COUNTABLE STRUCTURE DOES NOT HAVE A FREE UNCOUNTABLE AUTOMORPHISM GROUP

SAHARON SHELAH

Abstract It is proved that the automorphism-group of a countable structure cannot be a free uncountable groupi. The idea is that instead proving that every countable set of equations of a certain form has a solution, we prove that this holds for a co-meagre family of appropriate countable sets of equations.

Can a countable structure have an automorphism group, which a free uncountable group? This was a well known problem in group theory at least in England y David Evans, posed the question at the Durham meeting on model theory and groups which Wilfrid Hodges organised in Durham in 1987 and we thank Simon Thomas for telling us about it. Later and independently in descriptive set theory, Howard and Kekeris [BK96] asked if there is an uncountable free Polish group, i.e. which is on a complete separable metric space and the operations are continuous. Motivated by this, Solecki [Sol99] proved that the group of automorphisms of a countable structure cannot be an uncountable free Abelian group. See more in Just, Shelah and Thomas [JST99] where, as a byproduct, we can say something on uncountable structures.

Here, we prove

Theorem 1. *If* $\mathbb{A}$ *is a countable model, then* $\mathrm{Aut}(\mathbb{A})$ *cannot be a free uncountable group.*

The proof follows from the following two claims, one establishing a property of G and the other proving that free groups does not have it.

A similar result for general Polish groups is under preparation (see [She11]) (it also gives more on the Remark 5(B)).

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Notation 2.
1. Let ω denote the set of natural numbers, and let $x < \omega$ mean "x is a natural number".
2. Let a, b, c, d denote members of G (the group).
3. Let $\mathbf{d}$ denote the ω–sequence $\langle d_n : n < \omega\rangle$, and similarly in other cases.

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(4) Let $k, \ell, m, n, i, j, r, s, t$ denote natural numbers (and so also elements of the structure $\mathbb{A}$, which we assume is the set of natural number for notational simplicity).

Proposition 3. *Assume $\mathbb{A}$ a countable structure with automorphism group G, and for notational simplicity its set of elements is ω (and of course it is infinite, otherwise trivial).*

We define a metric $\mathfrak{d}$ on G by

$$\mathfrak{d}(f, g) = \sup\{2^{-n} : f(n) \neq g(n) \text{ or } f^{-1}(n) \neq g^{-1}(n)\}.$$

Then:

(1) *G is a complete separable metric space under $\mathfrak{d}$, in fact a separable topological group.*

(2) *If $\mathbf{d}$ is an ω–sequence of members of $G \setminus \{e_G\}$ converging to e_G, then for some (strictly increasing) ω–sequence $\mathbf{j}$ of natural numbers the pair $(\mathbf{d}, \mathbf{j})$ satisfies*

(*) *for any sequence $\langle w_n(x_1, x_2, \dots, x_{\ell_{1,n}}; y_1, y_2, \dots, y_{\ell_{2,n}}) : n < \omega\rangle$ (i.e., an ω–sequence of non-degenerate group words) obeying $\mathbf{j}$ (see below) we can find a sequence $\mathbf{b}$ from G, that is $b_n \in G$ for $n < \omega$ such that*

$$b_n = w_n(d_{n+1}, d_{n+2}, \dots, d_{n+\ell_{1,n}}; b_{n+1}, b_{n+2}, \dots, b_{n+\ell_{2,n}}) \qquad \text{for any } n.$$

$(*)_1$ *We say that $\langle w_n(x_1, x_2, \dots, x_{\ell_{1,n}}; y_1, y_2, \dots, y_{\ell_{2,n}}) : n < \omega\rangle$ obeys $\mathbf{j}$ whenever:*
if $m < j_n$, then $m + \ell_{1,m} < j_{n+1}$ and $m + \ell_{2,m} < j_{n+1}$, and:

$(*)_2$ *for any $n^*, m^* < \omega$ we can find $i(0), i(1)$ such that*

$(*)_3$ *$m^* < i(0)$, $n^* < i(0)$, $i(0) < i(1)$, and w_t is trivial (which means $w_t = y_1$) for $t = j_{i(0)}, j_{i(0)} + 1, \dots, j_{i(1)}$, and $f_2(j_{i(0)}, n^*, j_{i(0)}) < i(1) - i(0)$, where*

$(*)_4$ (i) *the length of a word $w = w(z_1, \dots, z_r)$ which in canonical form is $z_{\pi(1)}^{t(1)} z_{\pi(2)}^{t(2)} \dots z_{\pi(s)}^{t(s)}$, where $t(i) \in \mathbb{Z}$ and π is a function from $\{1, 2, \dots, s\}$ into $\{1, \dots, r\}$, is*

$$\text{length}(w) = \sum_{i=1,\dots,s} |t(i)|,$$

(ii) *for $s \leq i$ let $f_1(i, s) = \prod_{t=s,\dots,i-1} \text{length}(w_t)$; note that this is ≥ 1 as all w_t's are non-degenarate*

(iii) *for $n \leq s \leq i$ we let*

$$f_2(i, n, s) = \sum_{r=n,\dots,s-1} f_1(i, r+1) \times \text{length}(w_r),$$

and $f_2(i, n, s) = f_2(i, n, i)$ if $n \leq i < s$.

Proof. (1) Should be clear
(2) So we are given the sequence $\mathbf{d}$. We choose the increasing sequence $\mathbf{j}$ of natural numbers by letting $j_0 = 0$, j_{n+1} be the first $j > j_n$ such that

$(*)_5$ for every $\ell \le n$ and $m < j_n$ we have $d_\ell(m) < j$ and $(d_\ell)^{-1}(m) < j$, and $[k \ge j \ \Rightarrow \ d_k(m) = m]$,

Note that j_{n+1} is well defined as the sequence $\mathbf{d}$ converges to e_G, so for each $m < \omega$ for every large enough $k < \omega$ we have $d_k(m) = m$.

We shall prove that $\mathbf{j} = \langle j_n : n < \omega \rangle$ is as required in part (2) of the proposition. So let a sequence

$$\mathbf{w} = \langle w_n(x_1, x_2, \dots, x_{\ell_{1,n}}; y_1, y_2, \dots, y_{\ell_{2,n}}) : n < \omega \rangle$$

of group words obeying $\mathbf{j}$ be given (see $(*)_1 + (*)_2 + (*)_3$ above). Then $f_1(i,s)$ and $f_2(i,n,i)$ are well defined.

For each $k < \omega$ we define the sequence $\langle b^k_n : n < \omega \rangle$ of members of G as follows. For $n > k$ we let b^k_n be e_G and now we define b^k_n by downward induction on $n \le k$ letting

$(*)_6$ $b^k_n = w_n(d_{n+1}, d_{n+2}, \dots, d_{n+\ell_{1,n}}; b^k_{n+1}, b^k_{n+2}, \dots, b^k_{n+\ell_{2,n}})$.

Now we shall work on proving

$(*)_7$ for each $n^*, m^* < \omega$ the sequence $\langle b^k_{n^*}(m^*) : k < \omega \rangle$ is eventually constant.

Why does $(*)_7$ hold? By the definition of "$\mathbf{w}$ obeys $\mathbf{j}$" we can find $i(0), i(1)$ such that

$(*)_8$ $m^* < i(0)$, $n^* < i(0)$, $i(0) < i(1)$, and w_t is trivial for $t = j_{i(0)}, j_{i(0)} + 1, \dots, j_{i(1)}$, and $f_2(j_{i(0)}, n^*, j_{i(0)}) < i(1) - i(0)$; actually $m^* < j_{i(0)}$ suffices.

Now let $k(*) =^{\text{df}} j_{i(1)+1}$; we claim that:

$(*)_9$ if $k \ge k(*)$ and $s \ge j_{i(1)}$, then b^k_s restricted to the interval $[0, j_{i(1)-1})$ is the identity.

[Why? If $s > k$, this holds by the choice of the b^k_s as the identity everywhere. Now we prove $(*)_9$ by downward induction on $s \le k$ (but of course $s \ge j_{i(1)}$). But by the definition of composition of permutations it suffices to show

$(*)_{9a}$ every permutation mentioned in the word

$$w_s(d_{s+1}, \dots, d_{s+\ell_{1,s}}, b^k_{s+1}, \dots, b^k_{s+\ell_{2,s}})$$

maps every $m < j_{i(1)-1}$ to itself.

Let us check this criterion. The $d_{s+\ell}$ for $\ell = 1, \dots, \ell_{1,s}$ satisfy this as the indexes $(s+\ell)$ are $\ge j_{i(1)}$ and $m < j_{i(1)-1}$; now apply the choice of $j_{i(1)}$.

The $b^k_{s+1}, \dots, b^k_{s+\ell_{2,s}}$ satisfy this by the induction hypothesis on s. So the demands in $(*)_{9a}$ holds, hence we complete the downward induction on s. So $(*)_9$ holds.]

$(*)_{10}$ If $k \ge k(*)$ and $s \in [j_{i(0)}, j_{i(1)}]$, then b^k_s is the identity on the interval $[0, j_{i(1)-1})$.

[Why? We prove this by downward induction; for $s = j_{i(1)}$ this holds by $(*)_9$, if it holds for $s+1$, recall that w_s is trivial, i.e., $w_s = y_1$, and hence $b^k_s = b^k_{s+1}$, so this follows.]

$(*)_{11}$ For every $k \geq k(*)$ we have: for every $s \geq j_{i(0)}$, the functions $b^k_s, b^{k(*)}_s$ agree on the interval $[0, j_{i(1)-1})$, and they both are the identity on it, and also $(b^k_s)^{-1}, (b^{k(*)}_s)^{-1}$ agree on this interval and both are the identity on it.

[Why? For $s \geq j_{i(1)}$ by $(*)_9$; for $s \in [j_{i(0)}, j_{i(1)})$ by $(*)_{10}$.]

$(*)_{12}$ For any $s \in [n^*, \omega)$ and $m < j_{i(0)+f_2(j_{i(0)}, n^*, s)}$ we have:

$$\begin{aligned}
&k \geq k(*)(= j_{i(1)+1}) \quad \text{implies:}\\
&b^k_s(m) = b^{k(*)}_s(m) \text{ and } (b^k_s)^{-1}(m) = (b^{k(*)}_s)^{-1}(m)\\
&\text{and if in addition } t \text{ satisfies}\\
&\qquad t \leq i(0) + f_2(j_{i(0)}, n^*, s) \text{ and } m < j_t,\\
&\text{then } b^{k(*)}_s(m), (b^{k(*)}_s)^{-1}(m) \text{ are } < j_{t+f_1(j_{i(0)}, s)}.
\end{aligned}$$

Before proving, note that, as we assume

$$f_2(j_{i(0)}, n^*, s) \leq f_2(j_{i(0)}, n^*, j_{i(0)}) < i(1) - i(0),$$

necessarily $m < j_{i(0)+(i(1)-i(0)-1)} = j_{i(1)-1}$.

Case 1: s is $\geq j_{i(0)}$.

[Why? This holds by $(*)_{11}$ because, as said above $m < j_{i(1)-1}$.]

We prove this by downward induction on s (for all m and k as there).

Case 2: Proving for $s < j_{i(0)}$, assuming we have it for all relevant $s' > s$ (and $s \geq n^*$ of course).

Let $k \geq k(*)$ and $t \leq i(0) + f_2(j_{i(0)}, n^*, s)$ and $m < j_t$. Note that

$$\begin{aligned}
t + f_1(j_{i(0)}, s+1) \times \mathrm{length}(w_s) \leq i(0) + f_2(j_{i(0)}, n^*, s) + f_1(j_{i(0)}, s) = \\
i(0) + f_2(j_{i(0)}, n^*, s+1),
\end{aligned}$$

and hence for $s' = s+1, s+2, \ldots, s+\ell_{1,s}$ the induction hypothesis applies to every

$$t' \leq t + f_1(j_{i(0)}, s+1) \times \mathrm{length}(w_s)$$

(recall that $f_1(j_{i(0)}, s')$ is non-increasing in s').

We concentrate on proving that $b^k_s(m) = b^{k(*)}_s(m) < j_{t+f_1(j_{i(0)}, s)}$, as the proof of $(b^k_s)^{-1}(m) = (b^{k(*)}_s)^{-1}(m) < j_{t+f_1(j_{i(0)}, s)}$ is the same. So

$$b^k_s(m) = w_s(d_{s+1}, \ldots, d_{s+\ell_{1,s}}, b^k_{s+1}, \ldots, b^k_{s+\ell_{2,s}}).$$

Let us write this group expression as the product $u^k_{s,1} \ldots u^k_{s,\mathrm{length}(w_s)}$, where each $u^k_{s,r}$ is one of $\{d_{s+1}, \ldots, d_{s+\ell_{1,s}}, b^k_{s+1}, \ldots, b^k_{s+\ell_{2,s}}\}$, or is an inverse of one of them.

For $r = 0, 1, \ldots, \mathrm{length}(w_s)$ let

$$v^k_{s,r} = u^k_{s,\mathrm{length}(w_s)+1-r} \cdots u^k_{s,\mathrm{length}(w_s)},$$

so $v^k_{s,r} \in G$ is the identity permutation for $r = 0$ and is

$$w_s(d_{s+1}, \ldots, d_{s+\ell_{1,s}}, b^k_{s+1}, \ldots, b^k_{\ell_{2,s}}) = b^k_s$$

for $r = \text{length}(w_s)$ and $j_{t+f_1(j_{i(0)},s+1)\times\text{length}(w_s)} = j_{t+f_1(j_{i(0)},s)}$ by the definition of f_1. Hence it suffices to prove the following

$(*)_{12a}$ if $r \in \{0, \ldots, \text{length}(w_s)\}$ and $m < j_t$, then $v^k_{s,r}(m) = v^{k(*)}_{s,r}(m) < j_{t+f_1(j_{i(0)},s+1)\times r}$.

[Why does $(*)_{12a}$ hold? We do it by induction on r; now for $r = 0$ the permutation is the identity so trivial. For $r + 1$ we have

$$v^k_{s,r+1}(m) = u^k_{s,\text{length}(w_s)-r}(v^k_{s,r}(m)).$$

Note that if

$$u^k_{s,\text{length}(w_s)-r} \in \{b^k_{s+1}, \ldots, (b^k_{s+1})^{-1}, \ldots\}.$$

then $u^k_{s,\text{length}(w_s)-r}$ can map any $m' < j_{t+f_1(j_{n(0)},s+1)\times r}$ only to numbers $m'' < j_{t+f_1(j_{i(0)},s+1)\times r+f_1(j_{i(0)},s+1)}$ because $(*)_{12}$ has been proved for $b^k_{s'}, b^{k(*)}_{s'}$ when $s' > s$ is appropriate. Together with the induction hyopthesis, this gives the conclusion of $(*)_{12a}$ if

$$u^k_{s,\text{length}(w_s)-r} \in \{b^k_{s+1}, \ldots, (b^k_{s+1})^{-1}, \ldots\}.$$

Otherwise

$$u^k_{s,\text{length}(w_s)-r} \in \{d_{s+1}, \ldots, d^{-1}_{s+1}, \ldots\},$$

so the equality

$$u^k_{s,\text{length}(w_s)-r}(m') = u^{k(*)}_{s,\text{length}(w_s)-r}(m')$$

is trivial, and for the inequality $(< j_{t+f_1(j_{i(0)},s+1)\times(r+1)})$ just remember the definition of j_i's; here adding 1 was enough.]

So we have proved $(*)_{12a}$, and hence $(*)_{12}$ and thus using $(*)_{12}$ for $s = n^*$ we have

$$k \geq k(*) \& m < j_{i(0)} \quad \Rightarrow \quad b^k_{n^*}(m) = b^{k(*)}_{n^*}(m);$$

in particular this holds for $m = m^*$ (as $m^* < j_{i(0)}$). Hence $(*)_7$ holds true.

Lastly

$(*)_{13}$ for each n^* and $m^* < \omega$ the sequence $\langle (b^k_{n^*})^{-1}(m^*) : k < \omega\rangle$ is eventually constant.

Why? Same as the proof of $(*)_7$.

Together, we can define for any $m, n < \omega$ the natural number $b^*_n(m)$ as the eventual value of $\langle b^k_n(m) : k < \omega\rangle$. So b^*_n is a well defined function from the natural numbers to themselves (by $(*)_7$), in fact it is one-to-one (as each b^k_n is) and is onto (by $(*)_{13}$), so it is a permutation of $\mathbb{A}$. Clearly the sequence $\langle b^k_n : k < \omega\rangle$ converges to b^*_n as a permutation, the metric is actually defined on the group of permutations of the family of members of

$\mathbb{A}$, and G is a closed subgroup; so b^*_n actually is an automorphism of $\mathbb{A}$. Similarly the required equations

$$b^*_n = w_n(d_{n+1}, \dots, d_{n+\ell_{1,n}}, b^*_{n+1}, \dots, b^*_{n+\ell_{2,n}})$$

hold. $\square$

Proposition 4. *The conclusion ((1)+(2)) of Proposition 3 fails for any uncountable free group G.*

Proof. So let Y be a free basis of G and as G is a separable metric space there is a sequence $\langle c_n : n < \omega\rangle$ of (pairwise distinct) members of Y with $\mathfrak{d}(c_n, c_{n+1}) < 2^{-n}$. Let $d_n = (c_{2n})^{-1}c_{2n+1}$, so $\langle d_n : n < \omega\rangle$ converges to e_G and $d_n \neq e_G$. Assume $\mathbf{j} = \langle j_n : n < \omega\rangle$ is as in the conclusion of Proposition 3, and we shall eventually get a contradiction. Let H be a subgroup of G generated by some countable $Z \subseteq Y$ and including $\{c_n : n < \omega\}$.

Now

$(*)_1$ $\langle d_n : n < \omega\rangle$ satisfies the conclusion of Proposition 3 also in H.

[Why? As there is a projection from G onto H and $d_n \in H$.]

For each $\nu \in {}^\omega\omega$ let $\mathbf{w}_\nu = \langle w^\nu_n : n < \omega\rangle$, where

$$w^\nu_n = w^\nu_n(x_1, y_1) = \begin{cases} x_1(y_1)^{k+1} & \text{if } \nu(n) = 2k+1, \\ (y_1)^{k+1} & \text{if } \nu(n) = 2k; \end{cases}$$

so this is a sequence of words as mentioned in Proposition 3, and $\nu(n) = 0$ implies that w_n is trivial, i.e., equals y_1. Recalling that we consider ${}^\omega\omega$ as a Polish space in the standart way,

$(*)_2$ The set of $\nu \in {}^\omega\omega$ for which $\mathbf{w}_\nu$ obeys $\mathbf{j}$ is co-meagre.

[Why? Easy; for each $n^*, m^* < \omega$ the set of $\nu \in {}^\omega\omega$ for which $\mathbf{w}_\nu$ fail the demand for n^*, m^* is nowhere dense (and closed), hence the set of those failing it is the union of countably many nowhere dense sets, hence is meagre.]

$(*)_3$ For each $a \in H$ the family of $\nu \in {}^\omega\omega$ such that there is a solution $\mathbf{b}$ for $(\mathbf{d}, \mathbf{w}_\nu)$ in H satisfying $b_0 = a$ is nowhere dense.

[Why? Given a finite sequence ν of natural numbers note that for any sequence $\rho \in {}^\omega\omega$ of which ν is an initial segment and solution $\mathbf{b}$ for $(\mathbf{d}, \mathbf{w}_\rho)$ satisfying $b_0 = a$, we can show by induction on $n \leq \lg(\nu)$ that b_n is uniquely determined (i.e., does not depend on ρ), call it $b[n, \nu, \mathbf{d}]$; we use that: in a free group, for every $k \geq 1$, any member of the group has at most one k-th root. Now, if $b[\lg(\nu), \nu, \mathbf{d}]$, which is a member of G, is not e_G, then for some $t < \omega$ it has no t-th root and we let $\nu_1 = \nu^\frown\langle 2t-2\rangle$ and we are done. If not, letting $\nu_0 = \nu^\frown\langle 1\rangle$, also $b[\lg(\nu)+1, \nu_0, \mathbf{d}]$ is well defined and equal to $d_{\lg(\nu)}$, hence (by the choice of the d_n's) is not e_G. Therefore, for some $t < \omega$ has no t-th root, so $\nu_1 = \nu_0{}^\frown\langle 2t-2\rangle$ is as required.]

Now we can finish the proof of Proposition 4: just by $(*)_2 + (*)_3+$ Baire Theorem, for some $\nu \in {}^\omega\omega$, the sequence $\mathbf{w}_\nu$ of group words obeys $\mathbf{j}$, and there is no solution for $(\mathbf{d}, \mathbf{w}_\nu)$ in H, hence no solution in G. $\square$

Proof of Theorem 1. Follows by Propositions 3, 4. $\square$

Concluding Remarks 5. (A) In the proof of Proposition 4 we do not use all the strength of "G is free". E.g., it is enough to assume:

(a) if $g \in G$, $g \neq e_G$, then for some $t > 1$, g has no t-th root, and for every $t > 1$ it has at most one t-th root (in G),

(b) if X is a countable subset of G, then there is a countable subgroup H of G which includes X and there is a projection from G onto H,

(c) G is uncountable.

The uncountable free Abelian group fall under this criterion; in fact by Proposition 3, G is "large", "rich".

(B) What about uncountable structures? Sometimes a parallel result holds, an approximation to: if $\lambda = \beth_\omega$, replacing countable by "of cardinality $\leq \beth_\omega$". More generally, assume $\aleph_0 < \lambda = \sum\limits_{n<\omega} \lambda_n$ and $2^{\lambda_n} < 2^{\lambda_{n+1}}$ for $n < \omega$, hence $\mu =^{\text{df}} \sum\limits_{n<\omega} 2^{\lambda_n} < 2^\lambda$; and we have

($*$) if $\mathbb{A}$ is a structure with exactly λ elements, $\mathbb{A} = \bigcup\limits_{n<\omega} P_n^{\mathbb{A}}$ and $|P_n^{\mathbb{A}}| < \lambda$ for $n < \omega$, and G is its group of automorphisms, then G cannot be a free group of cardinality $> \mu$.

The proof is similar, but now w.l.o.g the set of elements of $\mathbb{A}$ is $\lambda = \{\alpha : \alpha < \lambda\}$ and we define $\mathfrak{d}$ by

$$\mathfrak{d}(f,g) = \sup\{2^{-n} : \begin{array}{l} \text{there is } \alpha < \lambda_n \text{ such that} \\ \text{for some } (f',g') \in \{(f,g),(f^{-1},g^{-1}),(g,f),(g^{-1},f^{-1})\} \\ \text{one of the following possibilities holds} \\ \text{(a)}\ \ \text{for some } m < \omega \text{ we have } f'(m) < \lambda_m \leq g'(m), \\ \text{(b)}\ \ f'(n) < g'(n) < \lambda_n\}. \end{array}$$

Under this metric, G is a complete metric space with density $\leq \sum\limits_{n<\omega} 2^{\lambda_n} = \mu$, and the conclusion of Proposition 3 holds.

(C) By [JST99], for $\kappa = \kappa^{<\kappa} > \aleph_0$ there is a forcing forcing adding such a group and not changing cardinalities or cofinalities. A parallel ZFC result is in preparation.

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Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel, and Institute Mittag-Leffler, The Royal Swedish Academy of Sciences, Auravagen 17, S182 62, Jursholm, Sweden, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

Email address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah