On regular reduced products*

Juliette Kennedy †
Department of Mathematics
University of Helsinki
Helsinki, Finland

Saharon Shelah ‡
Institute of Mathematics
Hebrew University
Jerusalem, Israel

October 5, 2020

Abstract

Assume $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$. Assume M is a model of a first order theory T of cardinality at most λ^+ in a vocabulary $\mathcal{L}(T)$ of cardinality $\leq \lambda$. Let N be a model with the same vocabulary. Let Δ be a set of first order formulas in $\mathcal{L}(T)$ and let D be a regular filter on λ. Then M is Δ-embeddable into the reduced power N^λ/D, provided that every Δ-existential formula true in M is true also in N. We obtain the following corollary: for M as above and D a regular ultrafilter over λ, M^λ/D is λ^{++}-universal. Our second result is as follows: For $i < \mu$ let M_i and N_i be elementarily equivalent models of a vocabulary which has has cardinality $\leq \lambda$. Suppose D is a regular filter on μ and $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ holds. We show that then the second player has a winning strategy in the Ehrenfeucht-Fraisse game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$. This yields the following corollary: Assume GCH and λ regular (or just $\langle \aleph_0, \aleph_1 \rangle \rightarrow \langle \lambda, \lambda^+ \rangle$ and $2^\lambda = \lambda^+$). For L, M_i and N_i be as above, if D is a regular filter on λ, then $\prod_i M_i/D \cong \prod_i N_i/D$.

*This paper was written while the authors were guests of the Mittag-Leffler Institute, Djursholm, Sweden. The authors are grateful to the Institute for its support.

†Research partially supported by grant 1011049 of the Academy of Finland.

‡Research partially supported by the Binational Science Foundation. Publication number 769.

1 Introduction

Suppose M is a first order structure and F is the Frechet filter on ω. Then the reduced power M^ω/F is $\aleph_1$-saturated and hence $\aleph_2$-universal ([6]). This was generalized by Shelah in [10] to any filter F on ω for which B^ω/F is $\aleph_1$-saturated, where B is the two element Boolean algebra, and in [8] to all regular filters on ω. In the first part of this paper we use the combinatorial principle $\Box_\lambda^{b*}$ of Shelah [11] to generalize the result from ω to arbitrary λ, assuming $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. This gives a partial solution to Conjecture 19 in [3]: if D is a regular ultrafilter over λ, then for all infinite M, the ultrapower M^λ/D is λ^{++}-universal.

The second part of this paper addresses Problem 18 in [3], which asks if it is true that if D is a regular ultrafilter over λ, then for all elementarily equivalent models M and N of cardinality $\leq \lambda$ in a vocabulary of cardinality $\leq \lambda$, the ultrapowers M^λ/D and N^λ/D are isomorphic. Keisler [7] proved this for good D assuming $2^\lambda = \lambda^+$. Benda [1] weakened "good" to "contains a good filter". We prove the claim in full generality, assuming $2^\lambda = \lambda^+$ and $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$.

Regarding our assumption $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$, by Chang's Two-Cardinal Theorem ([2]) $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$ is a consequence of $\lambda = \lambda^{<\lambda}$. So our Theorem 2 settles Conjecture 19 of [3], and Theorem 13 settles Conjecture 18 of [3], under GCH for λ regular. For singular strong limit cardinals $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$ follows from $\Box_\lambda$ (Jensen [5]). In the so-called Mitchell's model ([9]) $\langle\aleph_0,\aleph_1\rangle \not\to \langle\aleph_1,\aleph_2\rangle$, so our assumption is independent of ZFC.

2 Universality

Definition 1 *Suppose Δ is a set of first order formulas of vocabulary L. The set of* Δ-existential *formulas is the set of formulas of the form*

$$\exists x_1 ... \exists x_n(\phi_1 \wedge ... \wedge \phi_n),$$

where each ϕ_i is in Δ. The set of weakly Δ-existential *formulas is the set of formulas of the above form, where each ϕ_i is in Δ or is the negation of a formula in Δ. If M and N are L-structures and $h : M \to N$, we say that h is a* Δ-homomorphism *if h preserves the truth of Δ-formulas. If h preserves also the truth of negations of Δ-formulas, it is called a* Δ-embedding.

Theorem 2 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. Let M be a model of a first order theory T of cardinality at most λ^+, in a language L of cardinality $\leq\lambda$ and let N be a model with the same vocabulary. Let Δ be a set of first order formulas in L and let D be a regular filter on λ. We assume that every weakly Δ-existential sentence true in M is true also in N. Then there is a Δ-embedding of M into the reduced power N^λ/D.*

By letting Δ be the set of all first order sentences, we get from Theorem 2 and Łoś' Lemma:

Corollary 3 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. If M is a model with vocabulary $\leq\lambda$, and D is a regular ultrafilter over λ, then M^λ/D is λ^{++}-universal, i.e. if M' is of cardinality $\leq\lambda^+$, and $M'\equiv M$, then M' is elementarily embeddable into the ultrapower M^λ/D.*

We can replace "weakly Δ-existential" by "Δ-existential" in the Theorem, if we only want a Δ-homomorphism.

The proof of Theorem 2 is an induction over λ and λ^+ respectively, as follows. Suppose $M=\{a_\zeta:\zeta<\lambda^+\}$. We associate to each $\zeta<\lambda^+$ finite sets u_i^ζ, $i<\lambda$, and represent the formula set Δ as a union of finite sets Δ_i. At stage i, for each $\zeta<\lambda^+$ we consider the Δ_i-type of the elements a_ζ of the model whose indices lie in the set u_i^ζ, $\zeta<\lambda^+$. This will yield a witness $f_\zeta(i)$ in N at stage i,ζ. Our embedding is then given by $a_\zeta\mapsto\langle f_\zeta(i):i<\lambda\rangle/D$.

We need first an important lemma, reminiscent of Proposition 5.1 in [11]:

Lemma 4 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. Let D be a regular filter on λ. There exist sets u_i^ζ and integers n_i for each $\zeta<\lambda^+$ and $i<\lambda$ such that for each i,ζ*

(i) $|u_i^\zeta|<n_i<\omega$

(ii) $u_i^\zeta\subseteq\zeta$

(iii) *Let B be a finite set of ordinals and let ζ be such that $B\subseteq\zeta<\lambda^+$. Then $\{i:B\subseteq u_i^\zeta\}\in D$*

(iv) *Coherency: $\gamma\in u_i^\zeta\Rightarrow u_i^\gamma=u_i^\zeta\cap\gamma$*

Assuming the lemma, and letting $M=\{a_\zeta:\zeta<\lambda^+\}$ we now define, for each ζ, a function $f_\zeta:\lambda\mapsto N$.

Let $\Delta = \{\phi_\alpha : \alpha < \lambda\}$ and let $\{A_\alpha : \alpha < \lambda\}$ be a family witnessing the regularity of D. Thus for each i, the set $w_i = \{\alpha : i \in A_\alpha\}$ is finite. Let $\Delta_i = \{\phi_\alpha : \alpha \in w_i\}$, and let u_i^ζ, n_i be as in the lemma.

We define a sequence of formulas essential to the proof: suppose $\zeta < \lambda^+$ and $i < \lambda$. Let $m_i^\zeta = |u_i^\zeta|$ and let

$$u_i^\zeta = \{\xi_{\zeta,i,0}, ..., \xi_{\zeta,i,m_i^\zeta-1}\}$$

be an increasing enumeration of u_i^ζ. (We adopt henceforth the convention that any enumeration of u_i^ζ that is given is an increasing enumeration.) Let $\bar{\theta}_i^\zeta$ be the Δ_i-type of the tuple $\langle a_{\xi_{\zeta,i,0}}, ..., a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta\rangle$ in M. (So every $\phi(x_{\xi_{\zeta,i,0}}, ..., x_{\xi_{\zeta,i,m_i^\zeta-1}}, x_\zeta) \in \Delta_i$ or its negation occurs as a conjunct of $\bar{\theta}_i^\zeta$, according to whether $\phi(a_{\xi_{\zeta,i,0}}, ..., a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta)$ or $\neg\phi(a_{\xi_{\zeta,i,0}}, ..., a_{\xi_{\zeta,i,m_i^\zeta-1}}, a_\zeta)$ holds in M.) We define the formula θ_i^ζ for each i by downward induction on m_i^ζ as follows:

Case 1: $m_i^\zeta = n_i$. Let $\theta_i^\zeta = \bigwedge \bar{\theta}_i^\zeta$.

Case 2: $m_i^\zeta < n_i$. Let θ_i^ζ be the conjunction of $\bar{\theta}_i^\zeta$ and all formulas of the form $\exists x_{m_i^\epsilon} \theta_i^\epsilon(x_0, ..., x_{m_i^\zeta-1}, x_{m_i^\epsilon})$, where ϵ satisfies $u_i^\epsilon = u_i^\zeta \cup \{\zeta\}$ and hence $m_i^\epsilon = m_i^\zeta + 1$.

An easy induction shows that for a fixed $i < \lambda$, the cardinality of the set $\{\theta_i^\zeta : \zeta < \lambda^+\}$ is finite, using n_i.

Let $i < \lambda$ be fixed. We define $f_\zeta(i)$ by induction on $\zeta < \lambda^+$ in such a way that the following condition remains valid:

(IH) If $\zeta^* < \zeta$ and $u_i^{\zeta^*} = \{r_{\epsilon_1}, ..., r_{\epsilon_k}\}$, then $N \models \theta_i^{\zeta^*}(f_{\epsilon_1}(i), ..., f_{\epsilon_k}(i), f_{\zeta^*}(i))$.

To define $f_\zeta(i)$, we consider different cases:

Case 1: $n_i = m_i^\zeta$.

Case 1.1: $n_i = 0$. Then θ_i^ζ is the Δ_i type of the element a_ζ. But then

$$\begin{aligned} M &\models \theta_i^\zeta(a_\zeta) \Rightarrow \\ M &\models \exists x_0 \theta_i^\zeta(x_0) \Rightarrow \\ N &\models \exists x_0 \theta_i^\zeta(x_0), \end{aligned}$$

where the last implication follows from the assumption that N satisfies the weakly Δ-existential formulas holding in M. Now choose an element $b \in N$ to witness this formula and set $f_\zeta(i) = b$.
Case 1.2: $n_i > 0$. Let $u_i^\zeta = \{\xi_1, \ldots, \xi_{m_i^\zeta}\}$. Since $m_i^\zeta = n_i$, the formula θ_i^ζ is the Δ_i-type of the elements $\{\xi_1, \ldots, \xi_{m_i^\zeta}\}$. By assumption $\gamma = \xi_{m_i^\zeta}$ is the maximum element of u_i^ζ. Thus by coherency, $u_i^\gamma = u_i^\zeta \cap \gamma = \{\xi_1, \ldots, \xi_{m_i^\zeta - 1}\}$. Since $\gamma < \zeta$, we know by the induction hypothesis that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta}}(i)).$$

By the formula construction θ_i^γ contains the formula $\exists x_{m_i^\zeta} \theta_i^\zeta(x_1, \ldots, x_{m_i^\zeta})$, since $u_i^\zeta = u_i^\gamma \cup \{\gamma\}$ and since $m_i^\gamma < n_i$. Thus

$$N \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta}}(i), x_{m_i^\zeta+1}).$$

As before choose an element $b \in N$ to witness this formula and set $f_\zeta(i) = b$.
Case 2: $m_i^\zeta < n_i$. Let $u_i^\zeta = \{\xi_1, \ldots, \xi_{m_i^\zeta}\}$. We have that $M \models \theta_i^\zeta(a_{\xi_1}, \ldots, a_{\xi_{m_i^\zeta}}, a_\zeta)$, and therefore $M \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(a_{\xi_1}, \ldots, a_{\xi_{m_i^\zeta}}, x_{m_i^\zeta+1})$. Let $\gamma = \max(u_i^\zeta) = \xi_{m_i^\zeta}$. By coherency, $u_i^\gamma = u_i^\zeta \cap \gamma$ and therefore since $\gamma < \zeta$ by the induction hypothesis we have that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta - 1}}(i), f_\gamma(i)).$$

But then as in case 1.2 we can infer that

$$N \models \exists x_{m_i^\zeta+1} \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta}}(i), x_{m_i^\zeta+1}).$$

As in case 1 choose an element $b \in N$ to witness this formula and set $f_\zeta(i) = b$.

It remains to be shown that the mapping $a_\zeta \mapsto \langle f_\zeta(i) : i < \lambda \rangle / D$ satisfies the requirements of the theorem, i.e. we must show, for all ϕ which is in Δ, or whose negation is in Δ,

$$M \models \phi(a_{\xi_1}, \ldots, a_{\xi_k}) \Rightarrow \{i : N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))\} \in D.$$

So let such a ϕ be given, and suppose $M \models \phi(a_{\xi_1}, \ldots, a_{\xi_k})$. Let $I_\phi = \{i : N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))\}$. We wish to show that $I_\phi \in D$. Let $\alpha < \lambda$ so that

ϕ is ϕ_α or its negation. It suffices to show that $A_\alpha \subseteq I_\phi$. Let $\zeta < \lambda^+$ be such that $\{\xi_1, ..., \xi_n\} \subseteq \zeta$. By Lemma 4 condition (iii), $\{i : \{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta\} \in D$. So it suffices to show

$$A_\alpha \cap \{i : \{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta\} \subseteq I_\phi.$$

Let $i \in A_\alpha$ such that $\{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta$. By the definition of θ_i^ζ we know that $N \models \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))$. But the Δ_i-type of the tuple $\langle \xi_1, \ldots, \xi_k \rangle$ occurs as a conjunct of θ_i^ζ, and therefore $N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))$ $\Box$

3 Proof of Lemma 4

We now prove Lemma 4. We first prove a weaker version in which the filter is not given in advance:

Lemma 5 *Assume $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$. There exist sets $\langle u_i^\zeta : \zeta < \lambda^+, i < cof(\lambda) \rangle$, integers n_i and a regular filter D on λ, generated by λ sets, such that (i)-(iv) of Lemma 4 hold.*

Proof. By [11, Proposition 5.1, p. 149] the assumption $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$ is equivalent to:

$\Box_\lambda^{b*}$: There is a λ^+-like linear order L, sets $\langle C_a^\zeta : a \in L, \zeta < cf(\lambda) \rangle$, equivalence relations $\langle E^\zeta : \zeta < cf(\lambda) \rangle$, and functions $\langle f_{a,b}^\zeta : \zeta < \lambda, a \in L, b \in L \rangle$ such that

(i) $\bigcup_\zeta C_a^\zeta = \{b : b <_L a\}$ (an increasing union in ζ).

(ii) If $b \in C_a^\zeta$, then $C_b^\zeta = \{c \in C_a^\zeta : c <_L b\}$.

(iii) E^ζ is an equivalence relation on L with $\leq \lambda$ equivalence classes.

(iv) If $\zeta < \xi < cf(\lambda)$, then E^ξ refines E^ζ.

(v) If $aE^\zeta b$, then $f_{a,b}^\zeta$ is an order-preserving one to one mapping from C_a^ζ onto C_b^ζ such that for $d \in C_a^\zeta, dE^\zeta f_{a,b}^\zeta(d)$.

(vi) If $\zeta < \xi < cf(\lambda)$ and $aE^\xi b$, then $f_{a,b}^\zeta \subseteq f_{a,b}^\xi$.

(vii) If $f_{a,b}^\zeta(a_1) = b_1$, then $f_{a_1,b_1}^\zeta \subseteq f_{a,b}^\zeta$.

(viii) If $a \in C_b^\zeta$ then $\neg E^\zeta(a, b)$.

This is not quite enough to prove Lemma 5, so we have to work a little more. Let

$$\Xi_\zeta = \{a/E^\zeta : a \in L\}.$$

We assume, for simplicity, that $\zeta \neq \xi$ implies $\Xi_\zeta \cap \Xi_\xi = \emptyset$. Define for $t_1, t_2 \in \Xi_\zeta$:

$$t_1 <_\zeta t_2 \iff (\exists a_1 \in t_1)(\exists a_2 \in t_2)(a_1 \in C^\zeta_{a_2}).$$

Proposition 6 *$\langle \Xi_\zeta, <_\zeta \rangle$ is a tree order with $cf(\lambda)$ as the set of levels.*

Proof. We need to show (a) $t_1 <_\zeta t_2 <_\zeta t_3$ implies $t_1 <_\zeta t_3$, and (b) $t_1 <_\zeta t_3$ and $t_2 <_\zeta t_3$ implies $t_1 <_\zeta t_2$ or $t_2 <_\zeta t_1$ or $t_1 = t_2$. For the first, $t_1 <_\zeta t_2$ implies there exists $a_1 \in t_1$ and $a_2 \in t_2$ such that $a_1 \in C^\zeta_{a_2}$. Similarly $t_2 <_\zeta t_3$ implies there exists $b_2 \in t_2$ and $b_3 \in t_3$ such that $b_2 \in C^\zeta_{b_3}$. Now $a_2 E^\zeta b_2$ and hence we have the order preserving map $f^\zeta_{a_2,b_2}$ from $C^\zeta_{a_2}$ onto $C^\zeta_{b_2}$. Recalling $a_1 \in C^\zeta_{a_2}$, let $f^\zeta_{a_2,b_2}(a_1) = b_1$. Then by (vi), $a_1 E^\zeta b_1$ and hence $b_1 \in t_1$. But then $b_1 \in C^\zeta_{b_2}$ implies $b_1 \in C^\zeta_{b_3}$, by coherence and the fact that $b_2 \in C^\zeta_{b_3}$. But then it follows that $t_1 <_\zeta t_3$.

Now assume $t_1 <_\zeta t_3$ and $t_2 <_\zeta t_3$. Let $a_1 \in t_1$ and $a_3 \in t_3$ be such that $a_1 \in C^\zeta_{a_3}$, and similarly let b_2 and b_3 be such that $b_2 \in C^\zeta_{b_3}$. $a_3 E^\zeta b_3$ implies we have the order preserving map $f^\zeta_{a_3,b_3}$ from $C^\zeta_{a_3}$ to $C^\zeta_{b_3}$. Letting $f^\zeta_{a_3,b_3}(a_1) = b_1$, we see that $b_1 \in C^\zeta_{b_3}$. If $b_1 <_L b_2$, then we have $C^\zeta_{b_2} = C^\zeta_{b_3} \cap \{c | c < b_2\}$ which implies $b_1 \in C^\zeta_{b_2}$, since, as $f^\zeta_{a_3,b_3}$ is order preserving, $b_1 <_L b_2$. Thus $t_1 <_\zeta t_2$. The case $b_2 <_L b_1$ is proved similarly, and $b_1 = b_2$ is trivial. $\square$

For $a <_L b$ let

$$\xi(a,b) = \min\{\zeta : a \in C^\zeta_b\}.$$

Denoting $\xi(a,b)$ by ξ, let

$$tp(a,b) = \langle a/E^\xi, b/E^\xi \rangle.$$

If $a_1 <_L \ldots <_L a_n$, let

$$tp(\langle a_1, \ldots, a_n \rangle) = \{\langle l, m, tp(a_l, a_m)\rangle | 1 \leq l < m \leq n\}$$

and

$$\Gamma = \{tp(\vec{a}) : \vec{a} \in {}^{<\omega}L\}.$$

For $t = tp(\vec{a}), \vec{a} \in {}^nL$ we use n_t to denote the length of $\vec{a}$.

Proposition 7 *If $a_0 <_L \dots <_L a_n$, then*

$$\max\{\xi(a_l, a_m) : 0 \le l < m \le n\} = \max\{\xi(a_l, a_n) : 0 \le l < n\}.$$

Proof. Clearly the right hand side is $\le$ the left hand side. To show the left hand side is $\le$ the right hand side, let $l < m < n$ be arbitrary. If $\xi(a_l, a_n) \le \xi(a_m, a_n)$, then $\xi(a_l, a_m) \le \xi(a_m, a_n)$. On the other hand, if $\xi(a_l, a_n) > \xi(a_m, a_n)$, then $\xi(a_l, a_m) \le \xi(a_l, a_n)$. In either case $\xi(a_l, a_m) \le \max\{\xi(a_k, a_n) : 0 \le k < n\}$. $\square$

Let us denote $\max\{\xi(a_l, a_n) : 0 \le l < n\}$ by $\xi(\vec{a})$. We define on Γ a two-place relation $\le_\Gamma$ as follows:

$$t_1 <_\Gamma t_2$$

if there exists a tuple $\langle a_0, \dots a_{n_{t_2}-1} \rangle$ realizing t_2 such that some subsequence of the tuple realizes t_1.

Clearly, $\langle \Gamma, \le_\Gamma \rangle$ is a directed partial order.

Proposition 8 *For $t \in \Gamma$, $t = tp(b_0, \dots b_{n-1})$ and $a \in L$, there exists at most one $k < n$ such that $b_k E^{\xi(b_0,\dots,b_{n-1})} a$.*

Proof. Let $\zeta = \xi(b_0, \dots, b_{n-1})$ and let $b_{k_1} \ne b_{k_2}$ be such that $b_{k_1} E^\zeta a$ and $b_{k_2} E^\zeta a$, $k_1, k_2 \le n-1$. Without loss of generality, assume $b_{k_1} < b_{k_2}$. Since E^ζ is an equivalence relation, $b_{k_2} E^\zeta b_{k_1}$ and thus we have an order preserving map $f^\zeta_{b_{k_2}, b_{k_1}}$ from $C^\zeta_{b_{k_2}}$ to $C^\zeta_{b_{k_1}}$. Also $b_{k_1} \in C^\zeta_{b_{k_2}}$, by the definition of ζ and by coherence, and therefore $f^\zeta_{b_{k_2}, b_{k_1}}(b_{k_1}) E^\zeta b_{k_1}$. But this contradicts (viii), since $f^\zeta_{b_{k_2}, b_{k_1}}(b_{k_1}) \in C^\zeta_{b_{k_1}}$. $\square$

Definition 9 *For $t \in \Gamma$, $t = tp(b_0, \dots b_{n-1})$ and $a \in L$ suppose there exists $k < n$ such that $b_k E^{\xi(b_0,\dots,b_{n-1})} a$. Then let $u^a_t = \{f^{\zeta(b_0,\dots,b_{n-1})}_{a, b_k}(b_l) : l < k\}$ Otherwise, let $u^a_t = \emptyset$.*

Finally, let D be the filter on Γ generated by the λ sets

$$\Gamma_{\ge t^*} = \{t \in \Gamma : t^* <_L t\}.$$

We can now see that the sets u^a_t, the numbers n_t and the filter D satisfy conditions (i)-(iv) of Lemma 4 with L instead of λ^+: Conditions (i) and (ii) are trivial in this case. Condition (iii) is verified as follows: Suppose B is finite. Let $a \in L$ be such that $(\forall x \in B)(x <_L a)$. Let $\vec{a}$ enumerate $B \cup \{a\}$ in increasing order and let $t^* = tp(\vec{a})$. Clearly

$$t \in \Gamma_{\geq t^*} \Rightarrow B \subseteq u^a_t.$$

Condition (iv) follows directly from Definition 9 and Proposition 8.

To get the Lemma on λ^+ we observe that since L is λ^+-like, we can assume that $\langle \lambda^+, < \rangle$ is a submodel of $\langle L, <_L \rangle$. Then we define $v^\alpha_t = u^\alpha_t \cap \{\beta : \beta < \alpha\}$. Conditions (i)-(iv) of Lemma 5 are still satisfied. Also having D a filter of Γ instead of λ is immaterial as $|\Gamma| = \lambda$. $\square$

Now back to the proof of Lemma 4. Suppose u^ζ_i, n_i and D are as in Lemma 5, and suppose D' is an arbitrary regular filter on λ. Let $\{A_\alpha : \alpha < \lambda\}$ be a family of sets witnessing the regularity of D', and let $\{Z_\alpha : \alpha < \lambda\}$ be the family generating D. We define a function $h : \lambda \to \lambda$ as follows. Suppose $i < \lambda$. Then let

$$h(i) \in \bigcap \{Z_\alpha | i \in A_\alpha\}.$$

Now define $v^\zeta_\alpha = u^\zeta_{h(\alpha)}$. Define also $n_\alpha = n_{h(\alpha)}$. Now the sets v^ζ_α and the numbers n_α satisfy the conditions of Lemma 4. $\square$

4 Is $\square^{b*}_\lambda$ needed for Lemma 5?

In this section we show that the conclusion of Lemma 5 (and hence of Lemma 4) implies $\square^{b*}_\lambda$ for singular strong limit λ. By [11, Theorem 2.3 and Remark 2.5], $\square^{b*}_\lambda$ is equivalent, for singular strong limit λ, to the following principle:

$\mathcal{S}_\lambda$: There are sets $\langle C^i_a : a < \lambda^+, i < cf(\lambda) \rangle$ such that

(i) If $i < j$, then $C^i_a \subseteq C^j_a$.

(ii) $\bigcup_i C^i_a = a$.

(iii) If $b \in C^i_a$, then $C^i_b = C^i_a \cap b$.

(iv) $\sup\{otp(C^i_a) : a < \lambda^+\} < \lambda$.

Thus it suffices to prove:

Proposition 10 *Suppose the sets u_i^ζ and the filter D are as given by Lemma 5 and λ is a limit cardinal. Then $\mathcal{S}_\lambda$ holds.*

Proof. Suppose $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ is a family of sets generating D. W.l.o.g., $\mathcal{A}$ is closed under finite intersections. Let λ be the union of the increasing sequence $\langle \lambda_\alpha : \alpha < cf(\lambda) \rangle$, where $\lambda_0 \geq \omega$. Let the sequence $\langle \Gamma_\alpha : \alpha < cf(\lambda) \rangle$ satisfy:

(a) $|\Gamma_\alpha| \leq \lambda_\alpha$

(b) Γ_α is continuously increasing in α with λ as union

(c) If $\beta_1, ..., \beta_n \in \Gamma_\alpha$, then there is $\gamma \in \Gamma_\alpha$ such that

$$A_\gamma = A_{\beta_1} \cap ... \cap A_{\beta_n}.$$

The sequence $\langle \Gamma_\alpha : \alpha < cf(\lambda) \rangle$ enables us to define a sequence that will witness $\mathcal{S}_\lambda$. For $\alpha < cf(\lambda)$ and $\zeta < \lambda^+$, let

$$V_\zeta^\alpha = \{\xi < \zeta : (\exists \gamma \in \Gamma_\alpha)(A_\gamma \subseteq \{i : \xi \in u_i^\zeta\})\}.$$

Lemma 11 *(1) $\langle V_\zeta^\alpha : \alpha < \lambda \rangle$ is a continuously increasing sequence of subsets of ζ, $|V_\zeta^\alpha| \leq \lambda_\alpha$, and $\bigcup \{V_\zeta^\alpha : \alpha < cf(\lambda)\} = \zeta$.*

(2) If $\xi \in V_\zeta^\alpha$, then $V_\xi^\alpha = V_\zeta^\alpha \cap \xi$.

Proof. (1) is a direct consequence of the definitions. (2) follows from the respective property of the sets u_i^ζ. □

Lemma 12 $\sup\{otp(V_\zeta^\alpha) : \zeta < \lambda^+\} \leq \lambda_\alpha^+$.

Proof. By the previous Lemma, $|V_\zeta^\alpha| \leq \lambda_\alpha$. Therefore $otp(V_\zeta^\alpha) < \lambda_\alpha^+$ and the claim follows. □

The proof of the proposition is complete: (i)-(iii) follows from Lemma 11, (iv) follows from Lemma 12 and the assumption that λ is a limit cardinal. □

More equivalent conditions for the case λ singular strong limit, D a regular ultrafilter on λ, are under preparation.

5 Ehrenfeucht-Fraïssé-games

Let M and N be two first order structures of the same vocabulary L. All vocabularies are assumed to be relational. The *Ehrenfeucht-Fraïssé-game of length* γ *of* M *and* N denoted by EFG_γ is defined as follows: There are two players called I and II. First I plays x_0 and then II plays y_0. After this I plays x_1, and II plays y_1, and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$ has been played and $\alpha < \gamma$, then I plays x_α after which II plays y_α. Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ has been played. The rules of the game say that both players have to play elements of $M \cup N$. Moreover, if I plays his x_β in M (N), then II has to play his y_β in N (M). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ determines a relation $\pi \subseteq M \times N$. Player II wins this round of the game if π is a partial isomorphism. Otherwise I wins. The notion of winning strategy is defined in the usual manner. We say that a player *wins* EFG_γ if he has a winning strategy in EFG_γ.

Note that if II has a winning strategy in EFG_γ on M and N, where M and N are of size $\leq |\gamma|$, then $M \cong N$.

Assume L is of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i are elementarily equivalent L-structures. Shelah proved in [12] that if D is a regular filter on λ, then Player II has a winning strategy in the game EFG_γ on $\prod_i M_i/D$ and $\prod_i N_i/D$ for each $\gamma < \lambda^+$. We show that under a stronger assumption, II has a winning strategy even in the game EFG_{λ^+}. This makes a big difference because, assuming the models M_i and N_i are of size $\leq \lambda^+$, $2^\lambda = \lambda^+$, and the models $\prod_i M_i/D$ and $\prod_i N_i/D$ are of size $\leq \lambda^+$. Then by the remark above, if II has a winning strategy in EFG_{λ^+}, the reduced powers are actually isomorphic. Hyttinen [4] proved this under the assumption that the filter is, in his terminology, semigood.

Theorem 13 *Assume* $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$*. Let* L *be a vocabulary of cardinality* $\leq \lambda$ *and for each* $i < \lambda$ *let* M_i *and* N_i *be two elementarily equivalent* L*-structures. If* D *is a regular filter on* λ*, then Player II has a winning strategy in the game* EFG_{λ^+} *on* $\prod_i M_i/D$ *and* $\prod_i N_i/D$*.*

Proof. We use Lemma 4. If $i < \lambda$, then, since M_i and N_i are elementarily equivalent, Player II has a winning strategy σ_i in the game EFG_{n_i} on M_i and N_i. We will use the set u_i^ζ to put these short winning strategies together into one long winning strategy.

A "good" position is a sequence $\langle (f_\zeta, g_\zeta) : \zeta < \xi \rangle$, where $\xi < \lambda^+$, and for all $\zeta < \xi$ we have $f_\zeta \in \prod_i M_i$, $g_\zeta \in \prod_i N_i$, and if $i < \lambda$, then $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\zeta \cup \{\zeta\} \rangle$ is a play according to σ_i.

Note that in a good position the equivalence classes of the functions f_ζ and g_ζ determine a partial isomorphism of the reduced products. The strategy of player II is to keep the position of the game "good", and thereby win the game. Suppose ξ rounds have been played and II has been able to keep the position "good". Then player I plays f_ξ. We show that player II can play g_ξ so that $\langle (f_\zeta, g_\zeta) : \zeta \leq \xi \rangle$ remains "good". Let $i < \lambda$. Let us look at $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\xi \rangle$. We know that this is a play according to the strategy σ_i and $|u_i^\xi| < n_i$. Thus we can play one more move in EF_{n_i} on M_i and N_i with player I playing $f_\xi(i)$. Let $g_\xi(i)$ be the answer of II in this game according to σ_i. The values $g_\xi(i)$, $i < \lambda$, constitute the function g_ξ. We have showed that II can maintain a "good" position. $\square$

Corollary 14 *Assume GCH and λ regular (or just $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$ and $2^\lambda = \lambda^+$). Let L be a vocabulary of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i be two elementarily equivalent L-structures. If D is a regular filter on λ, then $\prod_i M_i/D \cong \prod_i N_i/D$.*

References

[1] M. Benda, On reduced products and filters. Ann.Math.Logic 4 (1972), 1-29.

[2] C. C. Chang, , A note on the two cardinal problem, Proc. Amer. Math. Soc., 16, 1965, 1148–1155,

[3] C.C. Chang and J.Keisler, Model Theory, North-Holland.

[4] T. Hyttinen, On κ-complete reduced products, Arch. Math. Logic, Archive for Mathematical Logic, 31, 1992, 3, 193–199

[5] R. Jensen, The fine structure of the constructible hierarchy, With a section by Jack Silver, Ann. Math. Logic, 4, 1972, 229–308

[6] B. Jónsson and P. Olin, Almost direct products and saturation, Compositio Math., 20, 1968, 125–132

[7] J. Keisler, Ultraproducts and saturated models. Nederl.Akad.Wetensch. Proc. Ser. A 67 (=Indag. Math. 26) (1964), 178-186.

[8] J. Kennedy and S. Shelah, On embedding models of arithmetic of cardinality $\aleph_1$ into reduced powers, to appear.

[9] W. Mitchell, Aronszajn trees and the independence of the transfer property, Ann. Math. Logic, 5, 1972/73, 21–46

[10] S. Shelah, For what filters is every reduced product saturated?, Israel J. Math., 12, 1972, 23–31

[11] S. Shelah, "Gap 1" two-cardinal principles and the omitting types theorem for $L(Q)$. Israel Journal of Mathematics vol 65 no. 2, 1989, 133–152.

[12] S. Shelah, Classification theory and the number of non-isomorphic models, Second, North-Holland Publishing Co., Amsterdam, 1990, xxxiv+705