Outer Automorphism Groups of Ordered Permutation Groups

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Abstract

An infinite linearly ordered set $(S, \leq)$ is called doubly homogeneous if its automorphism group $A(S)$ acts 2-transitively on it. We show that any group G arises as outer automorphism group $G \cong \mathrm{Out}(A(S))$ of the automorphism group $A(S)$, for some doubly homogeneous chain $(S, \leq)$.

1 Introduction

An infinite linearly ordered set ("chain") $(S, \leq)$ is called *doubly homogeneous*, if its automorphism group, i.e. the group of all order-preserving permutations, $A(S) = \mathrm{Aut}(S, \leq)$ acts 2-transitively on it. Chains $(S, \leq)$ of this type and their automorphism groups $A(S)$ have been intensively studied. They have been used e.g. for the construction of infinite simple torsion free groups (Higman [8]) or, in the theory of lattice-ordered groups (ℓ-groups), for embedding arbitrary ℓ-groups into simple divisible ℓ-groups (Holland [9]). The normal subgroup lattices of the groups $A(S)$ have been determined in [1, 3]. Obviously, all linearly ordered fields are examples for such chains. For a variety of further results, see Glass [6]. Here, we will be concerned with outer

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automorphism groups $\mathrm{Out}(A(S)) = \mathrm{Aut}(A(S))/\mathrm{Inn}(A(S))$ of the automorphism groups $A(S)$ for doubly homogeneous chains $(S, \leq)$.

In the literature, many authors have dealt with the problem of determining which groups G can arise as $G \cong \mathrm{Out}(H)$ for H in a given class of groups. Already Schreier and Ulam [14] showed that each automorphism of the infinite symmetric groups is inner, i.e. $\mathrm{Out}(\mathrm{Sym}(\mathbb{N})) = \{1\}$. It is known that each outer automorphism of $A(\mathbb{Q})$ or of $A(\mathbb{R})$ arises via conjugation by an anti-automorphism of $(\mathbb{R}, \leq)$, hence $\mathrm{Out}(A(\mathbb{Q})) \cong \mathrm{Out}(A(\mathbb{R})) \cong \mathbb{Z}_2$, and it is known how to construct doubly homogeneous chains $(S, \leq)$ for which $\mathrm{Out}(A(S))$ is trivial (Holland [10], Weinberg [16], Droste [2]). Solving a problem which had been open for quite some time, Holland [10] constructed a doubly homogeneous chain $(S, \leq)$ for which $A(S)$ has an outer automorphism *not* arising from an anti-automorphism of the Dedekind-completion $(\overline{S}, \leq)$ of $(S, \leq)$. In his example, $\mathrm{Out}(A(S)) \cong V_4$, Klein's four-group. Assuming the generalized continuum hypothesis (GCH), McCleary [13] constructed further examples of this type. However, to date V_4, $\mathbb{Z}_2$ and the trivial group are the only groups realized as $\mathrm{Out}(A(S))$, where $(S, \leq)$ is a doubly homogeneous chain. In fact, a realization of any group G even just as the outer automorphism group $G \cong \mathrm{Out}(H)$ of some group H was established more recently in Matumoto [11].

With the pointwise ordering of functions, $A(S)$ becomes an ℓ-group, and Holland [9] showed that any ℓ-group H can be ℓ-embedded (i.e. embedded as an ℓ-group) into $A(S)$, for some doubly homogeneous chain $(S, \leq)$. Here, we will show the following generalization of the previously mentioned results:

Theorem 1.1 *Let G be any group, H any ℓ-group and λ a regular uncountable cardinal with $\lambda \geq |G|$ and $\lambda > |H|$. Then there exists a doubly homogeneous chain $(S, \leq)$ of cardinality λ such that $G \cong \mathrm{Out}(A(S))$ and H ℓ-embeds into $A(S)$.*

Here, the realization result $G \cong \mathrm{Out}(A(S))$ involves constructing a doubly homogeneous chain $(S, \leq)$ such that $A(\overline{S})$ acts on the set of orbits of $A(S)$ in $\overline{S}$ just like G. Using codings of the group action of G through a system of suitable stationary subsets of λ inside $\overline{S}$, we will first describe a class of doubly homogeneous chains $(S, \leq)$ for which $G \cong \mathrm{Out}(A(S))$ follows. Then, in Section 4, we will actually construct these chains. As often for homogeneous structures, this could be done by suitable amalgamations of linear orderings, but here we will use methods of [3] for a more explicit construction. For the simultaneous embedding of H into $A(S)$, we will use Holland's result [9].

By a Löwenheim-Skolem argument, we obtain as a consequence:

Corollary 1.2 *Let H be any ℓ-group, G any group and λ a regular uncountable cardinal with $\lambda \geq |G|$ and $\lambda > |H|$. Then there exists an ℓ-group K with $H \subseteq K$ (as ℓ-groups) and $|K| = \lambda$ such that* $\mathrm{Out}(K) \cong G$.

Here, each group automorphism of K is also a lattice automorphism of (the ℓ-group) K.

Similar realization results have been established in the literature for various classes of groups. Dugas and Göbel [5] showed that any countable group arises as the outer automorphism group of some locally-finite p-group, and Göbel and Paras [7] established the corresponding result for the class of torsion-free metabelian groups. Note that the groups $A(S)$ of Theorem 1.1 are not simple. In Droste, Giraudet and Göbel [4], Theorem 1.1 will be used to show that *any group can be realized as outer automorphism group of a simple group*, which in turn arises as the automorphism group of a suitable homogeneous circle.

2 Outer automorphisms

In this section, we will present a class of doubly homogeneous chains $(S, \leq)$ for which $\mathrm{Out}(A(S)) \cong G$, for a given group G.

For any chain $(S, \leq)$, we denote by $\overline{S}$ its Dedekind-completion. Clearly, each automorphism f of $(S, \leq)$ extends uniquely to an automorphism of $(\overline{S}, \leq)$ which we will also denote by f; hence $A(S) \subseteq A(\overline{S})$. We recall that $(S, \leq)$ is *doubly homogeneous*, if for all $u, v, x, y \in S$ with $u < v$ and $x < y$ there is $g \in A(S)$ such that $u^g = x$ and $v^g = y$. If $x \in \overline{S}$, the set $x^{A(S)} = \{x^f : f \in A(S)\}$ is called an *orbit* of $A(S)$ in $\overline{S}$. The following result relates outer automorphisms of $A(S)$ to automorphisms of $\overline{S}$ permuting the orbits of $A(S)$.

Proposition 2.1 ([6]) *Let $(S, \leq)$ be a doubly homogeneous chain. Each automorphism φ of $A(S)$ corresponds bijectively to an automorphism or anti-automorphism f of $(\overline{S}, \leq)$, which permutes the orbits of $A(S)$ in $\overline{S}$, such that $g^\varphi = f^{-1} \circ g \circ f$ for all $g \in A(S)$.*

Now assume $(\overline{S}, \leq)$ is not anti-isomorphic to itself, and let $Z := \{U \subseteq \overline{S} :\ U \text{ orbit of } A(S) \text{ in } \overline{S}, \text{ and } (S, \leq) \cong (U, \leq)\}$. By Proposition **??**, each automorphism φ of $A(S)$ induces an element $f \in A(\overline{S})$ which permutes Z, and we obtain an epimorphism from $\mathrm{Aut}(A(S))$ onto the group $G_Z = \{p \in \mathrm{Sym}(Z) :\ \exists f \in A(\overline{S}).\ f \text{ induces } p\}$ with kernel $\mathrm{Inn}(A(S))$. Hence $\mathrm{Out}(A(S)) \cong G_Z$. In order to prove Theorem 1.1, given any group G we therefore have to construct a doubly homogeneous chain $(S, \leq)$ with $(\overline{S}, \leq)$

not anti-isomorphic to itself such that G represents the action of $A(\overline{S})$ on the orbits of $A(S)$ in $\overline{S}$, i.e. $G \cong G_Z$.

We recall some notation, which is mostly standard. Let $(S, \leq)$ be any chain. If $x \in \overline{S}$ has no immediate predecessor, we define the *cofinality* of x to be

$$\mathrm{cof}(x) := \min\{|A| : A \subseteq \overline{S}, x \notin A, x = \sup A\}.$$

We adopt the convention that if x has an immediate predecessor, then $\mathrm{cof}(x) = \aleph_0$. We define the *coinitiality* $\mathrm{coi}(x)$ of x dually. If $\mathrm{cof}(x) = \mathrm{coi}(x)$ then this cardinal is called the *coterminality* of x, denoted $\mathrm{cot}(x)$. The ordered pair $(\mathrm{cof}(x), \mathrm{coi}(x))$ is called the *character* of x, denoted $\mathrm{char}(x)$, and we let $\mathrm{char}(S) = \{\mathrm{char}(x) : x \in S\}$. If $X \subseteq \overline{S}$, we let $\mathrm{char}_{\overline{S}}(X) = \{\mathrm{char}(x) : x \in X\}$, where $\mathrm{char}(x)$ is determined in $\overline{S}$. If $a, b \in \overline{S}$ with $a < b$ and $X \subseteq \overline{S}$, let $[a, b]_X = \{x \in X : a \leq x \leq b\}$. If $A, B \subseteq \overline{S}$, we write $A < B$ to denote that $a < b$ for all $a \in A$, $b \in B$, and $A < x$ abbreviates $A < \{x\}$. The chain $(S, \leq)$ has *countable coterminality*, denoted $\mathrm{cot}(S) = \aleph_0$, if it contains a countable subset which is unbounded above and below in S. We say that $(S, \leq)$ is *dense*, if for all $a < b$ in S there is $s \in S$ with $a < s < b$, and *unbounded*, if S does not contain a greatest or smallest element.

Clearly, all points in a given orbit of $A(S)$ in $\overline{S}$ have the same character. Also, it is well-known and easy to see by piecewise patching of automorphisms of S (cf. [6] or below the argument for Lemma **??**), that if $(S, \leq)$ is doubly homogeneous, then all elements $x \in \overline{S} \setminus S$ with $\mathrm{cot}(x) = \aleph_0$ form a single orbit of $A(S)$ in $\overline{S}$. Hence when constructing the chain $(S, \leq)$ for Theorem 1.1, we have to resort to orbits of $A(S)$ whose elements do not have countable coterminality.

As usual, we identify cardinals with the least ordinal of their cardinality. A subset A of a cardinal λ is called *stationary*, if $A \cap C \neq \emptyset$ for each closed unbounded subset C of λ. A sequence $\langle x_i : i \in \lambda \rangle \subseteq \overline{S}$ is called *continuously increasing* if $x_i < x_j$ for each $i < j < \lambda$, and $x_j = \sup_{i<j} x_i$ in $\overline{S}$ for each limit ordinal $j < \lambda$. We define *continuously decreasing* dually.

Now we make the following

General assumption. In all of this section let G be a group with neutral element e, and let $\aleph_0 \leq \kappa < \lambda$ be two regular cardinals such that $\lambda \geq |G|$. Let $S^\lambda_\kappa = \{j < \lambda : \mathrm{cof}(j) = \kappa\}$, a stationary subset of λ. By Solovay's theorem [15], we split $S^\lambda_\kappa = \dot{\bigcup}_{g \in G} S_g$ into $|G|$ pairwise disjoint stationary subsets S_g $(g \in G)$.

Now we define a class of chains which will be crucial in all of our subsequent considerations.

Definition 2.2 *Let $\mathcal{K}$ be the class of all structures $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G})$ with the following properties:*

1. *$(S, \leq)$ is a dense unbounded chain;*
2. *P_g, Q_g $(g \in G)$ are pairwise disjoint non-empty subsets of $\overline{S} \setminus S$; let $P = \dot{\bigcup}_{g\in G} P_g$ and $Q = \dot{\bigcup}_{g\in G} Q_g$;*
3. *whenever $x \in P_g$ $(g \in G)$, then $\mathrm{char}(x) = (\lambda, \kappa)$, and there is a continuously increasing sequence $\langle x_i : i < \lambda\rangle$ in $\overline{S}$ such that $x = \sup_{i<\lambda} x_i$ and whenever $h \in G$ and $j \in S_h$, then $x_j \in Q_{hg}$;*
4. *whenever $y \in Q_g$ $(g \in G)$, then $\mathrm{char}(y) = (\kappa, \lambda)$, and there is a continuously decreasing sequence $\langle y_i : i < \lambda\rangle$ in $\overline{S}$ such that $y = \inf_{i<\lambda} y_i$ and whenever $h \in G$ and $j \in S_h$, then $y_j \in P_{hg}$;*
5. *whenever $a, b, c, d \in \overline{S}$ with $a < b$, $c < d$, $x \in P \cap (a, b)$, $y \in Q \cap (a, b)$ and $f : [a, b]_{\overline{S}} \longrightarrow [c, d]_{\overline{S}}$ is an order-isomorphism, then $x^f \in P$ and $y^f \in Q$.*

We will refer to the sets P_g, Q_g $(g \in G)$ as the *colours of* $\mathcal{S}$, and the sets P_g $(g \in G)$ are the *P-colours*. Condition 2.2(5) means that we can "locally recognize" points belonging to P or to Q. It ensures that then

$$(*) \qquad P^f \subseteq P \text{ and } Q^f \subseteq Q \text{ for each } f \in A(\overline{S}).$$

If $\aleph_0 < \kappa < \lambda$, we note that we will construct our structures $\mathcal{S}$ such that

$$(1) \qquad \mathrm{char}_{\overline{S}}(P) = \{x \in \overline{S} : \mathrm{char}(x) = (\lambda, \kappa)\} \text{ and}$$
$$(2) \qquad \mathrm{char}_{\overline{S}}(Q) = \{y \in \overline{S} : \mathrm{char}(y) = (\kappa, \lambda)\};$$

this obviously implies condition 2.2(5). Next we note:

Lemma 2.3 *Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$. Let $f \in \mathrm{Aut}(\overline{S}, \leq)$ satisfy $P_e^f \subseteq P_g$ for some $g \in G$. Then $P_h^f \subseteq P_{hg}$ and $Q_h^f \subseteq Q_{hg}$ for each $h \in G$. In particular, if $P_e^f \subseteq P_e$, then each P_h and each Q_h $(h \in G)$ is invariant under f.*

Proof: Let $h \in G$. We first show that $Q_h^f \subseteq Q_{hg}$. Choose any $x \in Q_h$. By requirement 2.2(5), clearly $x^f \in Q$. So, $x^f \in Q_k$ for some $k \in G$. By condition 2.2(4), we can find two continuously decreasing sequences $\langle y_i : i < \lambda\rangle$ and $\langle z_i : i < \lambda\rangle$ in $\overline{S}$ such that $x = \inf_{i<\lambda} y_i$, $x^f = \inf_{i<\lambda} z_i$, and whenever $h' \in G$ and $j \in S_{h'}$, then $y_j \in P_{h'h}$ and $z_j \in P_{h'k}$. Standard back-and-forth arguments involving closed unbounded subsets of uncountable cardinals show that $C = \{i < \lambda : \ y_i^f = z_i\}$ is a closed unbounded subset of λ. Since $S_{h^{-1}}$ is stationary, there is $j \in S_{h^{-1}}$ with $y_j^f = z_j$. So, $y_j \in P_e$ and $z_j = y_j^f \in P_g \cap P_{h^{-1}k}$. Thus $k = hg$ as claimed.

By a similar argument, it now follows that $P_h^f \subseteq P_{hg}$ for each $h \in G$. The final statement is then immediate. $\square$

Now let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G})$ and $\mathcal{S}' = (S', \leq, (P'_g)_{g\in G}, (Q'_g)_{g\in G}) \in \mathcal{K}$ and $A \subseteq \overline{S}$, $B \subseteq \overline{S'}$. Let $h \in G$ and $f : (A, \leq) \longrightarrow (B, \leq)$ be an order-isomorphism. We say that f *maps the colours as prescribed by* h, if

$$(A \cap P_g)^f = B \cap P'_{gh} \text{ and } (A \cap Q_g)^f = B \cap Q'_{gh} \quad \text{for each } g \in G.$$

If here $\mathcal{S} = \mathcal{S}'$, we also say that f *permutes the colours as prescribed by* h.

We call $\mathcal{S}$ *doubly homogeneous in the P-colours*, if for any $g, h \in G$ and $u, v \in P_g$, $x, y \in P_{gh}$ with $u < v$ and $x < y$, there is $f \in A(S)$ such that $\{u, v\}^f = \{x, y\}$ and f permutes the colours as prescribed by h.

Now we can prove:

Theorem 2.4 *Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$ be doubly homogeneous in the P-colours such that $(\overline{S}, \leq)$ is not anti-isomorphic to itself, and let $X = P_e$. Then $(X, \leq)$ is a doubly homogeneous chain and* $\mathrm{Out}(A(X)) \cong G$.

PROOF: Clearly, $(X, \leq)$ is doubly homogeneous and dense in $\overline{S}$. We identify $\overline{X} = \overline{S}$. By Lemma 2.3, each P_g and each Q_g $(g \in G)$ is invariant under $A(X)$. By homogeneity we get $P_g = P_g^{A(X)}$ for each $g \in G$. Hence each P_g $(g \in G)$ is an orbit of $A(X)$ in $\overline{X}$. By Proposition 2.1, any outer automorphism ψ of $A(X)$ determines an automorphism f of $\overline{X}$ which permutes the orbits of $A(X)$ in $\overline{X}$ and hence by requirement 2.2(3), permutes the orbits P_g $(g \in G)$ among themselves. By Lemma 2.3, this permutation f determines an element h of G. Conversely, by homogeneity of $\mathcal{S}$, any element h of G can be realized in this way by an automorphism f of $\overline{X}$ permuting the sets P_g $(g \in G)$, and hence h is realized by an outer automorphism ψ of $A(X)$. This correspondence constitutes the required isomorphism. $\square$

3 Isomorphisms between intervals

It is well-known that a chain $(S, \leq)$ is doubly homogeneous if and only if any two of its intervals $[a, b]_S$ and $[c, d]_S$ $(a, b, c, d \in S$ with $a < b$, $c < d)$ are order-isomorphic. In this section we will derive a similar result for particular structures in $\mathcal{K}$. In all of this section, we make the *general assumption* of Section 2.

Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$. We say that $\mathcal{S}$ is *G-homogeneous for S-intervals*, if for any $u < v$ and $x < y$ in S and any $h \in G$, there is an isomorphism $f : [u, v]_S \longrightarrow [x, y]_S$ permuting the colours as prescribed by h.

We call $\mathcal{S}$ *G-homogeneous for P-intervals*, if for any $g, h \in G$ and $u, v \in P_g$, $x, y \in P_{gh}$ with $u < v$ and $x < y$, there is an isomorphism $f : [u, v]_S \longrightarrow [x, y]_S$ permuting the colours as prescribed by h.

Clearly, if $\mathcal{S}$ is doubly homogeneous in the P-colours, $\mathcal{S}$ is also G-homogeneous for P-intervals. For the converse we need additional assumptions on $\mathcal{S}$. In general, there seems to be no relationship between G-homogeneity for S-intervals and for P-intervals, respectively. However, in this section we will show for particular structures $\mathcal{S} \in \mathcal{K}$ that G-homogeneity for S-intervals implies G-homogeneity for P-intervals. First we note:

Lemma 3.1 *Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$ be G-homogeneous for S-intervals. Let $u, v, x, y \in \overline{S}$ such that $u < v$, $x < y$ and* $\operatorname{cot}(u, v) = \aleph_0 = \operatorname{cot}(x, y)$. *Let $h \in G$. Then there is an isomorphism $f : (u, v)_S \longrightarrow (x, y)_S$ permuting the colours as prescribed by h.*

PROOF: Choose $\mathbb{Z}$-sequences $(a_i)_{i\in\mathbb{Z}} \subseteq (u, v)_S$ and $(b_i)_{i\in\mathbb{Z}} \subseteq (x, y)_S$ such that $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for each $i \in \mathbb{Z}$ and $u = \inf_{i\in\mathbb{Z}} a_i$, $v = \sup_{i\in\mathbb{Z}} a_i$, $x = \inf_{i\in\mathbb{Z}} b_i$, $y = \sup_{i\in\mathbb{Z}} b_i$. For each $i \in \mathbb{Z}$, there is an isomorphism $f_i : [a_i, a_{i+1}]_S \longrightarrow [b_i, b_{i+1}]_S$ permuting the colours as prescribed by h. Patching the f_i's together, we obtain the required isomorphism f. $\square$

Now we turn to the definition of structures $\mathcal{S} \in \mathcal{K}$ for which there is a closer relationship between the three types of homogeneity we defined.

A partially ordered set $(T, \preceq)$ is called a *tree*, if it contains a smallest element and for each $x \in T$, the set $(\{t \in T : t \prec x\}, \preceq)$ is well-ordered; the cardinality of this set will also be called the *height* of x. We write M_x for the set of immediate successors (= minimal strict upper bounds) of x in $(T, \preceq)$. We will now consider particular trees together with a partial ordering which will later on be extended to a linear ordering.

Definition 3.2 *Let $\mathcal{T}$ be the class of all structures $(T, \preceq, \leq)$ with the following properties:*

1. *$(T, \preceq)$ is a non-singleton tree and $\leq$ is a partial order on T;*
2. *each element $x \in T$ has finite height and if the height of x is even (odd), then $(M_x, \leq)$ is either empty or a chain isomorphic (anti-isomorphic, respectively) to λ;*

3. *whenever* $y \in M_x$ *for some* $x \in T$ *of even (odd) height, then* $M_y \neq \emptyset$ *if and only if in* $(M_x, \leq)$ *we have* $\mathrm{cof}(y) = \kappa$ $(\mathrm{coi}(y) = \kappa$*, respectively).*

Observe that each such tree has size λ.

Let $(T, \preceq, \leq) \in \mathcal{T}$. We denote the smallest element of $(T, \preceq)$ by t_0. Now we extend the partial order $\leq$ to a linear order $\leq'$ on T as follows. First, we put $M_{t_0} <' t_0$. Now assume $x \in T$ has even (odd, respectively) height and $y \in M_x$ with $M_y \neq \emptyset$. Then put $y <' M_y <' y^+$ $(y^+ <' M_y <' y)$ where y^+ denotes the immediate successor (predecessor) of y in $(M_x, \leq)$, respectively. For simplicity, we denote this extension $\leq'$ also by $\leq$.

Observe that in the Dedekind-completion $(\overline{T}, \leq)$, each element $x \in \overline{T} \setminus T$ corresponds uniquely to a maximal path $(a_i)_{i\in\omega}$ in $(T, \leq)$ such that $a_0 = t_0$, $a_{i+1} \in M_{a_i}$, and $a_{2i+1} < a_{2i+3} < a_{2i+2} < a_{2i}$ for each $i \in \omega$, and $x = \sup_{i\in\omega} a_{2i+1} = \inf_{i\in\omega} a_{2i}$ in $(\overline{T}, \leq)$. In particular, $\mathrm{cot}(x) = \aleph_0$. Hence $\mathrm{char}(\overline{T}) = \{(\mu, \aleph_0), (\aleph_0, \mu) : \aleph_0 \leq \mu \leq \lambda,\ \mu \neq \kappa,\ \mu \text{ regular}\} \cup \{(\lambda, \kappa), (\kappa, \lambda), (\aleph_0, \aleph_0)\}$.

Next, we define for each $k \in G$ a *coloured linear ordering* $\mathcal{L}_k = (L_k, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G})$ as follows. Let $(A, \leq)$ be a fixed linear ordering anti-isomorphic to κ and put, irregardless of k, $(L_k, \leq) = (\overline{T}, \leq) + (A, \leq)$, the disjoint sum with $(\overline{T}, \leq)$ "to the left" of $(A, \leq)$. Thus $t_0 = \inf A$ in $(L_k, \leq)$. We define the sets $P_g, Q_g \subseteq T$ now. Only elements of even (odd) height will belong to $\bigcup_{g\in G} P_g$ $(\bigcup_{g\in G} Q_g)$, respectively. First, put $t_0 \in P_k$. By induction, assume $t \in T$ satisfies $t \in P_g$ and $(M_t, \leq) \cong \lambda$, thus $M_t = \{x_i : i \in \lambda\}$ with $x_i < x_j$ if $i < j$ in λ. Then for each $h \in G$ and $j \in S_h$ put $x_j \in Q_{hg}$. Next, let $t \in Q_g$ and $(M_t, \leq)$ be anti-isomorphic to λ, thus $M_t = \{y_i : i \in \lambda\}$ with $y_j < y_i$ if $i < j$ in λ. Then for each $h \in G$ and $j \in S_h$ put $y_j \in P_{hg}$.

The reader will notice the similarity of $\mathcal{L}_k$ to the structures in $\mathcal{K}$. In fact, the chains $\mathcal{L}_k$ will be basic building blocks for particular chains in $\mathcal{K}$. Observe that $|P_g| = |Q_g| = \lambda$ for each $g \in G$, since any stationary subset of λ has size λ. First we note:

Lemma 3.3 *Let* $h, k \in G$.

(a) *Let* $\mathcal{L}_h = (L_h, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G})$ *and* $\mathcal{L}_{kh} = (L_{kh}, \leq, (P'_g)_{g\in G}, (Q'_g)_{g\in G})$. *Then* $f = id : L_h \longrightarrow L_{kh}$ *satisfies* $P_g^f = P'_{gh}$ *and* $Q_g^f = Q'_{gh}$ *for each* $g \in G$.

(b) *Let* $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$ *and let* $\pi : (S, \leq) \longrightarrow (X, \leq)$ *be an order-isomorphism. Then we can find* $P'_g, Q'_g \subseteq \overline{X}$ $(g \in G)$ *such that* $\mathcal{X} = (X, \leq, (P'_g)_{g\in G}, (Q'_g)_{g\in G}) \in \mathcal{K}$ *and* π *maps the colours as prescribed by* h.

PROOF: (a) Observe that $t_0 \in P_h$ in $\mathcal{L}_h$ and $t_0 \in P_{kh}$ in $\mathcal{L}_{kh}$. Now continue by induction through $(T, \leq)$ and construction of $\mathcal{L}_h$, $\mathcal{L}_{kh}$.

(b) Put $P'_{gh} = P_g^\pi$ and $Q'_{gh} = Q_g^\pi$ $(g \in G)$ to obtain the result. □

If $(C, \leq)$ is a chain and $a, b \in C$ are such that $a < b$ and there is no $c \in C$ with $a < c < b$, we call the pair (a, b) a *gap* of C, denoted $(a|b)$.

Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$, let $h \in G$ and let $\mathcal{L}_h = (L_h, \leq, (P'_g)_{g\in G}, (Q'_g)_{g\in G})$. An embedding $\varphi : (L_h, \leq) \longrightarrow (\overline{S}, \leq)$ is called *nice*, if it satisfies:

1. φ preserves all suprema and infima, i.e. $(\sup A)^\varphi = \sup A^\varphi$ and $(\inf A)^\varphi = \inf A^\varphi$ in $(\overline{S}, \leq)$ for each non-empty subset $A \subseteq L_h$;
2. $L_h^\varphi \subseteq \overline{S} \setminus S$ and $(P'_g)^\varphi \subseteq P_g$, $(Q'_g)^\varphi \subseteq Q_g$ for each $g \in G$;
3. whenever $x < y$ form a gap in L_h, then $\operatorname{cot}(x^\varphi, y^\varphi) = \aleph_0$ in $(\overline{S}, \leq)$;
4. $\operatorname{cot}(x) = \aleph_0$ if x is the smallest or the largest element of $(L_h, \leq)$.

A structure $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}$ is called *nice*, if for each $h \in G$ and $x \in P_h$ in $\mathcal{S}$ there exists a nice embedding $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}$ with $t_0^\varphi = x$. We denote by $\mathcal{K}_{\text{nice}}$ the class of all nice structures in $\mathcal{K}$.

Proposition 3.4 *Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}_{\text{nice}}$ be G-homogeneous for S-intervals. Then $\mathcal{S}$ is G-homogeneous for P-intervals.*

PROOF: Choose $g, h \in G$ and $u, v \in P_g$, $x, y \in P_{gh}$ with $u < v$ and $x < y$. By assumption, there are nice embeddings $\varphi, \varphi' : \mathcal{L}_g \longrightarrow \mathcal{S}$ and $\psi, \psi' : \mathcal{L}_{gh} \longrightarrow \mathcal{S}$ with $t_0^\varphi = v$, $t_0^{\varphi'} = u$, $t_0^\psi = y$, $t_0^{\psi'} = x$.

First, from φ', ψ' we obtain two continuously decreasing sequences $(a_i)_{i\in\kappa} \subseteq (u, v)$ and $(b_i)_{i\in\kappa} \subseteq (x, y)$ such that:

(i) $u = \inf_{i\in\kappa} a_i$ and $x = \inf_{i\in\kappa} b_i$;

(ii) $a_i, b_i \in \overline{S} \setminus (S \cup P \cup Q)$ for each $i \in \kappa$;

(iii) $\operatorname{cot}(a_{i+1}, a_i) = \aleph_0 = \operatorname{cot}(b_{i+1}, b_i)$ for each $i \in \kappa$.

By Lemma 3.1, for each $i \in \kappa$ there is an isomorphism $f_i : (a_{i+1}, a_i)_S \longrightarrow (b_{i+1}, b_i)_S$ permuting the colours as prescribed by h. Patching these isomorphisms f_i $(i \in \kappa)$ together, we obtain an isomorphism $f : [u, a_0) \longrightarrow [x, b_0)$ which permutes the colours as prescribed by h.

Secondly, let $m = \min(T, \leq)$. Then $m^\varphi < v$ and $m^\psi < y$, and we may assume that $a_0 < m^\varphi$ and $b_0 < m^\psi$ (otherwise consider appropriate upper segments of $\mathcal{L}_g^\varphi$ and $\mathcal{L}_{gh}^\psi$ subsequently). Observing Lemma 3.3, we see that $\varphi^{-1}\psi$ maps $[m^\varphi, v] \cap L_g^\varphi$ onto $[m^\psi, y] \cap L_{gh}^\psi$ permuting the colours in these subsets of $\mathcal{S}$ as prescribed by h. We have $[m^\varphi, v] \setminus L_g^\varphi = \bigcup (a^\varphi, b^\varphi)$ and $[m^\psi, y] \setminus L_{gh}^\psi = \bigcup (a^\psi, b^\psi)$ where the two unions are taken over all gaps $(a|b)$ in $(T, \leq)$. Moreover, for each gap $(a|b)$ in $(T, \leq)$, we have $\mathrm{cot}(a^\varphi, b^\varphi) = \aleph_0 = \mathrm{cot}(a^\psi, b^\psi)$, and by Lemma 3.1 there is an isomorphism $\rho_{(a|b)} : (a^\varphi, b^\varphi) \longrightarrow (a^\psi, b^\psi)$ permuting the colours as prescribed by h. Patching all these isomorphism $\rho_{(a|b)}$ together with $\varphi^{-1}\psi$ above, we obtain an isomorphism $f' : [m^\varphi, v] \longrightarrow [m^\psi, y]$ which permutes the colours as prescribed by h.

Again by Lemma 3.1, there is also such an isomorphism $f'' : (a_0, m^\varphi) \longrightarrow (b_0, m^\psi)$. Now $f \cup f'' \cup f'$ maps $[u, v]_S$ isomorphically onto $[x, y]_S$ and permutes the colours as prescribed by h. $\square$

4 Construction of doubly homogeneous chains

In this section, we wish to prove Theorem 1.1 first without considering H, i.e. for $H = \{1\}$. Afterwards, we will point out how to change our constructions in order to accommodate arbitrary ℓ-groups H.

Therefore, until Theorem 4.3 we make the *general assumption* of Section 2. We will construct a structure $\mathcal{S} \in \mathcal{K}$ which is G-homogeneous for S-intervals. By Proposition 3.4 it will follow that $\mathcal{S}$ is doubly homogeneous in the P-colours as needed for Theorem 2.4.

The building blocks of our chain $(S, \leq)$ are the following orderings, which were already defined and used in [3].

Definition 4.1 ([3]) *A chain $(L, \leq)$ is called a* good λ-set *if the following conditions are satisfied:*

1. *$|L| = \lambda$ and $(L, \leq)$ is dense and unbounded;*
2. *each $x \in L$ has countable coterminality;*
3. *whenever $x, y \in L$ with $x < y$, there is a set $A \subseteq [x, y]_{\overline{L}} \setminus L$ such that $|A| = \lambda$ and each $a \in A$ has countable coterminality.*

We let $\mathcal{K}^\lambda$ comprise all structures $\mathcal{S} = (S, \leq, (P_g)_{g \in G}, (Q_g)_{g \in G}) \in \mathcal{K}$ for which $(S, \leq)$ is a good λ-set, and $|P_g| = |Q_g| = \lambda$ for each $g \in G$. We let $\mathcal{K}^\lambda_{\mathrm{nice}}$ consist of all nice $\mathcal{S} \in \mathcal{K}^\lambda$. The following clarifies the existence of good λ-sets.

Lemma 4.2 ([3, Lemma 4.2]) *There exists a good λ-set $(L, \leq)$ of countable coterminality such that* $\mathrm{char}(\overline{L}) = \{(\mu, \aleph_0) : \ \aleph_0 \leq \mu \leq \lambda, \ \mu \text{ regular}\}$.

Here the final statement on $\mathrm{char}(\overline{L})$ follows easily from the construction for [3, Lemma 4.2].

The proof of Theorem 1.1 uses two basic construction techniques from [3] which we now describe.

Basic Construction A (defining a colour-permuting isomorphism)

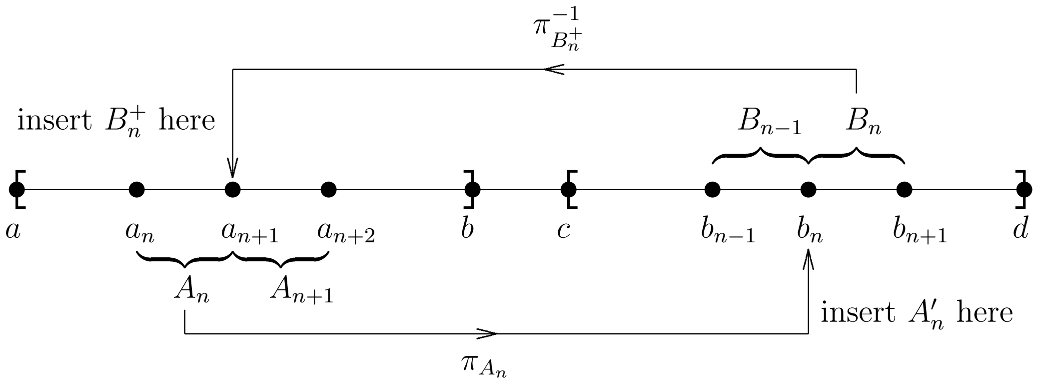

Figure 1: Defining a colour-permuting isomorphism.

Let $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}^\lambda$. Let $h \in G$ and $a, b, c, d \in S$ with $a < b < c < d$. We will enlarge $\mathcal{S}$ to a superstructure $\mathcal{S}^* \supseteq \mathcal{S}$ with $\mathcal{S}^* = (S^*, \leq, (P^*_g)_{g\in G}, (Q^*_g)_{g\in G})$ such that there is an isomorphism $f : [a, b]_{S^*} \longrightarrow [c, d]_{S^*}$ permuting the colours of $\mathcal{S}^*$ as prescribed by h. This will be obtained by splitting both $(a, b)_S$ and $(c, d)_S$ into countably many subintervals, inserting copies of these intervals into $(c, d)_S$ and $(a, b)_S$, respectively, to obtain S^*, hereby changing the colours as prescribed by h, and defining the isomorphism f correspondingly.

First, we choose $\mathbb{Z}$-sequences $(a_i)_{i\in\mathbb{Z}} \subseteq [a, b]_{\overline{S}\setminus S}$ and $(b_i)_{i\in\mathbb{Z}} \subseteq [c, d]_{\overline{S}\setminus S}$ such that $\mathrm{cot}(a_i) = \mathrm{cot}(b_i) = \aleph_0$, $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for each $i \in \mathbb{Z}$ and $a = \inf_{i\in\mathbb{Z}} a_i$, $b = \sup_{i\in\mathbb{Z}} a_i$, $c = \inf_{i\in\mathbb{Z}} b_i$, $d = \sup_{i\in\mathbb{Z}} b_i$. For each $i \in \mathbb{Z}$ let A'_i be a copy of $A_i := (a_i, a_{i+1})_S$, let B^+_i be a copy of $B_i := (b_i, b_{i+1})_S$, and let $\pi_{A_i} : A_i \longrightarrow A'_i$, $\pi_{B^+_i} : B^+_i \longrightarrow B_i$ be isomorphisms. Put $S^* = S \cup \bigcup_{i\in\mathbb{Z}} (A'_i \cup B^+_i)$.

We define a linear order $\leq$ on S^* in the natural way so that it extends the orders of S and of each A'_i, B^+_i and the Dedekind-completions of these sets satisfy in $(S^*, \leq)$

$$\overline{B_{i+1}} < \overline{A'_i} < \overline{B_i} \quad \text{and} \quad \overline{A_i} < \overline{B^+_i} < \overline{A_{i+1}} \quad \text{for each } i \in \mathbb{Z}.$$

We define a mapping $f : [a,b]_{S^*} \longrightarrow [c,d]_{S^*}$ as in Figure 1 by putting $a^f = c$, $b^f = d$, $f|_{A_i} = \pi_{A_i}$ and $f|_{B_i^+} = \pi_{B_i^+}$. Then f is an order-isomorphism.

Now define colours by putting, for each $g \in G$ and $i \in \mathbb{Z}$,

$$\begin{aligned}
P'_{gh,i} &= ((a_i, a_{i+1}) \cap P_g)^{\pi_{A_i}} \subseteq \overline{A'_i},\\
Q'_{gh,i} &= ((a_i, a_{i+1}) \cap Q_g)^{\pi_{A_i}} \subseteq \overline{A'_i},\\
P^+_{g,i} &= ((b_i, b_{i+1}) \cap P_{gh})^{\pi^{-1}_{B_i^+}} \subseteq \overline{B_i^+},\\
Q^+_{g,i} &= ((b_i, b_{i+1}) \cap Q_{gh})^{\pi^{-1}_{B_i^+}} \subseteq \overline{B_i^+},\\
P^*_g &= P_g \cup \bigcup_{i\in\mathbb{Z}} (P'_{g,i} \cup P^+_{g,i}),\\
Q^*_g &= Q_g \cup \bigcup_{i\in\mathbb{Z}} (Q'_{g,i} \cup Q^+_{g,i}).
\end{aligned}$$

Then $\mathcal{S}^* = (S^*, \leq, (P^*_g)_{g\in G}, (Q^*_g)_{g\in G}) \in \mathcal{K}^\lambda$, and it follows that f permutes the colours as prescribed by h. Note that if $\mathcal{S}$ is nice, then so is $\mathcal{S}^*$. $\square$

Basic Construction B (extension of a colour-permuting isomorphism)

Let $\mathcal{U} = (U, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G})$, $\mathcal{V} = (V, \leq, (P'_g)_{g\in G}, (Q'_g)_{g\in G}) \in \mathcal{K}^\lambda$ such that $\mathcal{U}$ is a substructure of $\mathcal{V}$ in the usual sense. Assume that

(*) whenever $v \in V \setminus S$, then the set

$$D = \{x \in V \cup P' \cup Q' : \ \forall u \in U \ : \ (u < v \longleftrightarrow u < x)\}$$

has no greatest or smallest element and has countable coterminality, the set $E = \{x \in U \cup P \cup Q : \ x < v\}$ has no greatest element, and the set $F = \{x \in U \cup P \cup Q : \ v < x\}$ has no smallest element.

Now let $a,b,c,d \in U$ with $a < b < c < d$, let $h \in G$ and let $f : [a,b]_U \longrightarrow [c,d]_U$ be an isomorphism permuting the colours of $\mathcal{U}$ as prescribed by h. We want to enlarge $\mathcal{V}$ to a superstructure $\mathcal{W} \supseteq \mathcal{V}$ with $\mathcal{W} = (W, \leq, (P^*_g)_{g\in G}, (Q^*_g)_{g\in G})$ such that f extends to an isomorphism $\bar{f} : [a,b]_W \longrightarrow [c,d]_W$ permuting the colours in $\mathcal{W}$ as prescribed by h. This is achieved by inserting, for certain decompositions $[a,b]_U = A \cup B$ with $A < B$, points into V between the sets A and B and also between A^f and B^f.

Consider a decomposition $[a,b]_U = A \cup B$ with $A < B$. We distinguish between four cases.

Case 1. There is no $x \in V$ with $A < x < B$ and no $y \in V$ with $A^f < y < B^f$.

In this case, no point is inserted, either between A and B or between A^f and B^f.

Case 2. There is $x \in V$ with $A < x < B$ but no $y \in V$ with $A^f < y < B^f$.

In this case, let $X = \{x \in V : A < x < B\}$, let $(Y, \leq)$ be a copy of $(X, \leq)$, and let $\pi : (X, \leq) \longrightarrow (Y, \leq)$ be an isomorphism. We insert Y into V between A^f and B^f. Next, using Lemma 3.3(b), we define colours in $\overline{Y}$ such that π permutes the colours as prescribed by h.

Case 3. There is $y \in V$ with $A^f < y < B^f$ but no $x \in V$ with $A < x < B$.

This case is dual to Case 2.

Case 4. There are $x, y \in V$ with $A < x < B$ and $A^f < y < B^f$.

In this case, let $X = \{x \in V : A < x < B\}$ and $Y = \{y \in V : A^f < y < B^f\}$ and define $a' = \inf_{\overline{V}} X$, $b' = \sup_{\overline{V}} X$, $c' = \inf_{\overline{V}} Y$, $d' = \sup_{\overline{V}} Y$. By $(*)$, X and Y contain no greatest or smallest element and have countable coterminality. Hence we can deal with the intervals $[a', b']_V$ and $[c', d']_V$ precisely as we dealt with the intervals $[a, b]_S$ and $[c, d]_S$ in Basic Construction A, the only difference being that the endpoints a', b', c', d' all belong to $\overline{V}$. By $(*)$, in fact we have $a', b', c', d' \in \overline{V} \setminus (V \cup P' \cup Q')$. Hence, using Basic Construction A we enlarge the intervals $[a', b']_V$ and $[c', d']_V$ to intervals $[a', b']_W$ and $[c', d']_W$, respectively, define colours in them appropriately and obtain an isomorphism $\pi : [a', b']_W \longrightarrow [c', d']_W$ which permutes the colours as prescribed by h.

If these constructions are carried out for each decomposition $[a, b]_U = A \cup B$ with $A < B$, we clearly obtain the required extension $\mathcal{W}$ and, by patching together all the "local" isomorphisms, also the isomorphism $\overline{f}$ as required. Note that if $\mathcal{U}, \mathcal{V} \in \mathcal{K}_{\text{nice}}$, then also $\mathcal{W} \in \mathcal{K}_{\text{nice}}$. Moreover, we have $\mathcal{V} \in \mathcal{K}^\lambda$, and if there are at most λ decompositions $[a, b]_U = A \cup B$ as described, then again $\mathcal{W} \in \mathcal{K}^\lambda$. $\square$

Theorem 4.3 *Let G be any group and $\lambda \geq |G|$ a regular uncountable cardinal. Then there exists $\mathcal{S} \in \mathcal{K}^\lambda_{\text{nice}}$ with the following properties:*

1. *$\mathcal{S}$ is doubly homogeneous in the P-colours;*
2. *$(S, \leq)$ has countable coterminality;*
3. *$(\overline{S}, \leq)$ is not anti-isomorphic to itself.*

Proof: We will first consider the case $\aleph_0 < \kappa < \lambda$ (although the case $\aleph_0 = \kappa < \lambda$ will be quite similar). We first construct $\mathcal{S} \in \mathcal{K}^\lambda_{\text{nice}}$ which is G-homogeneous for S-intervals. Let $\mathcal{L}_e = (L_e, \leq, (P^0_g)_{g\in G}, (Q^0_g)_{g\in G})$ be the coloured linear ordering defined above.

Let $\mathcal{P}$ be the set of all gaps of $(L_e, \leq)$. For each gap $(a|b)$ of $(L_e, \leq)$, choose a copy $L_{(a|b)}$ of the good λ-set given by Lemma 4.2. Define

$$S_0 = \bigcup\{L_{(a|b)} : \ (a|b) \in \mathcal{P}\}.$$

Next we define a linear order on $L_e \cup S_0$ in the unique way so that it extends the given orders of L_e and each $L_{(a|b)}$ and so that $a < L_{(a|b)} < b$ for each $(a|b) \in \mathcal{P}$. By construction of L_e, whenever $c, d \in L_e$ with $c < d$, there is $(a|b) \in \mathcal{P}$ with $c \leq a < b \leq d$ and so there is $s \in S_0$ with $a < s < b$. Therefore we regard $C := L_e \setminus \{\max L_e, \min L_e\}$ as a subset of $\overline{S_0} \setminus S_0$, and we put

$$\mathcal{S}_0 = (S_0, \leq, (P^0_g)_{g\in G}, (Q^0_g)_{g\in G}).$$

Clearly, $\mathcal{S}_0$ satisfies conditions 2.2(1)–(4). By Lemma 4.2, observe that $\text{char}(\overline{L_{(a|b)}}) = \{(\mu, \aleph_0) : \ \aleph_0 \leq \mu \leq \lambda, \ \mu \text{ regular}\}$ for each gap $(a|b) \in P$. Hence, if $x \in \overline{L_{(a|b)}}$ with $\text{cof}(x) = \lambda$, there is a continuously increasing sequence $(a_i)_{i<\lambda} \subseteq \overline{L_{(a|b)}} \subseteq \overline{S_0}$ such that $x = \sup_{i<\lambda} a_i$ and $\text{coi}(a_i) = \aleph_0$ for each $i < \lambda$. However, note that since $\mathcal{S}_0$ satisfies conditions **??**(3),(4), the above is impossible for points $x \in \overline{S_0}$ with $x \in P^0$. Since $\text{char}_{\overline{S_0}}(Q^0) = \{y \in \overline{S_0} : \ \text{char}(y) = (\kappa, \lambda)\}$, it follows that $\mathcal{S}_0$ also satisfies condition 2.2(5) and hence $\mathcal{S}_0 \in \mathcal{K}^\lambda_{\text{nice}}$. Moreover, the set $L_{(a|b)}$ with $b = \max L_e$ ($a = \min L_e$) is a final (initial) segment of $\mathcal{S}_0$, respectively, and has countable coterminality, so $\text{cot}(S_0) = \aleph_0$.

We wish to obtain the required structure $\mathcal{S} = (S, \leq, (P_g)_{g\in G}, (Q_g)_{g\in G}) \in \mathcal{K}^\lambda_{\text{nice}}$ as the union of a tower (indexed by λ) of structures $\mathcal{S}_i = (S_i, \leq, (P^i_g)_{g\in G}, (Q^i_g)_{g\in G}) \in \mathcal{K}^\lambda_{\text{nice}}$. For this, we employ the construction in the proof of Theorem 2.11 (parts (I)–(III)) of [3]; here our description will be less formal.

We let M_i be the set of all quintuples (h, a, b, c, d) such that $h \in G$, $a, b, c, d \in S_i$ and $a < b < c < d$, and enumerate M_i by a suitable subset $\mu_i \subseteq \lambda$. If we deal during the construction with such a quintuple $(h, a, b, c, d) \in M_i$ for the first time at step $i + 1$, we employ Basic Construction A to obtain $\mathcal{S}_{i+1}$ and an isomorphism from $[a, b]_{S_{i+1}}$ onto $[c, d]_{s_{i+1}}$ permuting the colours as prescribed by h. Later on, we employ Basic Construction B to extend isomorphisms constructed at previous stages.

For limit ordinals j, we just put $\mathcal{S}_j = \bigcup_{i<j} \mathcal{S}_i$, in the natural way. Since we perform the extension of a constructed isomorphism λ many times, for each $i < \lambda$ and $(h, a, b, c, d) \in M_i$ we finally obtain an isomorphism f :

$[a,b]_S \longrightarrow [c,d]_S$ permuting the colours as prescribed by h. This shows that $\mathcal{S}$ is G-homogeneous for S-intervals.

Also since Constructions A and B are carried out only at points $x \in \overline{S_i} \setminus S_i$, i.e. "inside" $\overline{S_i}$, the set S_0 remains unbounded above and below in each S_j ($j < \lambda$) and in S. So, $(S, \leq)$ has countable coterminality.

We have to ensure that all structures $\mathcal{S}_i$ ($i < \lambda$) and $\mathcal{S}$ belong to $\mathcal{K}^\lambda_{\text{nice}}$. To see that each element $s \in S$ has countable coterminality, observe that this is true in S_0, and that if $i < \lambda$, then in the construction of S_{i+1} new elements only get inserted at points of $\overline{S_i} \setminus S_i$; so, in particular, each $s \in S_i$ has the same character in S_{i+1} as in S_i.

Moreover, if $i < \lambda$ and $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}_i$ is a nice embedding with $t_0^\varphi \in P_h^i$, then $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}_j$ should remain nice for each $i < j < \lambda$. This is the case, if all insertion processes of Basic Construction A and B are only carried out at points $x \in \overline{S_j} \setminus (S_j \cup L_h^\varphi)$. Indeed we have $C \subseteq \overline{S_0} \setminus S_0$, and we declare all points of C "forbidden points" in the terminology of [3]. This means that we are never allowed, later on, to perform insertion processes at the cuts $x \in C$. This ensures that $C \subseteq \overline{S_i} \setminus S_i$ for each $i < \lambda$, and $\text{id} : L_e \longrightarrow \overline{S}$ is a nice embedding. Furthermore, during the Basic Constructions A and B, to construct $\mathcal{S}_{i+1}$ we insert copies of the whole intervals of $\mathcal{S}_i$ only into particular points $x \in \overline{S_i} \setminus S_i$ with $\text{cot}(x) = \aleph_0$ which are not "forbidden" in $\overline{S_i}$. Then we declare in the copy I' of I all elements $x' \in \overline{I'} \setminus I'$ which correspond to a forbidden point $x \in \overline{I} \setminus I$ is $\mathcal{S}_i$ as forbidden points in $\mathcal{S}_{i+1}$ (cf. [3], pp. 256–258). This ensures that our isomorphisms also preserve forbidden points, and if $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}_i$ is a nice embedding, then $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}_j$ remains nice for each $i < j < k$, and so is $\varphi : \mathcal{L}_h \longrightarrow \mathcal{S}$.

Also, $\{x \in \overline{S_0} \setminus S_0 : \text{ cot}(x) = \aleph_0\}$ is λ-dense in $\overline{S_0}$. Now assume that $i < \lambda$ and $x \in \overline{S_i} \setminus S_i$ with $\text{cot}(x) = \aleph_0$. Suppose that in the construction of $\mathcal{S}_{i+1}$ the chain $(X, \leq)$ gets inserted into $\overline{S_i}$ at x. Then by property $(*)$ of Basic Construction B, we may assume that $(X, \leq)$ has countable coterminality and the elements $\inf X$, $\sup X$ of $\overline{S_{i+1}}$ become forbidden points. This ensures that $\inf X$, $\sup X$ retain their countable character in each $\overline{S_j}$ ($i < j < \lambda$) and in $\overline{S}$.

We have to show that the forbidden points do not prevent our constructions. By induction, we may assume that if $i < j < \lambda$, then there are only $< \lambda$ many points $x \in \overline{S_i} \setminus S_i$ into which elements have been inserted in order to construct S_j. Hence at stage j, for any $a < b$ in S_i there are still λ many cuts $x \in [a,b]_{\overline{S_j} \setminus S_j}$ with $\text{cot}(x) = \aleph_0$ into which no element got inserted and which are not forbidden in $\mathcal{S}_j$, and these can be used when we deal again with a quintuple in M_i. Moreover, then at stage j again only $< \lambda$ many new forbidden points are created, keeping the above induction hypothesis. Since λ is regular, also for limit ordinals j for any $a < b$ in S_j we have $a, b \in S_i$ for some $i < j$, and it follows that the set $\{x \in [a,b]_{\overline{S_j}\setminus S_j} :\ \mathrm{cot}(x) = \aleph_0\}$ has size λ.

Next we consider the characters of elements of $\overline{S}$. As noted before, each inserted good λ-set $L_{(a|b)}$ satisfies $\mathrm{char}(L_{(a|b)}) = \{(\mu, \aleph_0) :\ \aleph_0 \leq \mu \leq \lambda,\ \mu \text{ regular}\}$. Also, for $L_e = \overline{L_e} \subseteq \overline{S} \setminus S$ we have (inside $\overline{S}$) $\mathrm{char}(L_e) = \{(\mu, \aleph_0), (\aleph_0, \mu) :\ \aleph_0 \leq \mu \leq \lambda,\ \mu \neq \kappa,\ \mu \text{ regular}\} \cup \{(\lambda, \kappa), (\kappa, \lambda)\}$. So, $\mathrm{char}(\overline{S_0}) = \mathrm{char}(L_e) \cup \{(\kappa, \aleph_0)\}$. We want to ensure that our construction neither destroys these characters nor adds new ones. The first part is clear, since we insert sets with countable coterminality only at points $x \in \overline{S_i} \setminus S_i$ with $\mathrm{cot}(x) = \aleph_0$ and add a forbidden point to the left and to the right of the inserted set, thereby preventing further insertions of sets at these points.

To ensure the second part, we may proceed as in [2, p. 231, Cor. 2.5]. That is, suppose $(j_i)_{i\in\mathbb{N}}$ is a countable sequence of ordinals $< \lambda$ and $a_i, b_i \in S_{j_i+1} \setminus S_{j_i}$ with $a_i < a_{i+1} < b_{i+1} < b_i$ for each $i \in \mathbb{N}$. Let $j = \sup_{i\in\mathbb{N}} j_i$. Then in $S_j = \bigcup_{i\in\mathbb{N}} S_{j_i}$ we have $\sup\{a_i :\ i \in \mathbb{N}\} = \inf\{b_i :\ i \in \mathbb{N}\} =: x \in \overline{S_j} \setminus S_j$ and $\mathrm{cot}(x) = \aleph_0$ in $\overline{S_j}$. In order to prevent sets getting inserted at later stages at x, we declare x (and any point arising in this way) as a forbidden point of S_j. This ensures that also $\mathrm{cot}(x) = \aleph_0$ in $\overline{S}$. Therefore $\mathrm{char}(\overline{S}) = \mathrm{char}(\overline{S_0})$.

In particular, since $\aleph_0 < \kappa < \lambda$, we have $(\kappa, \aleph_0) \in \mathrm{char}(\overline{S})$ but $(\aleph_0, \kappa) \notin \mathrm{char}(\overline{S})$, so $(\overline{S}, \leq)$ is not anti-isomorphic to itself.

By similar remarks as above about $\mathcal{S}_0$, we also obtain that each $\mathcal{S}_j$ $(j < \lambda)$ and $\mathcal{S}$ satisfy condition 2.2(5).

Now $\mathcal{S} \in \mathcal{K}_{\mathrm{nice}}$ is G-homogeneous for S-intervals and hence, by Proposition 3.4, also G-homogeneous for P-intervals. Since $\mathrm{cot}(S) = \aleph_0$, a patching argument similar to the one used for Lemma 3.1 shows that $\mathcal{S}$ is doubly homogeneous in P-colours.

It remains to consider the case $\lambda = \aleph_1$ and so $\kappa = \aleph_0$. The above construction would also work to give $\mathcal{S} \in \mathcal{K}^{\lambda}_{\mathrm{nice}}$ with (1), (2) and $\mathrm{char}(\overline{S}) = \{(\aleph_0, \aleph_0), (\aleph_1, \aleph_0), (\aleph_0, \aleph_1)\}$, but now this does not imply that $(\overline{S}, \leq)$ is not anti-isomorphic to itself. To remedy this, we can proceed as follows. Let $(D, \leq)$ be the chain obtained by inserting in the ordinal $\omega_1 + 1$ between each element and its successor a copy of ω_1^*, the converse of ω_1, and also adding a copy of ω_1^* to the right of $\omega_1 + 1$. Then $D = \overline{D}$, and there is a unique element $z \in D$ with $\mathrm{char}(z) = (\aleph_1, \aleph_1)$. Now put $(D, \leq)$ to the right of $(L_e, \leq)$ and proceed, with L_e replaced by $L_e \cup D$, as above, inserting copies of good $\aleph_1$-sets into each gap of the chain $(L_e \cup D, \leq)$ to obtain $\mathcal{S}_0$.

The construction above now produces a chain $(S, \leq)$ with $D \subseteq \overline{S} \setminus S$. In particular, $z \in D$ also satisfies $\mathrm{cot}(z) = \aleph_1$ in $\overline{S_0}$. By construction, there is a continuously increasing sequence $(a_i)_{i\in\omega_1} \subseteq D \subseteq \overline{S_0}$ such that $z = \sup_{i\in\omega_1} a_i$ and $\mathrm{coi}(a_i) = \aleph_1$ for each $i \in \omega_1$. There is also a continuously decreasing sequence $(b_j)_{j\in\omega_1} \subseteq D \subseteq \overline{S_0}$ such that $z = \inf_{j\in\omega_1} b_j$ and $\mathrm{cof}(b_j) = \aleph_0$ for each $j \in \omega_1$. But there is no element $z' \in \overline{S_0}$ with $\mathrm{cot}(z') = \aleph_1$ and these two asymmetric ascending respectively descending approximation properties interchanged, so $(\overline{S_0}, \leq)$ is not anti-isomorphic to itself. Since $z \in D \subseteq \overline{S}$ keeps these properties in $(\overline{S}, \leq)$, but the construction produces no $z' \in \overline{S}$ with the interchanged properties, it follows that $(\overline{S}, \leq)$ is not anti-isomorphic to itself. □

We note that we could have ensured that $(\overline{S}, \leq)$ is not anti-isomorphic to itself easier, without analysing the "interior" character of (elements of) $(\overline{S}, \leq)$, by constructing the set $(S, \leq)$ with uncountable cofinality and countable coinitiality. However, for use of Theorem 4.3 in [4] it will be essential that S has countable coterminality. Now we have:

Proof of Theorem 1.1 in case $H = \{1\}$: Immediate by Theorems 4.3 and 2.4. □

Next we wish to show how to change the above construction in order to embed an arbitrary ℓ-group into $A(S)$. First we recall:

Proposition 4.4 (Holland [9]) *Let H be any ℓ-group. Then there exists a chain $(C, \leq)$ with $|C| = |H|$ such that H ℓ-embeds into $A(C)$.*

Next we want to embed $A(C)$ into $A(S)$, for some suitable doubly homogeneous chain $(S, \leq)$. The following tool uses ideas already contained in Holland [9, proof of Theorem 4].

Proposition 4.5 ([2, Theorem 4.3]) *Let $(S, \leq)$ be a doubly homogeneous chain. Let $C \subseteq \overline{S} \setminus S$ such that C contains all suprema and infima (taken in $\overline{S}$) of bounded subsets of C and $\mathrm{cot}(a, b)_S = \aleph_0$ for each gap (a, b) of $(C, \leq)$. Then there exists an ℓ-embedding of $A(C)$ into $A(S)$.*

Now we obtain:

Proof of Theorem 1.1 in the general case: By Proposition 4.4, we can choose a chain $(Y, \leq)$ with $|Y| = |H|$ such that $H \subseteq A(Y)$ (as ℓ-groups). Let $Y \times \{1, 2\}$ be ordered lexicographically, let $(X, \leq)$ be the chain $\overline{Y \times \{1, 2\}}$ with a copy of ω added to the right of $Y \times \{1, 2\}$. Observe that $\overline{Y \times \{1, 2\}}$ is the same as the chain $\overline{Y}$ with each element of Y replaced by a 2-element chain. We identify $A(Y)$ with $A(Y \times \{1\}) \subseteq A(X)$. Now perform the construction of the above proof of Theorem 4.3, with L_e replaced by $L_e \cup X$, where $(X, \leq)$ is placed to the right of L_e if $\lambda \geq \aleph_2$, and with $L_e \cup D$ replaced by $L_e \cup D \cup X$ with X placed to the right of $L_e \cup D$, if $\lambda = \aleph_1$. The points of X also become "forbidden" points in $\mathcal{S}_0$. Since $\mathrm{cof}(x), \mathrm{coi}(x) \leq |Y| < \lambda$ for each $x \in X$, condition 2.2(5) is not disturbed and we obtain $\mathcal{S}_0 \in \mathcal{K}^\lambda_{\mathrm{nice}}$. Hence we can continue our construction as before, obtaining $\mathcal{S} \in \mathcal{K}^\lambda_{\mathrm{nice}}$ such that (cf. the proof of Theorem 2.4) the chain $(P_e, \leq)$ is doubly homogeneous and $\mathrm{Out}(A(P_e)) \cong G$. Since $X \cap P = \emptyset$ and by construction $\mathrm{cot}((y,1),(y,2)) = \aleph_0$ for each $y \in Y$, Proposition 4.5 now yields an ℓ-embedding of $A(X)$, hence of H, into $A(P_e)$. $\square$

Next we wish to turn to the proof of Corollary 1.2. Let $(S, \leq)$ be an infinite chain and $H \subseteq A(S)$ a subgroup. Then $(H, \mathcal{S})$ is called a *triply transitive ordered permutation group*, if whenever $A, B \subseteq S$ each have three elements, there exists $h \in H$ with $A^h = B$. An element $h \in H$ is called *strictly positive*, if $h > e$ in $A(S)$, and *bounded*, if there are $a, b \in S$ with $a < b$ such that each $x \in S$ with $x^h \neq x$ satisfies $a < x < b$. For the proof of Corollary 1.2 we need the following special case of McCleary [12]. It generalizes one implication of Proposition 2.1.

Proposition 4.6 (McCleary [12, Main Theorem 4]) *Let (H, S) be a triply transitive ordered permutation group such that H contains a strictly positive bounded element and $(\overline{S}, \leq)$ is not anti-isomorphic to itself. Then each automorphism φ of H is induced by a unique automorphism $f \in A(\overline{S})$, which permutes the orbits of H in $\overline{S}$, such that*

$$h^\varphi = f^{-1} \cdot h \cdot f \quad \textit{for each } h \in H.$$

Using this and Theorem 1.1 and 4.3, we can give the

Proof of Corollary 1.2. By the proof of Theorem 1.1, in particular by Theorems 4.3 and 2.4, there exists $\mathcal{S} = (S, \leq, (P_g)_{g \in G}, (Q_g)_{g \in G}) \in \mathcal{K}^\lambda$ with the properties of Theorem 2.4 and such that $H \subseteq A(X)$ (as ℓ-groups, where $X = P_e$). In particular, $G \cong \mathrm{Out}(A(X))$, the sets P_g $(g \in G)$ are all the $A(X)$-orbits in $\overline{X}$ which are isomorphic to X, and for each $g \in G$ there is $f_g \in A(\overline{X})$ permuting the $A(X)$-orbits such that $X^{f_g} = P_g$. So, $A(X)^{f_g} = A(X)$ for each $g \in G$. By Löwenheim-Skolem, we can find an ℓ-group K of size λ such that $H \subseteq K \subseteq A(X)$ (as ℓ-groups), (K, X), just like $(A(X), X)$, is triply transitive and contains a strictly positive bounded element, and $K^{f_g} = K$ for each $g \in G$. Hence $f_g \in \mathrm{Aut}(K)$, and by Proposition 4.6, f_g permutes the orbits of H in $\overline{X}$. So, X^{f_g} is an orbit of H, showing that each P_g $(g \in G)$ is also an H-orbit. Now let U be any H-orbit in $\overline{X}$ with $U \cong X$. By condition (2.2.5) it follows that $U \subseteq P$. Thus $U = P_g$ for some $g \in G$, showing that H and $A(X)$ have the same orbits in $\overline{X}$ which are isomorphic to X. Also, $\mathrm{Aut}(H)$ and $\mathrm{Aut}(A(X))$ induce the same permutations of these orbits. Thus $\mathrm{Out}(H) \cong \mathrm{Out}(A(X)) \cong G$. □

Finally we just remark that in a similar way, by Löwenheim-Skolem arguments it follows that Theorem 1.1 also holds for singular cardinals $\lambda > |G|$, and hence so does Corollary 1.2.

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