

## RADICALS AND PLOTKIN'S PROBLEM CONCERNING GEOMETRICALLY EQUIVALENT GROUPS

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ABSTRACT. If  $G$  and  $X$  are groups and  $N$  is a normal subgroup of  $X$ , then the  $G$ -closure of  $N$  in  $X$  is the normal subgroup  $\overline{X}^G = \bigcap \{ \ker \varphi \mid \varphi : X \rightarrow G, \text{ with } N \subseteq \ker \varphi \}$  of  $X$ . In particular,  $\overline{1}^G = R_G X$  is the  $G$ -radical of  $X$ . Plotkin [2, 6, 3] calls two groups  $G$  and  $H$  geometrically equivalent, written  $G \sim H$ , if for any free group  $F$  of finite rank and any normal subgroup  $N$  of  $F$  the  $G$ -closure and the  $H$ -closure of  $N$  in  $F$  are the same. Quasiidentities are formulas of the form  $(\bigwedge_{i \leq n} w_i = 1 \rightarrow w = 1)$  for any words  $w, w_i$  ( $i \leq n$ ) in a free group. Generally geometrically equivalent groups satisfy the same quasiidentities. Plotkin showed that nilpotent groups  $G$  and  $H$  satisfy the same quasiidentities if and only if  $G$  and  $H$  are geometrically equivalent. Hence he conjectured that this might hold for any pair of groups; see the Kourovka Notebook [2]. We provide a counterexample.

In a series of paper, B. I. Plotkin and his collaborators [6, 3, 4, 5] investigated radicals of groups and their relation to quasiidentities. If  $G$  is a group, then the  $G$ -radical  $R_G X$  of a group  $X$  is defined by

$$R_G X = \bigcap \{ \ker \varphi; \varphi : X \rightarrow G \text{ any homomorphism } \}.$$

Clearly,  $R_G X$  is a characteristic, hence a normal subgroup of  $X$ . The radical  $R_G$  can also be used to define the  $G$ -closure  $\overline{U}^G = \overline{U}$  of a normal subgroup  $U$  of  $X$ , by saying that  $\overline{U}/U = R_G(X/U)$ . This immediately leads to Plotkin's definition of geometrically equivalent groups, see [6, 3, 4, 5] and [2, p. 113].

**Definition 0.1.** *Let  $G$  and  $H$  be two groups. Then  $G$  and  $H$  are geometrically equivalent, written  $G \sim H$ , if for any free group  $F$  of finite rank and any normal subgroup  $U$  of  $F$  the  $G$ - and  $H$ -closure of  $U$  in  $F$  are the same, i.e. for any normal subgroup  $U$  we have  $\overline{U}^G = \overline{U}^H$ .*

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It is easy to see that  $G \sim H$  if and only if  $R_G K = R_H K$  for all finitely generated groups  $K$ . Plotkin notes that geometrically equivalent groups satisfy the same quasi-identities. The well-known notion of quasi-identities relates to quasivarieties of groups. A *quasiidentity* is an expression of the form

$$w_1 = 1 \wedge \cdots \wedge w_n = 1 \rightarrow w = 1 \text{ where } w_i, w \in F \text{ (} i \leq n \text{) are words.}$$

Moreover the following was shown in [6], see [2, p.113].

**Theorem 0.2.** (a) *If  $G \sim H$ , and  $G$  is torsion-free, then  $H$  is torsion-free.*  
 (b) *If  $G, H$  are nilpotent, then  $G \sim H$  if and only if  $G$  and  $H$  satisfy the same quasi-identities.*

This lead Plotkin to conjecture that two groups might be geometrically equivalent if and only if they satisfy the same quasi-identities, see the Kourovka Notebook [2, p.113, problem 14.71]. In this note we refute this conjecture. Clearly there are only countably many finitely presented groups which we enumerate as the set  $\mathfrak{K} = \{K_n : n \in \omega\}$  and let  $G = \prod_{n \in \omega} K_n$  be the restricted direct product. Then  $G$  satisfies only those quasi-identities satisfied by all groups and so if  $H$  is any group with  $G \leq H$ ,  $G$  satisfies the same quasi-identities as  $H$ .

R. Camm [1, p. 68, p. 75 Corollary] proved there are  $2^{\aleph_0}$  non-isomorphic, two-generator, simple groups, see also Lyndon, Schupp [7, p. 188, Theorem 3.2]. So there exists a 2-generated simple group  $L$  which cannot be mapped nontrivially into  $G$ . We consider the pair  $G, H = L \times G$  and show the following:

**Theorem 0.3.** *If  $G, H$  and  $L$  are as above,  $R_G L = L$  and  $R_H L = 1$ . In particular  $G$  and  $H$  are not geometrically equivalent. Since  $G \leq H$  satisfy the same quasi-identities, this is the required counterexample.*

*Proof.* Since  $L$  is a two-generated simple group,  $L$  is an epimorphic image of a free group of rank 2. So it is enough to prove that  $R_G L = L$  and  $R_H L = 1$ . The first equality follows since there is no nontrivial homomorphism of  $L$  into  $G$ . On the other hand, there is a canonical embedding  $L \rightarrow H = L \times G$ , so  $R_H L = 1$ .  $\square$

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