ in ω_2 such that for $i \in C$ we have that $\langle \underset{\sim}{K}_j[G_i] : j < \omega_1\rangle$ is a cofinality witness in $\mathbf{V}[G_i]$. By $\diamondsuit(S^2_1)$ there is some $i \in S^2_1$ such that X_i is a code of a P_i name of $\langle \underset{\sim}{K}_j[G_i] : j < \omega_1\rangle$. By (the analogues of) Claims 1.4 and 1.6 for Q'_E and because of $\mathfrak{b} = \aleph_2$ and because of clause (ζ) we get that the sequence $\langle \underset{\sim}{K}_j[G_i] : j < \omega_1\rangle$ does not lift to a cofinality witness in $\mathbf{V}[G_{\omega_2}]$ such that for all $j < \omega_1$ we have that $\underset{\sim}{K}_j[G_i] = \underset{\sim}{K}_j[G_{\omega_2}] \cap \mathbf{V}[G_i]$. Hence $\langle \underset{\sim}{K}_j[G_{\omega_2}] : j < \omega_1\rangle$ was no cofinality witness in $\mathbf{V}[G_{\omega_2}]$. $\square$

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