QUITE COMPLETE REAL CLOSED FIELDS SH757

SAHARON SHELAH

ABSTRACT. We prove that any ordered field can be extended to one for which every decreasing sequence of bounded closed intervals, of any length, has a nonempty intersection; equivalently, there are no Dedekind cuts with equal cofinality from both sides. Here we strengthen the results from the published version.

Date: April 13, 2016.

2010 *Mathematics Subject Classification.* Primary: 03C64, 03C60; Secondary: 03C55, 03C98, 13L05.

Key words and phrases. Real closed fields, ordered fields, cuts, completeness, cofinality, symmetric cuts, symmetric closure.

First typed: March 2000.

Research supported by the United States-Israel Binational Science Foundation. Publication 757.

§ 1. Introduction

Laszlo Csirmaz raised the question of the existence of non-archimedean ordered fields with the following completeness property: any decreasing sequence of closed bounded intervals, of any ordinal length, has nonempty intersection. We will refer to such fields as *symmetrically complete* for reasons indicated below.

Theorem 1.1. *1) Let K be an arbitrary ordered field. Then there is a symmetrically complete real closed field K^+ containing K such that any asymmetric cut of K is not filled so if K is not embeddable into $\mathbb{R}$ then K^+, K necessarily have an asymmetric cut.*
2) Moreover it is embeddable over K into K' for any symmetrically closed $K' \supseteq K$ and is unique up to isomorphisms over K.

The construction shows that there is even a "symmetric-closure" in a natural sense, and that the cardinality may be taken to be at most $2^{|K|^+ + \aleph_1}$.

I thank the referee for rewriting the paper as appeared in the Israel Journal.

In September 2005 (after the paper appeared), lecturing on it in the Rutgers logic seminar without details, Cherlin asked where the bound $\kappa \geq d(K)$ for the number of steps needed in the construction was the true one. Checking the proof, it appears that this was used (in the published version) and eventually we show that this is the right bound.

Note that by [She94], consistently with ZFC (i.e., after forcing extension, in fact just adding enough Cohen reals) for some non-principal ultrafilter D on $\mathbb{N}, \mathbb{R}^{\mathbb{N}}/D$, which is (an $\aleph_1$-saturated) ultrapower of the field of the reals (hence a real closed field), is Scott complete. Also compared to the published version we expand §5 dealing with other related closures; note also that if K is an order field which is symmetrically closed or just has no cut of cofinality $(\mathrm{cf}(K), \mathrm{cf}(K))$ then K is real closed.

Note also that being symmetrically complete is dual to being quite saturated because if K (a real closed field or just linear order) which is κ-saturated and have a (θ_1, θ_2)-cut then $\theta_1 < \kappa \Rightarrow \theta_2 \geq \kappa$.

Our problem of constructing such fields translate to considering cuts of K and their pair of cofinalities. Our strategy is:

- (a) we consider some properties of cuts (of a real closed field), namely being Dedekind, being Scott, being positive, being additive, being multiplicative,
- (b) we define dependence relation on the set of cuts of K, which satisfies the Steinitz assumptions,
- (c) realizing a maximal independent family of cuts with the right pairs of cofinality, we get a "one step symmetric closure".

It is fine: we can show existence and uniqueness. But will iterating this "atomic closure" eventually terminate?

- (d) For a field K we define a similar chain inside K, its minimal length being called $h(K)$
- (e) we define the depth $d(K)$ of K

(f) we show that $h(K)$, and $d(K)$ are quite closed and show that our iterated closure from (c) does not increase the cardinality too much

(g) we finally show that after $d(K)+1$ steps the iterated closure from (c) terminate.

Two later works (around 2012) seem relevant. One is by Kuhlmann-Kuhlman-Shelah [KKS15] deal with symmetrically complete ordered sets for generalizations of Banach fix point theorem.

The other, Malliaris-Shelah [MS16] prove that for $T = \mathrm{Th}(\mathbb{N})$ this is false. Malliaris ask (2013) whether we can generalize Theorem 1.1 to any o-minimal theory T. This is very reasonable but by the proof of [She94, §2] for, e.g. the theory of $(\mathbb{R}, e^x)$ this fails, but it is o-minimal by the celebrated theorem of Wilkie [Wil96].

More fully

Theorem 1.2. *1) Any model of T has a symmetric cut* <u>*when*</u>*:*

(a) *let T be a first order theory extending the theory of ordered semi-rings (so have 0,1 order, addition and multipliation with the usual rules but $x - y$ does not necessarily exist), which may have additional symbols (e.g. e^x)*

(b) *T implies some first order formula $\varphi(x,y)$ define a function f_φ such that:*

  (α) *for $x > 0, f_\varphi(x)$ increases and is $> x$*

  (β) *for $x > 1$ we have $x f_\varphi(x-1) < f_\varphi(x)$.*

2) We can weaken (b) to:

(b)′ *for some formula $\varphi(x,y) \in \mathbb{L}(\tau_T)$ the theory T implies:*

  (α) $0 < x \rightarrow (\exists y)\varphi(x,y)$

  (β) $\varphi(x,y) \rightarrow x < y$

  (γ) $1 < x \wedge \varphi(x, y_1) \wedge \varphi(x_1, y_1) \rightarrow x y_1 < y_2$.

Proof. By the proof of [She94, Th.2.2,pg.377-8] noticing:

($*$) the proof was written for $T = \mathrm{PA}$, Peano Arithmetic, but we use only:

  (a) any $M \models T$ is an ordered semi-ring

  (b) a function called exp which in [She94] is x^x, but only the properties listed in the theorem are used.

Alternatively it follows from part (2).
2) Similarly, but see details in 6.1. $\square_{1.2}$

Conclusion 1.3. *The theory $T = \mathrm{Th}(\mathbb{R}, e^x)$ is o-minimal (by* [Wil96] *and) any model of T has a symmetric cut.*

Proof. The function $x \mapsto e^{e^x}$ satisfies the requirement from 1.2. $\square_{1.3}$

Still

Claim 1.4. *Let T be an o-minimal (complete first order) theory, for notational transparency with elimination of quantifiers, and let K, L denote models of it.*
1) All the Definitions and Claims not mentioning Scott/additive/multiplicative cuts hold, that is, we have 2.1, 2.3(1)-(4), 2.4, 2.7 and 5.11 - 2.18 and 3.1 - 3.8.

2) So in particular the symmetric hull of a model K of T 2.17 and $\alpha(K)$ (see 2.18) are well defined.
3) But 1.3 says that $\alpha(T) = \infty$ for $T = \mathrm{Th}(\mathbb{R}, e^x)$.
4) If $\lambda > |K|$ is a Ramsey cardinal or just $\lambda \to (\omega_1)^{<\omega}_{|K|}$ and $\alpha(K) \geq \lambda$ then for some K of cardinality $|T|, \alpha(K) = \infty$.

Discussion 1.5. Note we conjecture that λ can be chosen as $\beth_{|K|^+}$ or so in 1.4(4).

§ 2. Real Closed Fields

Any ordered field embeds into a real closed field, and in fact has a unique real closure. We will find it convenient to work mainly with real closed fields throughout. Accordingly, we will need various properties of real closed fields. We assume some familiarity with quantifier elimination, real closure, and the like, and we use the following consequence of o-minimality. (Readers unfamiliar with o-minimality in general may simply remain in the context of real closed fields, or in geometrical language, *semialgebraic geometry*.)

Fact 2.1. Let K be a real closed field, and let f be a parametrically definable function of one variable defined over K. Then f is piecewise monotonic, with each piece either constant or strictly monotonic; that is we can find finitely many intervals (allowing ∞ and $-\infty$ as end points), f is constant or strictly increasing or strictly decreasing on each interval. Moreover this holds uniformly and definably in definable families, with a bound on the number of pieces required, and with each piece an interval whose endpoints are definable from the defining parameters for the function.

Notation 2.2. 1) K, L are ordered fields, usually real closed.
2) If $K \subseteq L, X \subseteq L$, then $K(X)$ is the (ordered) subfield of L generated by $K \cup X$.
3) Let $K_+ = \{c \in K : c > 0\}$.
4) For K an ordered field and $A, B \subseteq K$ let $A \subseteq B$ mean $a \in A \wedge b \in B \rightarrow a <_K b$.

§ 2(A). Cuts.

Definition 2.3. 1) A cut in a real closed field K is a pair $C = (C^-, C^+)$ with K the disjoint union of C^- and C^+, and $C^- < C^+$. The cut is a *Dedekind cut* if both sides are nonempty, and C^- has no maximum, while C^+ has no minimum. For $L \subseteq K$ let $C{\restriction}L = (C^- \cap L, C^+ \cap L)$, it is a cut of L.
2) The *cofinality* of a cut C is the pair (κ, λ) with κ the cofinality of C^- and λ the coinitiality of C^+ (i.e., the "cofinality to the left"). If the cut is not necessarily a Dedekind cut, then one includes 0 and 1 as possible values for these invariants.
3) A cut of cofinality (κ, λ) is *symmetric* if $\kappa = \lambda$.
4) A real closed field is *symmetrically complete* if it has no symmetric cuts.
5) A cut is *positive* if $C^- \cap K_+$ is nonempty.

Conclusion 2.4. *1) If L is a real closed field extending K and $a, b \in L \setminus K$ realizes the same cut of K <u>then</u> a, b realizes the same types over K in L.*
2) If C is a non-Dedekind cut of K and $\theta = \mathrm{cf}(K)$ <u>then</u> $\mathrm{cf}(C) \in \{(\theta, 0), (0, \theta), (\theta, 1), (1, \theta)\}$ and each of those cases occurs.

Proof. By Fact 2.1. □

We will need to consider some more specialized properties of cuts.

Definition 2.5. Let K be a real closed field, C a cut in K.
1) The cut C is a *Scott cut* if it is a Dedekind cut, and for all $r > 0$ in K, there are elements $a \in C^-$, $b \in C^+$ with $b - a < r$.
2) The cut C is *additive* if C^- is closed under addition and contains some positive element.
3) The cut C is *multiplicative* if $C^- \cap K_+$ is closed under multiplication and contains 2.

4) $C_{\rm add}$ is the cut with left side $\{r \in K : r + C^- \subseteq C^-\}$.
5) For C a positive cut, $C_{\rm mlt}$ is the cut with left side $\{r \in K : r \cdot (C^- \cap K_+) \subseteq C^-\}$.

Observe that

Observation 2.6. *1) Scott cuts are symmetric, in fact both cofinalities are equal to* $\mathrm{cf}(K)$.
2) If C is a Dedekind cut which is not a Scott cut, then $C_{\rm add}$ is a positive additive cut and note: $C^-_{\rm add} = \{c : c \le 0\}$ is impossible as "not scott".
3) If C is an additive cut which is not a multiplicative cut, then $C_{\rm mlt}$ is a multiplicative cut.
4) If C is a Dedekind cut of cofinality (κ, λ) then $\kappa, \lambda \ge \aleph_0$.

Definition 2.7. 1) If $K \subseteq L$ are ordered fields, then a cut C in K is said to be *realized*, or *filled*, by an element a of L iff the cut induced by a on K is the cut C.
2) If $C_1, C_2 \subseteq K$ and $C_1 < C_2$ but no $a \in K$ satisfies $C_1 < a < C_2$ then the cut of K defined (or induce or canonically extends) by (C_1, C_2) is $(\{a \in K : a \le c$ for some $c \in C_1\}, \{b \in K : c \le b$ for some $c \in C_2\})$, e.g., (C_1, C_2) may be a cut of a subfield of K.
3) If $K \subseteq L$ and C is a cut of L then $C{\restriction}K = (C^- \cap K, C^+{\restriction}K)$, a cut of K, is called the cut of K induced by C.

By Scott [Sco69] we know that

Lemma 2.8. *Let K be a real closed field. Then there is a real closed field L extending K in which every Scott cut has a unique realization, and no other Dedekind cuts are filled.*

This is called the *Scott completion* of K, and is strictly analogous to the classical Dedekind completion. The statement found in [Sco69] is worded differently, without referring directly to cuts, though the relevant cuts are introduced in the course of the proof. The result is also given in greater generality there.

Lemma 2.9. *Let K be a real closed field, C a multiplicative cut in K, and L the real closure of $K(x)$, where x realizes the cut C. Then for any $y \in L$ realizing the same cut, we have $x^{1/n} < y < x^n$ for some n.*

Proof. Let $\mathscr{O}_K$ be $\{a \in K : |a| \in C^-\}$, and let $\mathscr{O}_L$ be the convex closure in L of $\mathscr{O}_K$. Then these are valuation rings, corresponding to valuations on K and L which will be called v_K and v_L respectively.

The value group Γ_K of v_K is a divisible ordered abelian group, and the value group of the restriction of v_L to $K(x)$ is $\Gamma_K \oplus \mathbb{Z}\gamma$ where $\gamma := v_L(x)$ is negative, and infinitesimal relative to Γ_K. The value group of v_L is the divisible hull of $\Gamma_K \oplus \mathbb{Z}\gamma$.

Now if $y \in L$ induces the same cut C on K, then $v_L(y) = qv_L(x)$ for some positive rational q. Hence $u = y/x^q$ is a unit of $\mathscr{O}_L$, and thus $u, u^{-1} < x^\epsilon$ for all positive rational ϵ. So $x^{q-\epsilon} < y < x^{q+\epsilon}$ and the claim follows. $\square_{2.9}$

Lemma 2.10. *Let $K \subseteq L$ be real closed fields, and C an additive cut in L. Let C' and $C'_{\rm mlt}$ be the cuts induced on K by C and $C_{\rm mlt}$ respectively. Suppose that $C'_{\rm mlt} = (C')_{\rm mlt}$, and that $x, y \in L$ are two realizations of the cut C', with $x \in C^-$ and $y \in C^+$. Then y/x induces the cut $C'_{\rm mlt}$ on K.*

Proof. If $a \in K_+$ and $ax \ge y$, then $a \in (C_{\rm mlt})^+$, by definition, working in L.

On the other hand if $a \in K_+$ and $ax < y$, then $a \in [(C')_{\rm mlt}]^- \cap \kappa$, which by hypothesis is $(C'_{\rm mlt})^-$. $\square_{2.10}$

Lemma 2.11. *Let $K \subseteq L$ be real closed fields, and C a positive Dedekind cut in L which is not additive. Let C' and $C'_{\rm add}$ be the cuts induced on K by C and $C_{\rm add}$ respectively. Suppose that $C'_{\rm add} = (C')_{\rm add}$. Suppose that $x, y \in L$ are two realizations of the cut C', with $x \in C^-$ and $y \in C^+$. Then $y - x$ induces the cut $C'_{\rm add}$ on K.*

Proof. If $a \in K$ and $a + x \geq y$, then $a \in (C_{\rm add})^+$, by definition, working in L.

On the other hand if $a \in K$ and $a + x < y$, then $a \in [(C')_{\rm add}]^- \cap K$, which by hypothesis is $(C'_{\rm add})^-$. $\square_{2.11}$

§ 2(B). **Independent cuts.**

We will rely heavily on the following notion of independence.

Definition 2.12. Let K be a real closed field, and $\mathscr{C}$ a set of cuts in K. We say that the cuts in $\mathscr{C}$ are *dependent* if for every real closed field L containing realizations a_C $(C \in \mathscr{C})$ of the cuts over K, the set $\{a_C : C \in \mathscr{C}\}$ is algebraically dependent over K.

The following merely rephrases the definition (recalling 2.4(1))

Lemma 2.13. *Let K be a real closed field and $\mathscr{C}$ a set of cuts over K.*
1) The following are equivalent:

- *(A) $\mathscr{C}$ is independent*
- *(B) For each set $\mathscr{C}_0 \subseteq \mathscr{C}$, and each ordered field L containing K, if $a_C \in L$ is a realization of the cut C for each $C \in \mathscr{C}_0$, <u>then</u> the real closure of $K(a_C : C \in \mathscr{C}_0)$ does not realize any cuts in $\mathscr{C} \setminus \mathscr{C}_0$.*

2) For $\mathscr{C}_0$ and L as in clause (B) above, every $C \in \mathscr{C} \setminus \mathscr{C}_0$ define a (unique) cut C' of L (see Definition 2.3(2)) and $\{C' : C \in \mathscr{C} \setminus \mathscr{C}_0\}$ is an independent set of cuts of L.
3) Assume $\langle K_\alpha : \alpha \leq \delta \rangle$ is an increasing sequence of real closed fields, $\mathscr{C}$ a set of cuts of K_0, and $C \in \mathscr{C} \wedge \alpha < \delta \Rightarrow \mathscr{C}$ define a cut $C^{[\alpha]}$ of K_α. <u>Then</u> each $C \in \mathscr{C}$ define a cut of K_δ which we call $C^{[\delta]}$ and if $\{C^{[\alpha]} : C \in \mathscr{C}\}$ is an independent set of cuts of K_α for each $\alpha \in \delta$ then $\{C^{[\delta]} : C \in \mathscr{C}\}$ is an independent set of cuts of K_δ.
4) This dependence relation satisfies the Steinitz axioms for a dependence relation.

We will make use of it to realize certain sets of types in a controlled and canonical way.

Lemma 2.14. *Let K be a real closed field, and $\mathscr{C}$ a set of cuts over K.*
1) There is a real closed field L generated over K (as a real closed field) by a set of realizations of some independent family of cuts included in $\mathscr{C}$, in which all of the cuts $\mathscr{C}$ are realized.
2) Furthermore, such an extension is unique up to isomorphism over K.
3) Assume $\mathscr{C} = \bigcup\limits_{i<\alpha} \mathscr{C}_i$ is $\subseteq$-increasing continuous. There is a sequence $\langle K_i : i \leq \alpha \rangle$ of real closed fields, $K_0 = K, K_{i+1}$ is gotten as in (1) for $(K_i, \mathscr{C}'_i)$ where $\mathscr{C}'_i = \{C' : C'$ is a cut of K_i which is induced by $C' \restriction K_0$ and $C' \restriction K_0 \in \mathscr{C}_i\}$. Moreover, this sequence is unique up to isomorphism and K_α is gotten from the pair $(K, \mathscr{C})$ as in part (1).

Proof. 1) Clearly we must take L to be the real closure of $K(a_C : C \in \mathscr{C}_0)$, where $\mathscr{C}_0$ is some maximal independent subset of $\mathscr{C}$; and equally clearly, this works.

It remains to check the uniqueness. This comes down to the following: for any real closed field L extending K, and for any choice of independent cuts $C_1, \dots, C_n$ in K which are realized by elements $a_1, \dots, a_n$ of L, the real closure of the field $K(a_1, \dots, a_n)$ is uniquely determined by the cuts. One proceeds by induction on n. The real closure $\hat{K}$ of $K(a_n)$ is determined by the cut C_n by 2.4; and as none of the other cuts are realized in it, they extend canonically to cuts $C'_1, \dots, C'_{n-1}$ over $\hat{K}$, which are independent over $\hat{K}$. At this point induction applies.
2),3) Easy, too. $\square_{2.14}$

Lemma 2.15. *Let K be a real closed field, and $\mathscr{C}$ a set of Dedekind cuts in K. Suppose that C is a Dedekind cut of K of cofinality (κ, λ) which is dependent on $\mathscr{C}$, and let $\mathscr{C}_0$ be the set $\{C' \in \mathscr{C} : \mathrm{cof}(C') = (\kappa, \lambda)$ or $(\lambda, \kappa)\}$. Then[1] C is dependent on $\mathscr{C}_0$, and in particular $\mathscr{C}_0$ is non-empty.*

Proof. It is enough to prove this for the case that $\mathscr{C}$ is independent. If this fails, we may replace the base field K by the real closure $\hat{K}$ over K of a set of realizations of $\mathscr{C}_0$. Then since none of the cuts in $\mathscr{C} \setminus \mathscr{C}_0$ are realized, and C is not realized, these cuts extend canonically to cuts over $\hat{K}$, and hence we may suppose $\mathscr{C}_0 = \emptyset$. We may also suppose $\mathscr{C}$ is finite, and after a second extension of K we may even assume that $\mathscr{C}$ consists of a single cut C_0. This is the essential case.

So at this point we have $L \supseteq K$ and in it a realization a of C_0 over the real closed field K, and a realization b of C over K, with b algebraic, and hence definable, over a, relative to K. Thus b is the value at a of a K-definable function (being the ℓ-th root for some ℓ and polynomial over K), not locally constant near a, and by Fact 2.1 it follows that there is an interval I_0 of L containing b with endpoints in K and a K-definable function which is order isomorphism or anti-isomorphism from the interval I_0 to an interval I_1 including a, with the cuts corresponding. So the cuts have same (or inverted) cofinalities. This contradicts the supposition that $\mathscr{C}_0$ has become empty, and proves the claim. $\square_{2.15}$

For our purposes, the following case is the main one. We combine our previous lemma with the uniqueness statement.

Proposition 2.16. *Let K be a real closed field, and $\mathscr{C}$ a maximal independent set of symmetric cuts in K. Let L be an ordered field containing K together with realizations a_C of each $C \in \mathscr{C}$. Then the real closure K' of $K(a_C : C \in \mathscr{C})$ realizes the symmetric cuts of K and no others. Furthermore, the result of this construction is unique up to isomorphism over K. Moreover if L' is a real closed field extending K which realizes every cut in $\mathscr{C}$* **then** *K' can be embedded into L' over K.*

Proof. Clear (the "no other" by 2.15). $\square_{2.16}$

Evidently, this construction deserves a name.

Definition 2.17. 1) Let K be a real closed field. A *symmetric hull* of K is a real closed field generated over K by a set of realizations of a maximal independent set of symmetric cuts.

[1]If we allow $\{\kappa, \lambda\} \cap \{0, 1\} \neq \emptyset$ then we should equate 0 and 1.

2) We say that $\bar{K} = \langle K_\alpha : \alpha \le \alpha^* \rangle$ is an associated symmetric α_*-chain over K when:

- (a) $K_0 = K$,
- (b) K_α is an order field,
- (c) K_α is increasing continuous with α
- (d) $K_{\alpha+1}$ is a symmetric hull of K_α for $\alpha < \alpha^*$.

3) In (2) we replace "α_*-chain" by "chain" if K_{α_*} is symmetrically complete but $\alpha < \alpha_* \Rightarrow K_{\alpha+1} \neq K_\alpha$.
4) Let $\alpha(K)$ be the minimal α_* such that for some $\langle K_\alpha : \alpha \le \alpha_* \rangle$ as in part (2) is a chain, and ∞ if there is no such α.

Conclusion 2.18. *1) For every K, some L is a symmetric hull of K, and it is unique up to isomorphisms over K.*
2) For every K and α_* there is an associated symmetric α_*-chain $\langle K_\alpha : \alpha \le \alpha_* \rangle$ over K. It is unique up to isomorphism over K, that is, if $\langle K'_\alpha : \alpha \le \alpha_* \rangle$ is another such α^*-chain, then there is an isomorphism f from K_{α_*} onto K'_{α_*} such that $f{\restriction}K = \mathrm{id}_K$ and f maps K_α onto K'_α for $\alpha \le \alpha_*$.
3) $\alpha_* = \alpha(K) < \infty$ iff for every $\beta > \alpha_*$ and associated symmetric β-chain $\bar{K}$ over K we have $(\forall\gamma)(\alpha_* \le \gamma \le \beta \to K_\gamma = K_{\gamma_+ + 1}) \wedge (\forall\gamma)(\gamma < \alpha_* \to K_\gamma \neq K_{\gamma+1})$.

While the "symmetric hull" (from 2.17(1)) is unique up to isomorphism, there is certainly no reason to expect it to be symmetrically complete, and the construction will need to be iterated. The considerations of the next section will help to prove that the construction eventually terminates and to bound the length of the iteration.

Lemma 2.19. *1) For regular $\kappa < \lambda$ there is a real closed field with an (κ, λ)-cut.*
2) Let K be a real closed field, and L its symmetric hull. Then every Scott cut in K has a unique realization in L.
3) Assume L is the real closure of $K \cup \{a_C : C \in \mathscr{C}\}, \mathscr{C}$ an independent set of cuts of K and a_C realizing the cut C in L for $C \in \mathscr{C}$

- (a) *If every $C \in \mathscr{C}$ is a Dedekind cut then every cut of K realized in L is a Dedekind cut*
- (b) *if every $C \in \mathscr{C}$ is a non-Dedekind cut of L then every cut of K is realized in L is non-Dedekind*
- (c) *if some $C \in \mathscr{C}$ is a non-Dedekind cut of K then every non-Dedekind cut of K is realized in L.*

Proof. 1) First we choose K_i a real closed field K_i increasing continuous with $i \le \kappa, K_0 = K$ and for $i < \kappa$ the element $a_i \in K_{i+1} \setminus K_i$ is above all members of K_i. Second we choose a real closed field K^i increasing continuous with $i \le \lambda$ such that $K^0 = K_\kappa$, and for $i < \lambda, b_i \in K_i^{i+1} \setminus K_i$ is above a_j for $j < \kappa$ and below any $b \in K^i$ such that $(\forall j < \kappa)(a_j < b)$, in K^i. Lasly in K^λ, $(\{a_j : j < \kappa\}, \{b_i : i < \lambda\})$ determine a (κ, λ)-cut, i.e., $(\{a : a < a_j \text{ for some } j < \kappa\}, \{b : b_i < b \text{ for } i < \lambda\})$, in K_λ, is such a cut.
2) Recall that every Scott cut is symmetric. One can form the symmetric hull of K by first taking its Scott completion K_1, realizing only the Scott cuts (uniquely), and then taking the symmetric hull of K_1; this is equivalent by 2.14(3). By part (3) we are done.

3) Easy by now; Clause (a) is really from [Sco69]. $\square_{2.19}$

Observation 2.20. 1) For every linear order I there is a real closed field L and order preserving function from I into K such that: for every Dedekind cut (C_1, C_2) of I, the pair $(\{f(s) : s \in C_1\}, \{f(s) : s \in C_2\})$ induce a Dedekind cut of K; also $|L| = |I|$.
2) So, e.g. for every μ for some K of cardinality μ, K has a (θ_1, θ_2)-cut whenever $\theta_1, \theta_2 \leq \mu$ are regular.

Proof. 1) Let K be any field. We can find $L \supseteq K$ such that $L = K(\{a_s : s \in I\})$ such that $s < t \Rightarrow \bigwedge_n L \models$ "$(a_s)^n < a_t$". Now L is as required and $|L| = |K| + |I|$.

2) Easy. $\square_{2.20}$

§ 3. Height and Depth

Definition 3.1. Let K be a real closed field.
1) The *height* of K, $h(K)$, is the least ordinal α for which we can find a continuous increasing sequence K_i $(i \le \alpha)$ of real closed fields with K_0 countable, $K_\alpha = K$, and K_{i+1} generated over K_i, as a real closed field, by a set of realizations of a family of cuts which is independent.
2) Let $h^+(K)$ be $\max(|h(K)|^+, \aleph_1)$

Remark 3.2. 1) $\aleph_1$ is the first uncountable cardinal.
2) $h^+(K)$ is the first uncountable cardinal strictly greater than $h(K)$), so regular.
3) We could have chosen K_0 as the algebraic members of K, but this is not enough to make α unique. The point is that there may be, e.g. $\langle x_q : q \in \mathbb{Q}\rangle$ in K such that $q_1 < q_2 \Rightarrow \bigwedge_n (x_{q_1})^n < x_{q_2}$, so for every $\alpha < \aleph_1$ there is an increasing sequence $\langle q_\beta : \beta < \alpha\rangle$ of rationals, so may be $x_{q_\beta} \in K_\gamma \leftrightarrow \beta < \gamma$. Similarly for any $\lambda > \aleph_0$.
4) Observe that the height of K is an ordinal of cardinality at most $|K|$ (or is undefined, you can let it be ∞, a case which by 3.5 does not occur). We need to understand the relationship of the height of K and of $\alpha(K)$ with its order-theoretic structure, which for our purposes is controlled by the following parameter.

Definition 3.3. Let K be a real closed field. The *depth* of K, denoted $d(K)$, is the least cardinal κ greater than the length of every strictly increasing sequence in K.

Observation 3.4. *If K is a real closed field, <u>then</u> $d(K)$ is a regular uncountable cardinal.*

Proof. Uncountable because there is an infinite increasing sequence: $1, 2, \ldots$. Regular as any interval of K is order isomorphic to K. $\square_{3.4}$

The following estimate is straightforward, and what we really need is the estimate in the other direction, which will be given momentarily.

Lemma 3.5. *Let K be a real closed field. <u>Then</u> $h(K) \le d(K)$.*

Proof. One builds a continuous strictly increasing tower K_α of real closed subfields of K starting with any countable subfield of K. If α is limit, we define $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$. For successor ordinals, $K_{\alpha+1}$, is the real closure of $K_\alpha \cup \{a^\alpha_C : C \in \mathscr{C}_\alpha\}$ inside K where $\mathscr{C}_\alpha$ is a maxmimal independent set of cuts of K_α realized in K, and $a^\alpha_C \in K$ realized C. We stop when $K_\alpha = K$. Now if $K_\alpha \ne K$ then every $a \in K \backslash K_\alpha$ realizes some cut of K_α so by 2.13(4), there is $\mathscr{C}_\alpha$ as required (and it is not empty) hence $K_{\alpha+1}$ as required can be chosen. As K_α's definition implies $|\alpha| \le |K_\alpha| \le |K|$, necessarily for some $\alpha, K_\alpha = K$ and then we stop. If this continues past $\kappa = d(K)$, then there is a cut over K_κ filled at stage κ by an element $x \in K$. Then the cut determined by x over each K_α for $\alpha < \kappa$ is filled at stage $\alpha + 1$ by an element y_α. Those y_α's lying below x form an increasing sequence, by construction, which is therefore of length less than κ; and similarly (using the $\langle y_\alpha : \alpha < \kappa\rangle$) there are fewer than κ elements $y_\alpha > x$, so we arrive at a contradiction. $\square_{3.5}$

Proposition 3.6. *Let K be a real closed field. Then $d(K) \le h^+(K)$.*

Proof. Let $\kappa > h(K)$ be regular and uncountable, so it suffices to prove $d(K) \le \kappa$. Let K_α $(\alpha \le h(K))$ be a continuous increasing chain of real closed fields, with K_0

countable, $K_{h(K)} = K$, and K_{i+1} generated over K_i, as a real closed field, by a set of realizations of an independent family of cuts.

For $\alpha \le h(K)$ and $X \subseteq K$, let $K_{\alpha,X}$ be the real closure of $K_\alpha(X)$ inside K. We recast our claim as follows to allow an inductive argument

⊛ For $X \subseteq K$ with $|X| < \kappa$, and any $\alpha \le h(K)$, we have $d(K_{\alpha,X}) \le \kappa$.

Now this claim gives the promised result for $\alpha = h(K)$, is trivial for $\alpha = 0$ as K_0 is countable so $K_{\alpha,X}$ has cardinality $< \kappa$ (for $X \subseteq K, |X| < \kappa$), and the claim passes smoothly through limit ordinals up to $h(K)$ (because $\kappa = \mathrm{cf}(\kappa) > h(K)$), so we need only to consider the passage from α to $\beta = \alpha + 1$. So K_β is $K_{\alpha,S}$ with S a set of realizations of an independent family of cuts over K_α, (no two realizing the same cut, of course), and similarly $K_{\beta,X}$ is $K_{\alpha,X\cup S}$.

Consider the claim in the following form:

$$d(K_{\alpha,X\cup S_0}) \le \kappa \text{ for } S_0 \subseteq S$$

In this form for $S_0 = S$ we get the desired inductive step, and it clearly holds if $|S_0| < \kappa$, as it is included in the inductive hypothesis for α, and the case $|S_0| \ge \kappa$ reduces at once to the case $|S_0| = \kappa$. So we now assume that S_0 is a set of realizations of an independent family of cuts of K_α of cardinality κ (one element per cut).

By 2.13(2),(3),(4) we can find a subset S_1 of S_0 of cardinality $\aleph_0 + |X|$ such that:

(a) if $s \in S_0 \setminus S_1$ then the cut C which s induce on K_α is not realized in the real closure $K'_\alpha(\subseteq K)$ of $K_\alpha(X \cup S_1)$

(b) the cuts which the $s \in S_0 \setminus S_1$ induce on K'_α form an independent family.

Let $Y = X \cup S_1$, so $K'_\alpha = K_\alpha(Y), |Y| < \kappa$.

Let $\{s_\epsilon : \epsilon < \kappa\}$ list $S_0 \setminus S_1$. For $\zeta \le \kappa$, let $L_\zeta = K_{\alpha,Y\cup\{s_\epsilon:\epsilon<\zeta\}}$ and let $L = L_\kappa$. By the induction hypothesis (for α and $X \cup \{s_\epsilon : \epsilon < \zeta\}$) we have $d(L_\zeta) \le \kappa$ for $\zeta < \kappa$, and we shall prove $d(L) \le \kappa$ thus finishing the proof.

Let C_i be the cut realized by s_i over L_0. Note that C_i extends canonically to a cut C_i^j on K_j for all $j \le i$, and for fixed j, the set $\{C_i^j : i \in [j,\kappa)\}$ of cuts of K_j is independent.

Now suppose, toward a contradiction, that there is $\langle a_i : i < \kappa\rangle$ an increasing sequence in L. For each $i < \kappa$ let the ordinal $f(i) < \kappa$ be minimal such that $a_i \in L_{f(i)}$, so necessarily $f : \kappa \to \kappa$ is well defined and for each $j < \kappa$ the set $\{i < \kappa : a_i \in L_j$ equivalently $f(i) \le j\}$ is a bounded subset of κ (because $d(L_j) \le \kappa$). Now for $\varepsilon, i < \kappa$ with $f(i) > \epsilon$ let B_i^ϵ denote the cut induced on L_ϵ by a_i. For $i_1 < i_2 < n$ such that $\varepsilon < f(i_1), f(i_2)$ clearly $a_{i_1} <_K a_{i_2}$ hence $(B_{i_1}^\varepsilon)^- \subseteq (B_{i_2}^\varepsilon)^-$. With ϵ held fixed, and with i varying, as $d(L_\epsilon) \le \kappa$ we find that the cuts B_i^ϵ stabilize for large enough $i < \kappa$ (and furthermore, $a_i \notin L_\epsilon$). Accordingly, for each ϵ we may select $j_\epsilon < \kappa$ above ϵ such that the cuts B_i^ϵ coincide for all $i \ge j_\epsilon$.

Now fix a limit ordinal $\delta < \kappa$ such that for all $\epsilon < \delta$ we have $j_\epsilon < \delta$. We may also require that $a_i \in L_\delta$ for $i < \delta$.

Let $\zeta < \kappa$ be such that $a_\zeta \notin K_\delta$, it is well defined as $d(K_\delta) \le \kappa$ and is $\ge \delta$ as $i < \delta \Rightarrow a_i \in L_\delta$.

Now a_ζ is algebraic over $L_\delta(s_i : i \in I_0)$ for some finite subset I_0 of $[\delta, \kappa)$ and $I_0 \neq \emptyset$ because $a_\zeta \notin L_\delta$. Hence a_ζ is algebraic also over $L_\epsilon(s_i : i \in I_0)$ for some $\epsilon < \delta$. Thus the cut B^ϵ_ζ depends on the cuts C^ϵ_i $(i \in I_0)$ over L_ϵ. As $j_\epsilon < \delta < \zeta$ necessarily $B^\epsilon_\zeta = B^\epsilon_{j_\epsilon}$ is realized in L_δ and it follows that this cut is also dependent on the sets $\{C^\epsilon_i : i < \delta \text{ and } i \geq \varepsilon\}$ of cuts over L_ϵ. But the cuts C^ϵ_i for $i \geq \epsilon$ are supposed to be independent over L_ϵ, a contradiction. $\square_{3.6}$

Proposition 3.7. *Let K be a real closed field. Then $|h(K)| \leq |K| \leq 2^{|h(K)|}$.*

Proof. The first inequality is clear. For the second, let $\alpha = h(K)$, $\kappa = |\alpha| + \aleph_0$, and let K_i $(i \leq \alpha)$ be a chain of the sort afforded by the definition of the height. Note that $h^+(K) = \kappa^+$. We show by induction on i that $|K_i| \leq 2^\kappa$. Only successor ordinals $i = j + 1$ require consideration, where we suppose $|K_j| \leq 2^\kappa$.

Each generator a of K_i over K_j corresponds to a cut C_a in K_j, and each such cut is determined by the choice of some cofinal sequence S_a in C^-_a. Such a sequence S_a may be taken to have order type a regular cardinal, and will have length less than $d(K)$. Since $d(K) \leq h^+(K)$ by 3.6, we find that the order type of S_a is at most κ. So the number of such sequences is at most $\sum_{\lambda \leq \kappa} |K_j|^\lambda \leq \kappa \times (2^\kappa)^\kappa = 2^\kappa$. $\square_{3.7}$

Claim 3.8. *1) Assume L is the Scott completion of K (i.e., in our terms as in definition 2.5(1)). Let $\mathscr{C}$ be a maximal set of Scott cuts of K which is independent, then K is dense in L.*
1A) Moreover for every Dedekind cut C of L the cut $C \restriction K$ induce C hence $\mathrm{cf}(C) = \mathrm{cf}(C \restriction K)$.
1B) Moreover every non-Scott cut C_1 of K induce a non-Scott cut C_2 of L so $C_1 = C_1 \restriction K, \mathrm{cf}(C_2) = \mathrm{cf}(C_1)$.
2) If K has no symmetric cut except possibly Scott cuts, and L is a Scott completion of K then L is symmetrically complete hence L is a symmetric closure of K.

Proof. Part (1) by [Sco69]; the others clear, too. $\square_{3.8}$

§ 4. Proof of the Theorem

We now consider the following construction. Given a real closed field K, we form a continuous increasing chain K_α by setting $K_0 = K$, taking $K_{\alpha+1}$ to be the symmetric hull of K_α in the sense of Definition 2.17, and taking unions at limit ordinals.

If at some stage K_α is symmetrically complete, that is $K_\alpha = K_{\alpha+1}$, then we have the desired symmetrically complete extension of K, and furthermore our extension is prime in a natural sense. We claim in fact:

Proposition 4.1. *1) Assume K is a real closed field, $\kappa = h^+(K) + \aleph_2$ and K_α $(\alpha \leq \kappa + 1)$ is an associated continuous symmetric κ-chain, then*

- *(a) if $\mathrm{cf}(K) \neq \kappa$ then K_κ is symmetrically complete so $K_{\kappa+1} = K_\kappa$*
- *(b) $K_{\kappa+1}$ is symmetrically closed*
- *(c) if $\mathrm{cf}(K) = \kappa$ then every Dedekind cut of K_κ is a Scott cut.*

2) Also

- *(i) $|K_\kappa| \leq 2^{h^+(K)+\aleph_1}$, and $|K_{\kappa+1}| \leq 2^{h^+(K)+\aleph_1}$*
- *(ii) if K' is a symmetrically complete extension of K then $K_{\kappa+1}$ can be embedded into K' over K*
- *(iii) K is unbounded in $K_{\kappa+1}$ and no non-symmetric Dedekind cut of K is realized in K_κ*
- *(iv) any two real closed fields extending K which are symmetrically complete and embeddable into $K_{\kappa+1}$, are isomorphism over K (so we can say $K_{\kappa+1}$ is the closure)*
- *(v) for some unique $\alpha^* \leq \kappa+1$ there is an associated continuous chain over K of length α^*.*

The proof of Proposition 4.1 occupies the remainder of this section.

Lemma 4.2. *Suppose that K is a real closed field, and that $\langle K_\alpha : \alpha \leq \alpha(*)\rangle$ is a continuous chain of iterated symmetric hulls of any length. Let $x \in K_\alpha \setminus K_\gamma$ with $\alpha > \gamma \geq 0$ arbitrary. Then the cut induced on K_γ by x is symmetric.*

Remark 4.3. If we use $\kappa = \max\{h^+(K), \aleph_2\}$ then $\kappa \geq \aleph_2$ is regular and greater than $h(K)$; in particular $\kappa \geq d(K)$ by 3.6. Furthermore, as $\kappa > h(K)$, we can view the chain K_α as a continuation of a chain $\hat{K}_i$ $(i \leq h(K))$ of the sort occurring in the definition of $h(K)$, with $\hat{K}_{h(K)} = K_0$; then the concatenated chain gives a construction of K_α of length at most $h(K) + \alpha < \kappa$, and hence $h(K_\alpha) < \kappa$ for all $\alpha < \kappa$, and in particular $d(K_\alpha) \leq \kappa$ for all $\alpha < \kappa$ by 3.6.

Proof. Let $\beta < \alpha$ be minimal such that the cut in question is filled in $K_{\beta+1}$. Then the cut induced on K_β by x is the canonical extension of the cut induced on K_γ by x, and is symmetric by Proposition 2.16. $\square_{4.2}$

We now begin the proof by contradiction of Proposition 4.1(1). First, we assume (this does not contradict 4.1(1)) that:

$\boxplus_1$ (a) the chain $\bar{K} = \langle K_\alpha : \alpha \leq \kappa + 1\rangle$ is strictly increasing at every step up to K_κ, and

(b) C is a Dedekind cut of K_κ.

Now let

$\boxplus_2$ for $\alpha < \kappa$, let C_α denote the cut induced on K_α by C.

Lemma 4.4. *Assume (in addition to $\boxplus_1$) that C_α does not define C for $\alpha < \kappa$.*
1) For any $\alpha < \kappa$, the cut C_α is symmetric, in particular, a Dedekind cut.
2) For every $\alpha < \kappa, C^-_\alpha$ is bounded in C^- and C^+_α is bounded in C^+ from below.

Proof. 1) Suppose C_α is not symmetric. Then the cut C_α is not realized in K_κ, by Lemma 4.2. Hence the cut C is the canonical extension of C_α to K_κ, contradicting the Lemma's assumption.
2) Toward contradiction assume C^-_α is unbounded from below in C^-; so necessarily $\beta \in [\alpha, \kappa) \Rightarrow \mathrm{cf}(C^-_\beta) = \mathrm{cf}(C^-_\alpha)$.

For every $\beta \in [\alpha, \kappa)$, as $K_{\beta+1}$ is a symmetric hull of K_β, by part (1) it follows that some $a_\beta \in K_{\beta+1}$ realizes the cut C_β hence by the present assumption toward contradiction, $a_\beta \in C^+$ and $a_\beta \in C^+_\gamma$ for $\gamma \in (\beta, \kappa)$. So for every limit $\delta < \kappa$ which is $> \alpha, \{a_\beta : \beta \in [\alpha, \delta)\}$ is a subset of C^+_δ unbounded from below, hence $\mathrm{cf}(C_\delta) = (\mathrm{cf}(C^-_\delta), \mathrm{cf}(C^+_\delta)) = (\mathrm{cf}(C^-_\alpha), \mathrm{cf}(\delta))$. As $\aleph_0 \neq \aleph_1$ are regulars $< \kappa$ for some $\delta \in (\alpha, \kappa)$ we have $\mathrm{cf}(C^-_\alpha) \neq \mathrm{cf}(\delta)$ hence C_δ is not symmetric, contradicting part (1). So indeed for $\alpha < \kappa, C^-_\alpha$ is bounded in C^-; similarly C^+_α is unbounded from below in C^+. $\square_{4.4}$

After these preliminaries, (continue the proof of 4.1, for this) we prove:

$\boxplus_2$ if C is symmetric then C is a Scott cut and $\mathrm{cf}(K) = \kappa$.

We divide the analysis of the supposed cut C into a number of cases, each of which leads to a contradiction or to the desired conclusion.

So assume C is symmetric.

Case I: C is a Scott cut

If C is (a Scott cut and) $\mathrm{cf}(K) = \kappa$, there is nothing to be proved, so assume $\mathrm{cf}(K) \neq \kappa$.

By 4.4(2), we can find $\langle a^-_\alpha, a^+_\alpha : \alpha < \kappa \rangle$, such that $a^-_\alpha \in C^-, a^+_\alpha \in C^+$ both realizing the cut C_α. For some club E of κ consisting of limit ordinals we have $\alpha < \delta \in E \Rightarrow a^-_\alpha, a^+_\alpha \in K_\delta$. As C is a Scott cut of K_κ by the case assumption necessarily $\langle a^+_\alpha - a^-_\alpha : \alpha < \kappa \rangle$ is a decreasing sequence of positive members of K_κ with no positive lower bound, so $\langle 1/(a^+_\alpha - a^-_\alpha) : \alpha < \kappa \rangle$ is increasing cofinal in K_κ, so $\mathrm{cf}(K_\kappa) = \kappa$. But K is cofinal in K_κ hence $\mathrm{cf}(K) = \kappa$, contradicting what we have assumed in the beginning of the case.

Case II: C is a multiplicative cut

Let $\alpha < \kappa$ have uncountable cofinality (recall $\kappa \geq \aleph_2$).

The cut C_α is realized in $K_{\alpha+1}$ by some element a. As C is multiplicative, either all positive rational powers of a lie in C^-, or all positive rational powers of a lie in C^+.

On the other hand, $K_{\alpha+1}$ may be constructed in two stages as follows. First, let $K_{\alpha+1}$ be the real closure of $K_\alpha(a_{C'} : C' \in \mathscr{C}\rangle$ where $a_{C'} \in K_{\alpha+1}$ realizes C' for $C' \in \mathscr{C}$ and $\mathscr{C}$ is an independent set of symmetric cuts in K_α such that $C_\alpha \in \mathscr{C}$ and $a_{C_\alpha} = a$. Let $\mathscr{C}' = \mathscr{C} \backslash \{C\}$ and let K'_α be the real closure of $K_\alpha(a_{C'} : C' \in \mathscr{C}')$;

then take the real closure of $K'_\alpha(a)$, noting that a fills the canonical extension of the cut C_α to K'_α. By the choice of $\langle a_{C'} : C' \in \mathscr{C}\rangle$ clearly the real closure of $K'_\alpha(a)$ is $K_{\alpha+1}$. As seen in Lemma 2.9, there are only two cuts which may possibly be induced by C on $K_{\alpha+1}$ (which has to be multiplicative), one has lower part $C^- \cap K'_\alpha$ and the other has upper $C^+ \cap K'_\alpha$. Now each of those cuts has countable cofinality from one side, and uncountable cofinality from the other.

So $C_{\alpha+1}$ is not symmetric, and this is a contradiction to 4.4.

<u>Case III: C is an additive cut</u>

By 2.6(3) we know that $C_{\rm mlt}$ is a multiplicative cut (of C_κ).

Let $\langle b^-_\alpha : \alpha < \kappa\rangle$ be increasing cofinal in $C^- \cap K_+$ and let $\langle b^+_\alpha : \alpha < \kappa\rangle$ be decreasing unbounded from below in C^+. Clearly $\alpha < \kappa \Rightarrow b^+_\alpha / b^-_\alpha \in (C_{\rm mlt})^+$ and easily $\langle b^+_\alpha/b^-_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence of members of $(C_{\rm mlt})^+$ unbounded from below in it

According to the property of $\langle b^+_\alpha/b^-_\alpha : \alpha < \kappa\rangle$ the cofinality of $(C_{\rm mlt})$ from the right is κ.

Now if the cofinality of $C_{\rm mlt}$ from the left is also κ, then by 2.6(3) we contradict Case II. On the other hand if the cofinality of $C_{\rm mlt}$ from the left is θ which is less than κ, then from some point downward this cofinality stabilizes. Hence for some closed unbounded set $E \subseteq \kappa$ we have $\delta \in E \Rightarrow {\rm cf}(C_{\rm mdt} \restriction K_\beta) = (\theta, {\rm cf}(\delta))$; but then we can choose δ large and of some other cofinality (again, since $\kappa \geq \aleph_2$ there is such δ with ${\rm cf}(\delta) \in \{\aleph_0, \aleph_1\} \backslash \{\emptyset\}$). Now $C_{\rm mlt}$ is clearly a Dedekind cut of K_κ hence Lemma 4.4 applies to it, too, but its first conclusion fails for $\alpha = \delta$ hence its assumption fails. So for some $\beta < \kappa$, the cut $(C_{\rm mlt})\restriction K_\beta$ of K_β induces $C_{\rm mlt}$. So for some increasing sequence $\langle \alpha_\varepsilon : \varepsilon < \kappa\rangle$ of ordinals $< \kappa$ and $c^+_\varepsilon \in K_\beta$ we have $b^+_{\alpha_{\varepsilon+1}}/b^-_{\alpha_{\varepsilon+1}} < c^+_\varepsilon \leq b^+_{\alpha_\varepsilon}/b^-_{\alpha_\varepsilon}$ so $\langle c^+_\varepsilon : \varepsilon < \kappa\rangle$ is decreasing unbounded from below in $(C_{\rm mlt})^+ \cap K_\beta$, so unbounded in $(C_{\rm mlt})^+$.

This exemplifies $\kappa < d(K_\beta)$ but by 3.6 we have $d(K_\beta) \leq h^+(K_\beta)$ but clearly $h(K_\beta) \leq h(K) + \beta$ hence $|h(K_\beta)| \leq |h(K) + (\beta) < h^+(K) + \kappa = \kappa$, a contradiction.

<u>Case IV: C is a positive Dedekind cut, but not a Scott cut</u>

One argues as in the preceding case, considering $C_{\rm add}$ and using Lemma 2.11, which leads to a symmetric additive cut and thus a contradiction to the previous case. In details choose $\langle b^-_\alpha : \alpha < \kappa\rangle, \langle b^+_\alpha : \alpha < \kappa\rangle$ as in case III, so $\langle b^+_\alpha - b^-_\alpha : \alpha < \kappa\rangle$ is a decreasing sequence in $C^+_{\rm add}$ unbounded from below in it.

If the cofinality of $C_{\rm add}$ from below is also κ, recall that $C_{\rm add}$ is an additive cut by 2.6(2) contradiction to case III. If not, we repeat the argument in the end of Case III.

<u>Case V: C is a (Dedekind not Scott) cut of K_κ</u>

Choose $a \in C^+$ hence $(a + C^-, a + C^+)$ is a positive cut of K_κ so we get a contradiction by Case IV.

As no case remains, Proposition 4.1(1) is proved, and thus the construction of a symmetrically complete extension terminates.

As for clause (i) of 4.1(2), to estimate the cardinality of the resulting symmetrically complete extension, recall that it has height at most $h(\kappa) + \kappa + 1$ hence $|h(K_{\kappa+1})| \leq \kappa' = \max(h^+(K), \aleph_2) \leq \max(|K|^+, \aleph_2)$ and hence $K_{\kappa+1}$ has cardinality at most $2^{\kappa'}$. Moreover, similarly for any $\alpha < \kappa'$, $|K_\alpha| \leq 2^{h^+(K)+\aleph_1}$ hence

$|K_\kappa| = |\bigcup\limits_{\alpha<\kappa} K_\alpha| \le \sum\limits_{\alpha<\kappa} |K_\alpha| \le \sum\limits_{\alpha<\kappa} 2^{h^+(K)+\aleph_1} = \kappa' + 2^{h^+(K)+\aleph_1} = 2^{h^+(K)+\aleph_1}$. As K_κ is dense in $K_{\kappa+1}$ and $d(K_\kappa) \le h^+(K)$ also $|K_{\kappa+1}| \le 2^{h^+(K)+\aleph_1}$.

For clause (ii) of 4.1, we define an embedding h_α of K_α into K', increasing continuous with α for $\alpha \le \kappa$. For $\alpha = 0, h_0$ is the identity, for α limit take union and for $\alpha = \beta + 1$ use 2.16.

Clauses (iii),(iv),(v) of 4.1 is easy too.

Discussion 4.5. How do we prove that the bound in 4.1 is right? It goes as in [She94, §2] but using a given decreasing sequence of length θ for $\theta < \kappa$. This is to be filled.

§ 5. Concluding remarks

Discussion 5.1. It should be clear that there are considerably more general types of closure that can be constructed in a similar manner. Let Θ be a class of possible cofinalities of cuts, that is pairs of regular cardinals, and suppose that Θ is symmetric in the sense that $(\theta_1, \theta_2) \in \Theta$ implies $(\theta_2, \theta_1) \in \Theta$. Then we may consider Θ-constructions in which maximal independent sets of cuts, all of whose cofinalities are restricted to lie in Θ, are taken. In order to get such a construction always to terminate, all that is needed is the following:

- (a) for all regular θ_1, there is θ_2 such that the pair (θ_1, θ_2) is not in Θ.

From this it follows that:

- (b) for some regular $\kappa \geq h^+(K) + \aleph_2$, for every θ_1 regular $< \kappa$ there is a regular $\theta_2 < \kappa$ such that $(\theta_1, \theta_2) \notin \Theta$.

The proof is as above; in the symmetric case, $\Theta_{\rm sym}$ consists of all pairs (θ, θ) of equal regular cardinals. Clearly we may have to make the closure as large as we need κ as in (b) above. Also in the proof of 4.1(1) in the multiplicative case we choose $\delta < \kappa$ such that $(\aleph_0, {\rm cf}(\delta)) \notin \Theta$ and in the additive case, we choose $\delta < \kappa$ such that $(\theta, {\rm cf}(\delta)) \notin \Theta$ (but of course change the cardinality bound).

Under the preceding mild conditions, such a Θ-construction provides an "atomic" extension of the desired type. So we have Θ-closure, and it is prime (as in clause (ii) of 4.1(2)). We also can change the cofinality of K. Below we elaborate.

Definition 5.2. Assume Θ is a set or class of pairs of regular cardinals which is symmetric i.e., $(\kappa, \lambda) \in \Theta \Rightarrow (\lambda, \kappa) \in \Theta$.
0) A Θ-cut of K is a cut C with ${\rm cf}(C) \in \Theta$.
1) We say that a real closed field is Θ-complete iff no Dedekind cut of K has cofinality from Θ.
2) We say that L is a Θ-hull of K when there is a maximal subset $\mathscr{C}$ of ${\rm cut}_\Theta(K) := \{C : C \text{ a Dedekind cut of } K \text{ with cofinality} \in \Theta\}$ which is independent and $a_C \in L$ for $C \in \mathscr{C}$ realizing C such that L is the real closure of $K \cup \{a_C : C \in \mathscr{C}\}$.
2A) We say that L is a weak Θ-hull of K when there is a subset $\mathscr{C}$ of ${\rm cut}_\Theta(K)$ which is independent and $a_C \in L$ for $C \in \mathscr{C}$ realizing C such that L is the real closure of $K \cup \{a_C : C \in \mathscr{C}\}$.
3) We say that $\langle K_\alpha : \alpha \leq \alpha^* \rangle$ is an associated $\Theta - \alpha^*$-chain over K when: K_α is a real closed field, increasing continuous with $\alpha, K_0 = K$ and $K_{\alpha+1}$ is a Θ-hull of K_α for $\alpha < \alpha^*$.
3A) We say that $\langle K_\alpha : \alpha \leq \alpha^* \rangle$ is a weak associated $\Theta - \alpha^*$-chain over K when: K_α is an increasing continuous sequence of real closed fields, $K = K_0$ and $K_{\alpha+1}$ a weak Θ-hull of K_α for $\alpha < \alpha^*$. We may omit Θ meaning $\{(\kappa, \lambda) : \kappa, \lambda \text{ are regular infinite cardinals}\}$. We may omit "over K".
4) We say that $\langle K_\alpha : \alpha \leq \alpha^* \rangle$ is an associated Θ-chain over K when: it is an associated $\Theta - \alpha^*$-chain over K, $\alpha < \alpha^* \Rightarrow K_{\alpha+1} \neq K_\alpha$ and K_{α^*} is Θ-complete.
5) Let $d'(K)$ be the minimal regular cardinal κ (so infinite) such that: for every non-Scott Dedekind cut C of K both cofinalities of C are $< \kappa$).

Theorem 5.3. *Let K be a real closed field and Θ be as in Definition 5.2.*
1) There is a Θ-hull L of K, see Definition 5.2(1).
2) L in (1) is unique up to isomorphism over K, and K is cofinal in it.

3) For every ordinal α^* there is an associated $\Theta - \alpha^*$-chain over K, see Definition 5.2(2) and it is unique up to isomorphism over K.
3A) If $\langle K_\alpha : \alpha \le \alpha^* \rangle$ is a weak associated $\Theta - \alpha^*$-chain over K then K is cofinal in all its members in particular in K_{α^*}.
4) If $\theta = \mathrm{cf}(\theta) < d(K)$ moreover K has a non-Scott Dedkind cut of lower cofinality θ and $(\forall \lambda)[(\theta, \lambda) \in \Theta]$ then there is no associated Θ-chain over K, moreover no Θ-complete extension of K.
5) $\aleph_0 \le d'(K) \le d(K)$.
6) If $d'(K) = \aleph_0$ then K is isomorphic to a sub-field of the field of reals.
7) If $\langle K_\alpha : \alpha \le \alpha^* \rangle$ is a weak associated $\Theta - \alpha^*$-chain over K then $d'(K) \le \mathrm{Max}\{|\alpha^*|^+, d'(K)\}$.
7A) In (7) also $|K_{\alpha^*}| \le |K| + |\alpha^*| + \sum\{|K|^\theta : (\theta, \theta) \in \theta$ and $\theta < d'(K)\}$.
8) Assume κ satisfies "$\kappa \ne \mathrm{cf}(K), \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and for every regular $\theta < d'(K) + \kappa$ for some regular $\lambda < \kappa$ we have $(\theta, \lambda) \notin \Theta$, then there is an associated Θ-chain over K of length $\le \kappa$ (compare with part (4)).
9) If ($\forall$ regular θ) ($\exists$ regular λ) $[(\theta, \lambda) \notin \Theta]$ then for every real closed field K' there is an associated Θ-chain over it.
10) If there is an associated Θ-chain $\langle K_\alpha : \alpha \le \alpha^* \rangle$ over K then

- (a) *K_{α^*} is a real closed field, Θ-complete extending K*
- (b) *[universally]] if K' is a Θ-complete real closed field extending K then K_{α^*} can be embedded into K' over K*
- (c) *[uniqueness] if L is a Θ-complete real closed field extending K which can be embeded over K into L' for every Θ-complete, Θ-complete real closed field extending K then L is isomorphism to K_{α^*} over K*
- (d) *in Clause (c), $|L| \le 2^{<\kappa}$ when κ minimal as in (8).*

Proof. 1) As in 2.18(1).
2) As in 2.18(1).
3) Follows form (1)+(2).
3A) Easily by induction on α^*.
4) Let C be a non-Scott cut of K such that $\mathrm{cf}(C) = (\theta, \lambda)$, without loss of generality C is positive and $\langle a_\alpha : \alpha < \theta \rangle$ be an increasing sequence of members of $C \cap K_+$ cofinal in it. As C is non-Scott clearly for some $c \in K_+$ we have $a \in C^- \Rightarrow a + c \in C^-$ hence without loss of generality

- $(*)$ $c \in K_+$ and $a_\alpha + c < a_{\alpha+1}$ for $\alpha < \theta$.

Now if L is a Θ-complete extension of K then there is a cut C_1 of L such that $C_1^- = \{a \in L : a < a_\alpha$ for some $\alpha < \theta\}$. Let $\mathrm{cf}(C_1) = (\lambda_1, \lambda_2)$, so necessarily $\lambda_1 = \theta$ and obviously C_1^+ cannot have a first element b as then $b - c \in C_1^+$ by (*), so $\lambda_2 \ge \aleph_0$. So by an assumption $(\lambda_1, \lambda_2) \in \Theta$, contradiction to "$L$ is $\Theta -$ complete".
5) Trivial, see Definition 5.2(5).
6) Easily K is complete hence it is well known to be isomorphic to the field of reals.
7) So

- $(*)$ (a) $\langle K_\alpha : \alpha \le \alpha^* \rangle$ is a weak Θ-sequence
  - (b) K_0 is unbounded in K_{α_*}.

Now we repeat the proof of 3.6. The only place we have to say more is why $\langle B_i^\varepsilon : i < \kappa, f(i) > \kappa \rangle$ has a constant end segment. Otherwise, by the induction

hypothesis, its limit is a Scott cut of L_ε, but then $\langle a_i : i < \kappa\rangle$ is cofinal if B^-, B^- a Scott cut of K_κ.
7A) Recall that $\text{cut}_{\{(\theta,\lambda):\theta\neq\lambda \text{regular}\}}(K)$ has cardinality $\leq |K|$.
8) So assume $\langle K_\alpha : \alpha \leq \kappa+1\rangle$ is an associated $\Theta-(\kappa+1)$-chain over K.
First

$(*)_1$ like 4.2, 4.4 replacing "symmetric" cut by "Θ-cut".

Let

$\boxplus$ C be a cut of K_κ which is a Θ-cut, we fix it for awhile.

If for some $\alpha < \kappa$ the $\text{cat}(C_\alpha) = C{\restriction}K_\alpha$ induce C on K_κ, then $C{\restriction}K_\alpha$ is a cut of K_α of cofinality the same as C, hence a Θ-cut of K_α hence is realized in $K_{\alpha+1}$ by the construction, say by a, contradiction to "$C{\restriction}K_\alpha$ induce C on K_κ".
Hence

$(*)_2$ for no $\alpha < \kappa$ does $C{\restriction}K_\alpha$ induce C on K_κ (so the assumption of 4.4 holds).

This means that for every $\alpha < \kappa$ some $a_\alpha \in K\backslash K_\alpha$ realizes C, so $a_\alpha \in C^- \vee a_\alpha \in C^+$ so as we can replace C by $|\{-b : b \in C^+\}, \{-b : b \in C^-\}|$ without loss of generality for arbitrarily large $\alpha < \kappa, a_\alpha \in C^-$, so

$(*)_3$ $a_\alpha \in C^- \subseteq K_\kappa$ realizes the cut $C{\restriction}K_\alpha$
$(*)_4$ $\text{cf}(C^-) = \kappa$ so let $\text{cf}(C) = (\kappa, \lambda)$.

<u>Case 1</u>: $\lambda \neq \kappa$
Let σ be regular $< \kappa$ such that $(\sigma, \lambda) \notin \Theta$. Let $\{b_\beta : \beta < \lambda\rangle$ be a decreasing sequence in C^+ unbounded from below in it so without loss of generality

$(*)$ $\{b_\beta : \beta < \lambda\} \subseteq K_{\alpha(*)}$.

Now for some club E of κ we have

$(*)$ if $\delta \in E$ then $\delta > \alpha(*)$ and $\{a_\alpha : \alpha < \delta\}$ is an unbounded subset of $C^- \cap K_\delta$ hence $\text{cf}(C{\restriction}K_\delta) = (\text{cf}(\delta), \lambda)$.

Choose $\delta \in E$, such that $\text{cf}(\delta) = \sigma$. But then a_δ realizes a $\{(\text{cf}(\delta), \lambda)\}$-cut i.e., $\{(\sigma, \lambda)\}$-cut i.e., $\{(\sigma, \lambda)\}$-cut which is a non Θ-cut by the choice of σ, contradiction to $(*)_1$.

<u>Case 2</u>: $\lambda = \kappa$
We repeat the proof of 4.1 after 4.4, in (Case I) using $\kappa \neq \text{cf}(K)$.
9) For the given field K' define κ_n by induction on $n < \omega$

$$\kappa_0 = |K'| \text{ (or } d'(K))$$

$\kappa_{n+1} = \text{Min}\{\kappa : \kappa \text{ regular and if } \theta < \kappa_n \text{ then for some } \lambda < \kappa \text{ we have } (\theta, \lambda) \notin \Theta\}$.
Now $K', (\sum\{\kappa_n : n < \omega\})^+$ satisfies the condition in (8).
10) Should be clear. $\square_{5.3}$

What about $\text{cf}(K)$, we have not changed it in all our completion. It doesn't make much difference because

Claim 5.4. *1) For K and regular $\kappa(\geq \aleph_0)$ there is L such that:*

(a) $K \subseteq L$
(b) $\mathrm{cf}(L) = \kappa$
(c) *if* $K \subseteq L'$ *and* $\mathrm{cf}(L') = \kappa$ *then we can embed* L *into* L' *over* K
(d) (α) *if* $\mathrm{cf}(K) = \kappa$ *then* $K = L$
(β) *if* $\mathrm{cf}(K) \neq \kappa$ *then in clause (c) we can add: there is an embedding* f *of* L *into* L' *over* K such that $\mathrm{Rang}(f)$ *is unbounded in* L'.

2) We can combine this with 5.3.

Proof. Should be clear. $\square_{5.4}$

* * *

We have concentrated on real closed fields. This is justified by

Claim 5.5. *1) Assume* K *is an ordered field,* $\theta = \mathrm{cf}(K)$ *and* K *has no* $\{(\theta, \theta)\}$*-cut. Then* K *is real closed.*
2) In Theorem 5.3 if we add $(\mathrm{cf}(K), \mathrm{cf}(K)) \notin \Theta$ *and deal with ordered fields, it still holds.*

Claim 5.6. *1) If* F *is an ordered field of cardinality* $\mu > \aleph_0$ *then there is* F' such that *:*

(a) F' *is a real closed field of cardinality* λ
(b) F' *extends* F
(c) *if* C *is a Dedekind cut of* F' *of cofinality* (θ_1, θ_2) *then* $\theta_1 = \theta_2$
(d) *if* F'' *is another real satisfying (a),(b),(c) then* F' *can be embedded into* F' *over* F.

Proof. By 5.3 applied to $\Theta = \{(\theta_1, \theta_1) : \theta_1 \neq \theta_2$ are regular (so infinite)$\}$. $\square_{5.6}$

Observe

Claim 5.7. *Assume* $\langle K_\alpha : \alpha \leq \gamma \rangle$ *is increasing and* C *a cut of* K_γ *and let* $C_\alpha = C \restriction K_\alpha$ *for* $\alpha \leq \gamma$.
1) If $1 \in C^-$ *(or just* $C^- \cap (K_\alpha)_+ \neq \emptyset$ *then* $C_{\mathrm{add}} \restriction K_\alpha = (C_\alpha)_{\mathrm{add}}$.
2) If C *is an additive cut then each* C_α *is an additive cut and* $C_{\mathrm{add}} \restriction K_\alpha = (C_\alpha)_{\mathrm{add}}$.

Proof. Should be clear. $\square_{5.7}$

Claim 5.8. *If* $\bar{K} = \langle K_\alpha : \alpha \leq \alpha_* \rangle$ *is chains over* K_0 *then* $h(K_{\alpha_*}) \leq h(K_0) + \alpha_*$.

Proof. Implicite in §3. $\square_{5.8}$

Remark 5.9. Used in the end of §4.

So (see §0) we wonder

Question 5.10. For T dependent model which are $|T|^+$-saturated κ-saturated for types which does not split over sets of cardinality $\leq |T|$:

(a) what [She04a, §5] gives
(b) when do we have symmetric cuts? (see [She94, §2]).

Question 5.11. Similarly for o-minimal theory T.

§ 6. Symmetric cuts exist for BPA

We like in a model of, e.g. BPA to immitate [She94, §2], see 1.2 in [She04b]. The answer is essentially that it suffices to have a function f where $f(x)$ behaves like x^x.

We still have to sort out the beginning of the induction. If we ask only to have some countable model M_* of T such that "if $M_* \prec M$ then M has a symmetric cut", then this is O.K.

But it seems too much to ask for the existence of such function. So we may try to rework the proof using an almost function: there is a formula giving a convex set of possible values, and its non-existent infinum (i.e. exists in the completion) is as required.

Theorem 6.1. *The model N has a symmetric cut when:*

- (a) (α) *N is a model of T*
  - (β) *T is a first order theory extending the theory of ordered semi-rings (so have 0,1 order, addition and multipliation with the usual rules but $x - y$ does not necessarily exist), which may have additional symbols (e.g. x^x)*
- (b) *for some formula $\varphi(x,y) \in \mathbb{L}(\tau_T)$ the theory T implies:*
  - (α) $0 < x \to (\exists y)\varphi(x,y)$
  - (β) $\varphi(x,y) \to x < y$
  - (γ) $0 < x \to (\exists y)(\forall z)(\varphi(x,z) \leftrightarrow z \leq y)$
  - (δ) $0 < x_1 < x_2 \wedge \varphi(x_1,y_1) \wedge \varphi(x_2,y_2) \to x_1y_1 < y_2$
- (c) *there are $e_n \in N$ for $n \in \mathbb{N}$ such that for every $n,m \in \mathbb{N}$ we have: $\varphi(e_{n+1},x) \to x \leq e_n$ and $me_{n+1} < e_n$.*

Remark 6.2. 1) If $T = \text{BPA}$ then:

- (a) $\varphi(x,y) = (y = x^x)$ satisfies clause (b) of 6.1
- (b) if $N \models T$ is non-standard, T is as above, then there are non-standard e_n as in clause (c): choose e_0 arbitrarily, then e_{n+1} is the minimal x such that $x^{(x^x)} \geq e_n$.

Proof. So let T, φ be as in the assumptions of 6.1(2) and let N be a model of T. As satisfaction is all this proof is in N we omit $N \models$; and a,b,c,d will denote members of N; we adopt the notation:

$\boxplus_1$ (a) $a <_\varphi b$ means $(\forall x)(\varphi(a,x) \to x \leq b)$
  - (b) $a \ll b$ means $n \cdot a < b$ for every n.

By induction on the limit ordinal β we will choose elements $a_{\alpha,n}, b_{\alpha,n}$ in N for $n \in \mathbb{N}$ and $\alpha < \beta$ such that:

$\boxplus^2_\beta$ for all n and for all $\alpha_1 < \alpha_2 < \beta$ we have
  - (a) $1 < a_{\alpha_1,n} < a_{\alpha_2,n} < b_{\alpha_2,n} < b_{\alpha_1,n}$
  - (b) $b_{\alpha_1,n+1} <_\varphi a_{\alpha_1+1,n} - a_{\alpha_1,n}$.

Case 1: $\beta = \omega$

By clause (c) of the assumptions of Theorem 6.1 there are e_n for $n < \omega$ as there, let $d_n = e_{2n+1}, d^*_n = e_{2n}$, so d_n, d^*_n for $n < \omega$ are such that $d_{n+1} <_\varphi d^*_{n+1} \ll d_n$ for n.

We let $a_{i,n} = d_{n+1} + i \cdot d^*_{n+1}$ and $b_{i,n} = d_n - i - 1$. We should check that $\boxplus^2_\beta$ holds. Now clause (a) holds because: $a_{i,n} < a_{i+1,n}$ as $d^*_{n+1} > 0$ and $a_{i+1,n} - a_{i,n} = d^*_{n+1}$; and $a_{i,m} < b_{i,m}$ as $a_{i,m} = d_{m+1} + id^*_{m+1} \le (i+1)d^*_{m+1} < (i+1+i)d^*_{m+1} - i - 1 < d_m - i - 1 = b_{i,m}$; and lastly $b_{i+1,n} < b_{i,n}$ as $d_{n+i+1} < \ldots < d_n$ hence $i+1 < d_n$.

Also clause (b) of $\boxplus^2_\beta$ holds $b_{i,n+1} = d_{n+1} - i - 1 \le d_{n+1} <_\varphi d^*_{n+1} = a_{i+1,n} - a_{i,n}$.

Case 2: β is a limit ordinal such that $\gamma < \beta \Rightarrow \gamma + \omega < \beta$.

There is nothing to do.

Case 3: $\beta = \gamma + \omega, \beta$ a limit ordinal

So the elements $a_{\alpha,n}$ and $b_{\alpha,n}$ have been chosen for $\alpha < \gamma$ with γ a limit ordinal. Let A_n, B_n be the ranges of the sequences $a_{\alpha,n}, b_{\alpha,n}$ (for $\alpha < \gamma$) respectively so $A_n < B_n$, i.e. $(\forall a \in A_n)(\forall b \in B_n)(a <_N b)$. If one of the pairs (A_n, B_n) determines a cut in N, then it defines a cut as a desired (i.e. the cut $(\{x : (\exists y \in A_n)(x < y)\}, \{x : (\exists y \in B_n)(y < x\})$.

Assume therefore:

$\boxplus_3$ for all n there is an element c_n with $A_n < c_n < B_n$, i.e. $(\forall x \in A_n)(\forall y \in B_n)[x < c_n < y]$.

Under this assumption we will continue the construction by defining $a_{\gamma+i,n}$ and $b_{\gamma+i,n}$ for all finite i, n thus finishing this case too (hence the proof).

We set

$(*)_1$ (a) let $b^*_{\alpha,n}$ be such that $\varphi(b_{\alpha,n}, b^*_{\alpha,n})$ for $\alpha < \gamma, n < \omega$

(b) let c^*_n be such that $\varphi(c_n - 1, c^*_n)$

(c) let c^{**}_n be such that $\varphi(c_n, c^{**}_n)$

(d) $c'_n := c_n - c^{**}_{n+1}$.

Now we observe that

$(*)_2$ $A_n < c'_n$ (i.e. $(\forall x \in A_n)(x < c'_n)$).

[Why? Since for every $n \in \mathbb{N}$ and $\alpha < \gamma$ we have: $a_{\alpha,n} < a_{\alpha+1,n} - b^*_{\alpha,n+1} < a_{\alpha+1,n} - c^{**}_{n+1} < c_n - c^{**}_{n+1} = c'_n$.]

Why? First inequality by $\boxplus^2_\gamma(b)$ recalling $\boxplus_1(a)$. Second inequality as $b_{\alpha,n+1} > c_{n+1}$ by $\boxplus_3$ and clause $(b)(\delta)$ of the theorem (check). Third inequality holds as $a_{\alpha+1,n} < c_n$ by $\boxplus_3$. Fourth equality by the choice of c'_n.]

We set (for $i, n < \omega$):

$(*)_3$ (a) $a_{\gamma+i,n} := c'_n + i \cdot c^*_{n+1}$

(b) $b_{\gamma+i,n} := c'_n + c_{n+2} \cdot c^*_{n+1} - i$.

So we have to check the inductive demands, this means

Clause (a) of $\boxplus^2_{\gamma+1}$: This means that the following inequalities hold below $i, n \in \mathbb{N}$ and $\alpha < \gamma$:

$a_{\alpha,n} < a_{\gamma+i,n}$:

Why? As $a_{\alpha,n} < c'_n$ by $(*)_2$ and $c'_n < c'_n + i \cdot (c^*_{n+1})$ because $c^*_{n+1} > 1$ and $c'_n + i \cdot c^*_{n+1} = a_{\gamma+i,n}$ by the choice of the latter, see $(*)_3(a)$.

$a_{\gamma+i,n} < a_{\gamma+i+1,n}$:

Why? Trivial by $(*)_3(a)$ as $c^*_{n+1} > 0$.

$a_{\gamma+i,n} < b_{\gamma+i,n}$:

Why? By $(*)_3$ this means $i \cdot c^*_{n+1} < c_{n+2} \cdot c^*_{n+1} - i$ but $i < c_{n+2}$ hence $i \cdot c^*_{n+1} < (c_{n+2} - i) \cdot c^*_{n+1} = c_{n+2} \cdot c^*_{n+1} - i \cdot c^*_{n+1} < c_{n+2} \cdot c^*_{n+1} - i$ is as required.

$b_{\gamma+i+1,n} < b_{\gamma+i,n}$:

Why? Trivial by $(*)_3(b)$ because $i + 1 < c'_n$.

$b_{\gamma+i,n} < b_{\alpha,n}$:

Why? Note that $c_n < b_{\alpha,n}$ by $\boxplus_3$ as $b_{\alpha,n} \in B_n$ hence it suffices to prove $b_{\gamma+i,n} < c_n$, which by $(*)_3(b)$ means $c'_n + c_{n+2} \cdot c^*_{n+1} - i < c_n$. By the choice of c'_n in $(*)_1(d)$ this means $c_{n+2} \cdot c^*_{n+1} - i < c^{**}_{n+1}$ and as $c_{n+2} < (c_{n+1} - 1)$ it suffices to prove $(c_{n+1} - 1) \cdot c^*_{n+1} < c^{**}_{n+1}$ which holds by the choice of c^*_{n+1}, c^{**}_{n+1} in $(*)_2(b),(c)$ see Clause $(b)(\delta)$ of the claim.

Clause (b) of $\boxplus^2_{\gamma+n}$:

We have to show $b_{\gamma+i,n+1} <_\varphi a_{\gamma+i+1,n} - a_{\gamma+i,n}$.

So assume $\varphi(b_{\gamma+i,n+1}, b'_{\gamma+i,n+1})$ holds and we have to show $b'_{\gamma+i,n+1} \le a_{\gamma+i+1,n} - a_{\gamma+i,n}$. By clause $(*)_3(a)$ this means $b'_{\gamma+i,n+1} < c^*_{n+1}$. By the choice of $b'_{\gamma+i,n+1}$ above and of c^*_{n+1} in $(*)_1(b)$ it suffices to prove (that the model N satisfies) $\varphi(b_{\gamma+i,n+1}, y_1) \wedge \varphi(c_{n+1} - 1, y_2) \rightarrow y_1 < y_2$.

By clause $(b)(\delta)$ of the claim, it suffices to prove $b_{\gamma+i,n+1} < c_{n+1} - 1$ which by $(*)_3(b)$ means $c'_n + c_{n+2} \cdot c^*_{n+2} - i < c_{n+1}$ which by $(*)_1(d)$ means $c_{n+2} \cdot c^*_{n+1} - i < c^{**}_{n+1}$ which is proved above by $(b)(\delta)$ of the claim. $\square_{6.1}$

References

[KKS15] Franz-Viktor Kuhlmann, Katarzyna Kuhlmann, and Saharon Shelah, *Symmetrically complete ordered sets abelian groups and fields*, Israel J. Math. **208** (2015), no. 1, 261–290, arXiv: 1308.0780. MR 3416920

[MS16] Maryanthe Malliaris and Saharon Shelah, *Cofinality spectrum theorems in model theory, set theory, and general topology*, J. Amer. Math. Soc. **29** (2016), no. 1, 237–297, arXiv: 1208.5424. MR 3402699

[Sco69] Dana Scott, *On completing ordered fields*, Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967), Holt, Rinehart and Winston, New York, 1969, pp. 274–278.

[She94] Saharon Shelah, *Vive la différence. II. The Ax-Kochen isomorphism theorem*, Israel J. Math. **85** (1994), no. 1-3, 351–390, arXiv: math/9304207. MR 1264351

[She04a] ______, *Classification theory for elementary classes with the dependence property—a modest beginning*, Sci. Math. Jpn. **59** (2004), no. 2, 265–316, arXiv: math/0009056. MR 2062198

[She04b] ______, *Quite complete real closed fields*, Israel J. Math. **142** (2004), 261–272, arXiv: math/0112212. MR 2085719

[Wil96] A. J. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no. 4, 1051–1094.

Institute of Mathematics The Hebrew University of Jerusalem Jerusalem 91904, Israel and Department of Mathematics Rutgers University New Brunswick, NJ 08854, USA

Email address: shelah@math.huji.ac.il