On Crawley Modules *

Rüdiger Göbel
FB 6, Mathematik, Universität Duisburg Essen,
45117 Essen, Germany
e-mail: R.Goebel@Uni-Essen.De
and
Saharon Shelah
Institute of Mathematics, Hebrew University, Jerusalem, Israel
and Rutgers University, New Brunswick, NJ, U.S.A
e-mail: shelah@math.huji.ac.il

Abstract

This continues recent work in a paper by Corner, Göbel and Goldsmith [2]. A particular question was left open: Is it possible to carry over the results concerning the undecidability of torsion–free Crawley groups to modules over the ring of p-adic integers? We will confirm this and also strengthen one of the results in [2] by replacing the hypothesis of $\diamondsuit$ by CH. For details see the introduction.

1 Introduction

This note is an extension of the recent work in [2]. Let R be a principal ideal domain with quotient field Q. In this paper we will consider torsion–free R–modules. Crawley p-groups were investigated intensively in the last two decades of the last century. Megibben [9] showed that parallel to Whitehead groups, the existence of 'proper' Crawley groups (those which are not direct sums of cyclics) is undecidable. Further

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fundamental results for Crawley p-groups were derived in two adjoint papers by Mekler, Shelah [10, 11]. Parallel to the case of abelian Crawley p–groups we can define Crawley modules in the torsion–free case as outlined in [2].

Definition 1.1 *An R-module G is a Crawley module if for any pair M, N of pure and dense submodules of corank 1 there is an automorphism $\Theta \in \mathrm{Aut}_R G$ with $M\Theta = N$.*

The hypothesis on M is equivalent to say that $G/M \cong Q$.

An old theorem of Jeno Erdös (see the early edition of Fuchs [5] and [2]) can be reformulated to give a first result on torsion–free Crawley modules.

Theorem 1.2 *Free R-modules are Crawley modules.*

Moreover, the problem whether $\aleph_1$-free Crawley groups ($R = \mathbb{Z}$) are free is undecidable. This follows from the main results of [2]:

Theorem 1.3 *Let R be a principal ideal domain and κ a regular cardinal $> |\,R\,|$.*

(i) (Assume $\diamondsuit_\kappa(E)$ for all stationary sets $E \subseteq \kappa$.) *Any κ–free R–Crawley module of cardinality κ is free.*

(ii) (Assume ZFC + MA.) *If $|\,R\,| < 2^{\aleph_0}$, then any $\aleph_1$-separable, $\aleph_1$-coseparable R-module of rank $\aleph_1$ is a Crawley module. Thus there are non-free Crawley R-modules of rank $\aleph_1$ if R is not a field and $\aleph_1 < 2^{\aleph_0}$.*

We just note, that any strongly $\aleph_1$-free R-module of cardinality $\aleph_1$ satisfies the hypothesis in Theorem 1.3 (*ii*), thus is a Crawley module. The existence of non-free but strongly $\aleph_1$-free R-modules of cardinality $\aleph_1$ under (ZFC + MA + $\aleph_1 < 2^{\aleph_0}$) is well known, see [4] or directly Shelah [12]. We want to replace $\diamondsuit_\kappa(E)$ for $\kappa = \aleph_1$ in Theorem 1.3 by $2^{\aleph_0} < 2^{\aleph_1}$ and add $|\,R\,| \leq 2^{\aleph_0}$ in (*i*) and in (*ii*), respectively; see the results in the appropriate sections. The idea of the proof of the second assertion may also be useful for other applications. In particular we will derive the following new

Corollary 1.4 *Let $R = J_p$ be the ring of p-adic integers.*

(i) If $2^{\aleph_0} < 2^{\aleph_1}$, then any reduced, torsion-free Crawley R-module of rank at most $\aleph_1$ is free.

(ii) (ZFC + MA + $\aleph_1 < 2^{\aleph_0}$) *There are non-free, torsion-free, reduced Crawley R-modules of rank $\aleph_1$.*

2 The Construction using the Weak Diamond.

The notion of the weak diamond principle (Φ) which is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$ (a weak form of the special continuum hypothesis) comes from [3]; it is stated in Eklof, Mekler [4, pp. 147 – 152].

$\Phi_\lambda(S)$: *Let S be a stationary subset of an uncountable, regular cardinal λ (thus $\operatorname{cf}\lambda = \lambda$) and let $D = \{f \in {}^\alpha 2 : \alpha < \lambda\}$ (where $2 = \{0, 1\}$). Then S has the weak diamond property $\Phi_\lambda(S)$ if the following holds.*

For any (coloring) function $c : D \longrightarrow 2 = \{0, 1\}$, there is a weak diamond function $\eta \in {}^\lambda 2$ (for c) such that for all $f : \lambda \longrightarrow 2$ the set $\{\delta \in S : (f \restriction \delta)c = \delta\eta\}$ is stationary in λ.

Here we will use a stronger version which still is equivalent to the weak diamond property.

Lemma 2.1 *Let S be a stationary subset of an uncountable, regular cardinal λ. Then S has the weak diamond property $\Phi_\lambda(S)$ if and only if the following holds for the cardinal $\mu =: 2^{<\lambda}$ and the set $D = \{f \in {}^\alpha\mu : \alpha < \lambda\}$.*
If $c : D \longrightarrow 2$ is a coloring of D, then for every function $f : \lambda \longrightarrow \mu$ there is a weak diamond sequence $\eta \in {}^\lambda 2$ (depending on c) such that $\{\delta \in S : (f \restriction \delta)c = \delta\eta\}$ is stationary in λ.

Proof. This is a special case of Shelah [13, Appendix, Theorem 1.10, see Section 3].
$\square$

We have a particular case for $\lambda = \aleph_1$.

If S is a stationary subset of $\aleph_1$, $\mu =: 2^{\aleph_0} < 2^{\aleph_1}$ and $c : \{f \in {}^\alpha\mu : \alpha < \omega_1\} \longrightarrow 2$, then there is $\eta : \omega_1 \longrightarrow 2$ such that for all $f : \omega_1 \longrightarrow \mu$ the set $\{\delta \in S : (f \restriction \delta)c = \delta\eta\} \subseteq \omega_1$ is stationary.

We derive from $\Phi_\lambda(S)$ a lemma which can be applied immediately for Crawley modules.

Lemma 2.2 *Let S be a stationary subset of a regular, uncountable cardinal λ such that $\Phi_\lambda(S)$ holds. Suppose that A is a set of cardinality $|A| = 2^{<\lambda}$ and for any $\eta \in {}^\lambda 2$ there are a cub C_η of λ and a function $\Theta_\eta : \lambda \longrightarrow A$, then we can find $\alpha \in \lambda$ and $\eta_0, \eta_1 \in {}^\lambda 2$ $(i = 0, 1)$ such that the following holds.*

(i) $\eta_0 \restriction \alpha = \eta_1 \restriction \alpha$

(ii) $\alpha\eta_0 = 0$ *and* $\alpha\eta_1 = 1$

(iii) $\Theta_{\eta_0} \restriction \alpha = \Theta_{\eta_1} \restriction \alpha$

(iv) $\alpha \in C_{\eta_0}$

Proof. Let $\mu = 2^{<\lambda}$ and choose a one to one (coding) function $g : A \times \{0, 1\} \longrightarrow \mu$. This is used to define for each $\eta \in {}^{\lambda}2$ a function $f_\eta : \lambda \longrightarrow \mu$ by

$$(i\Theta_\eta, i\eta)g = if_\eta \text{ for every } i \in \lambda.$$

In order to apply $\Phi_\lambda(S)$ we also define a coloring $c : D \longrightarrow 2$, where $D = \{f : \alpha \longrightarrow \mu, \alpha \in \lambda\}$. If $f : \alpha \longrightarrow \mu$ is in D, then define

$$fc = \begin{cases} 1 & \text{if there is } \eta \in {}^{\lambda}2 \text{ such that } f_\eta \restriction \alpha = f, \alpha \in C_\eta \text{ and } \alpha\eta = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If η is as in the first line of (2.1) we call η *a witness for* f. Now let $\eta \in {}^{\lambda}2$ be a weak diamond function for c given by $\Phi_\lambda(S)$. Hence $S_1 = \{\delta \in S : (f_\eta \restriction \delta)c = \delta\eta\}$ is a stationary subset of λ and since C_η is a cub, also $S_2 = S_1 \cap C_\eta$ is stationary in λ. Hence clearly $S_2 \neq \emptyset$ and there is some $\delta \in S_2$.

We first claim that

$$\text{If } \delta \in S_2 \text{ and } \eta \text{ is a weak diamond function, then } \delta\eta = 1. \tag{2.2}$$

From $\delta \in S_2$ follows $(f_\eta \restriction \delta)c = \delta\eta$ and $\delta \in C_\eta$, moreover f_η extends $f_\eta \restriction \delta$. If also $\delta\eta = 0$, then $(f_\eta \restriction \delta)c = 0$ by the last observation but also $(f_\eta \restriction \delta)c = 1$ by (2.1), which is a contradiction and necessarily $\delta\eta = 1$.

We now choose $\eta_1 = \eta$ for such a δ and by the last claim follows $\delta\eta = 1$. Now the first line of (2.1) gives us a witness η_0 for $f_{\eta_1} \restriction \delta$. By the first line of (2.1) follows $\delta\eta_0 = 0$ and $\delta \in C_{\eta_0}$. Hence *(ii)* and *(iv)* are shown.

Then $f_{\eta_0} \restriction \delta = f_{\eta_1} \restriction \delta$, hence $(i\Theta_{\eta_0}, i\eta_0)g = (i\Theta_{\eta_1}, i\eta_1)g$ for all $i < \delta$. We apply g^{-1} to get $\Theta_{\eta_0} \restriction \delta = \Theta_{\eta_1} \restriction \delta$ and $\eta_0 \restriction \delta = \eta_1 \restriction \delta$. Thus (i) and (iii) follow as well. $\square$

We now return to Crawley modules. If κ is a regular cardinal, then any κ-free R-module of rank κ has a well defined Γ-invariant which is an equivalence class of a subset E of κ (i.e. $E \equiv E' \iff E \cap C = E' \cap C$ for some cub C in κ), see Eklof, Mekler [4, p. 86].

Here we want to show the following

Theorem 2.3 *Let R be a principal ideal domain and let G be an $\aleph_1$-free R-module of rank $\aleph_1$ with Γ-invariant $\Gamma(E)$ coming from a stationary set $E \subseteq \aleph_1$. If the weak diamond holds for E, then G is not a Crawley module.*
Thus, assuming $2^{\aleph_0} < 2^{\aleph_1}$, an $\aleph_1$-free R-module of rank $\aleph_1$ is Crawley if and only if it is free.

Proof. Let $G = \bigcup\limits_{\alpha \in \aleph_1} G_\alpha$ be an $\aleph_1$-filtration of G with G_α pure in G of countable rank for each $\alpha \in \omega_1$. Then

$$E = \{\alpha \in \omega_1, G/G_\alpha \text{ is not } \aleph_1 - \text{free}\}.$$

If E is not stationary, then G is free and the claim follows from Erdös's Theorem 1.2. We may assume for contradiction that E is stationary and a set of limit ordinals without restrictions. If $\alpha \in E$, then $G_{\alpha+1}/G_\alpha$ is countable and not free. By Stein's theorem (see Fuchs [6]) we can decompose $G_{\alpha+1}/G_\alpha$ into a direct sum of a free module and a complement different from 0 with trivial dual (having no proper homomorphisms into R). We can absorb the free summand into the next $G_{\alpha+2}$ and thus assume that it is 0. Hence $G_{\alpha+1}/G_\alpha \neq 0$ and $(G_{\alpha+1}/G_\alpha)^* = 0$ ($G_{\alpha+1}/G_\alpha$ is not free of minimal rank $n_\alpha \leq \omega$). There is a free R-module $X' \subseteq G_{\alpha+1}/G_\alpha$ such that $X'_* = G_{\alpha+1}/G_\alpha$. Its preimage $X \subseteq G_{\alpha+1}$ is of the form

$$X = \bigoplus_{m < n_\alpha} R x_{\alpha m} \subseteq G_{\alpha+1} \text{ with } G_{\alpha+1} = (G_\alpha \oplus X)_* \text{ and } (G_{\alpha+1}/G_\alpha)^* = (X'_*)^* = 0. \quad (2.3)$$

We choose one of these pure and dense maximal submodules $M \subseteq G$ of corank 1 and create a large family of its relatives:

By definition of M we can find $z \in G \setminus M$ such that $G = \langle M, Rz\rangle_*$. Moreover (changing the filtration) we may assume that $z \in G_0$. By induction on $\alpha \in \omega_1$ we determine for each $\eta \in {}^\alpha 2$ an epimorphism

$$\varphi_\eta : G_\alpha \longrightarrow Q$$

.

If $\alpha = 0$, then $\eta = \langle\rangle$ is the empty map, we send z to $1 \in Q$. We also assume that G_0 has infinite rank and map infinitely many free generators of G_0 onto $\mathbb{Q}$. By injectivity this map extends further to an epimorphism $\varphi_{\langle\rangle} : G_0 \longrightarrow Q$. If $\nu \trianglelefteq \eta$ is an initial segment, then we want $\varphi_\nu \subseteq \varphi_\eta$. Suppose that φ_η is constructed for all $\eta \in {}^{\alpha >}2$ We distinguish three cases. If α is a limit ordinal, and $\eta \in {}^\alpha 2$, then let $\varphi_\eta = \bigcup\limits_{\beta < \alpha} \varphi_{\eta \restriction \beta}$. If $\beta \in \omega_1 \setminus E$ and $\alpha = \beta + 1$, then we assume that η takes only one value at α, say

$\alpha\eta = 0$. Now any $\eta \in {}^{\beta}2$ has a unique extension $\nu = \eta^\wedge\langle 0\rangle \in {}^{\alpha}2$. Since G_β is a summand of G_α with free complement, we can extend φ_η arbitrarily to $\varphi_\nu : G_\alpha \longrightarrow Q$. Finally we assume that $\beta \in E$ and select the elements $x_{\beta m}$ ($m \le n_\beta$) by the choice of X in (2.3). If $\eta \in {}^{\beta}2$ and $\varphi_\eta : G_\beta \longrightarrow Q$ is given, then we want to choose $\varphi_{\eta^\wedge\langle 0\rangle}$ and $\varphi_{\eta^\wedge\langle 1\rangle}$ such that

$$x_{\beta m}\varphi_{\eta^\wedge\langle 1\rangle} - x_{\beta m}\varphi_{\eta^\wedge\langle 0\rangle} = 1 \text{ for all } m \le n_\beta. \tag{2.4}$$

Using (2.3) write $G_\alpha = G_\beta \oplus X$ for $X = \bigoplus\limits_{m \le n_\beta} R x_{\beta m}$ and we can extend φ_η to both of $\varphi'_\eta, \varphi''_\eta \in \operatorname{Hom}(G_\beta \oplus X, Q)$ with $x_{\beta m}\varphi'_\eta = 0$ and $x_{\beta m}\varphi''_\eta = 1$ for all $m \le n_\beta$. These maps can be extended further to $\varphi_{\eta^\wedge\langle 0\rangle}, \varphi_{\eta^\wedge\langle 1\rangle} \in \operatorname{Hom}(G_\alpha, Q)$ by injectivity. We have $x_{\beta m}\varphi_{\eta^\wedge\langle 0\rangle} = 0$ and $x_{\beta m}\varphi_{\eta^\wedge\langle 1\rangle} = 1$ for all $m \le n_\beta$ and in particular (2.4) holds.

This finishes the inductive construction. If $\eta \in {}^{\omega_1}2$ (with $\eta \restriction \omega_1 \setminus E$ the zero map), then we obtain the relatives of M:

$$\varphi_\eta = \bigcup_{\alpha \in \omega_1} \varphi_{\eta\restriction\alpha} \text{ and } N_\eta = \operatorname{Ker}\varphi_\eta.$$

Hence $G/\operatorname{Ker}\varphi_\eta \cong Q$ and $\{N_\eta : \eta \in {}^{\omega_1}2\}$ is a family of pure and dense submodules of corank 1.

We claim that there is $\eta \in {}^{\omega_1}2$ such that no $\Theta \in \operatorname{Aut}_R G$ satisfies $M\Theta = N_\eta$. From this the Theorem follows.

Assume for contradiction that for any $\eta \in {}^{\omega_1}2$ there is $\Theta_\eta \in \operatorname{Aut}_R G$ with $M\Theta_\eta = N_\eta$. Now we specify the choice of the cubs $C_\eta \subseteq \lambda$ for Lemma 2.2: By the usual back and forth argument we may assume that $G_\alpha\Theta_\eta = G_\alpha$ for all $\alpha \in C_\eta$. By the weak diamond and Lemma 2.2 there are some $\eta_0, \eta_1 \in {}^{\omega_1}2$ and $\alpha \in \omega_1$ such that

(i) α is the branch point of η_0 and η_1: $\eta_0 \restriction \alpha = \eta_1 \restriction \alpha = \nu$ but $\alpha\eta_0 \ne \alpha\eta_1$

(ii) $\alpha\eta_0 = 0$ and $\alpha\eta_1 = 1$,

(iii) $\Theta_{\eta_0} \restriction G_\alpha = \Theta_{\eta_1} \restriction G_\alpha$.

(iv) $\alpha \in C_{\eta_0}$.

If we put $\Theta = \Theta_{\eta_1} - \Theta_{\eta_0}$, then $\Theta \in \operatorname{End} G$ induces $\theta : G/G_\alpha \longrightarrow G$. But necessarily $\alpha \in E$, hence $(G_{\alpha+1}/G_\alpha)^* = 0$ by (2.3). On the other hand G is $\aleph_1$-free and $(G_{\alpha+1}/G_\alpha)$ is countable, hence $G_{\alpha+1}\Theta = 0$. It follows that

$$\Theta_{\eta_1} \restriction G_{\alpha+1} = \Theta_{\eta_0} \restriction G_{\alpha+1}. \tag{2.5}$$

Since $\alpha\eta_0 = 0$ by (ii), we have $\eta_0 = \nu^\wedge\langle 0\rangle$ and $x_{\alpha 0}\varphi_{\nu^\wedge\langle 0\rangle} = 0$ by definition of the extension map. But $\varphi_{\nu^\wedge\langle 0\rangle} \subseteq \varphi_{\eta_0}$ and so $x_{\alpha 0}\varphi_{\eta_0} = 0$. We derive $x_{\alpha 0} \in N_{\eta_0}$.

By the choice of C_{η_0} it will follow $y^*\Theta_{\eta_0} \in G_{\alpha+1}$: Note that $\Theta_{\eta_0}^{-1}$ leaves G_α invariant and thus also maps $G_{\alpha+1}$ into itself because $G/G_{\alpha+1}$ is $\aleph_1$-free but $(G_{\alpha+1}/G_\alpha)^* = 0$ by (2.3). (We say that $G_{\alpha+1}$ is the Chase radical of G over G_α.) From $y^*\Theta_{\eta_0} \in G_{\alpha+1}$ and (2.5) also follows $x_{\alpha 0} = y^*\Theta_{\eta_0}$. But now $x_{\alpha 0} = y^*\Theta_{\eta_1} \in G_{\alpha+1} \cap N_{\eta_1} = \operatorname{Ker}\varphi_{\nu^\wedge\langle 1\rangle}$, so $x_{\alpha 0}\varphi_{\nu^\wedge\langle 1\rangle} = 0$. Hence $x_{\alpha 0}\varphi_{\nu^\wedge\langle 0\rangle} - x_{\alpha 0}\varphi_{\nu^\wedge\langle 1\rangle} = 0$, which contradicts (2.4). $\square$

The proof immediately gives the a stronger result:

If $M \subseteq G$ is a pure and dense submodule of corank 1, the we denote by $[M] = M\,\mathrm{Aut}_R\, G$ the orbit of M under the action of the automorphism group.

Corollary 2.4 (ZFC + $2^{\aleph_0} < 2^{\aleph_1}$): *If G is a module as in the Theorem 2.3 which is not free and of cardinality $\aleph_1$, then G has $\aleph_1$ distinct orbits of pure, dense submodules of corank* 1.

Proof. Partition E into $\aleph_1$ stationary sets. $\square$

3 Crawley modules under Martin's Axiom

The proof of our next theorem is inspired by the Löwenheim-Skolem argument. We first state and prove an observation mentioned by Brendan Goldsmith. This will be used to reduce the theorem to a result in [2].

Observation 3.1 *Every subring of a principal ideal domain R is a subring of principal ideal domain contained in R and of the same cardinality.*

Proof. Let $R_1 \subseteq R$ be the given rings, let Q be the field of fractions of R_1 and set $R_2 = R \cap Q$ which is a subring of R of cardinality $|R_1|$ containing R_1. We will show that R_2 is a principal ideal domain. If I is an ideal of R_2, then IR is an ideal of R and so $IR = aR$ for some $a \in R$. However $a \in IR$ and so $a = ir$ for some $i \in I, r \in R$. Now if x is an arbitrary element of I, then $x = ar_x$ for some $r_x \in R$ and so $x = irr_x = it_x$ say where $t_x = rr_x \in R$. However $t_x = x/i \in K \cap R = R_2$. So $x = it_x$ is a product of an element in I and in R_2. Since x was arbitrary in I we have $I \subseteq iR_2$ and since the reverse inclusion is trivial, we deduce that $I = iR_2$ is principal. $\square$

Theorem 3.2 (ZFC + MA) *Let R be a principal ideal domain. Any strongly $\aleph_1$-free R-module of rank $\aleph_1$ is a Crawley module.*

Proof. Let $N_1, N_2 \subseteq G$ be two pure and dense submodules of corank 1 of the strongly $\aleph_1$-free R-module G of rank $\aleph_1$. Moreover let $\varphi_i : G \longrightarrow Q$ be the canonical epimorphisms with kernel $\operatorname{Ker}\varphi_i = N_i$ $(i = 1, 2)$. Choose an $\aleph_1$-filtration $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ and let $N_{i\alpha} = N_i \cap G_\alpha$ be the induced $\aleph_1$-filtration on N_i $(i = 1, 2)$. We can assume that there are $z_i \in G_0$ with $z_i\varphi_i = 1 \in Q$ for $i = 1, 2$. Moreover, let each G_0, N_{i0} have infinite rank (w.l.o.g.). Also choose an R-basis for each $G_\alpha, N_{i\alpha}$, thus

$$G_\alpha = \bigoplus_{n\in\omega} Rx^0_{\alpha n},\ \ N_{i\alpha} = \bigoplus_{n\in\omega} Rx^i_{\alpha n} \text{ for } i = 1, 2.$$

If $\alpha < \beta \in \omega_1, n \in \omega$, then

$$x^i_{\alpha n} = \sum_{m\in\omega} a^{i\alpha}_{\beta nm} x^i_{\beta m} \text{ for some } a^{i\alpha}_{\beta nm} \in R.$$

Since $z_i \in G_0$, also $z_i = \sum_{m\in\omega} b^i_m x^0_{0m}$ for some $b^i_m \in R$. Now we can choose a subring R' of R of cardinality $\aleph_1$ containing all these coefficients $a^{i\alpha}_{\beta nm}, b^i_m$.

Let

$$G'_\alpha = \bigoplus_{n\in\omega} R'x^0_{\alpha n},\ \ N'_{i\alpha} = \bigoplus_{n\in\omega} R'x^i_{\alpha n} \text{ for } i = 1, 2$$

be the corresponding free R'-module. We have $\aleph_1$-filtrations $G' = \bigcup_{\alpha\in\omega_1} G'_\alpha$ and $N'_i = \bigcup_{\alpha\in\omega_1} N'_{i\alpha}$; this uses that R' is sufficiently saturated! Moreover, passing from G to G' it is easy to see that also G' is a strongly $\aleph_1$-free R'-module. Also $z_i \in G'_0$ and $\varphi'_i = \varphi_i \restriction G'$ is an R'-homomorphism $G' \longrightarrow Q$.

By another back and forth argument, using that $\varphi_i : G \longrightarrow Q$ are epimorphisms (so enlarging R') we may assume that φ'_i maps G' onto the quotient field Q' of R'. By Observation 3.1 we also assume that R' is a principal ideal domain. Thus $G'/N'_i \cong Q'$ for $i = 1, 2$, and we can apply Theorem 1.3(*ii*). The R'-module G' is a Crawley R'-module and there is $\Theta' \in \operatorname{Aut}_{R'} G'$ with $N'_1\Theta' = N'_2$.

Finally we extend Θ' (uniquely) to $\Theta \in \operatorname{Aut}_R G$ such that $N_1\Theta = N_2$:

There is a cub $C \subseteq \omega_1$ such that $\Theta'_\alpha = \Theta' \restriction G'_\alpha \in \operatorname{Aut}_{R'} G'_\alpha$ for all $\alpha \in C$; moreover $\Theta' = \bigcup_{\alpha\in C} \Theta'_\alpha$ and $G' = \bigcup_{\alpha\in C} G'_\alpha$. These R'-automorphisms Θ'_α act on the free R'-module $G'_\alpha = \bigoplus_{n\in\omega} R'x^0_{\alpha n}$; they extend naturally to R-automorphisms Θ_α of the free R-module $G_\alpha = \bigoplus_{n\in\omega} Rx^0_{\alpha n}$ by tensoring with R. Clearly $\Theta = \bigcup_{\alpha\in C} \Theta_\alpha$ and $N_1\Theta = N_2$.

(This also shows that G is $\aleph_1$-presented, i.e. it can be expressed as the quotient of two free R-modules of rank $\aleph_1$.) $\square$

If R is a principal ideal domain with quotient field Q countably generated over R, then there is a multiplicatively closed subset $\mathbb{S}$ of $R \setminus \{0\}$ such that $\mathbb{S}^{-1}R = Q$. We may assume that $1 \in \mathbb{S}$ and choose an enumeration $\{s_i : i \in \omega\} = \mathbb{S}$ with $s_0 = 1$. Moreover $q_n = \prod_{i \le n} s_i$. In particular $0 = \bigcap\limits_{i\in\omega} Rq_i$, so R is an $\mathbb{S}$-ring; see [7]. Examples are the principal ideal domains of p-adic integers J_p with $\mathbb{S} = \{p^n : n \in \omega\}$ and counter examples are polynomial rings $\mathbb{Z}[X]$ over $\mathbb{Z}$ in sets X of uncountably many, commuting variables. If R is such an $\mathbb{S}$-ring, then we can construct Griffith's strongly $\aleph_1$-free R-module of rank $\aleph_1$: Let $F = \bigoplus\limits_{i\in\omega_1} Re_i$ be the free R-module of rank $\aleph_1$ and $\widehat{F}$ its $\mathbb{S}$-adic completion; note that the $\mathbb{S}$-topology on F is Hausdorff by the above. Let $E \subseteq \omega_1$ be a stationary subset of limit ordinals and choose for any limit ordinal $\delta \in \omega_1$ a ladder, i.e. a strictly increasing sequence of successor ordinals δ_n $(n \in \omega)$ with limit δ. We consider the elements

$$v_\delta^k = \sum_{n \ge k} (q_n q_k^{-1}) e_{\delta_n} \in \widehat{F}$$

and let

$$G = \langle F, Rv_\delta^k : \delta \in E, k \in \omega \rangle$$

which is a pure submodule of $\widehat{F} \cap \prod\limits_{i\in\omega_1} Re_i$. It is easy to check, that G is strongly $\aleph_1$-free with Γ-invariant $\Gamma(E) \ne 0$, hence not free. Thus the next corollary follows from Theorem 3.2.

Corollary 3.3 (ZFC + MA + $\aleph_1 < 2^{\aleph_0}$) *If R is a principal ideal domain with quotient field $Q \ne R$ countably generated over R, then there are $\aleph_1$-free but not free Crawley R-modules of rank $\aleph_1$.*

Corollary 1.4 follow immediately from Corollary 3.3 and Theorem 3.2.

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