CONSTRUCTING SIMPLE GROUPS FOR LOCALIZATIONS

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Abstract. A group homomorphism $\eta : A \to H$ is called a *localization* of A if every homomorphism $\varphi : A \to H$ can be 'extended uniquely' to a homomorphism $\Phi : H \to H$ in the sense that $\Phi\eta = \varphi$. This categorical concept, obviously not depending on the notion of groups, extends classical localizations as known for rings and modules. Moreover this setting has interesting applications in homotopy theory, see the introduction. For localizations $\eta : A \to H$ of (almost) commutative structures A often H resembles properties of A, e.g. size or satisfying certain systems of equalities and non-equalities. Perhaps the best known example is that localizations of finite abelian groups are finite abelian groups. This is no longer the case if A is a finite (non-abelian) group. Libman showed that $A_n \to SO_{n-1}(\mathbb{R})$ for a natural embedding of the alternating group A_n is a localization if n is even and $n \geq 10$. Answering an immediate question by Dror Farjoun and assuming the generalized continuum hypothesis GCH we recently showed in [12] that any non-abelian finite simple has arbitrarily large localizations. In this paper we want to remove GCH so that the result becomes valid in ordinary set theory. At the same time we want to generalize the statement for a larger class of A's. The new techniques exploit abelian centralizers of free (non-abelian) subgroups of H which constitute a rigid system of cotorsion-free abelian groups. A known strong theorem on the existence of such abelian groups turns out to be very helpful, see [5]. Like [12], this shows (now in ZFC) that there is a proper class of distinct homotopy types which are localizations of a given Eilenberg–Mac Lane space $K(A, 1)$ for many groups A. The Main Theorem 1.3 is also used to answer a question by Philip Hall in [13].

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1. Introduction

A homomorphism $\eta : A \to H$ in some category is a *localization* if every homomorphism $\varphi : A \to H$ in the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & H \\ \varphi \downarrow & \swarrow_{\Phi} & \\ H & & \end{array} \tag{1.1}$$

extends *uniquely* to a homomorphism $\Phi : H \to H$.

Such localization functors $L_\eta A \cong H$ with respect to η derive from modules and rings, have there a long history and are considered in many recent papers in group theory for non-commutative cases and in connection with homotopy theory, see e.g. [2, 17, 3]. It turned out to be of special interest to investigate properties of A which carry over to $L_\eta A$ - or not. Examples for groups are the properties to be commutative, nilpotent of class at most 2, or the condition to be a ring. In particular cases the size of H relates to the size of A, see a summary in [12]. The relation to homotopical localizations can be looked up in [3], see also Dror Farjoun's book [6]. Here we want to concentrate on the just mentioned cardinality problem mentioned in the abstract:

If A is finite abelian, then every localization $\eta : A \to H$ is obviously epic, hence $|H| \leq |A|$. Moreover, if A is torsion abelian then $|H| \leq |A|^{\aleph_0}$ as shown in [17] by Libman. In contrast to this localizations of $\mathbb{Z}$ are the E-rings, see [3] and by Dugas, Mader and Vinsonhaler [8] (using [5]) there are arbitrarily large E-rings. The question about the size of $L_\eta A$ for finite, non abelian groups A still remains. As also mentioned in the abstract, Libman [18] has shown that for particular alternating groups $A = A_n$ there are localizations $L_\eta A$ of size $2^{\aleph_0}$. Moreover assuming GCH any finite non abelian simple group A has arbitrarily large localizations, as recently shown in [12]. From our new main result we will see that GCH can be removed. Using stronger algebraic arguments, like abelian centralizers of free (non-abelian) groups and the existence of large rigid families of cotorsion-free abelian groups, we are able to avoid the old combinatorial setting (the Hart Laflamme Shelah game from [15]), hence GCH. As in [12] we will use the following definition.

Definition 1.1. *Let $A \neq 1$ be any group with trivial center and view $A \subseteq \mathrm{Aut}(A)$ as inner automorphisms of A. Then A is called* suitable *if the following conditions hold:*

(1) A is a finite group.

(2) If $A' \subseteq \mathrm{Aut}(A)$ and $A' \cong A$ then $A' = A$.

(3) $\mathrm{Aut}(A)$ is complete.

Note that $\mathrm{Aut}(A)$ has trivial center because A has trivial center. Hence the last condition only requires that $\mathrm{Aut}(A)$ has no outer automorphisms. It also follows from this that any automorphism of A extends to an inner automorphism of $\mathrm{Aut}(A)$. A group A is complete if A has trivial center $\mathfrak{z}A$ and any automorphism is inner. If $h \in A$ then we denote by

$$h^* : A \to A \ (x \to xh^* = h^{-1}xh) \ \text{ the function which conjugation by } h.$$

We also recall the easy observation from [12] which is a consequence of the classification of finite simple groups:

All finite simple groups are suitable.

Also note that there are many well-known examples of suitable groups which are not simple.

If μ is a cardinal, then μ^+ is the successor cardinal of μ. A partial homomorphism between two groups is a homomorphism between subgroups accordingly. Moreover, if $U \subseteq G$ is a subgroup of G, then the centralizer of U in G is the subgroup

$$\mathfrak{c}_G U = \{h \in G : \ [h, U] = 1\},$$

where $[h, U] = \langle [h, u] : \ u \in U \rangle$ is the subgroup generated by the commutators $[h, u] = h^{-1}u^{-1}hu$.

Definition 1.2. *If $\mathcal{A}$ is a family of groups and G is any group, then $G[\mathcal{A}]$ denotes the* $\mathcal{A}$-socle *which is the subgroup of G generated by all copies of $A \in \mathcal{A}$ in G. If $\mathcal{A} = \{A\}$, we write $G[A]$.*

Then we have the following

Main Theorem 1.3. *Let $\mathcal{A}$ be a family of suitable groups and μ be an infinite cardinal such that $\mu^{\aleph_0} = \mu$. Then we can find a group H of cardinality $\lambda = \mu^+$ such that the following holds.*

1. *(1) H is simple. Moreover, if $1 \neq g \in H$, then any element of H is a product of at most four conjugates of g.*
2. *(2) Any $A \in \mathcal{A}$ is a subgroup of H and two different groups in $\mathcal{A}$ have only 1 in common when considered as subgroups of H. If $\mathcal{A}$ is not empty, then $H[\mathcal{A}] = H$.*
3. *(3) Any monomorphism $\varphi : A \to H$ for some $A \in \mathcal{A}$ is induced by some $h \in H$, that is there is some $h \in H$ such that $\varphi = h^* \restriction A$.*
4. *(4) If $A' \subseteq H$ is an isomorphic copy of some $A \in \mathcal{A}$, then the centralizer $\mathfrak{c}_H A' = 1$ is trivial.*
5. *(5) Any monomorphism $H \to H$ is an inner automorphism.*

Note that the second property of (2) follows from the first property of (2) together with (1). Also (5) can be virtually strengthened replacing monomorphism by nontrivial homomorphism, which is also due to (1). The group theoretical techniques derive from standard combinatorial group theory and can be found in the book by Lyndon and Schupp [19]. We will also use a theorem concerning the existence of complicated abelian groups from [5]. For clarity the proof will be restricted to the case when $\mathcal{A}$ is a singleton. The extension to arbitrary sets $\mathcal{A}$ is easy and left to the reader. The reader may also ponder about our hypothesis that all members of $\mathcal{A}$ are finite. In fact it turns out that there are many infinite groups A such that $\mathcal{A} = \{A\}$ can not be extended to H as in the Theorem 1.3, see [14].

We are now ready to answer Dror Farjoun's question in ordinary set theory ZFC.

Corollary 1.4. *Any finite simple group has localizations of arbitrarily large cardinality.*

The localization $A \to H$ induces a map between Eilenberg–Mac Lane spaces

$$K(A, 1) \to K(H, 1)$$

which turns out to be a localization in the homotopy category; [18]. Hence these examples show the following

Corollary 1.5. *Let A be a finite simple group. Then $K(A, 1)$ has localizations with arbitrarily large fundamental group.*

A discussion of these corollaries is given in [12], they easily derive from the Main Theorem 1.3, see also [12]. The Main Theorem 1.3 will also be used to answer a problem of Philip Hall from 1966 in [13] mentioned in the Kourovka notebook. There is a class of groups G such that any extension of G by a copy of G is isomorphic to G. Only some of the properties of the groups in our Main Theorem 1.3 will only be used for the Hall problem.

2. Free Products With Amalgam and HNN-Extensions

The following lemma was shown in [12, Lemma 2.1]. It was basic for the proof of the main theorem of [12] and it will be used here again. The non-trivial proof needs that A is finite.

Lemma 2.1. *Let $H = G_1 *_{G_0} G_2$ be the free product of G_1 and G_2 amalgamating a common subgroup $G_0 = G_1 \cap G_2$. If A is a finite subgroup of H, then there exist $i \in \{1, 2\}$ and $y \in H$ such that $A^y \subseteq G_i$.*

Hence we have a

Corollary 2.2. *Let G be any group, and $\phi : G_0 \to G_1$ be an isomorphism between two subgroups of G. Consider the HNN-extension $H = \langle G, t : t^{-1}ht = \phi(h), h \in G_0\rangle$. If A is a finite subgroup of H, then there exists a $y \in H$ such that A^y is contained in G.*

We want to refine the well-known notion malnormality and say

Definition 2.3. *If κ is a cardinal and $L \subseteq G$ are groups, then L is κ-malnormal in G if*

$$|L \cap L^g| < \kappa \text{ for all } g \in G \setminus L.$$

This is used in the following

Lemma 2.4. *Let $L \subseteq G$ be groups, $K = U \times L$ be a direct product and $H = G *_L K$ be a free product over L. Suppose that* *(i)* L *is* κ*-malnormal in* G*,*

(ii) $h \in H \setminus G$ *is an element such that* $\kappa \leq |G \cap G^h|$*, and*

(iii) if $e \in L$ *and* $\kappa \leq |\mathfrak{c}_L(e)|$*, then* $e = 1$*.*

Then the following holds.

(a) *There are* $1 \neq y \in U,\ x, z \in G$ *such that* $h = xyz$*.*

(b) *If* $\kappa \leq |\mathfrak{c}_G(h)|$ *then* $x = z^{-1}$ *and* $\mathfrak{c}_G(h) = L^z$*.*

(c) $G \cap G^h \subseteq L^z$*.*

Proof. We distinguish two cases depending on the position of h.

Case 1: Let $h \in K = U \times L$. Then we can write $h = xy = xyz$ with $x \in L, y \in U$ and $z = 1$. If $y = 1$, then $h = x \in L \subseteq G$ contradicting (ii), hence $y \neq 1$ and (a) follows.

Suppose $c \in \mathfrak{c}_G(h) \setminus L, h \in K \setminus G$ and recall $K \setminus G = K \setminus L$ then $h = c^{-1}hc$ is reduced of length 3 and of length 1 in $H = G *_L K$, a contradiction, hence

$$\mathfrak{c}_G(h) = \mathfrak{c}_L(h).$$

We have $h = xy$ from above. Since $[y, L] = 1$, $\mathfrak{c}_L(h) = \mathfrak{c}_L(xy) = \mathfrak{c}_L(x)$ follows. If $x = 1$ then $\mathfrak{c}_L(h) = L$, $x = z = 1$ and (b) holds in this case. If $x \neq 1$ then by (iii) follows $|\mathfrak{c}_L(x)| = |\mathfrak{c}_L(h)| = |\mathfrak{c}_G(h)| < \kappa$ and (b) holds trivially.

If $g \in G \cap G^h$, then $g = h^{-1}fh$ for some $f \in G$, hence $h = f^{-1}hg$. Note that $h \in K \setminus L$. If $g, f \in G \setminus L$ then $h = f^{-1}hg$ has length 1 and 3, a contradiction. If $g \in G \setminus L, f \in L$ (respectively $f \in G \setminus L, g \in L$) then $h = (f^{-1}h)g$ has length 1 and 2, which is impossible. If $f, g \in L$, then $h = xy$ and $xy = f^{-1}xyg = f^{-1}xgy$. Thus $g = x^{-1}fx = f^x$ and $G \cap G^h \subseteq L = L^z$.

Case 2: If $h \in H \setminus K$, then let $h = b_1 \cdot \dots \cdot b_n$ be in reduced form for $H = G *_L K$, hence $1 < n$ and alternately b_i is an element of $G \setminus L$ and $K \setminus L$. Let X_i be the element of $\{G \setminus L, K \setminus L\}$ with $b_i \in X_i$ and let X_i^* be the other element of $\{G \setminus L, K \setminus L\}$. If $b_i \in K$ we surely may assume that $b_i \in U$ as the L-part of b_i can be absorbed into the amalgam L. If $x \in G \cap G^h$ then $x = h^{-1}yh \in G$ for some $y \in G$, hence $hx = yh$ and if

$$w_1 = b_1 \cdot \dots \cdot b_n \quad w_2 = y^{-1}b_1 \cdot \dots \cdot b_n x, \quad x, y \in G \text{ and } w_1 = w_2 \tag{2.2}$$

then we claim that

$$x, y \in X_1 = X_n, \text{ and } 3 \leq n \text{ is odd.} \tag{2.3}$$

We distinguish various cases:

(1) If $x \in X_n^*, y \in X_1^*$ then w_2 is in reduced form and has length $n + 2$ and $l(w_1) = n$ contradicts (2.2)

(2) If $x \in X_n, y \in X_1^*$, $b_n x \notin L$, $(y^{-1}b_1)b_2 \cdots b_{n-1}(b_n x)$ is reduced of length $n+1$. So $l(w_1) = n$ and (2.2) is impossible.

(3) The dual case $x \in X_n^*, y \in X_1$, $y^{-1}b_1 \notin L$ is similar to (2).

(4) If $x \in X_n^*, y \in X_1$ and $y^{-1}b_1 \in L$ then $w_2 = (y^{-1}b_1b_2) \cdot \cdots \cdot b_n x$ and w_1 are both in reduced form of length n but $y^{-1}b_1b_2 \in X_2$ and from $w_1 = w_2$ follows $b_1 \in X_2$ hence $b_1 \in X_1$ is a contradiction.

(5) The dual case $x \in X_n, y \in X_1^*$ and $b_n x \in L$ is similar.

(6) If $x \in X_n^*, y \in L$ then $w_2 = (y^{-1}b_1)b_2 \cdot \cdots \cdot b_n x$ has length n and $l(w_1) = n$ but $x \in X_n^*$ and $b_n \in X_n$ is impossible for (2.2).

(7) the dual case $x \in L, y \in X_1^*$ is similar.

Finally we have the case

(8) $x \in X_1 \cup L$, $y \in X_n \cup L$, hence $b_1 \cdot \cdots \cdot b_n = (y^{-1}b_1)b_2 \cdot \cdots \cdot b_{n-1}(b_n x)$ and both sides are reduced of length n. By uniqueness we find $t_1, \dots, t_{n-1} \in L$ such that

$$b_1 t_1 = y^{-1}b_1,\ t_1^{-1}b_2t_2 = b_2,\ t_2^{-1}b_3t_3 = b_3, \dots, t_{n-1}^{-1}b_n = b_n x.$$

From $x, y \in G$ follows $X_1 = X_n$ and n is odd. We noted that $n \neq 1$, hence $3 \leq n$ and the claim (2.3) is shown.

Note that $t_i = t_{iy}$ depends on y in (2.2) and the last displayed equations give us

$$t_{1y} = (y^{-1})^{b_1},\ t_{2y} = t_{1y}^{b_2},\ \dots, x = (t_{n-1,y}^{-1})^{b_n}.$$

We consider the pairs $(y^{-1}, b_1), (t_{1y}, b_2), (t_{2y}, b_3), \dots$ of the last equalities. In the first pair the first element may not be in L, in the second pair the second element may not be in G, but the third pair has both these properties. If $5 \leq n$ then the third pair exists and $t_{3y} \in L$, the equation above shows that

$$t_{2y}^{b_3} = t_{3y} \in L^{b_3} \cap L \text{ for all } y \in G \cap G^h.$$

Hence

$$\kappa \leq |\{t_{3y} \in L \cap L^{b_3} :\ y \in G\}|,$$

by assumption (ii) of the lemma, so $\kappa \leq |L \cap L^{b_3}|$. Condition (i) of the lemma implies $b_3 \in L$, but this contradicts the reduced form of $w_1 = b_1 \cdot \cdots \cdot b_n$. Hence $n = 3$ and $h = b_1b_2b_3$ and from the last claim $b_1, b_3 \in G$ hence $b_2 \in K$ so, as mentioned above, without less of generality $b_2 \in U$, and if we let $x = b_1, y = b_2$ and $z = b_3$ then (a) of the lemma holds.

Now it is easy to show that (b) and (c) hold:

(b) We may assume (a) and that we are not in Case 1, hence $h = xyz \in H \setminus K$ with $y \in U$ and $x, z \in G \setminus L$. The element $h = xyz$ is in reduced normal form.

If $c \in \mathfrak{c}_G(h)$, then $h = c^{-1}hc$ and we have

$$xyz = (c^{-1}x)y(zc) \text{ both sides in reduced normal form.}$$

By uniqueness there are $t_1, t_2 \in L$ such that

$$xt_1 = c^{-1}x,\ t_1^{-1}yt_2 = y,\ t_2^{-1}z = zcx.$$

From $y \in U, t_2 \in L, K = U \times L$ follows $[y, t_2] = 1$, hence $t := t_1 = t_2$ and the last displayed equations become

$$xt = c^{-1}x,\ t^{-1}z = zc.$$

Hence $c = (t^{-1})^{x^{-1}} = (t^{-1})^z$ and $\mathfrak{c}_G(h) \subseteq L^{x^{-1}} \cap L^z$ equivalently $\mathfrak{c}_G(h)^x \subseteq L \cap L^{zx}$. If $zx \in G \setminus L$, then $|\mathfrak{c}_G(h)| < \kappa$ by (i), and (b) holds trivially.

If $zx = l \in L$ then $\mathfrak{c}_G(h) \subseteq L^z \cap L^{x^{-1}} = L^z$, the element h becomes $h = xyz = x(yl)x^{-1} = z^{-1}(ly)z$ and $[y, l] = 1$.

From $\mathfrak{c}_G(h) \subseteq L^z$, $zx = l$ and $h = z^{-1}(ly)z$ follows $L^z \supseteq \mathfrak{c}_G(z^{-1}(ly)z) = \mathfrak{c}_G(ly)^z$ or equivalently $\mathfrak{c}_G(ly) \subseteq L$. Hence $\mathfrak{c}_G(ly) = c_L(ly) = \mathfrak{c}_L(l)$ by $[L, y] = 1$. However $\kappa \leq |c_L(ly)| = |\mathfrak{c}_L(l)|$ and (iii) implies $l = 1$. We derive $h = z^{-1}yz$ and $\mathfrak{c}_G(y) \subseteq L$ from above. Obviously $L \subseteq \mathfrak{c}_G(y)$, so $\mathfrak{c}_G(h) = \mathfrak{c}_(y^z) = L^z$ and (b) follows.

(c) If $g \in G \cap G^h$, then $g^{-1} = h^{-1}ch$ for some $c \in G$, hence $h = chg$ and from (a) we have $h = xyz$. We get that

$$xyz = (cx)y(zg) \text{ and both sides in reduced normal form of length 3.}$$

Again there are $t_1, t_2 \in L$ with $xt_1 = cx,\ t_1^{-1}yt_2 = y,\ t_2^{-1}z = zg$ and $y \in U$. As before $t = t_1 = t_2 \in L$ and hence $t \in L,\ xt = cx,\ t^{-1}z = zg$. We get $g = z^{-1}t^{-1}z \in L^z$ and (c) is also shown. $\square$

We must extend κ-malnormal to sets of subgroups, as in the

Definition 2.5. *A set $\mathfrak{L}$ of subgroups is κ-disjoint in G if each $L \in \mathfrak{L}$ has size $|L| = \kappa$ and $|L^g \cap L'| < \kappa$ for all $L \neq L' \in \mathfrak{L}$ and $g \in G$.*

Iterating Lemma 2.4 we get a

Lemma 2.6. *Let $\mathfrak{L} = \{L_1, \dots, L_n\}$ be a finite collection of subgroups of G such that*

(a) *Each group in $\mathfrak{L}$ is κ-malnormal in G.*

(b) *$\mathfrak{L}$ is κ-disjoint in G.*

If $0 \leq m \leq n$, $K_i = U_i \times L_i,\ M_i = K_i *_{L_i} G\ (i \leq m)$ and

$$H_0 = G,\ H_m = *_G\{M_i : i \leq m\} \text{ for } m \neq 0,$$

then the following holds for $m \leq n$.

(i) *Each* L_i *is* κ*-malnormal in* H_m *for* $m < i \leq n$.

(ii) $\mathfrak{L}$ *is* κ*-disjoint in* H_m.

(iii) *If* $h \in H_m \setminus G$ *and* $\kappa \leq |\mathfrak{c}_G(h)|$*, then there are* $g \in G,\ 1 \leq l \leq m,\ r \in U_l$ *with* $h = r^g$.

(iv) *If* $h \in H_m \setminus G$ *and* $\kappa \leq |G \cap G^h|$*, then there is* $1 \leq l \leq m$ *such that* $h \in M_l$.

Proof. The proof is by induction on m. If $m = 0$, then $(i), \ldots, (iv)$ hold by hypothesis. Suppose $(i), \ldots, (iv)$ holds for m. From $M_{m+1} = K_{m+1} *_{L_{m+1}} G$ follows

$$H_{m+1} = H_m *_G M_{m+1} = H_m *_{L_{m+1}} K_{m+1} = H_m *_{L_{m+1}} (L_{m+1} \times U_{m+1}). \tag{2.4}$$

(i) If $h \in H_m \setminus L_k$ and $m + 2 \leq k \leq n$, then (i) holds by induction hypothesis. Hence we also may assume that $h \in H_{m+1} \setminus H_m$ and suppose for contradiction that

$$\kappa \leq |L_k \cap L_k^h|. \tag{2.5}$$

The assumptions of Lemma 2.4 hold, hence we may apply (a) of the lemma and can express $h = xyz$ with $x, z \in H_m \setminus L_{m+1}$ and $1 \neq y \in U_{m+1}$. From (2.5) and Lemma 2.4 (c) follows $L_k \cap L_k^h \subseteq L_{m+1}^z$, hence $L_k \cap L_k^h \subseteq L_{m+1}^z \cap L_k$. From $k \neq m+1$, (2.5) and hypothesis (b) we get the contradiction

$$\kappa \leq |L_k \cap L_k^h| \leq |L_k \cap L_{m+1}^z| < \kappa.$$

(ii) From (2.4) we have a canonical projection $\pi : H_{m+1} \to H_m$ with $\ker \pi = U_{m+1}$. If $1 \leq i \neq j \leq n$ and $h \in H_{m+1}$ such that $L_i^h \cap L_j \subseteq H_{m+1}$ has size at least κ, then also $\kappa \leq |L_i^{h\pi} \cap L_j|$. But $h\pi \in H_m$ contradicts the induction hypothesis for (ii).

(iii) Let

$$h \in H_{m+1} \setminus G \text{ such that } \kappa \leq |\mathfrak{c}_G(h)|. \tag{2.6}$$

If $h \in H_m \setminus G$, then the induction hypothesis applies and (iii) follows. We may assume that $h \in H_{m+1} \setminus H_m$. By Lemma 2.4(a) we have $h = xyz$ with $x, z \in H_m \setminus L_{m+1}$ and $1 \neq y \in U_{m+1}$. From Lemma 2.4(b) follows

$$x = z^{-1} \text{ and } \mathfrak{c}_G(h) = L_{m+1}^z, \tag{2.7}$$

hence $h = y^z$. If $z \in G$, then (iii) is shown. Otherwise $z = x^{-1} \in H \setminus G$. We want to derive a contradiction, showing that this case does not happen.

By Lemma 2.4(c), (2.6) and (2.7) we have $\mathfrak{c}_G(h) \subseteq G \cap G^h \subseteq L_{m+1}^z = \mathfrak{c}_G(h)$, hence

$$\mathfrak{c}_G(h) = G \cap G^h = L_{m+1}^z \tag{2.8}$$

Hence we have that

$$\kappa \leq |L_{m+1}^z| = |G \cap G^z| \tag{2.9}$$

and by induction hypothesis from (iv) for z in place of h we find an $l \leq m$ such that $z \in G *_{L_l} K_l \subseteq H_m$. Now we apply Lemma 2.4(a) to write $z = abc$ with $a, c \in G$ and $b \in U_l$. From (2.9) and Lemma 2.4(c) we get $G^x \cap G \subseteq L_l^c$. Using $\mathfrak{c}_G(h) \subseteq G^x \cap G \subseteq L_l^c$ and (2.6) we also have $\mathfrak{c}_G(h) \subseteq L_l^c \cap L_{m+1}^z$, hence $\kappa = |L_l^c \cap L_{m+1}^z|$ which contradicts (a).

(iv) Let $h \in H_{m+1} \setminus G$ and $\kappa \leq |G \cap G^h|$. Again, if $h \in H_m$ then (iv) follows by induction hypothesis, hence we may assume that $h \in H_{m+1} \setminus H_m$. By Lemma 2.4 (a) we have $h = xyz$ with $x, z \in H_m \setminus L_{m+1}$ and $1 \neq y \in U_{m+1}$. If $x, z \in G$ then $h = xyz \in G *_{L_{m+1}} K_{m+1}$ and by induction hypothesis also (iv) follows. We may assume that $x, z \in G$ is not the case, so without restriction let $z \notin G$. From Lemma 2.4(c) follows

$$G \cap G^h \subseteq L_{m+1}^z \subseteq H_m. \tag{2.10}$$

If $w \in G \cap G^h$ then $w^{z^{-1}} \in L_{m+1} \subseteq G$. By hypothesis on h we derive that also

$$\kappa \leq |G \cap G^{z^{-1}} \cap L_{m+1}| \tag{2.11}$$

Now, using (2.10),(2.11) and the induction hypothesis (iv) for H_m, we find $1 \leq l \leq m$ such that $z \in G *_{L_l} K_l \subseteq H_m$. Using Lemma 2.4(c) for c^{-1} in place of h and (2.11) there is z' such that $G \cap G^{c^{-1}} \subseteq L_l^{z'}$, hence

$$G \cap G^{c^{-1}} \cap L_{m+1} \subseteq L^{z'} \cap L_{m+1} \text{ and } l \neq m+1.$$

Finally we apply (2.11) once more. By Lemma 2.4(a) we get the contradiction on cardinals $\kappa \leq |L_l^{z'} \cap L_{m+1}| < \kappa$. $\square$

The last Lemma 2.6 extends to infinite sets $\mathfrak{L}$. We have an immediate

Corollary 2.7. *Let $\mathfrak{L}$ be a collection of subgroups of G such that*

(a) *Each group in $\mathfrak{L}$ is κ-malnormal in G.*
(b) *$\mathfrak{L}$ is κ-disjoint in G.*

If $K_L = U_L \times L,\ M_L = K_L *_L G$ and $H = *_G\{M_L : L \in \mathfrak{L}\}$, then the following holds.

(i) *Each L is κ-malnormal in H for $L \in \mathfrak{L}$.*
(ii) *$\mathfrak{L}$ is κ-disjoint in H.*
(iii) *If $h \in H \setminus G$ and $\kappa \leq |\mathfrak{c}_G(h)|$, then there are $g \in G, L \in \mathfrak{L}$ and $r \in U_L$ with $h = r^g$.*
(iv) *If $h \in H \setminus G$ and $\kappa \leq |G \cap G^h|$, then there is $L \in \mathfrak{L}$ such that $h \in M_L$.*

Similar to polynomials over a field K which are elements of $K[x]$, we will say for a group G that

Definition 2.8. *A word w over G in a free variable x is an element of $G * \langle x \rangle$. We will write $w = w(x)$ and may substitute elements of an over-group.*

Lemma 2.9. *Let $G = G_1 * G_2 * G_3$ be a free product of groups, let $w_i(x)$ be words over G_1 $(1 \leq i \leq 3)$ and let $x_2 \in G_2, x_3 \in G_3$. Then the following holds.*

(1) If $w_1(x_2x_3) = w_2(x_2)w_3(x_3)$, then $w_1(x_2x_3) = tx_2x_3u$, $w_2(x_2) = tx_2t'$ and $w_3(x_3) = t'^{-1}x_3u$ for some $t, u, t' \in G_1$.

(2) If also $w_2 = w_3$, then $w_1(x) = x^u$ for $u \in G_1$.

Proof. Note that it is enough to consider $G = G_1 * \langle x_2 \rangle * \langle x_3 \rangle$. Write $w_2(x_2) = t_1 \cdots t_n$ with $t_i \in G_1 \cup \langle x_2 \rangle$ in normal form (from alternate factors). Similarly, write $w_3(x_3) = u_1 \cdots u_m$ with $u_i \in G_1 \cup \langle x_3 \rangle$ in normal form. Then

$$w := w_2(x_2) \cdot w_3(x_3) = t_1 \cdots t_n \cdot u_1 \cdots u_m.$$

If $t_n \in \langle x_2 \rangle$ or $u_1 \in \langle x_3 \rangle$, then w is in normal form as well. Otherwise $t_nu_1 \in G_1$ and $w = t_1 \cdots t_{n-1}(t_n \cdot u_1)u_2 \cdots u_m$ is in normal form. If also $w_1(x) = v_1 \cdots v_k$ with $v_i \in G_1 \cup \langle x \rangle$ is in normal form. We also may assume that $v_2 \in \langle x \rangle$ without loss of generality. Then writing $v_i = x^{m_i}$ if $v_i \in \langle x \rangle$, we have that

$$w_1(x_2x_3) = v_1(x_2x_3)^{m_2}v_3 \cdots v_k$$

is in reduced normal form. Hence

$$w_1(x_2x_3) = v_1x_2x_3v_3 = w_2(x_2) \cdot w_3(x_3) = t_1x_2(t_3u_1)x_3u_3$$

and it follows that $t_3u_1 = 1$, $v_3 = u_3$ and $t_1 = v_1$. We get

$$w_1(x_2x_3) = t_1x_2x_3u_3,\ w_2(x_2) = t_1x_2t_3 \text{ and } w_3(x_3) = t_3^{-1}x_2u_3.$$

If we put $t_1 = t, t_3 = t'$ and $u_3 = u$, then (i) follows.

If also $w_2(x) = w_3(x)$, then $txt' = t'^{-1}xu$, hence $t' = u, t = t'^{-1}$. It follows that $w_1(x_2x_3) = u^{-1}x_2x_3u = (x_2x_3)^u$ as well as $w_2(x_2) = x_2^u$ and $w_3(x_3) = x_3^u$. ☐

The following lemma describes centralizers of finite subgroups in free products with amalgamation.

Lemma 2.10. *Let $H = G_1 *_{G_0} G_2$ be the free product of G_1 and G_2 amalgamating a common subgroup G_0. Let $A \subseteq G_1$ be a non trivial finite subgroup and let $x \in H$ be an element which commutes with all elements of A. Then either $x \in G_1$ or $A^g \subseteq G_0$ for some $g \in G_1$.*

We repeat the short proof from [12].

Proof. Suppose $[x, A] = 1$, $x \notin G_1$ and $h \in A$. Express x in a reduced normal form

$$x = g_1 g_1' \cdots g_n g_n',$$

that is, $g_i \in G_1 \setminus G_0$, $(1 < i \leq n)$ and $g_i' \in G_2 \setminus G_0$, $(1 \leq i < n)$. The relation $h^{-1}x^{-1}hx = 1$ yields the following

$$h^{-1} {g'}_n^{-1} g_n^{-1} \cdots {g'}_1^{-1} (g_1^{-1} h g_1) g_1' \cdots g_n g_n' = 1.$$

By the normal form theorem for free products with amalgamation [19, Theorem 2.6 p. 187], this is only possible if $g_1 \in G_1$ and $g_1^{-1} h g_1 \in G_0$ for all $h \in A$, so $A^{g_1} \subseteq G_0$. This concludes the proof. $\square$

By similar arguments we have

Lemma 2.11. *Let G be any group, and $\phi : G_0 \to G_1$ be an isomorphism between two subgroups of G. Consider the HNN-extension $H = \langle G, t : t^{-1}ht = \phi(h), h \in G_0 \rangle$. If A is a non trivial finite subgroup of H and $x \in H$ such that $[x, A] = 1$, then x is in a conjugate of G.*

Let $\mathrm{pInn}\,(G)$ denote the set of *partial inner automorphisms*, which are the isomorphisms $\phi : G_1 \to G_2$ where $G_1, G_2 \subseteq G$ such that ϕ can be extended to an inner automorphism of G. Hence $\mathrm{pInn}\,(G)$ are all restrictions of conjugations to subgroups of G.

Definition 2.12. In addition we will use Definition 1.1.

(1) Let $A \subseteq \widehat{A} = \mathrm{Aut}A$ be fixed groups such that A is suitable.
(2) $\mathcal{K}$ consists of all groups G such that $A \subseteq \widehat{A} \subseteq G$, and any isomorphic copy of A in G has trivial centralizer in G. That is,

$$\mathcal{K} = \{G : \widehat{A} \subseteq G, \text{ if } A \cong A' \subseteq G, x \in G \text{ with } [A', x] = 1, \text{ then } x = 1\}.$$

We have an easy lemma from [12].

Lemma 2.13. *If G and G' are in $\mathcal{K}$ then $G * G' \in \mathcal{K}$.*

By a well-known result of Schupp [21] any automorphism is partially inner for some group extension. We will refine this result below. If G is any group in $\mathcal{K}$ and ϕ is an isomorphism between two subgroups of G isomorphic to A, we will need that ϕ is an partially inner automorphism in some extension $G \subseteq H \in \mathcal{K}$. This follows by using HNN-extensions as we will show next.

Lemma 2.14. *Let $G \in \mathcal{K}$ and $B \subseteq G$ be a subgroup isomorphic to A. Then there is $H \in \mathcal{K}$ such that $G \subseteq H$ and $\mathrm{Aut}(B) \subseteq H$.*

Proof. Let $\widehat{B} = \mathrm{Aut}(B)$ and $N = \mathfrak{n}_G(B)$ the normalizer of B in G. If $\widehat{B} \subseteq G$ then let $H = G$. Suppose that $\widehat{B} \not\subseteq G$. Note that $N = G \cap \widehat{B}$, so we can consider the free product with amalgamation $H := G *_N \widehat{B}$. We shall show that $H \in \mathcal{K}$. Let $A' \subseteq H$ be a subgroup isomorphic to A and $1 \neq x \in H$ such that $[A', x] = 1$. By Lemma 2.1 we can suppose that $A' \subseteq G$ or $A' \subseteq \widehat{B}$. Suppose that $A' \subseteq G$, the other case is easier. Let $x = g_1 g_2 \cdots g_n$ be written in a reduced normal form. First suppose that $n = 1$. If $x = g_1 \in G$ then $x = 1$ since $G \in \mathcal{K}$, and this is a contradiction. Hence $x = g_1 \in \widehat{B} \setminus N$. As in Lemma 2.10 we deduce that $A' = (A')^{g_1} \subseteq N$, thus $A' = B$ since B is suitable. Hence $g_1 \in N$ is a contradiction. If $n = 2$, then we obtain $(A')^{g_1} = (A')^{g_2^{-1}}$ a contradiction unless $A' \subseteq N$, so $A' = B$. So both g_1 and g_2 are in N, which also is a contradiction. Similarly, if $n \geq 3$ we have g_{n-1} and g_n in N. This is again impossible. This concludes the proof. $\square$

By the previous lemma we can suppose that if $B \subseteq G \in \mathcal{K}$, and if $B \cong A$, then $\widehat{B} \subseteq G$ as well. If $C, B \subseteq G$, $A \cong B \cong C$ and $\widehat{C}$, $\widehat{B}$ are conjugate in G then C and B are also conjugate. Indeed, if $g \in G$ such that $g^* : \widehat{C} \longrightarrow \widehat{B}$, then $C^g \subseteq \widehat{B}$ is a subgroup isomorphic to B, hence $C^g = B$ by Definition 1.1

Lemma 2.15. *Let $G \in \mathcal{K}$ and $B \subseteq \widehat{B} \subseteq G$. Suppose that A and B are isomorphic but not conjugate in G. Let $\phi : \widehat{A} \to \widehat{B}$ be any isomorphism. Then the HNN-extension*

$$H = \langle G, t \, : \, t^{-1} h t = \phi(h) \ \ \textit{for all } h \in \widehat{A} \rangle$$

is also in $\mathcal{K}$.

Proof. see [12, proof of Lemma 3.5]. $\square$

Lemma 2.16. *Let C and B be isomorphic to A and suppose $C \subseteq \widehat{C} \subseteq G \in \mathcal{K}$ and $B \subseteq \widehat{B} \subseteq G$. If $\phi : C \longrightarrow B$ is any isomorphism, then there is $G \subseteq H \in \mathcal{K}$ such that $\phi \in \mathrm{pInn}\,(H)$. Moreover, H can be obtained from G by at most two successive HNN-extensions.*

Proof. see [12, proof of Lemma 3.6]. $\square$

Lemma 2.17. *Let $G \in \mathcal{K}$ and suppose that $G' \in \mathcal{K}$ or G' does not contain any subgroup isomorphic to A. Let $g \in G$ and $g' \in G'$ with $o(g) = o(g')$. Then $(G * G')/N \in \mathcal{K}$ where N is the normal subgroup of $G * G'$ generated by $g^{-1}g' \in G * G'$.*

Proof. The group $H = (G * G')/N$ is a free product with amalgamation, hence G and G' can be seen as subgroups of H respectively. Suppose that we have a subgroup $A' \subseteq H$ isomorphic to A and $x \in H$ such that $[A', x] = 1$. By Lemma 2.1 we can assume that

A' is already contained in G. Suppose that $x \neq 1$. By Lemma 2.10 it follows that either $x \in G$ or a conjugate of A' is contained in $\langle g \rangle$. In the first case $x = 1$ from $G \in \mathcal{K}$ is a contradiction. The second case is obviously impossible. Thus $H \in \mathcal{K}$. □

Lemma 2.18. *Let $H = G *_{G_0} G'$ be the free product of G and G' amalgamating a common subgroup G_0. If any $X \in \{G, G', G_0\}$ is in $\mathcal{K}$ such that monomorphisms from A to X are induced by inner automorphisms of X, then $H \in \mathcal{K}$ as well.*

Proof. Let $A' \subseteq H$ be a subgroup isomorphic to A, and $1 \neq x \in H$ such that $[A', x] = 1$. By Lemma 2.1 we can assume that $A' \subseteq G_0$ and $x = g_1 g_2 \cdots g_n$, is written in a reduced form of length bigger than two. Then we have $g_1^* : A' \longrightarrow (A')^{g_1}$ both of them inside G_0. By the choice of G_0 there exists $g \in G_0$ such that $g^* : A' \longrightarrow (A')^{g_1}$. We can also suppose that the automorphism group $\widehat{A'}$ is already in G_0 by Lemma 2.14. Hence the composition $(g_1^{-1} g)^* : A' \to A'$ is an automorphism, which is inner by completeness. Thus, $g_1^{-1} g \in G_0$ and $g_1 \in G_0$. This is a contradiction, since x was written in a reduced form. □

Proposition 2.19. *Let G be a group in $\mathcal{K}$. Let $g, f \in G$, where $o(f) = o(g) = \infty$ and g does not belong to the normal subgroup generated by f. Then there is a group $H \in \mathcal{K}$ such that $G \leq H$ and g is conjugate to f in H.*

Proof. Let $\alpha : \langle f \rangle \to \langle g \rangle$ be the isomorphism mapping f to g. By hypothesis $\alpha \notin \operatorname{pInn} G$. As in Lemma 2.16 consider the HNN-extension $H = \langle G, t : t^{-1} f t = g \rangle$. We must show that $H \in \mathcal{K}$. Clearly $|H| < \lambda$ and consider any A' with $A \cong A' \leq G$ and any $x \in H$ with $[A', x] = 1$. As above we may assume that $A' \subseteq G$ and $x \in H$ with $[A', x] = 1$. Now we apply Lemma 2.10. □

Recall Definition 1.2 of an A-socle $G[A]$.

Lemma 2.20. *If $g \in G \in \mathcal{K}$, then there is a group $H \in \mathcal{K}$, such that $G \subseteq H$, with $|H| = |G| \cdot \aleph_0$ and $g \in H[A]$.*

Proof. Suppose that $o(g) = \infty$ and that $g \notin G[A]$. Let A_1 and A_2 be two isomorphic copies of A. Choose a non trivial element $h \in A$ and let h_1 and h_2 be its copies in A_1 and A_2 respectively. Now define

$$H = (G * A_1 * A_2)/N$$

where N is the normal subgroup generated by $g^{-1} h_1 h_2$. Then $H \in \mathcal{K}$ by Lemma 2.17 and moreover $g \in H[A]$.

If $o(g) = n < \infty$ we first embed $G \subseteq (G * K)/N$ where K is defined by the representation $\langle x_1, x_2 : (x_1x_2)^n = 1\rangle$ and N is the normal closure of $g^{-1}x_1x_2$. Then by the Lemma 2.17 $(G * K)/N \in \mathcal{K}$. Now, since $o(x_1) = o(x_2) = \infty$, we can apply the first case. $\square$

3. Construction of rigid groups

We want to use the following natural definition where we slightly abuse the notion of a free product as customary for external and internal direct products.

Definition 3.1. *If G is a group and $U \subseteq G$, $g \in G$ then g is free over U if $\langle g, U\rangle = U * \langle g\rangle$.*

In this section we want to construct from a given suitable group A as in Definition 1.1 certain rigid groups G containing A. For the rest of the paper we make the following assumptions on the cardinals κ, μ and λ:

(1) κ is an infinite regular cardinal ($\mathrm{cf}\,\kappa = \kappa$).
(2) $\mu = \mu^\kappa$ and $\lambda = \mu^+$ is the successor cardinal of μ.

Moreover note that $\kappa = \aleph_0$ will be good enough in the following.

We want to apply (in Construction 3.4) the following theorem on torsion-free abelian groups.

Theorem 3.2. *For each subset $X \subseteq \kappa$ of the set (the cardinal) κ there is an $\aleph_1$-free abelian group G_X of cardinal κ such that the following holds.*

$$\mathrm{Hom}(G_X, G_Y) = \begin{cases} \mathbb{Z} & : \quad \text{if } X \subseteq Y \\ 0 & : \quad \text{if } X \not\subseteq Y \end{cases}$$

Remark 3.3. A proof of the theorem can be found in Corner, Göbel [5, p.465]. An abelian group is $\aleph_1$-free if all its countable subgroups are free abelian.

The next section is a short description for the construction of the group H of Main Theorem 1.3. Let $\lambda = \mu^+$ be the cardinal above and assign four disjoint stationary subsets $S_i \subseteq \lambda$ $(i = 0, 1, 2, 3)$ such that each ordinal α is a limit ordinal of cofinality $\mathrm{cf}\,(\alpha) = \omega$ if $\alpha \in S_0 \cup S_1 \cup S_2$ and $\mathrm{cf}\,(\alpha) = \kappa$ if $\alpha \in S_3$. Moreover, identify the group $\widehat{A}$ as a set with a fixed interval $[\mu, \alpha_0)$ of ordinals in λ. We also will need three lists of maps, elements and pairs of elements each with λ repetitions respectively. Let

$$L_0 = \{x_\alpha \in \lambda : \ \alpha \in S_0\}$$

and let

$$L_1 = \{h_\alpha : A \to A_\alpha \subset \lambda; \ \alpha \in S_1\}$$

where h_α runs through all bijective maps from A to subsets of λ with λ repetitions for each map. Finally choose an enumeration of pairs

$$L_2 = \{(y_\alpha, z_\alpha) \in \lambda \times \lambda : \ \alpha \in S_2\}$$

also with λ repetitions for each pair. From $|S_0| = |S_1| = |S_2| = \lambda$ follows that L_0, L_1 and L_2 exist. Now we are ready to define H. The definition is by transfinite induction. The inductive steps are also called approximations, see Shelah [23] or Göbel, Rodriguez, Shelah [12].

The set of approximations in the Construction 3.4 is just the collection $\mathfrak{P}$ of initial sequences $\mathfrak{p} = \{H_\alpha : \alpha < \beta_\mathfrak{p}\}$ for any $\beta_\mathfrak{p} < \lambda$ of the final group $H = \bigcup_{\alpha<\lambda} H_\alpha$. More generally, the members of $\mathfrak{p}$ could depend on $\mathfrak{p}$, i.e. $\mathfrak{p} = \{H_\alpha^\mathfrak{p} : \alpha < \beta_\mathfrak{p}\}$. Then $\mathfrak{P}$ becomes naturally a poset by component-wise inclusion, and any unbounded sequence in $\mathfrak{P}$ gives rise to a group H. This obvious generalization may be useful for other constructions, in this case it can also be applied for finding a family of 2^λ non-isomorphic groups like H. Again, for transparency we will restrict to only one group H and the ordering on $\mathfrak{P}$ is just extending the initial sequence $\mathfrak{p}$ by some members H_α satisfying the

Construction 3.4. *We define an ascending chain of subgroups H_α $(\alpha < \lambda)$ with universe a subset of $\mu(\alpha + 1)$ of cardinality μ whose union is $H = \bigcup_{\alpha<\lambda} H_\alpha$. The chain is constructed by transfinite induction subject to the following conditions.*

(i) *$(\alpha = 0)$ Let $H_0 = H_\beta = *_{\alpha\in\mu}\alpha\mathbb{Z} * \widehat{A}$ for all $\beta \leq \mu$ be the free product of μ infinite cyclic groups $\alpha\mathbb{Z}$ and $\widehat{A}$. Hence $A \subseteq \widehat{A} \subseteq H_0$ are prescribed subgroups of any H_α.*

(ii) *If $\alpha \in \lambda \setminus (S_0 \cup S_1 \cup S_2 \cup S_3)$, then let $H_{\alpha+1} = H_\alpha * \alpha\mathbb{Z}$.*

(iii) *If $\alpha \in S_0$ and $x_\alpha \in H_\alpha[A]$, then let $H_{\alpha+1} = H_\alpha * \alpha\mathbb{Z}$. Otherwise apply free products with amalgamation $H_{\alpha+1} = H_\alpha * A * A/N$ as in Lemma 2.20 to get that $x_\alpha \in H_{\alpha+1}[A]$.*

(iv) *If $\alpha \in S_1$ and $h_\alpha : A \to A_\alpha \subseteq H_\alpha$ is a partial inner automorphism mapping A to some subgroup A_α of H_α or h_α is not an isomorphism between A and A_α, then we also put $H_{\alpha+1} = H_\alpha * \alpha\mathbb{Z}$. Otherwise choose an HNN-extension $H_{\alpha+1} = \langle H_\alpha, t_\alpha\rangle$ such that $t_\alpha^* \restriction A = h_\alpha$ is inner on the extended groups, see [19].*

(v) *If $\alpha \in S_2$ and y_α, z_α from L_2 are two elements of infinite order in H_α such that y_α is not a conjugate of z_α, then choose an HNN-extension $H_{\alpha+1} = \langle H_\alpha, t_\alpha\rangle$ such that $y_\alpha = z_\alpha^{t_\alpha}$. Otherwise let $H_{\alpha+1} = H_\alpha * \alpha\mathbb{Z}$.*

(vi) *If $\alpha \in S_3$, then we apply the Black Box 5.1 $(i), (ii), (iii)$ in order to define a family $\mathfrak{F}_\alpha = \{F_{\alpha j} : \ j \in \kappa\}$ of free subgroups of rank κ of H_α: There are branches*

$$\eta_j^\alpha : \kappa \to \alpha \quad (j \in \kappa)$$

given by the traps and models $(H_j^\alpha, \varphi_j, \psi_j)$ which are triples of subgroups $H_j^\alpha \subseteq H_\alpha$, a unary function $\psi : H_j^\alpha \to \alpha$ and a partial two place function

$$\varphi : H_j^\alpha \times H_j^\alpha \to \alpha \text{ such that } \operatorname{Im} \eta_j^\alpha \subseteq H_j^\alpha.$$

We will say that α is useful *(for φ and ψ) if we can choose for any $j \in \kappa$ a strictly increasing, continuous sequence $\zeta_j^\alpha : \kappa \to \alpha$ such that the following holds.*
(1) $\psi(\eta_j^\alpha(\zeta_j^\alpha(\varepsilon)) < \eta_j^\alpha(\zeta_j^\alpha(\varepsilon+1))$ *and*
(2) $\varphi(\eta_j^\alpha(\zeta_j^\alpha(2\varepsilon)), \eta_j^\alpha(\zeta_j^\alpha(2\varepsilon+1))) < \eta_j^\alpha(\zeta_j^\alpha(2\varepsilon+2))$.
In this case φ is a total map and we define $x_j^\alpha(\varepsilon)$ by

$$\varphi(\eta_j^\alpha(\zeta_j^\alpha(2\varepsilon)), \eta_j^\alpha(\zeta_j^\alpha(2\varepsilon+1))) = x_j^\alpha(\varepsilon).$$

Let $F_{\alpha j} = \langle x_j^\alpha(\varepsilon) : \varepsilon < \kappa\rangle$ and $\mathfrak{F}_\alpha = \{F_{\alpha j} : \ j \in \kappa\}$.

If this is not possible, we say that α is useless and pick $\mathfrak{F}_\alpha$ 'trivially' from branches as in the first case but regardless of what φ and ψ do. In Lemma 3.6 we will show that $\mathfrak{F}_\alpha$ meets all requirements, in particular that each $F_{\alpha j}$ is free of rank κ. Now we define $H_{\alpha+1}$ in two steps:

Take a rigid family U_j $(j \in \kappa)$ of torsion-free abelian groups of cardinal κ from Theorem 3.2 such that

$$\mathrm{Hom}(U_i, U_j) = \delta_{ij}\mathbb{Z}$$

and let

$$K_{\alpha j} = U_j \times F_{\alpha j} \text{ and } M_{\alpha j} = H_\alpha *_{F_{\alpha j}} K_{\alpha j}.$$

In the second step choose

$$H_{\alpha+1} = *_{H_\alpha}\{M_{\alpha j} : \ j \in \kappa\}$$

be the free product with amalgamated subgroup H_α. Hence $H_\alpha \subseteq H_{\alpha+1}$ by the normal form theorem, see [19, p. 187, Theorem 2.6].

(vi) *Finally let $H = \bigcup_{\alpha\in\lambda} H_\alpha$.*

It remains to show that H meets the requirements of the Main Theorem 1.3. The proof of condition (vi), which is based on the Black Box 5.1, will be postponed to the next section, however all prerequisites will be established now using the following

Remarks and Notations 3.5. *If $\alpha \in S_3$ and $j \in \kappa$ from the Construction 3.4 (vi), then let $\alpha_j(\varepsilon) = \eta_j^\alpha(\zeta_j^\alpha(\varepsilon))$, hence $x_j^\alpha(\varepsilon) \in H_{\alpha_j(\varepsilon+1)} \setminus H_{\alpha_j(\varepsilon)}$ is free over $H_{\alpha_j}(\varepsilon)$ and the elements $x_j^\alpha(\varepsilon)$ $(\varepsilon < \kappa)$ freely generate $F_{\alpha j} \in \mathfrak{F}_\alpha$. Moreover $\sup_{\varepsilon<\kappa} \alpha_j(\varepsilon) = \alpha$ and $H_\alpha = \bigcup_{\varepsilon<\kappa} H_{\alpha_j(\varepsilon)}$ for each $j \in \kappa$. If $i \neq j < \kappa$, then $|\{\alpha_j(\varepsilon) : \varepsilon < \kappa\} \cap \{\alpha_i(\varepsilon) : \varepsilon < \kappa\}| < \kappa$.*

First we show the

Lemma 3.6. *Let $H = \bigcup_{\alpha\in\lambda} H_\alpha$ be as in the Construction 3.4.*

(a) *The groups $F_{\alpha j}$ $(j \in \kappa)$ defined in (vi) for $\alpha \in S_3$ are freely generated by the sets $\{x_j^\alpha(\varepsilon) : \varepsilon < \kappa\}$.*

(b) *Each $F_{\alpha j}$ is κ-malnormal in H_α and $\mathfrak{F}_\alpha$ is κ-disjoint.*
(c) *If $y, z \in H$ and $z \neq 1$, then y is a product of at most four conjugates of z.*
(d) *Any monomorphism $A \to H$ is induced by an inner automorphism of H.*
(e) $H = H[A]$.
(f) *If S_ω denotes the infinite symmetric group acting on countably many elements, then* $\mathrm{Hom}(H, S_\omega) = 0$.

Proof. (a) Comparing with Remark and Notations 3.5 we see that each $x_j^\alpha(\varepsilon) \in H_{\alpha_j(\varepsilon+1)} \setminus H_{\alpha_j(\varepsilon)}$ is free over $H_{\alpha_j(\varepsilon)}$, hence $H_{\alpha_j(\varepsilon)} * \langle x_j^\alpha(\varepsilon)\rangle \subseteq H_{\alpha_j(\varepsilon+1)} \subseteq H_\alpha$. An easy induction shows that $F_{\alpha j} \subseteq H_\alpha$ is freely generated by the set $\{x_j^\alpha(\varepsilon) : \varepsilon < \kappa\}$.

(b) If $g \in H_\alpha \setminus F_{\alpha j}$ for some $j < \kappa$, then $g \in H_{\alpha_j(\varepsilon_*)}$ for some minimal $\varepsilon_* < \kappa$ from $H_\alpha = \bigcup_{\varepsilon<\kappa} H_{\alpha_j(\varepsilon)}$. If $\varepsilon_* \le \varepsilon < \kappa$, then clearly by freeness - as shown next -

$$\langle x_j^\alpha(\nu) : \nu < \varepsilon\rangle \cap \langle x_j^\alpha(\nu) : \nu < \varepsilon\rangle^g = \langle x_j^\alpha(\nu) : \nu < \varepsilon_*\rangle \cap \langle x_j^\alpha(\nu) : \nu < \varepsilon_*\rangle^g$$

which is a set of cardinality less then κ as $|\varepsilon_*| < \kappa$, and the first part of (b) follows.

The proof of the displayed equation is by induction on $\varepsilon < \kappa$. If $\varepsilon = \varepsilon_*$ or ε is a limit ordinal the assertion obviously holds. So suppose $\varepsilon_* < \varepsilon < \kappa$ is not a limit and let $U = \langle H_{\alpha_j(\varepsilon_*)}, x_j^\alpha(\nu) : \nu < \varepsilon\rangle$. Hence

$$g \in \langle H_{\alpha_j(\varepsilon_*)}, x_j^\alpha(\nu) : \varepsilon_* < \nu \le \varepsilon\rangle = U * \langle x_j^\alpha(\varepsilon)\rangle \subseteq H_\alpha.$$

If w is an element of the left hand side of the displayed equality, then there are also two words $w_1(x), w_2(x)$ free over U such that $w = w_1(x_j^\alpha(\varepsilon)) = g^{-1}w_2^{-1}(x_j^\alpha(\varepsilon))g$. Hence $g = w_2(x_j^\alpha(\varepsilon))gw_1(x_j^\alpha(\varepsilon))$ has length 1 in $U * \langle x_j^\alpha(\varepsilon)\rangle$. Write $w_2 = a_1x^{t_1}a_2\cdots x^{t_{n-1}}a_n$ and $w_1 = f_1x^{s_1}f_2\cdots x^{s_{m-1}}f_m$ in normal form, hence

$$g = w_2gw_1 = a_1x^{t_1}a_2\cdots x^{t_{n-1}}(a_ngf_1)x^{s_1}\cdots x^{s_{m-1}}f_m$$

has length 1 which is only possible if $t_1 = s_1 = 0$, so $w = w_1(x_j^\alpha(\varepsilon)) \in U$ and the claim follows by the induction hypothesis.

Next we consider $g \in H_\alpha$ and $i \neq j < \kappa$. We must show that $|F_{\alpha i} \cap F_{\alpha j}^g| < \kappa$. If $w \in F_{\alpha i} \cap F_{\alpha j}^g$, we can choose $\gamma < \alpha$ such that $g \in H_\gamma$ and $\mathrm{Im}\,\eta_i^\alpha \cap \mathrm{Im}\,\eta_j^\alpha \subseteq \gamma$. Then $\{x_i^\alpha(\varepsilon) : \varepsilon < \kappa\} \cap H_\gamma = \{x_i^\alpha(\varepsilon) : \varepsilon < \varepsilon_1\}$ and $\{x_j^\alpha(\varepsilon) : \varepsilon < \kappa\} \cap H_\gamma = \{x_j^\alpha(\varepsilon) : \varepsilon < \varepsilon_2\}$ for some $\varepsilon_1, \varepsilon_2 < \kappa$. As before we have $F_{\alpha i} \cap F_{\alpha j}^g \cap H_\gamma = F_{\alpha i} \cap F_{\alpha j}^g \cap H_\nu$ for all ν with $\gamma < \nu < \alpha$. Hence $F_{\alpha i} \cap F_{\alpha j}^g \cap H_\alpha = F_{\alpha i} \cap F_{\alpha j}^g \cap H_\gamma$, which has cardinality $< \kappa$, and (b) is shown.

(c) If y and z have infinite order in H, then $(y, z) = (y_\alpha, z_\alpha)$ for some $\alpha \in S_2$ and it is $y = z^{t_\alpha}$ in $H_{\alpha+1}$, so (c) follows in this case. If y has finite order, then we can write $y = y'y''$ with both y', y'' of infinite order and it remains to show that y' is product of at most two conjugates of z. If z has infinite order, this is clear from above. If z has finite order, then we can find suitable elements $x_i \in H$ such that $w = z^{x_1} z^{x_2}$ has infinite order. By the first case $y' = w^t$ for some $t \in H$, hence $y' = z^{x_1 t} z^{x_2 t}$ is product of two conjugates and y is product of four.

(d) This is taken care of by the construction at stage (iv) for S_3.

(e) If $g \in H$, there is $\alpha \in S_0$ such that $g = g_\alpha$, hence $g \in H_{\alpha+1}[A]$ by construction and $H[A] = H$ follows.

(f) If S_ω is the infinite symmetric group acting on countably many elements, then $|S_\omega| = 2^{\aleph_0} \leq \mu^{\aleph_0} = \mu < \mu^+ = \lambda$. Hence $|S_\omega| < \lambda$ and (f) follows because H is simple by (c). □

Corollary 3.7. *H is simple and there is an element in H such that each other element is a product of at most four of its conjugates.*

Lemma 3.8. *Let H be as in the Construction 3.4.*

(a) *If $\alpha \in S_3$ and $\alpha < \beta < \lambda$, $j \in \kappa$, then $F_{\alpha j}$ is κ-malnormal in H_β and $\mathfrak{F}_\alpha$ is κ-disjoint in H_β.*

(b) *If $A' \subseteq H$ is an isomorphic copy of A, then $\mathfrak{c}_H A' = 1$.*

Proof. (a) follows from Lemma 3.6(b). (b) is based an the definition of $\mathcal{K}$ and also follows by induction on β for all $A' \subseteq H_\beta$ $(\beta < \lambda)$, using Lemma 2.15, Lemma 2.17 , Lemma 2.18 and Lemma 2.20. □

We now have an implication which follows from Corollary 2.4, a

Lemma 3.9. *If $\delta \in \lambda$ and H_δ from the Construction 3.4, $\alpha \leq \beta < \delta$ and $y \in H_\beta \setminus H_\alpha$ with a large centralizer $\kappa \leq |\mathfrak{c}_{H_\alpha(y)}|$, then $\alpha \in S_3$ and there are $j < \kappa$, $g \in H_\alpha$ and $x \in K_{\alpha j}$ such that $y = x^g$.*

Proof. Let β be minimal such that $y \in H_\beta \setminus H_\alpha$. The proof is now induction on β. Clearly $\alpha < \beta$ and β is not a limit ordinal, hence $\beta = \gamma + 1$ for some $\alpha \leq \gamma$. We write $D_y = \mathfrak{c}_{H_\gamma}(y)$ and similarly $C_y = \mathfrak{c}_{H_\alpha}(y)$. From $\alpha \leq \gamma$ follows $C_y \subseteq D_y$. For the first part of the lemma is enough to show that $|D_y| < \kappa$ if $\gamma \neq \alpha$ or if $\gamma = \alpha \notin S_3$. Recall that $y \in H_{\gamma+1} \setminus H_\gamma$. We must distinguish cases depending on the position of γ.

If $\gamma \in S_1$, then $H_{\gamma+1} = \langle H_\gamma, t\rangle$ is an HNN-extension. Let $y = g_0 t^{\varepsilon_1} g_1 \cdots g_{n-1} t^{\varepsilon_n} g_n$ be given in normal form with $g_i \in H_\gamma$ such that there is no subword $t^{-1} g_i t$ with $g_i \in A$ or $t g_i t^{-1} \in A' \subseteq H_{\gamma+1}$, see the Construction 3.4 and [19, p. 181]. Note that $1 \leq n$ from $y \notin H_\gamma$. Any $1 \neq x \in D_y$ is in H_γ and commutes with y, hence

$$x^{-1} g_n^{-1} t^{-\varepsilon_1} \cdots t^{-\varepsilon_1} (g_0^{-1} x g_0) t^{\varepsilon_1} g_1 \cdots g_{n-1} t^{\varepsilon_n} g_n = 1.$$

By the normal form theorem of HNN-extensions ([19, p. 182]) either $\varepsilon_1 = 1$ and $g_0^{-1}xg_0 \in A$ or $\varepsilon_1 = -1$ and $g_0^{-1}xg_0 \in A'$. By symmetry we may assume that $\varepsilon_1 = 1$, hence $g_0^{-1}xg_0 \in A$ for all $x \in D_y$. We have $D_y^{g_0} \subseteq A$ and $|D_y| \leq |A| < \kappa$ as desired.

If $\gamma \in S_2$, then $H_{\gamma+1} = \langle H_\gamma, t\rangle$ is another HNN-extension and the result follows as in the last case.

If $\gamma \in \alpha \setminus (S_0 \cap S_1 \cap S_2 \cap S_3)$ then $H_{\gamma+1} = H_\gamma * \langle t\rangle$ which is similar to the first cases but much easier.

If $\gamma \in S_0$ then $H_{\gamma+1}$ arrives from two extensions as before which settles this case.

We finally deal with $\gamma \in S_3$ and the free product of the $M_{\alpha j}$'s, which is

$$H_{\gamma+1} = *_{H_\gamma}\{M_{\alpha j} :\ j < \kappa\}.$$

Now apply Corollary 2.7 to find $y = x^g$ as in the lemma. $\square$

4. Proof Of The Main Theorem

The crucial part of this paper is the following

Main Lemma 4.1. *Any endomorphism of the group H from Construction 3.4 is an inner automorphism of H.*

Proof. If π is an endomorphism of H, then π is a monomorphism because H is simple. We will write $H = \bigcup_{\alpha\in\lambda} H_\alpha$ as in the construction.

Constructing modules with prescribed endomorphism rings, the most important condition is finding elements x of the module (say H) such that $x\pi \notin \langle H_\alpha, x\rangle_*$, see the 'strong case' in [5, p.455]. Here we will also say that an element $1 \neq x \in H$ is *strong* (for π) at $\alpha \in \lambda$ if x is free over H_α, hence

$$\langle H_\alpha, x\rangle = H_\alpha * \langle x\rangle \tag{4.12}$$

and

$$x\pi \notin \langle H_\alpha, x\rangle. \tag{4.13}$$

In this case we also say that α is strong for π. If x is free over H_α (i.e. (4.12) is true), but (4.13) does not hold, we call x *weak* (for π) at α, and if all free elements x over H_α are weak at α, we call α a weak ordinal for π.

We will distinguish two cases:

(A) *All ordinals are strong.*

(B) *There is a weak ordinal $\alpha_* < \lambda$.*

The case **(B)** is the complementary case of **(A)** . We first consider case **(A)**:

For each ordinal α there is a strong element $x_{*\alpha} \in H$ for π at α.

By a back and forth argument we can choose a closed and unbounded set $C \subseteq \lambda$ and an enumeration $C = \{\beta_\alpha : \alpha < \lambda\}$ such that the following holds for all $\alpha < \lambda$:

(1) $\alpha \leq \beta_\alpha$,

(2) $H_{\beta_\alpha}\pi \subseteq H_{\beta_\alpha}$ *and* $\pi^{-1}(\mathrm{dom}\,\pi \cap H_{\beta_\alpha}) \subseteq H_{\beta_\alpha}$

(3) $x_{*\alpha} \in H_{\beta_\alpha+1} \setminus H_{\beta_\alpha}$ *for all* $\alpha < \lambda$.

(4) $\{\beta_\alpha : \alpha < \lambda\}$ *is strictly increasing and continuous.*

Note that (2) is a purity condition for π, saying that $H\pi \cap H_{\beta_\alpha} \subseteq H_{\beta_\alpha}\pi$. Condition (3) follows by a new enumeration of the $x_{*\alpha}$'s and the H_{β_α}'s. Let U_α be the set of all ordinals $\gamma \geq \beta_\alpha + 1$ such that all elements in $H_{\gamma+1}$ are weak for π over α. First we claim

(4.14) if α is strong, then the set U_α is bounded in λ; (hence $|U_\alpha| \leq \mu$).

If U_α is unbounded and $x_{*\alpha} \in H_{\beta_\alpha+1} \setminus H_{\beta_\alpha}$ is strong for α, then choose any $\gamma \in U_\alpha$ and $y_\gamma \in H_{\gamma+1}$ free over H_γ. This is possible, because by construction often (on a stationary set) we choose $H_{\gamma+1} = H_\gamma * \langle y_\gamma \rangle$. The 'weak element' y_γ tells us

$$y_\gamma\pi \in \langle H_\alpha, y_\gamma\rangle = H_\alpha * \langle y_\gamma \rangle.$$

On the other hand the strong element $x_{*\alpha}$ makes

$$x_{*\alpha}\pi \notin \langle H_\alpha, x_{*\alpha}\rangle \text{ and } x_{*\alpha} \in H_{\beta_\alpha+1} \subseteq H_\gamma.$$

Hence $x_{*\alpha}y_\gamma \in H_{\gamma+1}$ is also free over H_α, and γ, being in U_α, requires

$$(x_{*\alpha}y_\gamma)\pi \in \langle H_\alpha, x_{*\alpha}y_\gamma\rangle.$$

Hence there is a word $w_\gamma(y)$ over H_α with free variable y such that $y_\gamma\pi = w_\gamma(y_\gamma)$. As we assume that U_α is unbounded in λ and $\lambda = \mu^+$, also $|U_\alpha| = \lambda > |H_\alpha|\mu$. By a pigeon hole argument we find an equipotent subset $U \subseteq U_\alpha$ such that for $\gamma \in U$ the word $w_\gamma(y) = w(y)$ does not depend on γ. We have $y_\gamma\pi = w(y_\gamma)$ for all $\gamma \in U$ and $|U| = \lambda$. Pick any $\gamma_1 < \gamma_2$ in U and consider $y = y_{\gamma_1} \cdot y_{\gamma_2}$. Clearly y is also free over H_α and $\gamma_i \in U$ implies $y\pi \in \langle H_\alpha, y\rangle$, hence $y\pi = w'(y)$ for another word w' over H_α. We can summarize

$$w'(y) = y\pi = (y_{\gamma_1} \cdot y_{\gamma_2})\pi = y_{\gamma_1}\pi \cdot y_{\gamma_2}\pi = w(y_{\gamma_1}) \cdot w(y_{\gamma_2}).$$

Also note that we have 'freeness'

$$H_\alpha * \langle y_{\gamma_1}\rangle * \langle y_{\gamma_2}\rangle \subseteq H_{\gamma_2+1}$$

and normal forms

$$w(y_{\gamma_i}) \in H_\alpha * \langle y_{\gamma_i}\rangle \quad (i = 1, 2),$$

and Lemma 2.9 applies. There is $g \in H_\alpha$ such that

$$y_{\gamma_i}\pi = w(y_{\gamma_i}) = y_{\gamma_i}^g \quad (i = 1, 2), \text{ and similarly } (x_{*\alpha}y_{\gamma_i})\pi = (x_{*\alpha}y_{\gamma_i})^g.$$

Using $\gamma = \gamma_1$ we derive

$$x_{*\alpha}\pi = (x_{*\alpha}y_\gamma y_\gamma^{-1})\pi = (x_{*\alpha}y_\gamma)\pi(y_\gamma\pi)^{-1} = (x_{*\alpha}y_\gamma)^g(y_\gamma^g)^{-1} = x_{*\alpha}^g,$$

hence $x_{*\alpha}\pi = x_{*\alpha}^g \in \langle x_{*\alpha}, H_\alpha\rangle$ contradicts that $x_{*\alpha}$ is strong for α. The claim (4.14) is shown.

By case **(A)** we may apply the last claim (4.14) to all $\alpha < \lambda$ and see that all U_α's are bounded. Hence there is a new increasing, continuous sequence $E = \{\nu_\xi : \xi < \lambda\} \subseteq C$ of ordinals such that in addition $U_{\nu_\xi} \subseteq \nu_\xi + 1$, hence $U_{\nu_\xi} \subseteq H_{\nu_\xi+1}$. Hence (1), (2), (3), (4) hold for ν_ξ in place of β_α there. In particular $\xi \leq \nu_\xi$ and $x_{*\xi} \in H_{\nu_\xi+1} \setminus H_{\nu_\xi}$ is a strong element at ν_ξ.

We now want to adjust the Black Box 5.1 for application in this case **(A)**. Let us define two maps φ_π (a partial map) and ψ_π on H which makes $(H, \psi_\pi, \varphi_\pi)$ into an L-model as mentioned in the Black Box 5.1.

Let $\psi_\pi(\xi) = \nu_\xi$ $(\xi < \lambda)$ and define $\varphi_\pi(\xi, \gamma)$ if and only if $\nu_\xi \leq \gamma < \lambda$ and let $\varphi_\pi(\xi, \gamma) = x_{*\xi}$ be from above. Hence ψ_π is a total function and φ_π is partial such that $\varphi_\pi(\xi, \gamma)$ exists if and only if $\psi_\pi(\xi) \leq \gamma$. The group H together with ψ_π, φ_π is an L-model M with universe λ. We want to consider L-submodels $M = (H', \psi', \varphi')$ of $M = (H, \psi_\pi, \varphi_\pi)$, hence M is a subgroup H' of H, ψ' a total function and $\varphi'(\alpha, \beta)$ is defined if and only if $\psi'(\alpha) \leq \beta$. As E is a cub and S_3 is a stationary set, we also find stationary many $\alpha \in S_3 \cap E$, hence H_α is closed under π and π^{-1}. The restriction $\psi_\pi \restriction \alpha$ (denoted by ψ_π again) is a total function, and H_α with these restrictions of ψ_π, φ_π is an L-model M' with universe a subset of λ.

By (iv) of the Black Box there is some $i < \kappa$ for such α such that $(H_i^\alpha, \varphi_\pi, \psi_\pi) \subseteq (H_\alpha, \varphi_\pi, \psi_\pi)$. Hence H_i^α is a subgroup of H with $\operatorname{Im}\eta_i^\alpha \subseteq H_i^\alpha$, $\psi_\pi \restriction H_i^\alpha$ a unary total function on H_i^α, $\varphi_\pi \restriction H_i^\alpha$ a two place function on H_i^α which is defined again for (ξ, γ) if and only if $\psi_\pi(\xi) = \nu_\xi \leq \gamma < \alpha$ and such that $\varphi_\pi(\xi, \gamma) = x_{*\xi} \in H_{\nu_\xi+1} \setminus H_{\nu_\xi}$ is strong for π at ν_ξ.

The Black Box 5.1 predicts some $j < \kappa$ and a strictly increasing sequence $\zeta_j^\alpha(\varepsilon)$ $(\varepsilon < \kappa)$ with $\sup_{\varepsilon<\kappa} \zeta_j^\alpha(\varepsilon) = \alpha$, $\zeta_j^\alpha(\varepsilon) + 2 < \zeta_j^\alpha(\varepsilon + 1)$ for all $\varepsilon < \kappa$ and

(4.15) $\alpha_j(\varepsilon) \in [\nu_{\zeta_\varepsilon}, \nu_{\zeta_{\varepsilon+1}})$ with $\varphi_\pi(\eta_j^\alpha(\zeta_j^\alpha(2\varepsilon)), \eta_j^\alpha(\zeta_j^\alpha(2\varepsilon + 1))) = x_j^\alpha(\varepsilon) = x_{*\alpha_j(\varepsilon)}$

for all $\varepsilon < \kappa$ as defined in the Construction 3.4. If $F_1 = F_{\alpha j}$, which is freely generated by the $x_j^\alpha(\varepsilon)$ $(\varepsilon < \kappa)$, also given by the Construction 3.4, then $F_1\pi \subseteq H_\alpha$ follows from

$F_1 \subseteq H_\alpha$ and $\alpha \in C$. Recall that $H_{\alpha+1} = *_{H_\alpha}\{M_{\alpha j} : j < \kappa\}$. If $1 \neq r \in U_j$, then $r \notin H_\alpha$ and $r\pi \notin H_\alpha$ by the closure property (2). However

$$M_{\alpha j} = H_\alpha *_{F_1} K_j = H_\alpha *_{F_1} (F_1 \times U_j) \subseteq H_{\alpha+1} \text{ and } U_j \cap H_\alpha = 1$$

hence $\mathfrak{c}_{H_\alpha}(r) \subseteq F_1$. Clearly $r \in U_j$ implies $F_1 \subseteq \mathfrak{c}_{H_\alpha}(r)$, we get the important centralizer condition

$$\mathfrak{c}_{H_\alpha}(r) = F_1.$$

Next we calculate the centralizer of the image $r\pi$, using $H_\alpha\pi \subseteq H_\alpha$:

$$F_1\pi = (\mathfrak{c}_{H_\alpha}(r))\pi = \mathfrak{c}_{H_\alpha\pi}(r\pi) \subseteq \mathfrak{c}_{H_\alpha}(r\pi). \tag{4.16}$$

By Corollary 3.9 there are $F_{\alpha i} = F_2 \in \mathfrak{F}_\alpha, g \in H_\delta$ and $f \in K_i$ such that

$$r\pi = f^g. \tag{4.17}$$

From $f \in K_i$ follows $r\pi = f^g \in H_{\alpha+1}$. We derive the invariance

$$H_{\alpha+1}\pi \subseteq H_{\alpha+1}. \tag{4.18}$$

Using (4.16), (4.17) and $g \in H_\alpha$, $f \in K_i$ we get

$$F_1\pi \subseteq \mathfrak{c}_{H_\alpha}(r\pi) = \mathfrak{c}_{H_\alpha}(f^g) = \mathfrak{c}_{H_\alpha}(f)^g = F_2^g. \tag{4.19}$$

Note that by definition $g = g_r$ depends on r and similarly $F_2 = F_{2r}$. We want to show that different r's give the same g and F_2:

If $r_1 \neq r_2$ and g_{r_1}, g_{r_2} are as above, then by (4.19) we have $F_1\pi \subseteq F_{2r_1}^{g_{r_1}} \cap F_{2r_2}^{g_{r_2}}$, hence $\kappa = |F_{2r_1} \cap F_{2r_2}^{g_{r_2}g_{r_1}^{-1}}|$ and by the choice of $\mathfrak{F}_\alpha$ (which is κ-disjoint by Lemma 3.6) it follows that $F_{2r_1} = F_{2r_2}$ and $g_{r_1} = g_{r_2}$. We obtain that for all $r \in U_{F_1}$ also $r\pi \in U_{F_2}^g$, hence $U_{F_1}\pi \subseteq U_{F_2}^g \cong U_{F_2}$. The family $\{U_i : i < \kappa\}$ of abelian groups is rigid, and this forces $F_1 = F_2$ and $\pi \restriction U_{F_1}$ is conjugation by g. The claim (4.19) becomes $F_1\pi \subseteq F_1^g$, and $g \in H_\alpha = \bigcup_{\varepsilon<\kappa} H_{\alpha_j(\varepsilon)}$ is an element of some $H_{\alpha_j(\varepsilon)}$. If $\rho \in (\varepsilon, \kappa)$, then $x_{*\alpha_j}(\rho)$ by (4.15) is a canonical free generator of F_1 and $x_{*\alpha_j}(\rho) \in H_{\alpha_j(\rho)+1} \setminus H_{\alpha_j(\rho)}$. Hence $x_{*\alpha_j}(\rho) \in H_{\alpha_j(\rho)+1} \subseteq H_{\alpha_j(\rho+1)}$ and also

$$x_{\alpha_j(\rho)}\pi \in F_{\nu_{\rho+1}} \cap \langle x_{\alpha_j(\tau)} : \tau < \kappa\rangle^g \subseteq \langle x_{\alpha_j(\tau)} : \tau < \rho+1\rangle^g$$

from $F_1\pi \subseteq F_1$ and Lemma 3.6. From $g \in H_{\alpha_j}(\varepsilon)$ and $x_{\alpha_j(\tau)} \in H_{\alpha_j}(\rho)$ for all $\tau < \rho$ follows $x_{\alpha_j(\rho)}\pi \in \langle x_{\alpha_j(\rho)}, H_{\alpha_j(\rho)}\rangle$. However $x_{\alpha_j(\rho)}$ is free over $H_{\alpha_j(\rho)}$ and strong for ρ by definition of ν_ρ, which is a contradiction. Hence case **(A)** does not come up.

Case **(B)** can be derived quite easily: There is an ordinal $\alpha_* < \lambda$ such that any free element $x \in H$ over H_{α_*} is weak for α, hence $x\pi \in \langle H_{\alpha*}, x\rangle$. So there is a word

$$w_x(y) \text{ over } H_{\alpha_*} \text{ with variable } y \text{ such that } x\pi = w_x(x).$$

Pick any two free elements b_1, b_2 over H_{α_*} such that b_2 is also free over $\langle H_{\alpha*}, b_1\rangle$. There are many b's! And choose $\gamma < \lambda$ such that $H_{\alpha_*} \cup \{b_1, b_2\} \subseteq H_\gamma$, also let $z \in H$ be free over H_γ. Hence $b_1 z, b_2 z$ and z are free over H_γ and we have

$$\langle H_{\alpha*}, b_i, z\rangle = H_{\alpha*} * \langle b_1\rangle * \langle z\rangle.$$

For the words $w_{b_i z}(y), w_z(y)$ we get $(b_i z)\pi = w_{b_i z}(b_i z)$ and $z\pi = w_z(z)$ and it follows

$$w_{b_i z}(b_i z) = (b_i z)\pi = (b_i \pi)(z\pi) = w_{b_i}(b_i) w_z(z)$$

and Lemma 2.9(a) applies. We can write $b_i\pi = e_i b_i d_i, z\pi = e_z z d_z$ with $e_i, d_i.e_z, d_z \in H_{\alpha_*}$ and $d_i e_z = 1$ for $i = 1, 2$. Hence $d_1 = d_2 = d$ and $e_z = d^{-1}$. We have $b_i\pi = e_i b_i d$ and $z\pi = d^{-1} z d_z$. The same argument for $(b_1 b_2 z)\pi = (b_1\pi)(b_2 z)\pi$ gives $d = d_z$. Similarly for $b_1^{-1} b_2^{-1} z^{-1}$ also $e_i = d^{-1}$, hence $b_i\pi = b_i^d,\ z\pi = z^d$. We conclude that $\pi = d^*$ for all elements $b \in H$ which are free over H_{α_*}. Obviously H is generated by such b's, hence $\pi = d^*$ on H is inner, which finishes the proof of the Main Lemma. □

Proof. (of the Main Theorem 1.3) From Main Lemma 4.1 follows that we only must check conditions $1., 2., 3.$ of the Main Theorem 1.3. This follows however from Lemma 3.6. □

5. Appendix: A Model Theoretic Version Of The Black Box

Let L be the language (of groups in our case) with a finite vocabulary of cardinality at most κ and with a unary function $\psi(\ )$ and a partial two place function $\varphi(\ ,\ )$. From Shelah [23, Chapter IV] we adopt the following prediction principle - a model theoretic version of the 'old' and often used Black Box form [24] which was also used and proved in the appendix of [5], see also [11, 7, 10] for other applications.

In order to match the setting to earlier ones, we will use terms from trees. Condition (i) below can be viewed as a tree embedding from κ^κ into a tree in λ of branches below δ and condition (ii) just says that distinct branches of length κ have only a small branch of length $< \kappa$ in common. Condition (iii) is the earlier requirement that the image of a tree κ^κ can be found in any submodel of the 'trap' (see [5] for instance) here called $(\eta_i^\delta, M_i^\delta :\ i < \delta)$ and (iv) is the prediction of a submodel of M, earlier ([5]) this was a module or a group **together** with an unwanted homomorphism φ, so a pair (H, φ). In our application it will be a group together with a unary map ψ and a partial two place map φ on H.

Another preliminary remark seems in order:

We will predict ordinals from a stationary subset of λ (and submodels), hence the following is actually a 'stationary' Black Box as used in [10] for instance. The reader can either find a proof of the group theoretic version of the Black Box by slight modification from these references, or adopt the model theoretic version, which then has the advantage that it is applicable in many different algebraic situation (including the old ones) without any further changes. Again the proof of the model theoretic version of the Black Box is a natural and easy modification of the existing proofs; a final reference will be [23].

Black Box 5.1. (in model theoretic terms) *Let L be a language just mentioned and suppose $\mu = \mu^\kappa < \lambda = \mu^+$ as before and let S be a stationary subset of*

$$\{\delta < \lambda : \quad \mathrm{cf}\, \delta = \kappa\}.$$

Then there is a sequence (of traps)

$$\langle((\eta_i^\delta, M_i^\delta) : \ i < \kappa), \delta \in S\rangle$$

such that the following holds.

(i) *$\eta_i^\delta : \kappa \to \delta$ $(\varepsilon \to \eta_i^\delta(\varepsilon))$ $(i < \kappa)$ is an increasing, continuous sequence with supremum δ (a branch):*

(ii) *Any two distinct branches are almost disjoint:*

$$\textit{If } i < j < \kappa, \ \textit{then } |\mathrm{Im}\, \eta_i^\delta \cap \mathrm{Im}\, \eta_j^\delta| < \kappa.$$

(iii) *M_i^δ is an L-model with a universe of cardinality κ which is a subset of δ and $\mathrm{Im}\, \eta_i^\delta \subseteq M_i^\delta$.*

(iv) *If M is such an L-model with universe λ, then there are stationary many $\delta \in S$ with some $i < \kappa$ such that $M_i^\delta \subseteq M$.*

References

[1] M. Aubry, *Homotopy theory and models, Based on lectures held at a DMV Seminar in Blaubeuren by H. J. Baues, S. Halperin and J.-M. Lemaire,* DMV Seminar, Vol. **24**, Birkhäuser Verlag, Basel, 1995.

[2] C. Casacuberta, *Recent advances in unstable localization*, CRM Proceedings and Lecture Notes **6**, Amer. Math. Soc., Providence, 1994, 1–22.

[3] C. Casacuberta, J. L. Rodríguez and J.-Y. Tai, *Localizations of abelian Eilenberg–Mac Lane spaces of finite type*, preprint 1998.

[4] C. Casacuberta, D. Scevenels and J. H. Smith, *Implications of large-cardinal principles in homotopical localization*, preprint 1998.

[5] A. L. S. Corner, R. Göbel, *Prescribing endomorphism algebras - a unified treatment,* Proc. London Math. Soc. (3) **50**, 447 – 479 (1985).

[6] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. vol. 1622, Springer-Verlag, Berlin–Heidelberg–New York 1996.

[7] M. Dugas, R. Göbel, *On locally finite p-groups and a problem of Philip Hall's*, Journal of Algebra **159** (1993), 115–138.

[8] M. Dugas, A. Mader and C. Vinsonhaler, *Large E-rings exist*, J. Algebra **108** (1987), 88–101.

[9] P. C. Eklof and A. H. Mekler, *Almost Free Modules. Set-theoretic Methods*, North-Holland, Amsterdam, New York 1990.

[10] B. Franzen, R. Göbel, *Prescribing endomorphism algebras. The cotorsion-free case.* Rend. Sem. Mat. Padova **80** (1989), 215 – 241.

[11] R. Göbel, A. Paras, *Outer automorphism groups of metabelian groups,* Journal of Pure and Applied Algebra **149** (2000) 251–266.

[12] R. Göbel, J. L. Rodríguez and S. Shelah, *Large localizations of finite simple groups*, to appear Crelle Journal 2001

[13] R. Göbel, S. Shelah, *Philip Hall's Problem On Non-Abelian Splitters*, submitted to Proc. Camb. Phil. Soc. 2000

[14] R. Göbel, S. Shelah, *Simple extensions of groups,* in preparation 2001

[15] B. Hart, C. Laflamme and S. Shelah, *Models with second order properties V: A general principle*, Ann. Pure Appl. Logic, **64** (1993), 169–194.

[16] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[17] A. Libman, *Cardinality and nilpotency of localizations of groups and G-modules*, Israel J. Math., to appear.

[18] A. Libman, *A note on the localization of finite groups,* preprint 1998.

[19] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer Ergebnisberichte **89**, Berlin–Heidelberg–New York 1977.

[20] D. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Math. vol **80**, Berlin–Heidelberg–New York 1996.

[21] P. E. Schupp, *A characterization of inner automorphisms*, Proc. Amer. Math. Soc. **101** (1987), 226–228.

[22] S. Shelah, *On a problem of Kurosh, Jonsson groups and applications*, S. I. Adian, W. W. Boone, G. Higman, eds., Word Problems II, North-Holland, Amsterdam, 1980, 373–394.

[23] S. Shelah, *Non Structure Theory,* Oxford University Press (2001) in preparation

[24] S. Shelah, *A combinatorial theorem and endomorphism rings of abelian groups II,* pp. 37 – 86, in *Abelian groups and modules*, CISM Courses and Lectures, **287**, Springer, Wien 1984

[25] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math. vol **61**, Springer-Verlag, Berlin–Heidelberg–New York, 1978.

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