INTERPRETING GROUPS AND FIELDS IN SOME NONELEMENTARY CLASSES

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ABSTRACT. This paper is concerned with extensions of geometric stability theory to some nonelementary classes. We prove the following theorem:

Theorem. Let $\mathfrak{C}$ be a large homogeneous model of a stable diagram D. Let $p, q \in S_D(A)$, where p is quasiminimal and q unbounded. Let $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$. Suppose that there exists an integer $n < \omega$ such that

$$\dim(a_1 \dots a_n / A \cup C) = n,$$

for any independent $a_1, \dots, a_n \in P$ and finite subset $C \subseteq Q$, but

$$\dim(a_1 \dots a_n a_{n+1} / A \cup C) \leq n,$$

for some independent $a_1, \dots, a_n, a_{n+1} \in P$ and some finite subset $C \subseteq Q$.

Then $\mathfrak{C}$ interprets a group G which acts on the geometry P' obtained from P. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n = 1, 2, 3$ and

- If $n = 1$ then G is abelian and acts regularly on P'.
- If $n = 2$ the action of G on P' is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K.
- If $n = 3$ the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.

We prove a similar result for excellent classes.

0. INTRODUCTION

The fundamental theorem of projective geometry is a striking example of interplay between geometric and algebraic data: Let k and ℓ distinct lines of, say, the complex projective plane $\mathbb{P}^2(\mathbb{C})$, with ∞ their point of intersection. Choose two distinct points 0 and 1 on $k \setminus \{\infty\}$. We have the *Desarguesian property*: For any 2 pairs of distinct points (P_1, P_2) and (Q_1, Q_2) on $k \setminus \{\infty\}$, there is an automorphism σ of $\mathbb{P}^2(\mathbb{C})$ fixing ℓ pointwise, preserving k, such that $\sigma(P_i) = Q_i$, for $i = 1, 2$. But for some triples (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) on $k \setminus \{\infty\}$, this property fails. From this, it is possible to endow k with the structure of a division ring, and another geometric property garantees that it is a field. Model-theoretically, in the language of *points* (written $P, Q, \dots$), *lines* (written $\ell, k, \dots$), and an *incidence relation* $\in$, we have a saturated structure $\mathbb{P}^2(\mathbb{C})$, and two strongly minimal types $p(x) = \{x \in$

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$k\}$ and $q(x) = \{x \in \ell\}$. The Desarguesian property is equivalent to the following statement in *orthogonality calculus*, which is the area of model theory dealing with the independent relationship between types: p^2 is weakly orthogonal to q^ω, but p^3 is not almost orthogonal to q^ω (see the abstract gives another equivalent condition in terms of dimension). From this, we can define a division ring on k. Model theory then gives us more: strong minimality guarantees that it is an algebraically closed field, and further conditions that it has characteristic 0; it follows that it must be $\mathbb{C}$.

A central theorem of geometric stability, due to Hrushovski [Hr1] (extending Zilber [Zi]), is a generalisation of this result to the context of stable first order theory: Let $\mathfrak{C}$ be a large saturated model of a stable first order theory. Let $p, q \in S(A)$ be stationary and regular such that for some $n < \omega$ the type p^n is weakly orthogonal to q^ω but p^{n+1} is not almost orthogonal to q^ω. Then $n = 1, 2, 3$ and if $n = 1$ then $\mathfrak{C}$ interprets an abelian group and if $n = 2, 3$ then $\mathfrak{C}$ interprets an algebraically closed field. He further obtains a description of the action for $n = 1, 2, 3$ (see the abstract).

Geometric stability theory is a branch of first order model theory that grew out of Shelah's classification theory [Sh]; it began with the discovery by Zilber and Hrushovski that certain model-theoretic problems (finite axiomatisability of totally categorical first order theories [Zi], existence of strictly stable unidimensionaly first order theories [Hr2]) imposed abstract (geometric) model-theoretic conditions implying the existence of definable classical groups. The structure of these groups was then invoked to solve the problems. Geometric stability theory has now developed into a sophisticated body of techniques which have found remarkable applications both within model theory (see [Pi] and [Bu]) and in other areas of mathematics (see for example the surveys [Hr3] and [Hr4]). However, its applicability is limited at present to mathematical contexts which are first order axiomatisable. In order to extend the scope of these techniques, it is necessary to develop geometric stability theory beyond first order logic. In this paper, we generalise Hrushovski's result to two non first order settings: homogeneous model theory and excellent classes.

Homogeneous model theory was initiated by Shelah [Sh3], it consists of studying the class of elementary submodels of a large homogeneous, rather than saturated, model. Homogeneous model theory is very well-behaved, with a good notion of stability [Sh3], [Sh54], [Hy1], [GrLe], superstability [HySh], [HyLe1], ω-stability [Le1], [Le2], and even simplicity [BuLe]. Its scope of applicability is very broad, as many natural model-theoretic constructions fit within its framework: first order, Robinson theories, existentially closed models, Banach space model theory, many generic constructions, classes of models with set-amalgamation (L^n, infinitary), as well as many classical non-first order mathematical objects like free groups or Hilbert spaces. We will consider the stable case (but note that this context may be unstable from a first order standpoint), *without* assuming simplicity, *i.e.* without assuming that there is a dependence relation with all the properties of forking in the first order stable case. (This contrasts with the work of Berenstein [Be], who carries out some group constructions under the assumption of stability, simplicity, and the existence of canonical bases.)

Excellence is a property discovered by Shelah [Sh87a] and [Sh87b] in his work on categoricity for nonelementary classes: For example, he proved that, under GCH, a sentence in $L_{\omega_1,\omega}$ which is categorical in all uncountable cardinals is excellent. On the other hand, excellence is central in the classification of almost-free algebras [MeSh] and also arises naturally in Zilber's work around complex exponentiation [Zi1] and [Zi2] (the structure $(\mathbb{C}, \exp)$ has intractable first order theory since it interprets the integers, but is manageable in an infinitary sense). Excellence is a condition on the existence of prime models over certain countable sets (under an ω-stability assumption). Classification theory for excellent classes is quite developed; we have a good understanding of categoricity ([Sh87a], [Sh87b], and [Le3] for a Baldwin-Lachlan proof), and Grossberg and Hart proved the Main Gap [GrHa]. Excellence follows from uncountable categoricity in the context of homogeneous model theory. However, excellence is at present restricted to ω-stability (see [Sh87a] for the definition), so excellent classes and stable homogeneous model theory, though related, are not comparable.

In both contexts, we lose compactness and saturation, which leads us to use various forms of homogeneity instead (model-homogeneity and only ω-homogeneity in the case of excellent classes). Forking is replaced by the appropriate dependence relation, keeping in mind that not all properties of forking hold at this level (for example extension and symmetry may fail over certain sets). Finally, we have to do without canonical bases.

Each context comes with a notion of monster model $\mathfrak{C}$ (homogeneous or full), which functions as a universal domain; all relevant realisable types are realised in $\mathfrak{C}$, and models may be assumed to be submodels of $\mathfrak{C}$. We consider a *quasiminimal* type p, *i.e.* every definable subset of its set of realisations in $\mathfrak{C}$ is either bounded or has bounded complement. Quasiminimal types are a generalisation of strongly minimal types in the first order case, and play a similar role, for example in Baldwin-Lachlan theorems. We introduce the natural closure operator on the subsets of $\mathfrak{C}$; it induces a pregeometry and a notion of dimension $\dim(\cdot/C)$ on the set of realisations of p, for any $C \subseteq \mathfrak{C}$. We prove:

Theorem 0.1. *Let $\mathfrak{C}$ be a large homogeneous stable model or a large full model in the excellent case. Let p, q be complete types over a finite set A, with p quasiminimal. Assume that there exists $n < \omega$ such that*

(1) *For any independent sequence $(a_0, \dots, a_{n-1})$ of realisations of p and any countable set C of realisations of q we have*

$$\dim(a_0, \dots, a_{n-1}/A \cup C) = n.$$

(2) *For some independent sequence* $(a_0, \dots, a_{n-1}, a_n)$ *of realisations of* p *there is a countable set* C *of realisations of* q *such that*

$$\dim(a_0, \dots, a_{n-1}, a_n / A \cup C) \leq n.$$

Then $\mathfrak{C}$ *interprets a group* G *which acts on the geometry* P' *induced on the realisations of* p*. Furthermore, either* $\mathfrak{C}$ *interprets a non-classical group, or* $n = 1, 2, 3$ *and*

- *If* $n = 1$*, then* G *is abelian and acts regularly on* P'*;*
- *If* $n = 2$*, the action of* G *on* P' *is isomorphic to the affine action of* $K^+ \rtimes K^*$ *on the algebraically closed field* K*.*
- *If* $n = 3$*, the action of* G *on* P' *is isomorphic to the action of* $\mathrm{PGL}_2(K)$ *on the projective line* $\mathbb{P}^1(K)$ *of the algebraically closed field* K*.*

As mentioned before, the phrasing in terms of dimension theory is equivalent to the statement in orthogonality calculus in Hrushovski's theorem. The main difference with the first order result is the appearance of the so-called *non-classical groups*, which are nonabelian ω-homogeneous groups carrying a pregeometry. In the first order case, it follows from Reineke's theorem [Re] that such groups cannot exist. Another difference is that in the interpretation, we must use invariance rather than definability; since we have some homogeneity in our contexts, invariant sets are definable in infinitary logic (in the excellent case, for example, they are type-definable).

The paper is divided into four sections. The first two sections are group-theoretic and, although motivated by model theory, contain none. The first section is concerned with generalising classical theorems on strongly minimal saturated groups and fields. We consider groups and fields whose universe carries an ω-homogeneous pregeometry. We introduce generic elements and ranks, but make no stability assumption. We obtain a lot of information on the structure of non-classical groups, for example they are not solvable, their center is 0-dimensional, and the quotient with the center is divisible and torsion-free. Nonclassical groups are very complicated; in addition to the properties above, any two nonidentity elements of the quotient with the center are conjugate. Fields carrying an ω-homogeneous pregeometry are more amenable; as in the first order case, we can show that they are algebraically closed.

In the second section, we generalise the theory of groups acting on strongly minimal sets. We consider groups G *n-acting* on a pregeometry P, *i.e.* the action of the group G respects the pregeometry, and further (1) the integer n is maximal such that for each pair of independent n-tuples of the pregeometry P, there exists an element of the group G sending one n-tuple to the other, and (2) two elements of the group G whose actions agree on an $(n+1)$-dimensional set are identical. As a nontriviality condition, we require that this action must be ω-homogeneous (in

[Hy2] Hyttinen considered this context under a stronger assumption of homogeneity, but in order to apply the results to excellent classes we must weaken it). We are able to obtain a picture very similar to the classical first order case. We prove (see the section for precise definitions):

Theorem 0.2. *Suppose G n-acts on a geometry P'. If G admits hereditarily unique generics with respect to the automorphism group Σ, then either there is an A-invariant non-classical unbounded subgroup of G (for some finite $A \subseteq P'$), or $n = 1, 2, 3$ and*

- *If $n = 1$ then G is abelian and acts regularly on P'.*
- *If $n = 2$ the action of G on P' is isomorphic to the affine action of $K \ltimes K^*$ on the algebraically closed field K.*
- *If $n = 3$ the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1$ of the algebraically closed field K.*

The last two sections are completely model-theoretic. In the third section, we consider the case of stable homogeneous model theory, and in the fourth the excellent case. In each case, the group we interpret is based on the automorphism group of the monster model $\mathfrak{C}$: Let p, q be unbounded types, say over a finite set A, and assume that p is quasiminimal. Let $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$. Bounded closure induces a pregeometry on P and we let P' be its associated geometry. In the stable homogeneous case, the group we interpret is the group of permutations of P' induced by automorphisms of $\mathfrak{C}$ fixing $A \cup Q$ pointwise. However, in the excellent case, we may not have enough homogeneity to carry this out. To remedy this, we consider the group G of permutations of P' which agree *locally* with automorphisms of $\mathfrak{C}$, *i.e.* a permutation g of P' is in G if for any finite $X \subseteq P$ and countable $C \subseteq Q$, there is an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that the permutation of P' induced by σ agrees with g on X. In each case, we show that the group n-acts on the geometry P' in the sense of Section 2. The interpretation in $\mathfrak{C}$ follows from the n-action.

Although the construction we provide for excellent classes works for the stable homogeneous case also, for expositional reasons we present the construction with the obvious group in the homogeneous case first, and then present the modifications with the less obvious group in the excellent case.

To apply Theorem 0.2 to G and obtain Theorem 0.1, it remains to show that G admits hereditarily unique generics with respect to some group of automorphisms Σ. For this, we deal with an invariant (and interpretable) subgroup of G, the connected component, and deal with the group of automorphisms Σ induced by the *strong automorphisms*, *i.e.* automorphisms preserving Lascar strong types. Hyttinen and Shelah introduced Lascar strong types for the stable homogeneous case in [HySh]; this is done without stability by Buechler and Lessmann in [BuLe]. In the excellent case, this is done in detail in [HyLe2]; we only use the results over finite sets.

1. Groups and fields carrying a homogeneous pregeometry

In this section, we study algebraic structures carrying an ω-homogeneous pregeometry. It is similar to the definition from [Hy2], except that the homogeneity requirement is weaker.

Definition 1.1. An infinite model M *carries an ω-homogeneous pregeometry* if there exists an invariant closure operator

$$\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M),$$

satisfying the axioms of a pregeometry with $\dim(M) = \|M\|$, and such that whenever $A \subseteq M$ is finite and $a, b \notin \mathrm{cl}(A)$, then there is an automorphism of M preserving cl, fixing A pointwise, and sending a to b.

Remark 1.2. In model-theoretic applications, the model M is generally uncountable, and $|\mathrm{cl}(A)| < \|M\|$, when A is finite. Furthermore, if $a, b \notin \mathrm{cl}(A)$ and $|A| < \|M\|$ one can often find an automorphism of M fixing $\mathrm{cl}(A)$ pointwise, and not just A. However, we find this phrasing more natural and in non first order contexts like excellence, ω_1-homogeneity may fail.

Strongly minimal $\aleph_0$-saturated groups are the simplest example of groups carrying an ω-homogeneous pregeometry. In this case, Reineke's famous theorem [Re] asserts that it must be abelian. Groups whose universe is a regular type are also of this form, and when the ambient theory is stable, Poizat [Po] showed that they are also abelian. We are going to consider generalisations of these theorems, but first, we need to remind the reader of some terminology.

Fix an infinite model M and assume that it carries an ω-homogeneous pregeometry. Following model-theoretic terminology, we will say that a set Z is A-*invariant*, where A and Z are subsets of the model M, if any automorphism of M fixing A pointwise, fixes Z setwise. In particular, if $f : M^m \to M^n$ is A-invariant and σ is an automorphism of M fixing A pointwise, then $f(\sigma(\bar{a})) = \sigma(f(\bar{a}))$, for any $\bar{a} \in M^m$. We use the term *bounded* to mean of size less than $\|M\|$.

The ω-homogeneity requirement has strong consequences. Obviously, any model carries the trivial pregeometry, but it is rarely ω-homogeneous; for example no group can carry a trivial ω-homogeneous pregeometry. We list a few consequences of ω-homogeneity which will be used repeatedly. First, if Z is A-invariant, for some finite A, then either Z or $G \setminus Z$ is contained in $\mathrm{cl}(A)$ and hence has finite dimension (if not, choose $x, y \notin \mathrm{cl}(A)$, such that $x \in Z$ and $y \notin Z$; then some automorphism of M fixing A sends x to y, contradicting the invariance of Z). Hence, if Z is an A-invariant set, for some finite A, and has bounded dimension, then $Z \subseteq \mathrm{cl}(A)$. It follows that if a has bounded orbit under the automorphisms of M fixing the finite set A, then $a \in \mathrm{cl}(A)$. This observation has the following consequence:

Lemma 1.3. *Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite. Let $f : M^n \to M^m$ be an A-invariant function. Then, for each $\bar{a} \in M^n$ we have* $\dim(f(\bar{a})/A) \leq \dim(\bar{a}/A)$.

Proof. Write $f = (f_0, \ldots, f_m)$ with A-invariant $f_i : M^n \to M$, for $i < m$. Let $\bar{a} \in M^n$. If $\dim(f(\bar{a}/A) > \dim(\bar{a}/A)$, then there is $i < m$ such that $f_i(\bar{a}) \notin \mathrm{cl}(\bar{a}A)$. But this is impossible since any automorphism M fixing $A\bar{a}$ pointwise fixes $f_i(\bar{a})$. □

We now introduce generic tuples.

Definition 1.4. Suppose that M carries an ω-homogeneous pregeometry. A tuple $\bar{a} \in M^n$ is said to be *generic over* A, for $A \subseteq M$, if $\dim(\bar{a}/A) = n$.

Since M is infinite dimensional, for any finite $A \subseteq M$ and any $n < \omega$, there exists a generic $\bar{a} \in M^n$ over A. Further, by ω-homogeneity, if $\bar{a}, \bar{b} \in M^n$ are both generic over the finite set A, then $\bar{a}$ and $\bar{b}$ are automorphic over A. This leads immediately to a proof of the following lemma.

Lemma 1.5. *Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite and let Z be an A-invariant subset of M^n. If Z contains a generic tuple over A, then Z contains all generic tuples over A.*

We now establish a few more lemmas in case when M is a group $(G, \cdot)$. Generic elements are particularly useful here. For example, let $\bar{a} = (a_0, \ldots, a_{n-1})$ and $\bar{b} = (b_0, \ldots, b_{n-1})$ belong to G^n. If $\bar{a}$ is generic over $A \cup \{b_0, \ldots, b_{n-1}\}$, then $(a_0 \dot{b}_0, \ldots, a_{n-1} b_{n-1})$ is generic over A. (This follows immediately from Lemma 1.3.) When $n = 1$, the next lemma asserts that if H is a proper A-invariant subgroup of G (A finite), then $H \subseteq \mathrm{cl}(A)$.

Lemma 1.6. *Let G be a group carrying an ω-homogeneous pregeometry. Suppose that H is an A-invariant subgroup of G^n (with A finite and $n < \omega$). If H contains a generic tuple over A, then $H = G^n$.*

Proof. Let $(g_0, \ldots, g_{n-1}) \in G^n$. By the previous lemma, H contains a generic tuple $(a_0, \ldots, a_{n-1})$ over $A \cup \{g_0, \ldots, g_{n-1}\}$. Then $(a_0 \cdot g_0, \ldots, a_{n-1} \cdot g_{n-1})$ is also generic over A and therefore belongs to H by another application of the previous lemma. It follows that $(g_0, \ldots, g_{n-1}) \in H$. □

The previous lemma implies that groups carrying an ω-homogeneous pregeometry are *connected* (see the next definition).

Definition 1.7. A group G is *connected* if it has no proper subgroup of bounded index which is invariant over a finite set.

We now introduce the *rank* of an invariant set.

Definition 1.8. Suppose that M carries an ω-homogeneous pregeometry. Let $A \subseteq M$ be finite and let Z be an A-invariant subset of M^n. The *rank of Z over A*, written $rk(Z)$, is the largest $m \leq n$ such that there is $\bar{a} \in Z$ with $\dim(\bar{a}/A) = m$.

Notice that if Z is A-invariant and if B contains A is finite, then the rank of Z over A is equal to the rank of Z over B. We will therefore omit the parameters A. The next lemma is interesting also in the case where $n = 1$; it implies that any invariant homomorphism of G is either trivial or onto.

Lemma 1.9. *Let G be a group carrying an ω-homogeneous pregeometry. Let $f : G^n \to G^n$ be an A-invariant homomorphism, for $A \subseteq G$ finite. Then*

$$rk(\ker(f)) + rk(\operatorname{ran}(f)) = n.$$

Proof. Let $k \leq n$ such that $rk(\ker(f)) = k$. Fix $\bar{a} = (a_0, \dots, a_{n-1}) \in \ker(f)$ be of dimension k over A. By a permutation, we may assume that $(a_0, \dots, a_{k-1})$ is independent over A.

Notice that by ω-homogeneity and A-invariance of $\ker(f)$, for each generic $(a'_0, \dots, a'_{i-1})$ over A (for $i < k$), there exists $(b_0, \dots, b_{n-1}) \in \ker(f)$ such that $b_i = a'_i$ for $i < k$. We now claim that for any $i < k$ and any $b \notin \operatorname{cl}(A)$, there is $\bar{b} = (b_0, \dots, b_{n-1}) \in \ker(f)$ such that $b_j = 1$ for $j < i$ and $b_i = b$. To see this, notice that $(a_0^{-1}, \dots, a_{i-1}^{-1})$ is generic over A (by Lemma 1.3). Choose $c \in G$ generic over $A\bar{a}$. Then there is $(d_0, \dots, d_{n-1}) \in \ker(f)$ such that $d_j = a_j^{-1}$ for $j < i$ and $d_i = c$. Let $(e_0, \dots, e_{n-1}) \in \ker(f)$ be the product of $\bar{a}$ with $(d_0, \dots, d_{n-1})$. Then $e_j = 1$ if $j < i$ and $e_i = a_i \cdot c^{-1} \notin \operatorname{cl}(A)$. By ω-homogeneity, there is an automorphism of G fixing A sending e_i to b. The image of $(e_0, \dots, e_{n-1})$ under this automorphism is the desired $\bar{b}$.

We now show that $rk(\operatorname{ran}(f)) \leq n - k$. Let $\bar{d} = f(\bar{c})$. Observe that by multiplying $\bar{c} = (c_0, \dots, c_{n-1})$ by appropriate elements in $\ker(f)$, we may assume that $c_i \in \operatorname{cl}(A)$ for each $i < k$. Hence $\dim(\bar{c}/A) \leq n-k$ so the conclusion follows from Lemma 1.3.

To see that $rk(\operatorname{ran}(f)) \geq n - k$, choose $\bar{c} \in G^n$ such that $c_i = 1$ for $i < k$ and $(c_k, \dots, c_{n-1})$ is generic over A. It is enough to show that $\dim(f(\bar{c})/A) \geq \dim(\bar{c}/A)$. Suppose, for a contradiction, that $\dim(f(\bar{c})/A) < \dim(\bar{c}/A)$. Then there is $i < n$, with $k \leq i$ such that $c_i \notin \operatorname{cl}(f(\bar{c})A)$. Let $d \in G \setminus \operatorname{cl}(Af(\bar{c})\bar{c})$ and choose an automorphism σ fixing $Af(\bar{c})$ such that $\sigma(c_i) = d$. Let $\bar{d} = \sigma(\bar{c})$. Then

$$f(\bar{d}) = f(\sigma(\bar{c})) = \sigma(f(\bar{c})) = f(\bar{c}).$$

Let $\bar{e} = (e_0, \dots, e_{n-1}) = \bar{c} \cdot \bar{d}^{-1}$. Then $\bar{e} \in \ker(f)$, $e_j = 1$ for $j < k$, and $e_i = c_i \cdot d^{-1} \notin \operatorname{cl}(A)$. By ω-homogeneity, we may assume that $e_i \notin \operatorname{cl}(A\bar{a})$. But $\bar{a} \cdot \bar{e} \in \ker(f)$, and $\dim(\bar{a} \cdot \bar{e}/A) \geq k + 1$ (since the i-th coordinate of $\bar{a} \cdot \bar{e}$ is not in $\operatorname{cl}(a_0, \dots, a_{k-1}A)$). This contradicts the assumption that $rk(\ker(f)) = k$. $\square$

The next theorem is obtained by adapting Reineke's proof to our context. For expository purposes, we sketch some of the proof and refer to reader to [Hy2] for details. We are unable to conclude that groups carrying an ω-homogeneous pregeometry are abelian, but we can still obtain a lot of information.

Theorem 1.10. *Let G be a nonabelian group which carries an ω-homogeneous pregeometry. Then the center $Z(G)$ has dimension 0, G is not solvable, any two nonidentity elements in the quotient group $G/Z(G)$ are conjugate, and $G/Z(G)$ is torsion-free and divisible. Also G contains a free subgroup on $\dim(G)$ many generators, and the first order theory of G is unstable.*

Proof. If G is not abelian, then the center of G, written $Z(G)$, is a proper subgroup of G. Since $Z(G)$ is invariant, Lemma 1.6 implies that $Z(G) \subseteq \mathrm{cl}(\emptyset)$.

We now claim that if H is an A-invariant proper normal subgroup of G then $H \subseteq Z(G)$.

By the previous lemma, H is finite dimensional. For $g, h \in H$, define $X_{g,h} = \{x \in G : g^x = h\}$. Suppose, for a contradiction, that $H \not\subseteq Z(G)$ and choose $h_0 \in H \setminus Z(G)$. If for each $h \in H$, the set $X_{h_0,h}$ is finite dimensional, then $X_{h_0,h_1} \subseteq \mathrm{cl}(h_0 h)$, and so $G \subseteq \bigcup_{h \in H} X_{h_0,h} \subseteq \mathrm{cl}(H)$, which is impossible since H has finite dimension. Hence, there is $h_1 \in H$, such X_{h_0,h_1} is infinite dimensional and has finite-dimensional complement. Similarly, there is $h_2 \in H$ such that X_{h_1,h_2} has finite dimensional complement. This allows us to choose $a, b \in G$ such that a, b, ab belong to both X_{h_0,h_1} and X_{h_1,h_2}. Then, $h_1 = h_0^{ab} = (h_0^b)^a = h_2$. This implies that the centraliser of h_1 has infinite dimension (since it is X_{h_1,h_2}) and must therefore be all of G by the first paragraph of this proof. Thus $h_1 \in Z(G)$, which is impossible, since it is the conjugate of h_0 which is not in $Z(G)$.

We now claim that $G^* = G/Z(G)$ is not abelian. Suppose, for a contradiction, that G^* is abelian. Let $a \in G \setminus Z(G)$. Then the sets $X_{a,k} = \{b \in G : a^b = ak\}$, where $k \in Z(G)$ form a partition of G, and so, as above, there is $k_a \in Z(G)$ such that X_{a,k_a} has finite-dimensional complement. Now notice that $k_a = 1$: Otherwise, for $c \in X_{a,k_a}$, the infinite dimensional sets cX_{a,k_a} and X_{a,k_a} are disjoint, which is impossible, since they each have finite-dimensional complements. But then, $X_{a,1}$ is a subgroup of G of infinite dimension and so is equal to G, which implies that $a \in Z(G)$, a contradiction.

Since $G/[G,G]$ is abelian, and $[G,G]$ is normal and invariant, then it cannot be proper (otherwise $[G,G] \leq Z(G)$). It follows that G is not solvable.

It follows easily from the previous claims that G^* is centerless. We now show that any two nonidentity elements in G^* are conjugate: Let $a^* \in G^*$ be a nonidentity element. Since G^* is centerless, the centraliser of a^* in G^* is a proper subgroup of G^*. Hence, the inverse image of this centraliser under the canonical homomorphism induces a proper subgroup of G, which must therefore be of finite dimension. Hence, the set of conjugates of a^* in G^* is all of G^*, except for a set of finite dimension. It then follows that the set of elements of G^* which are not conjugates of a^* must have bounded dimension.

Since this holds for any nonidentity $b^* \in G^*$, this implies that any two nonidentity elements of G^* must be conjugates. The instability of $Th(G)$ now follows as in the proof of [Po]: Since $G/Z(G)$ is not abelian and any two nonidentity elements of it are conjugate, we can construct an infinite strictly ascending chain of centralisers. This contradicts first order stability.

That G is torsion free and divisible is proved similarly (see [Hy2] for details). Finally, it is easy to check that any independent subset of G must generate a free group. $\square$

Hyttinen called such groups *bad* in [Hy2], but this conflicts with a standard notion, so we re-baptise them:

Definition 1.11. We say that a group G is *non-classical* if it is nonabelian and carries an ω-homogeneous pregeometry.

Question 1.12. Are there non-classical groups? And if there are, can they arise in the model-theoretic contexts we consider in this paper?

We now turn to fields. Here, we are able to adapt the proof of Macintyre's classical theorem [Ma] that ω-stable fields are algebraically closed.

Theorem 1.13. *A field carrying an ω-homogeneous pregeometry is algebraically closed.*

Proof. To show that F is algebraically closed, it is enough to show that any finite dimensional field extension K of F is perfect, and has no Artin-Schreier or Kummer extension.

Let K be a field extension of F of finite degree $m < \omega$. Let $P \in F[X]$ be an irreducible polynomial of degree m such that $K = F(\xi)$, where $P(\xi) = 0$. Let A be the finite subset of F consisting of the coefficients of P. We can represent K in F as follows: K^+ is the vector space F^m, *i.e.* $a = a_0 + a_1\xi + \dots a_{m-1}\xi^{m-1}$ is represented as $(a_0, \dots, a_{m-1})$. We can then easily represent addition in K and multiplication (the field product in K induces a bilinear form on $(F^+)^m$) as A-invariant operations. Notice that an automorphism σ of F fixing A pointwise induces an automorphism of K, via

$$(a_0, \dots, a_{m-1}) \mapsto (\sigma(a_0), \dots, \sigma(a_{m-1})).$$

We now consider generic elements of the field. For a finite subset $X \subseteq F$ containing A, we say that an $a \in K$ is *generic over* X if $\dim(a_0 \dots a_{m-1}/X) = m$ (that is $(a_0, \dots, a_{m-1})$ is generic over X), where $a_i \in F$ and $a = a_0 + a_1\xi +$

$\dots a_{m-1}\xi^{m-1}$. Notice that if $a, b \in K$ are generic over X (with $X \subseteq F$ finite containing A) then there exists an automorphism of K fixing X sending a to b. We prove two claims about generic elements.

Claim 1.14. *Assume that $a \in K$ is generic over the finite set X, with $A \subseteq X \subseteq F$. Then a^n, $a^n - a$ for $n < \omega$, as well as $a+b$ and ab for $b = b_0 + b_1\xi + \dots b_{m-1}\xi^{m-1}$, $b_i \in X$ ($i < m$) are also generic over X.*

Proof of the claim. We prove that a^n is generic over X. The other proofs are similar. Suppose, for a contradiction, that $a^n = c_0 + c_1\xi + \dots + c_{m-1}\xi^{m-1}$ and $\dim(c_0 \dots c_{m-1}/X) < m$. Then $\dim(a_0 \dots a_{m-1}/Xc_0 \dots c_{m-1}) \geq 1$, so there is $a_i \notin \mathrm{cl}(Xc_0 \dots c_{m-1})$. Since F is infinite dimensional, there are infinitely many $b \in F \setminus \mathrm{cl}(Xc_0 \dots c_{m-1})$, and by ω-homogeneity there is an automorphism of F fixing $Xc_0 \dots c_{m-1}$ sending a_i to b. It follows that there are infinitely many $x \in K$ such that $x^n = a^n$, a contradiction. □

Claim 1.15. *Let G be an A-invariant subgroup of K^+ (resp. of K^*). If G contains an element generic over A then $G = K^+$ (resp. $G = K^*$).*

Proof of the claim. We prove only one of the claims, as the other is similar. First, observe that if G contains an element of K generic over A, it contains all elements of K generic over A. Let $a \in K$ be arbitrary. Choose $b \in K$ generic over Aa. Then $b \in G$, and since $a + b$ is generic over Aa (and hence over A), we have also $a + b \in G$. It follows that $a \in G$, since G is a subgroup of K^+. Hence $G = K^+$. □

Consider the A-invariant subgroup $\{a^n : a \in K^*\}$ of K^*. Let $a \in K$ be generic over A. Since a^n is generic over A by the first claim, we have that $\{a^n : a \in K^*\} = K^*$ by the second claim. This shows that K is perfect (if the characteristics is a prime p, this follows from the existence of p-th roots, and every field of characteristics 0 is perfect).

Suppose F has characteristics p. The A-invariant subgroup $\{a^p - a : a \in K^+\}$ of K^+ contains a generic element over A and hence $\{a^p - a : a \in K^+\} = K^+$.

The two previous paragraphs show that K is perfect and has no Kummer extensions (these are obtained by adjoining a solution to the equation $x^n = a$, for some $a \in K$) or Artin-Schreier extensions (these are obtained by adjoining a solution to the equation $x^p - x = a$, for some $a \in K$, where p is the characteristics). This finishes the proof. □

Question 1.16. If there are non-classical groups, are there also division rings carrying an ω-homogeneous pregeometry which are not fields?

2. GROUP ACTING ON PREGEOMETRIES

In this section, we generalise some classical results on groups acting on strongly minimal sets. We recall some of the facts, terminology, and results from [Hy2], and then prove some additional theorems.

The main concept is that of *Σ-homogeneous group n-action of a group G on a pregeometry P*. This consists of the following:

We have a group G acting on a pregeometry (P, cl). We denote by $\dim$ the dimension inside the pregeometry (P, cl) and always assume that $\mathrm{cl}(\emptyset) = \emptyset$. We write

$$(g, x) \mapsto gx,$$

(or sometimes $g(x)$ for legibility) for the action of G on P. For a tuple $\bar{x} = (x_i)_{i<n}$ of elements of P, we write $g\bar{x}$ or $g(\bar{x})$ for $(gx_i)_{i<n}$. The group G acts on the universe of P and respects the pregeometry, *i.e.*

$$a \in \mathrm{cl}(A) \quad \text{if and only if} \quad ga \in \mathrm{cl}(g(A)),$$

for $a \in P$, $A \subseteq P$ and $g \in G$.

We assume that the action of G on P is an *n-action*, *i.e.* has the following two properties:

- The action has *rank* n: Whenever $\bar{x}$ and $\bar{y}$ are two n-tuples of elements of P such that $\dim(\bar{x}\bar{y}) = 2n$, then there is $g \in G$ such that $g\bar{x} = \bar{y}$. However, for some $(n+1)$-tuples $\bar{x}, \bar{y}$ with $\dim(\bar{x}\bar{y}) = 2n+2$, there is no $g \in G$ is such that $g\bar{x} = \bar{y}$.
- The action is *$(n+1)$-determined*: Whenever the action of $g, h \in G$ agree on an $(n+1)$-dimensional subset X of P, then $g = h$.

An *automorphism of the group action* is a pair of automorphisms (σ_1, σ_2), where σ_1 is an automorphism of the group G and σ_2 is an automorphism of the pregeometry (P, cl), which preserve the group action, *i.e.*

$$\sigma_2(gx) = \sigma_1(g)\sigma_2(x).$$

Following model-theoretic practice, we will simply think of (σ_1, σ_2) as a single automorphism σ acting on two disjoint structures (the group and the pregeometry) and write $\sigma(gx) = \sigma(g)\sigma(x)$.

We let Σ be a group of automorphisms of this group action. We assume that the group action is *ω-homogeneous with respect to* Σ, *i.e.* if whenever $X \subseteq P$ is *finite* and $x, y \in P \setminus \mathrm{cl}(X)$, then there is an automorphism $\sigma \in \Sigma$ such that $\sigma(x) = y$ and $\sigma \restriction X = id_X$. Notice that $x \in P$ is fixed under all automorphisms in Σ fixing the finite set X pointwise, then $x \in \mathrm{cl}(X)$.

This is essentially the notion that Hyttinen isolated in [Hy2]. There are two slight differences: (1) We specify the automorphism group Σ, whereas [Hy2] works with *all* automorphisms of the action (but there he allows extra structure on P, thus changing the automorphism group, so the settings are equivalent). (2) We require the existence of $\sigma \in \Sigma$ such that $\sigma(x) = y$ and $\sigma \restriction X = id_X$, when $x, y \notin \mathrm{cl}(X)$ only for *finite* X. All the statements and proofs from [Hy2] can be easily modified. Some of the results of this section are easy adaptation from the proofs in [Hy2]. To avoid unnecessary repetitions, we sometimes list some of these results as facts and refer the reader to [Hy2].

Homogeneity is a nontriviality condition; it actually has strong consequences. For example, if $\bar{x}$ and $\bar{y}$ are n-tuples each of dimension n, then there is $g \in G$ such that $g(\bar{x}) = \bar{y}$. Further, for no pair of $(n+1)$-tuples $\bar{x}$, $\bar{y}$ with $\dim(\bar{x}\bar{y}) = 2n+2$ is there a $g \in G$ sending $\bar{x}$ to $\bar{y}$. This implies that if $\bar{x}$ is an independent n-tuple and y is an element outside $\mathrm{cl}(\bar{x}g(\bar{x}))$, then necessarily $g(y) \in \mathrm{cl}(\bar{x}g(\bar{x})y)$.

We often just talk about *ω-homogeneous group acting on a pregeometry*, when the identity of Σ or n are clear from the context.

The classical example of homogeneous group actions on a pregeometry are definable groups acting on a strongly minimal sets inside a saturated model. Model theory provides important tools to deal with this situation; we now give generalisations of these tools and define *types*, *stationarity*, *generic elements*, *connected component*, and so forth in this general context.

From now until the end of this section, we fix an n-action of G on the pregeometry P which is ω-homogeneous with respect to Σ.

Let A be a k-subset of P with $k < n$. We can form a new homogeneous group action by *localising at* A: The group $G_A \subseteq G$ is the stabiliser of A; the pregeometry P_A is obtained from P by considering the new closure operator $\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X) \setminus \mathrm{cl}(A)$ on $P \setminus \mathrm{cl}(A)$; the action of G_A on P_A is by restriction; and let Σ_A be the group of automorphisms in Σ fixing A pointwise. We then have a Σ_A-homogeneous group $(n-k)$-action of G_A on the pregeometry P_A.

Generally, for $A \subseteq G \cup P$, we denote by Σ_A the group of automorphisms in Σ which fix A pointwise.

Using Σ, we can talk about *types* of elements of G: these are the orbits of elements of G under Σ. Similarly, the *type of an element* $g \in G$ *over* $X \subseteq P$ is the orbit of g under Σ_X. We write $\mathrm{tp}(g/X)$ for the type of g over X.

Definition 2.1. We say that $g \in G$ is *generic* over $X \subseteq P$, if there exists an independent n-tuple $\bar{x}$ of P such that

$$\dim(\bar{x}g(\bar{x})/X) = 2n.$$

It is immediate that if g is generic over X then so is its inverse. An important property is that given a finite set $X \subseteq P$, there is a $g \in G$ generic over X.

Notice also that genericity of g over X is a property of $\mathrm{tp}(g/X)$; we can therefore talk about *generic types over* X, which are simply types of elements generic over X. Finally, if $\mathrm{tp}(g/X)$ is generic over X, $X \subseteq Y$ are finite dimensional, then there is $h \in G$ generic over Y such that $\mathrm{tp}(h/X) = \mathrm{tp}(g/X)$.

We can now define stationarity in the natural way (notice the extra condition on the number of types; this condition holds trivially in model-theoretic contexts).

Definition 2.2. We say that G is *stationary* with respect to Σ, if whenever $g, h \in G$ with $\mathrm{tp}(g/\emptyset) = \mathrm{tp}(h/\emptyset)$ and $X \subseteq P$ is finite and both g and h are generic over X, then $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$. Furthermore, we assume that the number of types over each finite set is bounded.

The following is a strengthening of stationarity.

Definition 2.3. We say that G has *unique generics* if for all finite $X \subseteq P$ and $g, h \in G$ generic over X we have $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.

We now introduce the *connected component* G^0: We let G^0 be the intersection of all invariant, normal subgroups of G with bounded index. Recall that a set is *invariant* (or more generally A-*invariant*) if it is fixed setwise by any automorphism in Σ (Σ_A respectively). The proof of the next fact is left to the reader; it can also be found in [Hy2].

Fact 2.4. *If G is stationary, then G^0 is a normal invariant subgroup of G of bounded index. The restriction of the action of G on P to G^0 is an n-action, which is homogeneous with respect to the group of automorphisms obtained from Σ by restriction.*

We provide the proof of the next proposition to convey the flavour of these arguments.

Proposition 2.5. *If G is stationary then G^0 has unique generics.*

Proof. Let Q be the set of generic types over the empty set. For $q \in Q$ and $g \in G$, we define gq as follows: Let $X \subseteq P$ with the property that $\sigma \upharpoonright X = id_X$ implies $\sigma(g) = g$ for any $\sigma \in \Sigma$. Choose $h \models q$ which is generic over X. Define $gq = \mathrm{tp}(gh/\emptyset)$.

Notice that by stationarity of G, the definition of gq does not depend on the choice of X or the choice of h. Similarly, the value of gq depends on $\mathrm{tp}(g/\emptyset)$ only. We claim that

$$q \mapsto gq$$

is a group action of G on Q. Since $1q = q$, in order to prove that this is indeed an action on Q, we need to show that gq is generic and $(gh)(q) = g(hq)$.

This is implied by the following claim: If $X \subseteq P$ is finite containing $\bar{x}$ and $g(\bar{x})$, where $\bar{x}$ is an independent $(n+1)$-tuple of elements in P, and $h \models q$ is generic over X, then gh is generic over X.

To see the claim, choose $\bar{z}$ an n-tuple of elements of P such that

$$\dim(\bar{z}h(\bar{z})/X) = 2n.$$

Notice that $h(\bar{z}) \subseteq \mathrm{cl}(Xgh(\bar{z}))$, since any $\sigma \in \Sigma$ fixing $Xgh(\bar{z})$ pointwise fixes $h(\bar{z})$ (for any such σ, we have $\sigma(h(\bar{z})) = \sigma(g^{-1}gh(\bar{z})) = \sigma(g^{-1})\sigma(gh(\bar{z})) = g^{-1}gh(\bar{z}) = h(\bar{z})$). Thus, $\dim(\bar{z}gh(\bar{z})/X) \geq \dim(\bar{z}h(\bar{z})/X) = 2n$, so $\bar{z}$ demonstrates that gh is generic over X.

Now consider the kernel H of the action, namely the set of $h \in G$ such that $hq = q$ for each $q \in Q$. This is clearly an invariant subgroup, and since the action depends only on $\mathrm{tp}(h/\emptyset)$, H must have bounded index (this condition is part of the definition of stationarity). Hence, by definition, the connected component G^0 is a subgroup of H.

By stationarity of G, if G^0 does not have unique generics, there are $g, h \in G^0$ be generic over the empty set such that $\mathrm{tp}(g/\emptyset) \neq \mathrm{tp}(h/\emptyset)$. Without loss of generality, we may assume that h is generic over $\bar{x}g(\bar{x})$, where $\bar{x}$ is an independent $(n+1)$-tuple of P. Now it is easy to check that $hg^{-1}(\mathrm{tp}(g/\emptyset)) = \mathrm{tp}(h/\emptyset))$, so that $hg^{-1} \notin H$. But $hg^{-1}h \in G^0 \subseteq H$, since $g, h \in G^0$, a contradiction. $\square$

We now make another definition:

Definition 2.6. We say that G *admits hereditarily unique generics* if G has unique generics and for any independent k-set $A \subseteq P$ with $k < n$, there is a normal subgroup G' of G_A such that the action of G' on P_A is a homogeneous $(n-k)$-action which has *unique generics*.

If we have a Σ-homogeneous 1-action of a group G on a pregeometry P which has unique generics, then the pregeometry lifts up on the universe of the group in the natural way and so the group carries a homogeneous pregeometry: For $g \in G$ and $g_0, \ldots, g_k \in G$, we let

$$g \in \mathrm{cl}(g_0, \ldots, g_k),$$

if for some independent 2-tuple $\bar{y} \in P$ and some $x \in P \setminus \mathrm{cl}(\bar{y}g(\bar{y})g_0(\bar{y}) \ldots g_k(\bar{y}))$ then

$$g(x) \in \mathrm{cl}(xg_0(x), \ldots, g_k(x)).$$

Notice first that this definition does not depend on the choice of x and $\bar{y}$: Let $x' \notin \mathrm{cl}(\bar{y}'g(\bar{y}')g_0(\bar{y}') \ldots g_k(\bar{y}'))$ for another independent 2-tuple $\bar{y}'$. Let z be such that

$$z \notin \mathrm{cl}(\bar{y}g(\bar{y})g_0(\bar{y}) \ldots g_k(\bar{y})\bar{y}'g_0(\bar{y}') \ldots g_k(\bar{y}')).$$

Then by homogeneity, there exists $\sigma \in \Sigma_{\bar{y}g_0(\bar{y})\ldots g_k(\bar{y})}$ such that $\sigma(x) = z$, and $\tau \in \Sigma_{\bar{y}'g_0(\bar{y}')\ldots g_k(\bar{y}')}$ such that $\tau(z) = x'$. Notice that $\sigma(g) = \tau(g) = g$ and $\sigma(g_i) = \tau(g_i) = g_i$ for $i \leq k$ by 2-determinacy. Hence $g(x) \in \mathrm{cl}(xg_0(x), \ldots, g_k(x))$ if and only if $g(x') \in \mathrm{cl}(x'g_0(x), \ldots, g_k(x))$ by applying $\sigma \circ \tau$.

We define $g \in \mathrm{cl}(A)$ for g, A in G, where A may be infinite, if there are $g_0, \ldots, g_k \in G$ such that $g \in \mathrm{cl}(g_0, \ldots, g_k)$. It is not difficult to check that this induces a pregeometry on G.

The unicity of generics implies that the pregeometry is ω-homogeneous: Suppose $g, h \notin \mathrm{cl}(A)$, where $A \subseteq G$ is finite. For a tuple $\bar{z}$, write $A(\bar{z}) = \{f(\bar{z}) : f \in A\}$. Let $\bar{y}$ be an independent 2-tuple and choose $x \notin \mathrm{cl}(\bar{y}g(\bar{y})h(\bar{y})A(\bar{y}))$ with $g(x), h(x) \notin \mathrm{cl}(xA(x))$. Since G has unique generics, it is enough to show that g, h are generic over $\bar{y}A(\bar{y})$. Let $z \in P$ outside $\mathrm{cl}(xg(x)h(x)A(x))$. Then, since the action has rank 1, we must have $f(x) \in \mathrm{cl}(xA(x))$, for each $f \in A$. Hence $\mathrm{cl}(xzA(x)A(z) \subseteq \mathrm{cl}(xzA(x))$ and by exchange, this implies that $g(x), h(x) \notin \mathrm{cl}(xzA(x))$. Let z' be an element outside $\mathrm{cl}(xzA(x)g(x)h(x))$. It is easy to see that $\dim(z'g(z')/xzA(x)A(z)) = 2$ and so g is generic over $xzA(x)A(z)$ and hence over $xA(x)$. The same argument shows that h is generic over $xA(x)$. Hence, there is $\sigma \in \Sigma$ fixing A such that $\sigma(g) = h$.

We have just proved the following fact:

Fact 2.7. *If $n = 1$, G is stationary and has unique generics, then G carries an ω-homogeneous pregeometry.*

Admitting hereditarily unique generics is connected to n-determinacy and non-classical groups in the following way. The proof of the next fact is in [Hy2]; notice that the group $(G_A)^0$ 1-acts and so carries an ω-homogeneous pregeometry.

Fact 2.8. *Suppose that G admits hereditarily unique generics. Then either $(G_A)^0$ is non-classical, for some independent $(n-1)$-subset $A \subseteq P$ or the action of G on P is n-determined.*

So in the case of $n = 1$, either the connected component is non-classical, or it is abelian and the action of G on P is 1-determined. Hence, the action of G^0 on P is regular.

Again, see [Hy2] for the next fact.

Fact 2.9. *If the action is n-determined then $n = 1, 2, 3$.*

Following standard terminology, we set:

Definition 2.10. We say that the n-action of G on P is *sharp* if it is n-determined.

Notice that if G n-acts sharply on P, then the element of G sending a given independent n-tuple of P to another is unique.

From now, until theorem 2.15 we assume that the n-action of G on P is sharp. Hence $n = 1, 2, 3$ by Fact 2.9. We are interested in constructing a field so we may assume that $n \geq 2$. By considering the group G_a acting on the pregeometry P_a with Σ_a when a is an element of P, we may assume that $n = 2$. This part has not been done in [Hy2].

Following Hrushovski [Hr1], we now introduce some invariant subsets of G, which will be useful in the construction of the field. We first consider the set of involutions.

Definition 2.11. Let $I = \{g \in G : g^2 = 1\}$.

The set I may not be a group.

Definition 2.12. Let $a \in P$. We let $N_a \subseteq G$ consists of those elements $g \in G$ for which the set

$$\{h(a) : h \in I, gh \notin I\}$$

has bounded dimension in P.

We now establish a few facts about I and N_a; in particular that N_a is an abelian subgroup of G:

Lemma 2.13. *Let $a \in P$.*

1. *Let $g, h \in I$. If $g(a) = h(a)$ and $g(a) \notin \mathrm{cl}(a)$, then $g = h$.*
2. *Let $g, h \in I$. Assume that $g(a) \notin \mathrm{cl}(ah(a))$, and $h(a) \notin \mathrm{cl}(ag(a))$. Then $gh \in N_a$.*
3. *Let $g, h \in N_a$. If $g(a) = h(a)$, then $g = h$.*
4. *N_a is a subgroup of G.*

Proof. (1) Since $g^2 = h^2 = 1$, then $g(g(a)) = a$, and $h(g(a)) = a$, since $g(a) = h(a)$. Hence g and h agree on a 2-dimensional set so $g = h$ since the action of G is 2-determined.

(2) It is easy to see that $h(a) \notin \mathrm{cl}(agh(a))$. Now $ghh = g \in I$, since both $g, h \in I$. But then, $ghf \in I$ for all generic $f \in I$. Hence, $gh \in N_a$.

(3) Suppose first that $a \notin \mathrm{cl}(g(a))$. Choose $f \in I$ and $b \in P$ such that $b \notin \mathrm{cl}(ag(a))$ and $f(b) = a$. Then gf and hf belong to I and since $gf(b) = hf(b)$, we have $gf = hf$ by (1) so $g = h$.

Now if $g(a) = a$, we show that $g = 1$. If not, then since the action is 2-determined we have that $g(b) \neq b$, for any $b \in P$ with $b \notin \mathrm{cl}(a)$. Now let $f \in I$ be such that $f(a) = b$ for $b \notin \mathrm{cl}(a)$. Then $gfgf(a) = a$, since $g \in N_a$. But this implies that $g(b) = b$, a contradiction.

(4) Let $g, h \in N_a$: First we show that $gh \in N_a$. Choose $f \in I$ such that $f(a) \notin \mathrm{cl}(ag(a)h(a))$. Then $h(f(a)) \notin \mathrm{cl}(ag(a))$. Hence, since $h \in N_a$ we have

that $hf \in I$, so $ghf \in I$ since $g \in N_a$. This shows that $gh \in N$. Second, we show that $g^{-1} \in N_a$. If $g^2 = 1$, then it is clear. Otherwise by (3) $g(a) \notin \mathrm{cl}(a)$. Let $f \in I$ such that $f(a) \notin \mathrm{cl}(ag(a))$. Then $gf \in I$ and so $gfgf = 1$ so that $g^{-1} = fgf$. But, by (2) $fgf \in N_a$. □

Lemma 2.14. *For $a \in P$ the group N_a is abelian.*

Proof. By 2-determinacy and Lemma 2.13, it is easy to verify that N_a carries a homogeneous pregeometry (N_a, cl'): For $X \subseteq N_a$, let $g \in \mathrm{cl}'(X)$ if $g(a) \in \mathrm{cl}(a \cup \{f(a) : f \in X\})$. It is ω-homogeneous with respect to the restrictions of $\sigma \in \Sigma_a$ to N_a. If N_a were not abelian, then its center $Z(N_a)$ be 0-dimensional and using Lemma 2.13 (3) it follows that $Z(N_a)$ is trivial. Also there is $g \in N_a$ with $g \neq g^{-1}$. By Theorem 1.10, choose $f \in G$ such that $g^{-1} = f^{-1}gf$. Let $h \in I$ be independent from g and f (in the sense of the pregeometry cl' on N_a), and as in the proof of Lemma 2.13 we have $g = hg^{-1}h^{-1} = hf^{-1}gfh$. Then $fh \in I$ and since $hf^{-1} = (fh)^{-1} = fh$, there is $k \in I$ independent from g such that $g = kgk$. But $kgk = g^{-1}$, a contradiction. □

We can now state a proposition. Recall that we say that a group action is *regular* if it is sharply transitive.

Recall that a *geometry* is a pregeometry such that $\mathrm{cl}(a) = \{a\}$ for each $a \in P$ (we already assumed that $\mathrm{cl}(\emptyset) = \emptyset$.

Proposition 2.15. *Consider the Σ-homogeneous sharp 2-action of G on P. Then, G_a acts regularly on N_a by conjugation and $G = G_a \ltimes N_a$. Furthermore, either G_a is non-classical, or G_a is abelian and the action of G_a on N_a induces the structure of an algebraically closed field on N_a. Furthermore, if G_a is abelian, then the action of G on P is sharply 2-transitive (on the* set *P), and P is a geometry.*

Proof. We have already shown that N_a carries a homogeneous pregeometry, and G_a carries a homogeneous pregeometry by 1-action. Furthermore, N_a is abelian.

We now show that G_a acts on $N_a \setminus \{0\}$ by conjugation, *i.e.* if $g \in N_a$ and $f \in G_a$, then $g^f \in N_a$. To see this, choose $b \in P$ such that $b \notin \mathrm{cl}(ag(a)f(g(a)))$ and $f(b) \notin \mathrm{cl}(ag(a)f(g(a)))$. Let $h \in I$ such that $h(a) = b$. Since conjugation is a permutation of $I \setminus \{0\}$, and $X \cup f^{-1}(X)$ is finite, for each finite subset X of P, it suffices to show that $g^f h^f \in I$. But this is clear since $gh \in I$.

It is easy to see that the action of G_a is transitive, and even sharply transitive by 2-determinedness. Using 2-determinedness again, one shows that for each $g \in G$ there is $f \in N_a$ and $h \in G_a$ such that $g = fh$. Since also $G_a \cap N_a = 0$, we have that $G = G_a \ltimes N_a$.

If G_a is abelian, we define the structure of a field on N_a as follows: We let N_a be the additive group of the field, *i.e.* the addition $\oplus$ on N_a is simply the group operation of N_a and 0 its identity element. Now fix an arbitrary element in $N_a \setminus \{0\}$, which we denote by 1 and which will play the role of the identity. For each $g \in N_a \setminus 0$, let $f_g \in G_a$ be the unique element such that $1^{f_g} = g$. We define the multiplication $\otimes$ of elements $g, h \in N_a$ as follows: $g \otimes h = h^{f_g}$. It is easy to see that this makes N_a into a field K. This field carries an ω-homogeneous pregeometry, and hence it is algebraically closed by Theorem 1.13.

Now for the last sentence, let $b_i \in P$ for $i < 4$ be *distinct* elements. We must show $g \in G$ such that $h(b_0) = b_1$ and $h(b_2) = b_3$. Let $b'_i \in K(= N_a)$ such that $b'_i(a) = b_i$. Then there are $f, g \in K$ such that $f \cdot b'_0 + g = b'_1$ and $f \cdot b'_2 + g = b'_3$. Let $f' \in G_a$ such that $1^{f'}(a) = f(a)$. Then $g(f'(b_0)) = b_1$ and $g(f'(b_2)) = b_3$. Hence, for all $b \in P_a$, we have $\mathrm{cl}(b) \setminus \mathrm{cl}(a) = \{b\}$ by ω-homogeneity. Thus P is a geometry and the action of G on P (as a set) is sharply 2-transitive. □

We can obtain a geometry P' from a pregeometry by taking the quotient with the equivalence relation $E(x, y)$ given by $\mathrm{cl}(x) = \mathrm{cl}(y)$, for $x, y \in P$.

Proposition 2.16. *Assume that G 2-acts sharply on the* geometry P. *Then G_a acts regularly on $(P_a)'$ and N_a acts regularly on P.*

Proof. The fact that G_a acts regularly on $(P_a)'$ follows from the last sentence of the previous proposition. We now show that N_a acts transitively on P'. Suppose first that for some $x \in P' \setminus \{a\}$ the subgroup $Stab(x)$ of N_a has bounded index. Then, since N_a is connected (as it carries an ω-homogeneous pregeometry), we have $Stab(x) = N_a$, and so $N_a x = \{x\}$. Let $y \in P \setminus \{a\}$. G_a acts transitively on P_a, so there is $g \in G_a$ such that $gx = y$. Then $N_a y = N_a gy = gN_a x = gx = y$, since G_a normalises N_a. But the action of G on P is 2-determined, so the action of N_a on P is 2-determined and hence $N_a = \{0\}$, a contradiction. So, for each $x \in P \setminus \{a\}$, the stabiliser $Stab(x)$ is proper. An easy generalisation of Lemma 1.6 therefore shows that it is finite-dimensional (with respect to cl'). Since this holds for every $x \in P \setminus \{a\}$, there is exactly one orbit and N_a acts transitively on $P \setminus \{a\}$. But $N_a a \neq \{a\}$ since $G_a \cap N_a = \{0\}$. This implies that N_a acts transitively on P'.

Now to see that the action of N_a on P is sharp, suppose that $gx = x$ for some $x \in P_a$. Let $y \in P_a \setminus \mathrm{cl}(ax)$. By transitivity, there is $h \in N_a$ such that $hx = y$. Then $gy = ghx = hgx = hx = y$, since N_a is abelian. It follows that $g = 0$ by 2-determinedness, so the action is regular. □

We can now obtain the full picture for groups acting on *geometries*.

Theorem 2.17. *Let G be a group n-acting on a geometry P. Assume that G admits hereditarily unique generics with respect to Σ. Then, either there is an unbounded non-classical A-invariant subgroup of G, or $n = 1, 2, 3$ and*

(1) *If* $n = 1$*, then* G *is abelian and acts regularly on* P*.*
(2) *If* $n = 2$*, then* P *can be given the* A*-invariant structure of an algebraically closed field* K *(for* $A \subseteq P$ *finite), and the action of* G *on* P *is isomorphic to the affine action of* $K^* \ltimes K^+$ *on* K*.*
(3) *If* $n = 3$*, then* $P \setminus \{\infty\}$ *can be given the* A*-invariant structure of an algebraically closed field* K *(for some* $\infty \in P$ *and* $A \subseteq P$ *finite), and the action of* G *on* P *is isomorphic to the action of* $\mathrm{PGL}_2(K)$ *on the projective line* $\mathbb{P}^1(K)$*.*

Proof. Suppose that there are no A-invariant unbounded non-classical subgroup of P, for some finite A. Then $(G_a)^0$ must be abelian, so that the action of G on P is n-determined, by Fact 2.8. Thus $n = 1, 2, 3$ by Fact 2.9.

For $n = 1$, the G acts regularly on P, and hence carries a pregeometry and must therefore be abelian (otherwise it is nonclassical).

For $n = 2$, notice that since N_a acts regularly on P, we can endow P with the algebraically closed field structure of N_a by Proposition 2.15. The conclusion follows immediately.

For $n = 3$, we follow [Bu], where some of this is done in the strongly minimal case. Choose a point $b \in P$ and call it ∞. Then by Proposition 2.15 the G_∞ acts sharply 2-transitively on the set P_∞, which is also a geometry, *i.e.* for each $b \in P_\infty$, $\mathrm{cl}(b, \infty) = \{b\}$. Hence by ω-homogeneity of P, we have that $\mathrm{cl}(b, c) = \{b, c\}$ for any $b, c \in P$, from which it follows that the action of G on the set P is sharply 3-transitive.

By (2) we can endow P_∞ with the structure of an algebraically closed field K. Denote by 0 and 1 the identity elements of K. Then $\{\infty, 0, 1\}$ is a set of dimension 3.

Consider $G_{\infty,0}$ which consists of those elements fixing both ∞ and 0. Then $G_{\infty,0}$ carries an ω-homogeneous pregeometry. It is isomorphic to the multiplicative group K^*.

Now let α be the unique element of G sending $(0, 1, \infty)$ to $(\infty, 1, 0)$, which exists since the action of G on P is sharply 3-transitive. Notice that $\alpha^2 = 1$.

We leave it to the reader to check that conjugation by α induces an idempotent automorphism σ of $G_{\infty,0}$, which is not the identity. Furthermore, $\sigma g = g^{-1}$ for each $g \in G_{\infty,0}$: To see this, consider the proper definable subgroup $B = \{a \in G_{\infty,0} : \sigma(a) = a\}$ of $G_{\infty,0}$. Then B is 0-dimensional in the pregeometry cl' of $G_{\infty,0}$. Consider also $C = \{a \in G_{\infty,0} : \sigma(a) = a^{-1}\}$. Let $\tau : G_{\infty,0} \to G_{\infty,0}$ be the homomorphism defined by $\tau(x) = \sigma(x)x^{-1}$. Then for $x \in G_{\infty,0}$ we have

$$\sigma(\tau(x)) = \sigma^2(x)\sigma(x^{-1}) = x\sigma(x)^{-1} = \tau(x)^{-1},$$

so τ maps $G_{\infty,0}$ into C. If $\tau(x) = \tau(y)$, then $x \in yB$, so $x \in \mathrm{cl}'(y)$ (in the pregeometry of $G_{\infty,0}$). It follows that the kernel of τ is finite dimensional, and therefore $C = G_{\infty,0}$ (using essentially Lemma 1.9).

We can now complete the proof: Given $x \in K^*$, choose $h \in G_{\infty,0}$ such that $h1 = x$. Then $\alpha x = \alpha h1 = h^{-1}\alpha 1 = h^{-1}1 = x^{-1}$. So α acts like an inversion on K^*. It follows that G contains the group of automorphisms of $\mathbb{P}^1(K)$ generated by the affine transformations and inversion. Hence $\mathrm{PGL}_2(K)$ embeds in G. Since the action of $\mathrm{PGL}_2(K)$ and G are both sharp 3-actions, the embedding is all of G.

To see that $N_{0,\infty}$ now carries the field K and that the action is as desired, it is enough to check that the correspondence

$$N_{0,\infty} \leftrightarrow G_{0,\infty} \leftrightarrow P_\infty$$

commutes. This follows from the following computation: For $0, 1, x \in P$, $1' \in N_{0,\infty}$ chosen so that $1'(0) = 1$, and $h \in G_{0,\infty}$ such that $h1 = x$, we have

$$h1'h^{-1}(0) = h1'0 = h1 = x.$$

Finally, going back from P_∞ to P, one checks easily that the action of G on P is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$. □

3. The stable homogeneous case

We remind the reader of a few basic facts in homogeneous model theory, which can be found in [Sh3], [HySh], or [GrLe]. Let L be a language and let $\bar\kappa$ be a suitably big cardinal. Let $\mathfrak{C}$ be a *strongly $\bar\kappa$-homogeneous* model, *i.e.* any elementary map $f : \mathfrak{C} \to \mathfrak{C}$ of size less than $\bar\kappa$ extends to an automorphism of $\mathfrak{C}$. We denote by $\mathrm{Aut}_A(\mathfrak{C})$ or $\mathrm{Aut}(\mathfrak{C}/A)$ the group of automorphisms of $\mathfrak{C}$ fixing A pointwise. A set Z will be called *A-invariant* if Z is fixed setwise by any automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$. This will be our substitute for definability; by homogeneity of $\mathfrak{C}$ an A-invariant set is the disjunction of complete types over A.

Let D be the *diagram* of $\mathfrak{C}$, *i.e.* the set of complete L-types over the empty set realised by finite sequences from $\mathfrak{C}$. For $A \subseteq \mathfrak{C}$ we denote by

$$S_D(A) = \{p \in S(A) : \text{ For any } c \models p \text{ and } a \in A \text{ the type } \mathrm{tp}(ac/\emptyset) \in D\}.$$

The homogeneity of $\mathfrak{C}$ has the following important consequence. Let $p \in S(A)$ for $A \subseteq \mathfrak{C}$ with $|A| < \bar\kappa$. The following conditions are equivalent:

- $p \in S_D(A)$;
- p is realised in $\mathfrak{C}$;
- $p \restriction B$ is realised in $\mathfrak{C}$ for each finite $B \subseteq \mathfrak{C}$.

The equivalence of the second and third item is sometimes called *weak compactness*, it is the chief reason why homogeneous model theory is so well-behaved.

We will use $\mathfrak{C}$ as a universal domain; each set and model will be assumed to be inside $\mathfrak{C}$ of size less than $\bar{\kappa}$, satisfaction is taken with respect to $\mathfrak{C}$. We will use the term *bounded* to mean 'of size less than $\bar{\kappa}$' and *unbounded* otherwise. By abuse of language, a type is bounded if its set of realisations is bounded.

We will work in the *stable* context. We say that $\mathfrak{C}$ (or D) is *stable* if one of the following equivalent conditions are satisfied:

Fact 3.1 (Shelah)**.** *The following conditions are equivalent:*

(1) *For some cardinal λ, D is* λ-stable, i.e. $|S_D(A)| \leq \lambda$ *for each $A \subseteq \mathfrak{C}$ of size λ.*
(2) *D does not have the order property,* i.e. *there does not exist a formula $\phi(x, y)$ such that for arbitrarily large λ we have $\{a_i : i < \lambda\} \subseteq \mathfrak{C}$ such that*
$$\mathfrak{C} \models \phi(a_i, a_j) \quad \textit{if and only if} \quad i < j < \lambda.$$
(3) *There exists a cardinal κ such that for each $p \in S_D(A)$ the type p does not split over a subset $B \subseteq A$ of size less than κ.*

Recall that p *splits over* B if there is $\phi(x, y) \in L$ and $c, d \in A$ with $\mathrm{tp}(c/B) = \mathrm{tp}(d/B)$ such that $\phi(x, c) \in p$ but $\neg\phi(x, d) \in p$.

Note that a diagram D may be stable while the first order theory of $\mathfrak{C}$ is unstable. Further, in (3) the cardinal κ is bounded by the first stability cardinal, itself at most $\beth_{(2^{|L|})^+}$.

Nonsplitting provides a rudimentary independence relation in the context of stable homogeneous model theory, but we will work primarily inside the set of realisations of a quasiminimal type, where the independence relation has a simpler form. Recall that a type $p \in S_D(A)$ is *quasiminimal* (also called *strongly minimal*) if it is unbounded but has a unique unbounded (hence quasiminimal) extension to any $S_D(B)$, for $A \subseteq B$. Quasiminimal types carry a pregeometry:

Fact 3.2. *Let p be quasiminimal and let $P = p(\mathfrak{C})$. Then (P, bcl), where for $a, B \subseteq P$*
$$a \in \mathrm{bcl}_A(B) \quad \textit{if} \quad \mathrm{tp}(a/A \cup B) \textit{ is bounded},$$
satisfies the axioms of a pregeometry.

We can therefore define $\dim(X/B)$ for $X \subseteq P = p(\mathfrak{C})$ and $B \subseteq \mathfrak{C}$. This induces a dependence relation $\downarrow$ as follows:
$$a \underset{B}{\downarrow} C,$$
for $a \in P$ a finite sequence, and $B, C \subseteq \mathfrak{C}$ if and only if
$$\dim(a/B) = \dim(a/B \cup C).$$

We write $\not\downarrow$ for the negation of $\downarrow$. The following lemma follows easily.

Lemma 3.3. *Let $a, b \in P$ be finite sequences, and $B \subseteq C \subseteq D \subseteq E \subseteq \mathfrak{C}$.*

(1) *(Finite Character) If $a \underset{B}{\not\downarrow} C$, then there exists a finite $C' \subseteq C$ such that $a \underset{B}{\not\downarrow} C'$.*
(2) *(Monotonicity) If $a \underset{B}{\downarrow} E$ then $a \underset{C}{\downarrow} D$.*
(3) *(Transitivity) $a \underset{B}{\downarrow} D$ and $a \underset{D}{\downarrow} E$ if and only if $a \underset{B}{\downarrow} E$.*
(4) *(Symmetry) $a \underset{B}{\downarrow} b$ if and only if $b \underset{B}{\downarrow} a$.*

This dependence relation (though defined only some sets in $\mathfrak{C}$) allows us to extend much of the theory of forking.

From now until Theorem 3.21, we make the following hypothesis:

Hypothesis 3.4. Let $\mathfrak{C}$ be stable. Let p, $q \in S_D(A)$ be unbounded, with p quasi-minimal. Let $n < \omega$ be such that:

(1) For any independent sequence $(a_0, \ldots, a_{n-1})$ of realisations of p and any (finite) set C of realisations of q we have
$$\dim(a_0, \ldots, a_{n-1}/A) = \dim(a_0, \ldots, a_{n-1}/A \cup C).$$
(2) For some independent sequence $(a_0, \ldots, a_n)$ of realisations of p there is a finite set C of realisations of q such that
$$\dim(a_0, \ldots, a_n/A) > \dim(a_0, \ldots, a_n/A \cup C).$$

Remark 3.5. In case we are in the ω-stable [Le1] or even the superstable [HyLe1] case, there is a dependence relation on all the subsets, induced by a rank, which satisfies many of the properties of forking (symmetry and extension only over certain sets, however). This dependence relation, which coincides with the one defined when both make sense, allows us to develop orthogonality calculus in much the same way as the first order setting, and would have enabled us to phrase the conditions (1) and (2) in the same way as the one we phrased for Hrushovski's theorem. Without canonical bases, however, it is not clear that the, apparently weaker, condition that p^n is weakly orthogonal to q^ω implies (1).

We now make the pregeometry P into a *geometry* P/E by considering the equivalence relation E on elements of P given by
$$E(x, y) \quad \text{if and only if} \quad \mathrm{bcl}_A(x) = \mathrm{bcl}_A(y).$$

We now proceed with the construction. Before we start, recall that the notion of interpretation we use in this context is like the first order notion, except that we replace definable sets by invariant sets (see Definition 3.17).

Let $Q = q(\mathfrak{C})$. The group we are going to interpret is the following:

$$\mathrm{Aut}_{Q \cup A}(P/E).$$

The group $\mathrm{Aut}_{Q\cup A}(P/E)$ is the group of permutations of the geometry obtained from P, which are induced by automorphisms of $\mathfrak{C}$ fixing $Q \cup A$ pointwise. There is a natural action of this group on the geometry P/E. We will show in this section that the action has rank n, is $(n+1)$-determined. Furthermore, considering the automorphisms induced from $\mathrm{Aut}_A(\mathfrak{C})$, we have a group acting on a geometry in the sense of the previous section. By restricting the group of automorphism to those induced by the group of strong automorphisms $\mathrm{Saut}_A(\mathfrak{C})$, we will show in addition that this group is stationary and admits hereditarily unique generics. The conclusion will then follow easily from the last theorem of the previous section.

We now give the construction more precisely.

Notation 3.6. We denote by $\mathrm{Aut}(P/A \cup Q)$ the group of permutations of P which extend to an automorphism of $\mathfrak{C}$ fixing $A \cup Q$.

Then $\mathrm{Aut}(P/A \cup Q)$ acts on P in the natural way. Moreover, each $\sigma \in \mathrm{Aut}(P/A \cup Q)$ induces a unique permutation on P/E, which we denote by σ/E. We now define the group that we will interpret:

Definition 3.7. Let G be the group consisting of the permutations σ/E of P/E induced by elements $\sigma \in \mathrm{Aut}(P/A \cup Q)$.

Since Q is unbounded, $\mathrm{Aut}_{A\cup Q}(\mathfrak{C})$ could be trivial (this is the case even in the first order case if the theory is not stable). The next lemma shows that this is not the case under stability of $\mathfrak{C}$. By abuse of notation, we write

$$\mathrm{tp}(a/A \cup Q) = \mathrm{tp}(b/A \cup Q),$$

if $\mathrm{tp}(a/AC) = \mathrm{tp}(b/AC)$ for any bounded $C \subseteq Q$.

Lemma 3.8. *Let a, b be bounded sequences in $\mathfrak{C}$ such that*

$$\mathrm{tp}(a/A \cup Q) = \mathrm{tp}(b/A \cup Q).$$

Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{C})$ sending a to b which is the identity on $A \cup Q$.

Proof. By induction, it is enough to prove that for all $a' \in \mathfrak{C}$, there is $b' \in \mathfrak{C}$ such that $\mathrm{tp}(aa'/A \cup Q) = \mathrm{tp}(bb'/A \cup Q)$.

Let $a' \in \mathfrak{C}$. We claim that there exists a bounded $B \subseteq Q$ such that for all $C \subseteq Q$ bounded, we have $\mathrm{tp}(aa'/ABC)$ does not split over AB.

Otherwise, for any λ, we can inductively construct an increasing sequence of bounded sets $(C_i : i < \lambda)$ such that $\mathrm{tp}(aa'/C_{i+1})$ does not split over C_i. This contradicts stability (such a chain must stop at the first stability cardinal).

Now let $\sigma \in \mathrm{Aut}_{A\cup B}(\mathfrak{C})$ sending a to b and let $b' = \sigma(a')$. We claim that $\mathrm{tp}(aa'/A \cup Q) = \mathrm{tp}(bb'/A \cup Q)$. If not, let $\phi(x, y, c) \in \mathrm{tp}(aa'/A \cup Q)$ and $\neg\phi(x, y, c) \in \mathrm{tp}(bb'/A \cup Q)$. Then, $\phi(x, y, c), \neg\phi(x, y, \sigma(c)) \in \mathrm{tp}(aa'/A \cup Bc\sigma(c))$, and therefore $\mathrm{tp}(aa'/ABc\sigma(c))$ splits over AB, a contradiction. □

It follows that the action of $\mathrm{Aut}(P/A \cup Q)$ on P, and a fortiori the action of G on P/E, has some transitivity properties. The next corollary implies that the action of G on P has rank n (condition (2) in Hypothesis 3.4 prevents two distinct independent $(n+1)$-tuples of realisations of p from being automorphic over $A\cup Q$).

Corollary 3.9. *For any independent $a, b \in P^n$, there is $g \in G$ such that $g(a/E) = b/E$.*

Proof. By assumption (1) $\dim(a/A \cup C) = \dim(b/A \cup C) = n$, for each finite $C \subseteq Q$. By uniqueness of unbounded extensions, we have that $\mathrm{tp}(a/A \cup C) = \mathrm{tp}(b/A \cup C)$ for each finite $C \subseteq Q$. It follows that $\mathrm{tp}(a/A \cup Q) = \mathrm{tp}(b/A \cup Q)$ so by the previous lemma, there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that $\sigma(a) = b$. Then $g = \sigma/E$. □

The next few lemmas are in preparation to show that the action is $(n + 1)$-determined. We first give a condition ensuring that two elements of G coincide.

Lemma 3.10. *Let $\sigma, \tau \in \mathrm{Aut}(P/A \cup Q)$. Let $a_i, b_i \in P$, for $i < 2n$ be such that*

$$\sigma(a_i) = b_i = \tau(a_i), \quad \textit{for } i < 2n.$$

Assume further that

$$a_i \underset{A}{\perp} \{a_j, b_j : j < i\} \quad \textit{and} \quad b_i \underset{A}{\perp} \{a_j, b_j : j < i\}, \quad \textit{for } i < 2n.$$

Let $c \in P$ be such that $c, \sigma(c), \tau(c) \notin \mathrm{bcl}_A(\{a_i, b_i : i < 2n\})$. Then

$$\sigma(c)/E) = \tau(c)/E).$$

Proof. Let $\bar{a} = (a_i : i < 2n)$ and $\bar{b} = (b_i : i < 2n)$ satisfy the independence requirement, and $\sigma(a_i) = b_i = \tau(a_i)$, for $i < 2n$. Assume, for a contradiction, that $c \in P$ is as above but $\sigma(c)/E \neq \tau(c)/E$.

We now establish a few properties:

(1) $\sigma(c) \underset{A \cup \{a_i, b_i : i < n\}}{\not\perp} c$,
(2) $\sigma(c) \underset{A \cup \{a_i, b_i : n \leq i < 2n\}}{\not\perp} c$,
(3) $\tau(c) \underset{A \cup \{a_i, b_i : i < n\}}{\not\perp} c$,
(4) $\tau(c) \underset{A \cup \{a_i, b_i : n \leq i < 2n\}}{\not\perp} c$

All these statements are proved the same way, so we only show (1): Suppose, for a contradiction, that $\sigma(c) \underset{A \cup \{a_i, b_i : i < n\}}{\perp} c$. By Hypothesis 3.4, there is a finite $C \subseteq Q$ such that $\dim(ca_0 \dots a_{n-1}/A \cup C) \leq n$, *i.e.*

(*) $$ca_0 \dots a_{n-1} \underset{A}{\not\perp} C.$$

Let $c' \in P$ be such that $c' \notin \mathrm{bcl}_A(C \cup c \cup \{a_i, b_i : i < n\})$. By assumption, $\sigma(c) \notin \mathrm{bcl}_A(c \cup \{a_i, b_i : i < n\})$. Hence, there exists an automorphism f of $\mathfrak{C}$ such that $f(c') = \sigma(c)$ which is the identity on $A \cup c \cup \{a_i, b_i : i < n\}$. Then by using f on (*), we obtain

$$ca_0 \dots a_{n-1} \underset{A}{\not\perp} f(C)$$

On the other hand, $\sigma(c) \notin \mathrm{bcl}_A(f(C) \cup c \cup \{a_i, b_i : i < n\})$, since $\sigma(c) = f(c')$. But by Hypothesis 3.4 we have

$$b_0 \dots b_{n-1} \underset{A}{\perp} f(C).$$

Together these imply

$$\sigma(c) b_0 \dots b_{n-1} \underset{A}{\perp} f(C).$$

But this contradicts (*), since σ fixes $f(C) \subseteq Q$.

We now prove another set of properties:

(5) $\sigma(c) \underset{A \cup \{b_i : i < n\}}{\not\perp} \tau(c)$

(6) $\sigma(c) \underset{A \cup \{b_i : n \leq i < 2n\}}{\not\perp} \tau(c)$

These are again proved similarly using the fact that for all finite $C \subseteq Q$

$$\sigma(c) \cup \{b_i : i < n\} \underset{A}{\not\perp} C \quad \text{if and only if} \quad \tau(c) \cup \{b_i : i < n\} \underset{A}{\not\perp} C.$$

We can now finish the claim: By (6) we have that

$$\{b_i : n \leq i < 2n\} \underset{A}{\not\perp} \sigma(c)\tau(c).$$

This implies that

$$\{b_i : n \leq i < 2n\} \underset{A \cup \{b_i : i < n\}}{\not\perp} \sigma(c)\tau(c),$$

since $\{b_i : i < 2n\}$ are independent. By (5) using the fact that (P, bcl_A) is a pregeometry, we therefore derive that

$$\{b_i : n \leq i < 2n\} \underset{A \cup \{b_i : i < n\}}{\not\perp} \sigma(c).$$

But this contradicts the fact that $\sigma(c) \underset{A}{\perp} \bar{a}\bar{b}$. $\square$

Lemma 3.11. *Let $\sigma \in \mathrm{Aut}(P/A \cup Q)$ and $\bar{a} \in P^{n+1}$ be independent. If*

$$\sigma(a_i)/E = a_i/E, \quad \text{for each } i \leq n,$$

and $c \notin \mathrm{bcl}_A(\bar{a}\sigma(\bar{a}))$, then

$$\sigma(c)/E = c/E.$$

Proof. Suppose, for a contradiction, that the conclusion fails. Let $c \in P$, with $c \notin \mathrm{bcl}(A\bar{a}\sigma(\bar{a})$ such that $\sigma(c) \notin \mathrm{bcl}_A(c)$. Choose a_i, for $n < i < 2n+1$ such that a_i and $\sigma(a_i)$ satisfy the assumptions of Lemma 3.10 and

$$ca_0 \underset{A}{\perp} \{a_i, \sigma(a_i) : 0 < i < 2n+1\}.$$

This is possible: To see this, assume that we have found a_j and $\sigma(a_j)$, for $j < i$, satisfying the requirement. For each $k \leq j$, choose a'_k such that

$$a'_k \underset{A}{\perp} \{a'_\ell : \ell < k\} \cup \{a_j, \sigma(a_j) : j < i\}.$$

Then,

$$\dim(\{a_j : j < i\} \cup \{a'_k : k \leq i\}) = 2i + 1.$$

Since σ extends to an automorphism of $\mathfrak{C}$, we must have also

$$\dim(\{\sigma(a_j) : j < i\} \cup \{\sigma(a'_k) : k \leq i\}) = 2i + 1.$$

Hence, for some $k \leq i$ we have

$$\sigma(a'_k) \underset{A}{\perp} \{a_j, \sigma(a_j) : j < i\},$$

and we can let $a_i = a'_k$ and $\sigma(a_i) = \sigma(a'_k)$.

Then there is an automorphism f of $\mathfrak{C}$ which sends c to a_0 and is the identity on $A \cup \{a_i, \sigma(a_i) : 0 < i < 2n+1\}$. Then σ and $f^{-1} \circ \sigma \circ f$ contradict Lemma 3.10. □

We can now obtain:

Lemma 3.12. *Let $\sigma \in \mathrm{Aut}(P/A \cup Q)$. Assume that $(a_i)_{i \leq n} \in P^{n+1}$ is independent and $\sigma(a_i)/E = a_i/E$, for $i \leq n$. Then σ/E is the identity in G.*

Proof. Let $\bar{a} \in P^{n+1}$ be independent. Choose $a \in P$ arbitrary and $a'_i \in P$ for $i < n+1$ such that $a'_i \notin \mathrm{bcl}(Aa_0, \ldots, a_n, a'_0, \ldots a'_i)$ for each $i < n+1$. By the previous lemma, $\sigma(a'_i) \in \mathrm{bcl}_A(a'_i)$ for each $i < n+1$. Hence $\sigma(a) \in \mathrm{bcl}_A(a)$ by another application of the lemma. □

The next corollary follows by applying the lemma to $\tau^{-1} \circ \sigma$. Together with Corollary 3.9, it shows that the action of G on P/E is an n-action.

Corollary 3.13. *Let $\sigma, \tau \in \mathrm{Aut}(P/A \cup Q)$ and assume there is an $(n+1)$-dimensional subset X of P/E on which σ/E and τ/E agree. Then $\sigma/E = \tau/E$.*

We now consider automorphisms of this group action. Let $\sigma \in \mathrm{Aut}_A(\mathfrak{C})$. Then, f induces an automorphism σ' of the group action as follows: σ' is σ/E on P/E, and for $g \in G$ we let $\sigma'(g)(a/E) = \sigma(\tau(\sigma^{-1}(a)))/E$, where τ is such that $\tau/E = g$. It is easy to verify that

$$\sigma' : G \to G$$

is an automorphism of G (as $\sigma \circ \tau \circ \sigma^{-1} \in \mathrm{Aut}_{Q \cup A}(\mathfrak{C})$ if $\tau \in \mathrm{Aut}_{Q \cup A}(\mathfrak{C})$, and both P and Q are A-invariant). Finally, one checks directly that σ' preserves the action.

For stationarity, it is more convenient to consider strong automorphisms. Recall that two sequences $a, b \in \mathfrak{C}$ have the same *Lascar strong types* over C, written $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, if $E(a, b)$ holds for any C-invariant equivalence relation E with a bounded number of classes. An automorphism $f \in Aut(\mathfrak{C}/C)$ is called *strong* if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(f(a)/C)$ for any $a \in \mathfrak{C}$. We denote by $\mathrm{Saut}(\mathfrak{C}/C)$ or $\mathrm{Saut}_C(\mathfrak{C})$ the group of strong automorphisms fixing C pointwise. We let $\Sigma = \{\sigma' : \sigma \in \mathrm{Saut}_A(\mathfrak{C})\}$. The reader is referred to [HySh] or [BuLe] for more details.

First, we show that the action is ω-homogeneous with respect to Σ.

Lemma 3.14. *If $X \subseteq P/E$ is finite and $x, y \in P/E$ are outside $\mathrm{bcl}_A(X)$, then there is an automorphism $\sigma \in \Sigma$ of the group action sending x to y which is the identity on X.*

Proof. By uniqueness of unbounded extensions, there is an automorphism $\sigma \in \mathrm{Saut}(\mathfrak{C})$ fixing $A \cup X$ pointwise and sending x to y. The automorphism σ' is as desired. □

We are now able to show the stationarity of G.

Proposition 3.15. *G is stationary with respect to Σ.*

Proof. First, notice that the number of strong types is bounded by stability. Now, let $g \in G$ be generic over the bounded set X and let $\bar{x} \in P^n$ be an independent sequence witnessing this, *i.e.*

$$\dim(\bar{x} g(\bar{x})/X) = 2n.$$

If $x' \in P$ is such that $\dim(\bar{x}x'/X) = n+1$, then $\dim(\bar{x}x' g(\bar{x}) g(x)'/X) = 2n+1$. By quasiminimality of p, this implies that

$$\bar{x}x' g(\bar{x}) g(x') \underset{A}{\perp} X.$$

Now let $h \in G$ be also generic over X and such that $\sigma(g) = h$ with $\sigma \in \Sigma$. For $\bar{y}, y'$ witnessing the genericity of h as above, we have

$$\bar{y}y' h(\bar{y}) h(y') \underset{A}{\perp} X.$$

Hence, by stationarity of Lascar strong types we have $\mathrm{Lstp}(\bar{x}x'g(\bar{x})g(x')/AX) = \mathrm{Lstp}(\bar{y}y'h(\bar{y})h(y')/AX)$. Thus, there is τ, a strong automorphism of $\mathfrak{C}$ fixing $A \cup X$ pointwise, such that $\tau(\bar{x}x'g(\bar{x}g(x')) = \bar{y}y'h(\bar{y})h(y')$. Then, $\tau'(g) = h$ ($\tau' \in \Sigma$) since the action is $(n+1)$-determined. □

The previous proposition implies that G^0 has unique generics, but we can prove more:

Proposition 3.16. *G^0 admits hereditarily unique generics with respect to Σ.*

Proof. For any independent k-tuple $a \in P/E$ with $k < n$, consider the Σ_a-homogeneous $(n-k)$-action G_a on P/E. Instead of Σ_a, consider the smaller group Σ'_a consisting of σ' for strong automorphisms of $\mathfrak{C}$ fixing Aa and preserving strong types over Aa. Then, as in the proof of the previous proposition, G_a is stationary with respect to Σ'_a, which implies that the connected component G'_a of G_a (defined with Σ'_a) has unique generics with respect to restriction of automorphisms in Σ'_a by Theorem 2.5. But, there are even more automorphisms in Σ_a so G'_a has unique generics with respect to restriction of automorphisms in Σ_a. By definition, this means that G admits hereditarily unique generics. □

We now show that G is interpretable in $\mathfrak{C}$. We recall the definition of interpretable group in this context.

Definition 3.17. A group $(G, \cdot)$ *interpretable in* $\mathfrak{C}$ if there is a (bounded) subset $B \subseteq \mathfrak{C}$ and an unbounded set $U \subseteq \mathfrak{C}^k$ (for some $k < \omega$), an equivalence relation E on U, and a binary function $*$ on U/E which are B-invariant and such that $(G, \cdot)$ is isomorphic to $(U/E, *)$.

We can now prove:

Proposition 3.18. *The group G is interpretable in $\mathfrak{C}$.*

Proof. This follows from the $(n+1)$-determinacy of the group action. Fix a an independent $(n+1)$-tuple of elements of P/E. Let $B = Aa$.

We let $U/E \subseteq P^{n+1}/E$ consist of those $b \in P^{n+1}/E$ such that $ga = b$ for some G. Then, this set is B-invariant since if $b \in P^{n+1}/E$ and $\sigma \in \mathrm{Aut}_B(\mathfrak{C})$, then $\sigma'(g) \in G$ and sends a to $\sigma(b)$ (recall that σ' is the automorphism of the group action induced by σ).

We now define $b_1 * b_2 = b_3$ on U/E, if whenever $g_\ell \in G$ such that $g_\ell(a) = b_\ell$, then $g_1 \circ g_2 = g_3$. This is well-defined by $(n+1)$-determinacy and the definition of U/E. Furthermore, the binary function $*$ is B-invariant. It is clear that $(U/E, *)$ is isomorphic to G. □

Remark 3.19. As we pointed out, by homogeneity of $\mathfrak{C}$, any B-invariant set is equivalent to a disjunction of complete types over A. So, for example, if B is finite, E and U is expressible by formulas in $L_{\lambda^+,\omega}$, where $\lambda = |S^D(B)|$.

It follows from the same proof that G^0 is interpretable in $\mathfrak{C}$, and similarly G_a and $(G_a)^0$ are interpretable for any independent k-tuple a in P/E with $k < n$.

Remark 3.20. If we choose p to be regular (with respect to, say, strong splitting), we can still interpret a group G, exactly as we have in the case of p quasiminimal. We have used the fact that the dependence relation is given by bounded closure only to ensure the stationarity of G, and to obtain a field.

We can now prove the main theorem. We restate the hypotheses for completeness.

Theorem 3.21. *Let $\mathfrak{C}$ be a large, homogeneous model of a stable diagram D. Let $p, q \in S_D(A)$ be unbounded with p quasiminimal. Assume that there is $n \in \omega$ such that*

(1) *For any independent n-tuple $(a_0, \dots, a_{n-1})$ of realisations of p and any finite set C of realisations of q we have*
$$\dim(a_0, \dots, a_{n-1}/A \cup C) = n.$$
(2) *For some independent sequence $(a_0, \dots, a_n)$ of realisations of p there is a finite set C of realisations of q such that*
$$\dim(a_0, \dots, a_n/A \cup C) < n + 1.$$

Then $\mathfrak{C}$ interprets a group G which acts on the geometry P' obtained from P. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n \leq 3$ and

- *If $n = 1$, then G is abelian and acts regularly on P';*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

Proof. The group G is interpretable in $\mathfrak{C}$ by Proposition 3.18. This group acts on the geometry P/E; the action has rank n and is $(n+1)$-determined. Furthermore, G^0 admits hereditarily unique generics with respect to set of automorphisms Σ induced by strong automorphisms of $\mathfrak{C}$. Working now with the connected group G^0, which is invariant and therefore interpretable, the conclusion follows from Theorem 2.17. □

Question 3.22. The only point where we use quasiminimality is in showing that G admits hereditarily unique generics. Is it possible to do this for regular types, say in the superstable case?

4. THE EXCELLENT CASE

Here we consider a class $\mathcal{K}$ of atomic models of a countable first order theory, *i.e.* D is the set of isolated types over the empty set. We assume that $\mathcal{K}$ is *excellent* (see [Sh87a], [Sh87b], [GrHa] or [Le3] for the basics of excellence). We will use the notation $S_D(A)$ and splitting, which have been defined in the previous section.

Excellence lives in the ω-*stable* context, *i.e.* $S_D(M)$ is countable, for any countable $M \in \mathcal{K}$. This notion of ω-stability is strictly weaker than the corresponding notion given in the previous section; in the excellent, non-homogeneous case, there are countable atomic sets A such that $S_D(A)$ is uncountable. *Splitting* provides an dependence relation between sets, which satisfies all the usual axioms of forking, provided we only work over models in $\mathcal{K}$. For each $p \in S_D(M)$, for $M \in \mathcal{K}$, there is a finite $B \subseteq M$ such that p does not split over B. Moreover, if $N \in \mathcal{K}$ extends M then p has a unique extension in $S_D(N)$ which does not split over B. Types with a unique nonsplitting extension are called *stationary*.

Excellence is a requirement on the existence of primary models, *i.e.* a model $M \in \mathcal{K}$ is *primary over* A, if $M = A \cup \{a_i : i < \lambda\}$ and for each $i < \lambda$ the type $\mathrm{tp}(a_i/A \cup \{a_j : j < i\})$ is isolated. Primary models are prime in $\mathcal{K}$. The following fact is due to Shelah [Sh87a], [Sh87b]:

Fact 4.1 (Shelah)**.** *Assume that $\mathcal{K}$ is excellent.*

1. *If A is a finite atomic set, then there is a primary model $M \in \mathcal{K}$ over A.*
2. *If $M \in \mathcal{K}$ and $p \in S_D(M)$, then for each $a \models p$, there is a primary model over $M \cup a$.*

We will use *full models* as universal domains (in general $\mathcal{K}$ does not contain uncountable homogeneous models). The existence of arbitrarily large full models follows from excellence. They have the following properties (see again [Sh87a] and [Sh87a]):

Fact 4.2 (Shelah)**.** *Let M be a full model of uncountable size $\bar{\kappa}$.*

1. *M is ω-homogeneous.*
2. *M is* model-homogeneous, i.e. *if $a, b \in M$ have the same type over $N \prec M$ with $\|N\| < \bar{\kappa}$, then there is an automorphism of M fixing N sending a to b.*
3. *M realises any $p \in S_D(N)$ with $N \prec M$ of size less than $\bar{\kappa}$.*

We work inside a full $\mathfrak{C}$ of size $\bar{\kappa}$, for some suitably big cardinal $\bar{\kappa}$. All sets and models will be assumed to be inside $\mathfrak{C}$ of size less than $\bar{\kappa}$, unless otherwise specified. The previous fact shows that all types over finite sets, and all stationary types of size less than $\bar{\kappa}$ are realised in $\mathfrak{C}$.

Since the automorphism group of $\mathfrak{C}$ is not as rich as in the homogeneous case, it will be necessary to consider another closure operator: For all $X \subseteq \mathfrak{C}$ and $a \in M$, we define the *essential closure* of X, written $\mathrm{ecl}(X)$ by

$$a \in \mathrm{ecl}(X), \qquad \text{if } a \in M \text{ for each } M \prec \mathfrak{C} \text{ containing } X.$$

As usual, for $B \subseteq \mathfrak{C}$, we write $\mathrm{ecl}_B(X)$ for the closure operator on subsets X of $\mathfrak{C}$ given by $\mathrm{ecl}(X \cup B)$. Over finite sets, essential closure coincides with bounded closure, because of the existence of primary models. Also, it is easy to check that $X \subseteq \mathrm{ecl}_B(X) = \mathrm{ecl}_B(\mathrm{ecl}_B(X))$, for each $X, B \subseteq \mathfrak{C}$. Furthermore, $X \subseteq Y$ implies that $\mathrm{ecl}_B(X) \subseteq \mathrm{ecl}_B(Y)$.

Again we consider a *quasiminimal* type $p \in S_D(A)$, *i.e.* $p(\mathfrak{C})$ is unbounded and there is a unique unbounded extension of p over each subset of $\mathfrak{C}$. Since the language is countable in this case, and we have ω-stability, the bounded closure of a countable set is countable. Bounded closure satisfies exchange on the set of realisations of p (see [Le3]). This holds also for essential closure.

Lemma 4.3. *Let $p \in S_D(A)$ be quasiminimal. Suppose that $a, b \models p$ are such that $a \in \mathrm{ecl}_B(Xb) \setminus \mathrm{ecl}_B(X)$. Then $b \in \mathrm{ecl}_B(Xa)$.*

Proof. Suppose not, and let $M \prec \mathfrak{C}$ containing $A \cup B \cup X \cup a$ such that $b \notin M$. Let N containing $A \cup B \cup X$ such that $a \notin N$. In particular $a \notin \mathrm{bcl}_B(N)$ and $a \in \mathrm{ecl}_B(Nb)$. Let $b' \in \mathfrak{C}$ realise the unique free extension of p over $M \cup N$. Then $\mathrm{tp}(b/M) = \mathrm{tp}(b'/M)$ since there is a unique big extension of p over M. It follows that there exists $f \in \mathrm{Aut}(\mathfrak{C}/M)$ such that $f(b) = b'$. Let $N' = f(N)$. Then $b' \notin \mathrm{bcl}_B(N'a)$. On the other hand, we have $a \in \mathrm{ecl}_B(Nb) \setminus \mathrm{ecl}_B(N)$ by monotonicity and choice of N, so $a \in \mathrm{ecl}_B(N'b') \setminus \mathrm{ecl}_B(N')$. But, then $a \in \mathrm{bcl}_B(N'b') \setminus \mathrm{bcl}(N')$ (if $a \notin \mathrm{bcl}_B(N'b')$, then $a \notin N'(b')$, for some (all) primary models over $N' \cup b'$). But this is a contradiction. □

It follows from the previous lemma that the closure relation ecl_B satisfies the axioms of a pregeometry on the *finite* subsets of $P = p(\mathfrak{C})$, when p is quasiminimal.

Thus, for finite subsets $X \subseteq P$, and any set $B \subseteq \mathfrak{C}$, we can define $\dim(X/B)$ using the closure operator ecl_B. We will now use the independence relation $\downarrow$ as follows:

$$a \underset{B}{\downarrow} C,$$

for $a \in P$ a finite sequence, and $B, C \subseteq \mathfrak{C}$ if and only if

$$\dim(a/B) = \dim(a/B \cup C).$$

The following lemma follows easily.

Lemma 4.4. *Let $a, b \in P$ be finite sequences, and $B \subseteq C \subseteq D \subseteq E \subseteq \mathfrak{C}$.*

(1) *(Monotonicity) If* $a \underset{B}{\downarrow} E$ *then* $a \underset{C}{\downarrow} D$.
(2) *(Transitivity)* $a \underset{B}{\downarrow} D$ *and* $a \underset{D}{\downarrow} E$ *if and only if* $a \underset{B}{\downarrow} E$.
(3) *(Symmetry)* $a \underset{B}{\downarrow} b$ *if and only if* $b \underset{B}{\downarrow} a$.

From now until Theorem 4.19, we now make a hypothesis similar to Hypothesis 3.4, except that A is chosen finite and the witness C is allowed to be countable (the reason is that we do not have finite character in the right hand-side argument of $\downarrow$). Since we work over finite sets, notice that p and q below are actually equivalent to formulas over A.

Hypothesis 4.5. Let $\mathfrak{C}$ be a large full model of an excellent class $\mathcal{K}$. Let $A \subseteq \mathfrak{C}$ be finite. Let p, $q \in S_D(A)$ be unbounded with p quasiminimal. Let $n < \omega$. Assume that

(1) For any independent sequence $(a_0, \dots, a_{n-1})$ of realisations of p and any countable set C of realisations of q we have
$$\dim(a_0, \dots, a_{n-1}/A) = \dim(a_0, \dots, a_{n-1}/A \cup C).$$
(2) For some independent sequence $(a_0, \dots, a_n)$ of realisations of p there is a countable set C of realisations of q such that
$$\dim(a_0, \dots, a_n/A) > \dim(a_0, \dots, a_n/A \cup C).$$

Write $P = p(\mathfrak{C})$ and $Q = q(\mathfrak{C})$, as in the previous section. Then, P carries a pregeometry with respect to bounded closure, which coincides with essential closure over finite sets. Thus, when we speak about finite sets or sequences in P, the term independent is unambiguous. We make P into a geometry P/E by considering the A-invariant equivalence relation
$$E(x, y), \quad \text{defined by} \quad \operatorname{bcl}_A(x) = \operatorname{bcl}_A(y).$$

The group we will interpret in this section is defined slightly differently, because of the lack of homogeneity (in the homogeneous case, they coincide). We will consider the group G of all permutations g of P/E with the property that for each countable $C \subseteq Q$ and for each finite $X \subseteq P$, there exists $\sigma \in \operatorname{Aut}_{A \cup C}(\mathfrak{C})$ such that $\sigma(a)/E = g(a/E)$ for each $a \in X$. This is defined unambiguously since if $x, y \in P$ such that $x/E = y/E$ then $\sigma(x)/E = \sigma(y)/E$ for any automorphism $\sigma \in \operatorname{Aut}(\mathfrak{C}/A)$.

We will show first that for any $a, b \models p^n$ and countable $C \subseteq Q$ there exists $\sigma \in \operatorname{Aut}(\mathfrak{C}/A \cup C)$ sending a to b. Next, we will show essentially that the action of G on P/E is $(n+1)$-determined, which we will then use to show that the action has rank n. It will follow immediately that G is interpretable in $\mathfrak{C}$, as in the previous section. Finally, we will develop the theory of Lascar strong types and strong automorphisms (over finite sets) to show that G admits hereditarily unique generics, again, exactly like in the previous section.

We now construct the group more formally.

Definition 4.6. Let G be the group of permutations of P/E such that for each countable $C \subseteq Q$ and finite $X \subseteq P$ there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that $\sigma(a)/E = g(a/E)$ for each $a \in X$.

G is clearly a group. We now prove a couple of key lemmas that explain why we chose ecl rather than bcl; these will be used to show that G is not trivial.

Lemma 4.7. *Let $a = (a_i)_{i<k}$ be a finite sequence in P. Suppose that $\dim(a/C) = k$, for some $C \subseteq \mathfrak{C}$. Then there exists $M \prec \mathfrak{C}$ such that*

$$a_i \notin \mathrm{bcl}(Ma_0 \dots a_{i-1}), \quad \textit{for each } i < k.$$

Proof. We find models M_i^j, for $i \leq j < k$, and automorphisms $f_j \in \mathrm{Aut}(\mathfrak{C}/M_j^j)$ for each $j < k$ such that:

(1) $A \cup C \cup a_0 \dots a_{i-1} \subseteq M_i^j$ for each $i \leq j < k$.
(2) For each $i < j < n$, $M_i^{j-1} = f_j(M_i^j)$.
(3) $a_j \mathop{\downarrow}_{M_j^j} M_0^j \cup \dots \cup M_{j-1}^j$.

This is possible: Let $M_0^0 \prec \mathfrak{C}$ containing $A \cup C$ be such that $a_0 \notin M_0^0$, which exists by definition, and let f_0 be the identity on $\mathfrak{C}$. Having constructed M_i^j for $i \leq j$, and f_j, we let $M_{j+1}^{j+1} \prec \mathfrak{C}$ contain $A \cup C \cup a_0 \dots a_j$ such that $a_{j+1} \notin M_{j+1}^{j+1}$, which exists by definition. Let $b_{j+1} \in \mathfrak{C}$ realise $\mathrm{tp}(a_{j+1}/M_{j+1}^{j+1})$ such that

$$b_{j+1} \mathop{\downarrow}_{M_{j+1}^{j+1}} M_0^j \cup \dots \cup M_j^j.$$

Such b_{j+1} exists by stationarity of $\mathrm{tp}(a_{j+1}/M_{j+1}^{j+1})$. Let f_{j+1} be an automorphism of $\mathfrak{C}$ fixing $M_{j+1}^{j+1})$ sending b_{j+1} to a_{j+1}. Let $M_i^{j+1} = f_{j+1}(M_i^j)$, for $i \leq j$. These are easily seen to be as required.

This is enough: Let $M = M_0^{k-1}$. To see that M is as needed, we show by induction on $i \leq j < k$, that $a_i \notin \mathrm{bcl}(M_0^j a_0 \dots a_{i-1})$. For $i = j$, this is clear since $a_i \notin \mathrm{bcl}(M_0^i \cup \dots \cup M_i^i)$. Now if $j = \ell + 1$, $a_i \notin \mathrm{bcl}(M_0^\ell a_0 \dots a_{i-1})$ by induction hypothesis. Since $M_0^{\ell+1} = f_{\ell+1}(M_0^\ell)$ and $f_{\ell+1}$ is the identity on $a_0 \dots a_i$, the conclusion follows. □

It follows from the previous lemma that the sequence $(a_i : i < k)$ is a Morley sequence of the quasiminimal type p_M, and hence that (1) it can be extended to any length, and (2) that any permutation of it extends to an automorphism of $\mathfrak{C}$ over M (hence over C).

Lemma 4.8. *Let $a = (a_i)_{i<n}$ and $b = (b_i)_{i<n}$ be independent finite sequence in P and a countable $C \subseteq Q$. Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/C)$ such that $\sigma(a_i) = b_i$, for $i < n$.*

Proof. By assumption, we have $\dim(a/A \cup C) = \dim(b/A \cup C)$. By using a third sequence if necessary, we may also assume that $\dim(ab/A \cup C) = 2n$. Then, by the previous lemma, there exists $M \prec \mathfrak{C}$ containing $A \cup C$ such that ab is a Morley sequence of M. Thus, the permutation sending a_i to b_i extends to an automorphism σ of $\mathfrak{C}$ fixing M (hence C). □

The fact that the previous lemma fails for independent sequences of length $n + 1$ follows from item (2) of Hypothesis 4.5.

We now concentrate on the n-action. We first prove a lemma which is essentially like Lemma 3.10, Lemma 3.11 and Lemma 3.12. However, since we cannot consider automorphisms fixing all of Q, we need to introduce good pairs and good triples.

A pair (X, C) is a *good pair* if X is a countable infinite-dimensional subset of P with $X = \mathrm{ecl}_A(X) \cap P$; C is a countable subset of Q such that if $x_0, \dots, x_n \in X$ with $x_n \underset{A}{\perp} x_0 \dots x_{n-1}$, then there are $C' \subseteq C$ with

$$\dim(x_0 \dots x_n/A \cup C') \leq n,$$

$y \in P \setminus \mathrm{ecl}_A(C' x_0 \dots x_{n-1})$ and $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ such that

$$\sigma(x_n) = y \quad \text{and} \quad \sigma(C') \subseteq C.$$

Good pairs exist; given any countable X, there exists $X' \subseteq P$ countable and $C \subseteq Q$ such that (X', C) is a good pair.

A triple (X, C, C^*) is a *good triple* if (X, C) is a good pair, C^* is a countable subset of Q containing C, and whenever two tuples $\bar{a}, \bar{b} \in X$ are automorphic over A, then there exists $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$ with $\sigma(\bar{a}) = \bar{b}$ such that, in addition,

$$\sigma(C) \subseteq C^*.$$

Again, given a countable X, there are X', and $C \subseteq C^*$ such that (X', C, C^*) is a good triple.

Lemma 4.9. *Let (X, C, C^*) be a good triple. Suppose that $x_0, \dots, x_n \in X$ are independent and $\sigma(x_i)/E = x_i$ ($i \leq n$) for some $\sigma \in \mathrm{Aut}(P/A \cup C^*)$. Then $\sigma(x)/E = x/E$ for any $x \in X$.*

Proof. We make two claims, which are proved exactly like the stable case using the definition of good pair or good triple. We leave the first claim to the reader.

Claim 4.10. *Let (X, C) be a good pair. Suppose that $x_0, \dots x_{2n-1} \in X$ are independent and $\sigma(x_i)/E = x_i/E$ for $i < 2n$ for some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$. Then, for all $x \in X \setminus \mathrm{ecl}_A(x_0 \dots x_{2n-1})$ with $\sigma(x) \in X \setminus \mathrm{ecl}_A(x_0 \dots x_{2n-1})$ we have $\sigma(x)/E = x/E$.*

We can then deduce the next claim:

Claim 4.11. *Let (X, C, C^*) be a good triple. Suppose that $x_0, \dots, x_n \in X$ are independent and $\sigma(x_i)/E = x_i/E$ for $i < 2n$ with $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C^*)$. Then for each $x \in X \setminus \mathrm{cl}_A(x_0 \dots x_n)$ we have $\sigma(x)/E = x/E$.*

Proof of the claim. Suppose, for a contradiction, that $\sigma(x) \underset{A}{\perp} x$. Using the infinite-dimensionality of X and the fact that $\sigma(x_i) \in \mathrm{ecl}_A(x_i x_1 \dots x_n)$ we can find x_i for $n \leq i < 2n$ such that

$$x_i \underset{A}{\perp} x x_0 \dots x_{i-1} \sigma(x_1) \dots \sigma(x_{i-1})$$

and

$$\sigma(x_i) \underset{A}{\perp} x x_0 \dots x_{i-1} \sigma(x_1) \dots \sigma(x_{i-1}).$$

It follows that

$$x x_0 \underset{A}{\perp} x_1 \dots x_{2n} \sigma(x_1) \dots \sigma(x_{2n}),$$

so there is $\tau \in \mathrm{Aut}(\mathfrak{C}/A x_1 \dots x_{2n} \sigma(x_1) \dots \sigma(x_{2n}))$ such that $\tau(x) = x_0$. By definition of good triple, we may assume that $\tau(C) \subseteq C^*$. Then $\sigma^{-1} \circ \tau^{-1} \circ \sigma \circ \tau$ contradicts the previous claim. □

The lemma follows from the previous claim by choosing x'_i for $i \leq n$ such that $x'_i \notin \mathrm{ecl}_A(x x_0 \dots x_n x'_0 \dots x'_{i-1})$: First $\sigma(x'_i)/E = x'_i$ for $i \leq n$, and then $\sigma(x)/E = x/E$. □

We now deduce easily the next proposition.

Proposition 4.12. *Let $(a_i)_{i \leq n}$ and $(b_i)_{i \leq n}$ be in P such that $\dim((a_i)_{i \leq n}/A) = n+1$. Let $c \in P$. There exists a countable $C \subseteq Q$ such that if $\sigma, \tau \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ and*

$$\sigma(a_i)/E = b_i/E = \tau(a_i)/E, \quad \textit{for each } i \leq n$$

then $\sigma(c)/E = \tau(c)/E$.

The value of $\sigma(c)$ in the previous proposition is independent of C. It follows that the action of G on P/E is $(n+1)$-determined. We will now show that the action has rank n (so G is automatically nontrivial).

Proposition 4.13. *The action of G on P/E is an n-action.*

Proof. The $(n+1)$-determinacy of the action of G on P follows from the previous lemma. We now have to show that the action has rank n.

For this, we first prove the following claim: If $\bar{a} = (a_i)_{i<n}$ and $\bar{b} = (b_i)_{i<n}$ are in P such that $\dim(\bar{a}\bar{b}/A) = 2n$ and $c \notin \mathrm{ecl}(A\bar{a}\bar{b})$, then there is $d \in P$ such that for each countable $C \subseteq Q$ there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/AC)$ satisfying $\sigma(a_i) = b_i$ (for $i < n$) and $\sigma(c) = d$.

To see this, choose $D \subseteq Q$ such that $\dim(\bar{a}c/D) = n$ (this is possible by our hypothesis). Suppose, for a contradiction, that no such d exists. Any automorphism fixing D and sending $\bar{a}$ to $\bar{b}$ must send $c \in \mathrm{ecl}(AD\bar{b}) \cap P$. Thus, for each $d \in \mathrm{ecl}(AD\bar{b})$ a countable set $C_d \subseteq Q$ containing D with the property that no automorphism fixing C_d sending a to b also sends c to d. Since $\mathrm{ecl}(AD\bar{b})$ is countable, we can therefore find a countable $C \subseteq Q$ containing D such that any $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ sending $\bar{a}$ to $\bar{b}$ is such that $\sigma(c) \notin \mathrm{ecl}(AD\bar{b})$. By Lemma 4.8, there does exist $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cup C)$ such that $\sigma(\bar{a}) = \bar{b}$, and by choice of D we have $\sigma(c) \in \mathrm{ecl}(AD\bar{b})$. This contradicts the choice of C.

We can now show that the action of G on P/E has rank n. Assume that $\bar{a}, \bar{b}$ are independent n-tuples of realisations of p. We must find $g \in G$ such that $g(\bar{a}/E) = \bar{b}/E$. Let $c \in P \setminus \mathrm{ecl}_A(\bar{a}\bar{b})$ and choose $d \in P$ as in the previous claim. We now define the following function $g : P/E \to P/E$. For each $e \in P$, choose C_e as in the Proposition 4.12, *i.e.* for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{C}/C_e)$, such that $\sigma(\bar{a})/E = \bar{b}/E = \tau(\bar{a})/E$ and $\sigma(c)/E = d/E = \tau(c)/E$, we have $\sigma(e)/E = \tau(e)/E$. By the previous claim there is $\sigma \in \mathrm{Aut}(\mathfrak{C}/C_e)$ sending ac to bd. Let $g(e/E) = \sigma(e)/E$. The choice of C_e guarantees that this is well-defined. It is easily seen to induce a permutation of P/E. Further, suppose a countable $C \subseteq Q$ is given and a finite $X \subseteq P$. Choose C_e as in the previous proposition for each $e \in X$. There is $\sigma \in \mathrm{Aut}(\mathfrak{C})$ sending ac to bd fixing each C_e pointwise. By definition of g, we have $\sigma(e)/E = g(e/E)$. This implies that $g \in G$. Since this fails for independent $(n+1)$-tuples, by Hypothesis 4.5, the action of G on P has rank n. □

The next proposition is now proved exactly like Proposition 3.18.

Proposition 4.14. *The group G is interpretable in $\mathfrak{C}$ (over a finite set).*

Remark 4.15. Recall that in this case, any complete type over a finite set is equivalent to a formula (as $\mathcal{K}$ is the class of atomic models of a countable first order theory). By ω-homogeneity of $\mathfrak{C}$, for any finite B, any B-invariant is subset of $\mathfrak{C}$ is a countable disjunction of formulas over A. Since the complement of a B-invariant set is B-invariant, any B-invariant set over a finite set is actually type-definable over B. Hence, the various invariant sets in the above interpretation are all type-definable over a finite set.

It remains to deal with the stationarity of G. As in the previous section, this is done by considering *strong automorphisms* and *Lascar strong types*. We only need to consider the group of strong automorphisms over finite sets C, which makes the theory easier.

In the excellent case, indiscernibles do not behave as well as in the homogeneous case: on the one hand, some indiscernibles cannot be extended, and on the other hand, it is not clear that a permutation of the elements induce an automorphism. However, Morley sequences over models have both of these properties. Recall that $(a_i : i < \alpha)$ is the *Morley sequence* of $\mathrm{tp}(a_0/M)$ if $\mathrm{tp}(a_i/M\{a_j : j < i\})$ does not split over M. (In the application, we will be interested in Morley sequences inside P, these just coincide with independent sequences.)

We first define Lascar strong types.

Definition 4.16. Let C be a finite subset of $\mathfrak{C}$. We say that a and b have the same *Lascar strong type over* C, written $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, if $E(a, b)$ holds for any C-invariant equivalence relation E with a bounded number of classes.

Equality between Lascar strong types over C is clearly a C-invariant equivalence relation; it is the finest C-invariant equivalence relation with a bounded number of classes. With this definition, one can prove the same properties for Lascar strong types as one has in the homogeneous case. The details are in [HyLe2]; the use of excellence to extract good indiscernible sequences from large enough sequences is a bit different from the homogeneous case, but once one has the fact below, the details are similar.

Fact 4.17. *Let $I \cup C \subseteq \mathfrak{C}$ be such that $|I|$ is uncountable and C countable. Then there is a countable $M_0 \prec \mathfrak{C}$ containing C and $J \subseteq I$ uncountable such that J is a Morley sequence of some stationary type $p \in S_D(M_0)$.*

The key consequences are that (1) The Lascar strong types are the orbits of the group Σ of strong automorphisms, and (2) Lascar strong types are stationary. We can then show a proposition similar to Proposition 3.15 and Proposition 3.16.

Proposition 4.18. *G is stationary and admits hereditarily unique generics with respect to Σ.*

We have therefore proved:

Theorem 4.19. *Let $\mathcal{K}$ be excellent. Let $\mathfrak{C}$ be a large full model containing the finite set A. Let $p, q \in S_D(A)$ be unbounded with p quasiminimal. Assume that there exists an integer $n < \omega$ such that*

(1) *For each independent n-tuple $a_0, \ldots, a_{n-1}$ of realisations of p and countable $C \subseteq Q$ we have*

$$\dim(a_0 \ldots a_{n-1}/AC) = n.$$

(2) *For some independent $(n+1)$-tuple $a_0, \ldots, a_n$ of realisations of p and some countable $C \subseteq Q$ we have*

$$\dim(a_0 \ldots a_n / AC) \leq n.$$

Then $\mathfrak{C}$ interprets a group G acting on the geometry P' induced on the realisations of p. Furthermore, either $\mathfrak{C}$ interprets a non-classical group, or $n \leq 3$ and

- *If $n = 1$, then G is abelian and acts regularly on P';*
- *If $n = 2$, the action of G on P' is isomorphic to the affine action of $K \rtimes K^*$ on the algebraically closed field K.*
- *If $n = 3$, the action of G on P' is isomorphic to the action of $\mathrm{PGL}_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

Question 4.20. Again, as in the stable case, we can produce a group starting from a regular type only (see [GrHa] for the definition). Is it possible to get the field (*i.e.* hereditarily unique generics) starting from a regular, rather than quasiminimal type?

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