REASONABLY COMPLETE FORCING NOTIONS

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Abstract. We introduce more properties of forcing notions which imply that their λ–support iterations are λ–proper, where λ is an inaccessible cardinal. This paper is a direct continuation of Rosłanowski and Shelah [RS07, §A.2]. As an application of our iteration result we show that it is consistent that dominating numbers associated with two normal filters on λ are distinct.

0. Introduction

There are serious ZFC obstacles to easy generalizations of properness to the case of iterations with uncountable supports (see, e.g., Shelah [She98, Appendix 3.6(2)]). This paper belongs to the series of works aiming at localizing "good properness conditions" for such iterations and including Shelah [She03a], [She03b], Rosłanowski and Shelah [RS01], [RS07] and Eisworth [Eis03]. Our results continue Rosłanowski and Shelah [RS07, §A.2], but no familiarity with the previous paper is assumed and the current work is fully self-contained.

In Section 2 we introduce 3 bounding–type properties (A, B, C) and we essentially show that the first two are almost preserved in λ–support iterations (Theorems 2.5, 2.8). "Almost" as the limit of the iteration occurs to have a somewhat weaker property, but equally applicable. In the following section we show that *reasonably A–bounding* forcing notions are exactly the ones introduced in [RS07, §A.2], thus showing that Theorem 2.8 improves [RS07, Thm A.2.4]. In the fourth section of the paper, we give an example of an interesting *reasonably B–bounding* forcing notion and we use it to show that it is consistent that dominating numbers associated with two normal filters on λ are distinct (Corollary 4.13). Finally, in the last section we present two forcing notions that are not yet covered by existing iteration theorems. We hope that the further development of the theory will include also them.

Like in [RS07], we assume here that our cardinal λ is inaccessible. We do not know at the moment if any parallel work can be done for a successor cardinal, though some progress will be presented in a subsequent paper [RS11a].

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Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [Jec03]). In forcing we keep the older convention that *a stronger condition is the larger one.*

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(1) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha,\beta,\gamma,\delta\ldots$) and also by i,j (with possible sub- and superscripts).

Cardinal numbers will be called κ,λ,μ; λ *will be always assumed to be inaccessible* (we may forget to mention it).

By χ we will denote a *sufficiently large* regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ. Moreover, we fix a well ordering $<^*_\chi$ of $\mathcal{H}(\chi)$.

(2) For two sequences η,ν we write $\nu\vartriangleleft\eta$ whenever ν is a proper initial segment of η, and $\nu\trianglelefteq\eta$ when either $\nu\vartriangleleft\eta$ or $\nu=\eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.

(3) We will consider several games of two players. One player will be called *Generic* or *Complete* or just *COM*, and we will refer to this player as "she". Her opponent will be called *Antigeneric* or *Incomplete* or just *INC* and will be referred to as "he".

(4) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underaccent{\tilde}{\tau}$, $\underaccent{\tilde}{X}$). The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under consideration are atomless.

By "λ–support iterations" we mean iterations in which domains of conditions are of size $\leq\lambda$. However, we will pretend that conditions in a λ–support iteration $\bar{\mathbb{Q}}=\langle\mathbb{P}_\zeta,\underaccent{\tilde}{\mathbb{Q}}_\zeta:\zeta<\zeta^*\rangle$ are total functions on ζ^* and for a condition p in the limit $\lim(\bar{\mathbb{Q}})$ of the iteration $\bar{\mathbb{Q}}$ and $\alpha\in\zeta^*\setminus\mathrm{Dom}(p)$ we will let $p(\alpha)=\underaccent{\tilde}{\emptyset}_{\underaccent{\tilde}{\mathbb{Q}}_\alpha}$.

(5) For a filter D on λ, the family of all D–positive subsets of λ is called D^+. (So $A\in D^+$ if and only if $A\subseteq\lambda$ and $A\cap B\neq\emptyset$ for all $B\in D$.)

In this paper we assume the following.

Context 0.1. (a) λ is a strongly inaccessible cardinal,

(b) $\bar{\mu}=\langle\mu_\alpha:\alpha<\lambda\rangle$, each μ_α is a regular cardinal satisfying (for $\alpha<\lambda$)

$$\aleph_0\leq\mu_\alpha\leq\lambda\qquad\text{and}\qquad(\forall f\in{}^\alpha\mu_\alpha)\big(\big|\prod_{\xi<\alpha}f(\xi)\big|<\mu_\alpha\big),$$

(c) $\mathcal{U}$ is a normal filter on λ.

1. Preliminaries on λ–support iterations

Definition 1.1. Let $\mathbb{P}$ be a forcing notion.

(1) For a condition $r\in\mathbb{P}$ let $\Game^\lambda_0(\mathbb{P},r)$ be the following game of two players, *Complete* and *Incomplete*:

> the game lasts at most λ moves and during a play the players construct a sequence $\langle(p_i,q_i):i<\lambda\rangle$ of pairs of conditions from $\mathbb{P}$ in such a way that $(\forall j<i<\lambda)(r\leq p_j\leq q_j\leq p_i)$ and at the stage $i<\lambda$ of the game, first Incomplete chooses p_i and then Complete chooses q_i.

Complete wins if and only if for every $i<\lambda$ there are legal moves for both players.

(2) We say that the forcing notion $\mathbb{P}$ is *strategically* $({<}\lambda)$*–complete* if Complete has a winning strategy in the game $\Game^\lambda_0(\mathbb{P},r)$ for each condition $r\in\mathbb{P}$.

(3) Let $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ be a model such that ${}^{<\lambda}N\subseteq N$, $|N|=\lambda$ and $\mathbb{P}\in N$. We say that a condition $p\in\mathbb{P}$ is $(N,\mathbb{P})$*–generic in the standard sense* (or just: $(N,\mathbb{P})$*–generic*) if for every $\mathbb{P}$–name $\underset{\sim}{\tau}\in N$ for an ordinal we have $p\Vdash$“ $\underset{\sim}{\tau}\in N$ ”.

(4) $\mathbb{P}$ is λ*–proper in the standard sense* (or just: λ*–proper*) if there is $x\in\mathcal{H}(\chi)$ such that for every model $N\prec(\mathcal{H}(\chi),\in,<^*_\chi)$ satisfying

$$ {}^{<\lambda}N\subseteq N,\quad |N|=\lambda\quad\text{ and }\quad \mathbb{P},x\in N, $$

and every condition $q\in N\cap\mathbb{P}$ there is an $(N,\mathbb{P})$–generic condition $p\in\mathbb{P}$ stronger than q.

Proposition 1.2 ([RS07, Prop. A.1.4]). *Suppose that $\mathbb{P}$ is a $({<}\lambda)$–strategically complete (atomless) forcing notion, $\alpha^*<\lambda$ and $p_\alpha\in\mathbb{P}$ (for $\alpha<\alpha^*$). Then there are conditions $q_\alpha\in\mathbb{P}$ (for $\alpha<\alpha^*$) such that $p_\alpha\le q_\alpha$ and for distinct $\alpha,\alpha'<\alpha^*$ the conditions $q_\alpha,q_{\alpha'}$ are incompatible.*

Proposition 1.3 ([RS07, Prop. A.1.6]). *Suppose $\bar{\mathbb{Q}}=\langle\mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i:i<\gamma\rangle$ is a λ–support iteration and, for each $i<\gamma$,*

$$ \Vdash_{\mathbb{P}_i}\text{ “ }\underset{\sim}{\mathbb{Q}}_i\textit{ is strategically }({<}\lambda)\textit{–complete ”.} $$

Then, for each $\varepsilon\le\gamma$ and $r\in\mathbb{P}_\varepsilon$, there is a winning strategy $\mathbf{st}(\varepsilon,r)$ of Complete in the game $\Game^\lambda_0(\mathbb{P}_\varepsilon,r)$ such that, whenever $\varepsilon_0<\varepsilon_1\le\gamma$ and $r\in\mathbb{P}_{\varepsilon_1}$, we have:

(i) *if $\langle(p_i,q_i):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_0},r{\restriction}\varepsilon_0)$ in which Complete follows the strategy $\mathbf{st}(\varepsilon_0,r{\restriction}\varepsilon_0)$, then $\langle(p_i{}^\frown r{\restriction}[\varepsilon_0,\varepsilon_1),q_i{}^\frown r{\restriction}[\varepsilon_0,\varepsilon_1)):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1},r)$ in which Complete uses $\mathbf{st}(\varepsilon_1,r)$;*

(ii) *if $\langle(p_i,q_i):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1},r)$ in which Complete plays according to the strategy $\mathbf{st}(\varepsilon_1,r)$, then $\langle(p_i{\restriction}\varepsilon_0,q_i{\restriction}\varepsilon_0):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_0},r{\restriction}\varepsilon_0)$ in which Complete uses $\mathbf{st}(\varepsilon_0,r{\restriction}\varepsilon_0)$;*

(iii) *if ε_1 is limit and a sequence $\langle(p_i,q_i):i<\lambda\rangle\subseteq\mathbb{P}_{\varepsilon_1}$ is such that for each $\xi<\varepsilon_1$, $\langle(p_i{\restriction}\xi,q_i{\restriction}\xi):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_\xi,r{\restriction}\xi)$ in which Complete uses the strategy $\mathbf{st}(\xi,r{\restriction}\xi)$, then $\langle(p_i,q_i):i<\lambda\rangle$ is a play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1},r)$ in which Complete plays according to $\mathbf{st}(\varepsilon_1,r)$;*

(iv) *if $\langle(p_i,q_i):i<i^*\rangle$ is a partial play of $\Game^\lambda_0(\mathbb{P}_{\varepsilon_1},r)$ in which Complete uses $\mathbf{st}(\varepsilon_1,r)$ and $p'\in\mathbb{P}_{\varepsilon_0}$ is stronger than all $p_i{\restriction}\varepsilon_0$ (for $i<i^*$), then there is $p^*\in\mathbb{P}_{\varepsilon_1}$ such that $p'=p^*{\restriction}\varepsilon_0$ and $p^*\ge p_i$ for $i<i^*$.*

Definition 1.4 (Compare [RS07, Def. A.1.7], see also [She03a, A.3.3, A.3.2]). (1) Let γ be an ordinal, $\emptyset\ne w\subseteq\gamma$. *A standard $(w,1)^\gamma$–tree* is a pair $\mathcal{T}=(T,\mathrm{rk})$ such that

- $\mathrm{rk}:T\longrightarrow w\cup\{\gamma\}$,
- if $t\in T$ and $\mathrm{rk}(t)=\varepsilon$, then t is a sequence $\langle(t)_\zeta:\zeta\in w\cap\varepsilon\rangle$,
- $(T,\vartriangleleft)$ is a tree with root $\langle\rangle$ and such that every chain in T has a $\vartriangleleft$–upper bound it T,
- if $t\in T$, then there is $t'\in T$ such that $t\trianglelefteq t'$ and $\mathrm{rk}(t')=\gamma$.

We will keep the convention that $\mathcal{T}^x_y$ is (T^x_y,rk^x_y).

(2) Let $\bar{\mathbb{Q}}=\langle\mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i:i<\gamma\rangle$ be a λ–support iteration. *A standard tree of conditions in* $\bar{\underset{\sim}{\mathbb{Q}}}$ is a system $\bar p=\langle p_t:t\in T\rangle$ such that

- (T,rk) is a standard $(w,1)^\gamma$–tree for some $w\subseteq\gamma$,

- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T$, and
- if $s, t \in T$, $s \lhd t$, then $p_s = p_t \restriction \mathrm{rk}(s)$.

(3) Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^i = \langle p^i_t : t \in T\rangle$. We write $\bar{p}^0 \leq \bar{p}^1$ whenever for each $t \in T$ we have $p^0_t \leq p^1_t$.

Note that our standard trees and trees of conditions are a special case of $(w,\alpha)^\gamma$–trees introduced in [RS07, Def. A.1.7] (for $\alpha = 1$). Our notation preserves the redundant "1" to keep the compatibility with the established terminology. For the same reason we use $(t)_\zeta$ instead of $t(\zeta)$.

Proposition 1.5. *Assume that $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \gamma\rangle$ is a λ–support iteration such that for all $i < \gamma$ we have*

$$\Vdash_{\mathbb{P}_i} \text{“ } \underset{\sim}{\mathbb{Q}}_i \text{ is strategically } ({<}\lambda)\text{–complete ”}.$$

(1) [RS07, Prop. A.1.9] *Suppose that $\bar{p} = \langle p_t : t \in T\rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$, $|T| < \lambda$, and $\mathcal{I} \subseteq \mathbb{P}_\gamma$ is open dense. Then there is a standard tree of conditions $\bar{q} = \langle q_t : t \in T\rangle$ such that $\bar{p} \leq \bar{q}$ and $(\forall t \in T)(\mathrm{rk}(t) = \gamma \ \Rightarrow\ q_t \in \mathcal{I})$.*

(2) *If $\bar{p} = \langle p_t : t \in T\rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$ and $|T| < \lambda$, then there is a standard tree of conditions $\bar{q} = \langle q_t : t \in T\rangle$ such that $\bar{p} \leq \bar{q}$ and*
- *if $t_0, t_1 \in T$, $\mathrm{rk}(t_0) = \mathrm{rk}(t_1)$, $\xi \in \mathrm{Dom}(t_0)$ and $(t_0)_\xi \neq (t_1)_\xi$, $t_0 \restriction \xi = t_1 \restriction \xi$ then*

$$q_{t_0} \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“ the conditions } q_{t_0}(\xi), q_{t_1}(\xi) \text{ are incompatible in } \underset{\sim}{\mathbb{Q}}_\xi \text{ ”}.$$

(3) *Suppose that*
- *$w \subseteq \gamma$, $|w| < \lambda$, $1 < \mu^*_\zeta \leq \lambda$ for $\zeta \in w$, and $T = \bigcup\limits_{\xi \leq \gamma} \prod\limits_{\zeta \in w \cap \xi} \mu^*_\zeta$ (so $\mathcal{T} = (T, \mathrm{rk})$ is a standard $(w,1)^\gamma$–tree),*
- *$\bar{p} = \langle p_t : t \in T\rangle$ is a standard tree of conditions in $\bar{\mathbb{Q}}$,*
- *for $\xi \in w$, $\underset{\sim}{\varepsilon}_\xi$ is a $\mathbb{P}_\xi$–name for a non-zero ordinal below μ^*_ξ.*

Then there are a standard $(w,1)^\gamma$–tree $\mathcal{T}' = (T', \mathrm{rk}')$ and a tree of conditions $\bar{q} = \langle q_t : t \in T'\rangle$ such that
- *$T' \subseteq T$, $\mathrm{rk}' = \mathrm{rk} \restriction T'$, and for every $t \in T'$ such that $\mathrm{rk}'(t) = \xi \in w$, the condition q_t decides the value of $\underset{\sim}{\varepsilon}_\xi$, say $q_t \Vdash \underset{\sim}{\varepsilon}_\xi = \varepsilon^t_\xi$, and*
- *$p_t \leq q_t$ for $t \in T'$, and*
- *if $t \in T'$, $\mathrm{rk}(t) = \xi \in w$, then*

$$\big\{\alpha < \mu^*_\xi : t \cup \{\langle \xi, \alpha\rangle\} \in T'\big\} = \varepsilon^t_\xi.$$

Proof. (2) Straightforward application of 1.2.

(3) Note that we cannot apply the first part directly, as the tree T may be of size λ. So we will proceed inductively constructing initial levels of T' of size $< \lambda$ and applying (1) to them.

For $\varepsilon \leq \gamma$ and $r \in \mathbb{P}_\varepsilon$ let $\mathbf{st}(\varepsilon, r)$ be the winning strategy of Complete in $\Game^\lambda_0(\mathbb{P}_\varepsilon, r)$ given by 1.3 (so these strategies have the coherence properties listed there). Let $\langle \xi_\beta : \beta \leq \beta^*\rangle$ be the increasing enumeration of $w \cup \{\gamma\}$, $\beta^* < \lambda$. By induction on $\beta \leq \beta^*$ we will pick $\mathcal{T}_\beta, \bar{q}^\beta, \bar{r}^\beta$ and $\bar{\varepsilon}^\beta$ such that

(a) $\mathcal{T}_\beta = (T_\beta, \mathrm{rk}_\beta)$ is a standard $(w \cap \xi_\beta, 1)^\gamma$–tree, $T_\beta \subseteq T$, $|T_\beta| < \lambda$, and $\bar{q}^\beta = \langle q^\beta_t : t \in T_\beta\rangle$, $\bar{r}^\beta = \langle r^\beta_t : t \in T_\beta\rangle$ are trees of conditions, $\bar{q}^\beta \leq \bar{r}^\beta$ and $r^\beta_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for $t \in T_\beta$ (note: $\mathrm{rk}(t)$, not $\mathrm{rk}_\beta(t)$);

(b) if $\beta_0 < \beta_1 \leq \beta^*$, then $T_{\beta_0} = \{t{\restriction}\xi_{\beta_0} : t \in T_{\beta_1}\}$ and $r^{\beta_0}_{t{\restriction}\xi_{\beta_0}} \leq q^{\beta_1}_t{\restriction}\xi_{\beta_0}$ for $t \in T_{\beta_1}$;

(c) if $\beta < \beta^*$, $t \in T_\beta$ and $\mathrm{rk}_\beta(t) = \gamma$ (so $\mathrm{rk}(t) = \xi_\beta$), then

$$\big\langle \big(q^\alpha_{t{\restriction}\xi_\alpha}{}^\frown p_t{\restriction}[\xi_\alpha,\xi_\beta), r^\alpha_{t{\restriction}\xi_\alpha}{}^\frown p_t{\restriction}[\xi_\alpha,\xi_\beta)\big) : \alpha < \beta\big\rangle{}^\frown\big\langle\big(q^\alpha_t, r^\alpha_t\big) : \beta \leq \alpha < \beta^*\big\rangle$$

is a partial play of $\partial^\lambda_0(\mathbb{P}_{\xi_\beta}, p_t)$ in which Complete uses her winning strategy $\mathbf{st}(\xi_\beta, p_t)$;

(d) $\bar{\varepsilon}^\beta = \langle \varepsilon^\beta_t : t \in T_\beta,\ \mathrm{rk}_\beta(t) = \gamma\rangle \subseteq \mu^*_{\xi_\beta}$;

(e) if $\beta < \beta^*$, $t \in T_\beta$ and $\mathrm{rk}_\beta(t) = \gamma$ (so $\mathrm{rk}(t) = \xi_\beta$), then $p_t \leq q^\beta_t \in \mathbb{P}_{\xi_\beta}$ and $q^\beta_t \Vdash_{\mathbb{P}_{\xi_\beta}} \underset{\sim}{\varepsilon}_{\xi_\beta} = \varepsilon^\beta_t$;

(f) if $\beta < \beta^*$, $t \in T_\beta$ and $\mathrm{rk}_\beta(t) = \gamma$, then $\big\{\alpha < \mu^*_{\xi_\beta} : t \cup \{\langle \xi_\beta, \alpha\rangle\} \in T_{\beta+1}\big\} = \varepsilon^\beta_t$.

We let $T_0 = \{\langle\rangle\}$ and we choose $q^0_{\langle\rangle} \in \mathbb{P}_{\xi_0}$ and $\varepsilon^0_{\langle\rangle}$ so that $p_{\langle\rangle} \leq q^0_{\langle\rangle}$ and $q^0_{\langle\rangle} \Vdash_{\mathbb{P}_{\xi_0}} \underset{\sim}{\varepsilon}_{\xi_0} = \varepsilon^0_{\langle\rangle}$. Then we let $r^0_{\langle\rangle}$ be the answer given by $\mathbf{st}(\xi_0, p_{\langle\rangle})$ in $\partial^\lambda_0(\mathbb{P}_{\xi_0}, p_{\langle\rangle})$ to $q^0_{\langle\rangle}$. Now suppose that we have defined $\mathcal{T}_\alpha, \bar{q}^\alpha, \bar{r}^\alpha$ and $\bar{\varepsilon}^\alpha$ for $\alpha < \beta \leq \beta^*$.

If β is a limit ordinal then the demands (a) and (b) uniquely define the standard tree $\mathcal{T}_\beta$. Note that $|T_\beta| < \lambda$ as λ is inaccessible; remember also clause (f). It follows from the choice of $\mathbf{st}(\varepsilon, r)$ (see clause 1.3(iii)) and demand (c) at previous stages that

$(\oplus)_\beta$ if $t \in T_\beta$, $\mathrm{rk}_\beta(t) = \gamma$ (so $\mathrm{rk}(t) = \xi_\beta$), then the sequence

$$\big\langle \big(q^\alpha_{t{\restriction}\xi_\alpha}{}^\frown p_t{\restriction}[\xi_\alpha,\xi_\beta), r^\alpha_{t{\restriction}\xi_\alpha}{}^\frown p_t{\restriction}[\xi_\alpha,\xi_\beta)\big) : \alpha < \beta\big\rangle$$

is a partial play of $\partial^\lambda_0(\mathbb{P}_{\xi_\beta}, p_t)$ in which Complete uses her winning strategy $\mathbf{st}(\xi_\beta, p_t)$.

For $t \in T_\beta$ we define a condition $q_t \in \mathbb{P}_{\xi_\beta}$ as follows:

- $\mathrm{Dom}(q_t) = \bigcup\limits_{\alpha<\beta} \mathrm{Dom}(r^\alpha_{t{\restriction}\xi_\alpha}) \cup \mathrm{Dom}(p_t) \subseteq \mathrm{rk}(t)$,
- if $\zeta \in \mathrm{Dom}(q_t)$, then $q_t(\zeta)$ is the $<^*_\chi$–first $\mathbb{P}_\zeta$–name for an element of $\underset{\sim}{\mathbb{Q}}_\zeta$ such that

$$q_t{\restriction}\zeta \Vdash_{\mathbb{P}_\zeta} \text{“ if the set } \{r^\alpha_{t{\restriction}\xi_\alpha}(\zeta) : \zeta < \xi_\alpha\ \&\ \alpha < \beta\} \cup \{p_t(\zeta)\} \text{ has an upper bound, then } q_t(\zeta) \text{ is such an upper bound ”.}$$

It follows from $(\oplus)_\beta$ (and 1.3(iv)) that $p_t \leq q_t$ and $r^\alpha_{t{\restriction}\xi_\alpha} \leq q_t{\restriction}\xi_{\alpha+1}$ for $\alpha < \beta$. Now, by "the $<^*_\chi$–first", clearly $\bar{q} = \langle q_t : t \in T_\beta\rangle$ is a tree of conditions. Applying 1.5(1) we may choose a tree of conditions $\bar{q}^\beta = \langle q^\beta_t : t \in T_\beta\rangle$ such that $\bar{q} \leq \bar{q}^\beta$ and

- if $\beta < \beta^*$, $t \in T_\beta$ and $\mathrm{rk}_\beta(t) = \gamma$, then the condition q^β_t decides the value of $\underset{\sim}{\varepsilon}_{\xi_\beta}$ (and let $q^\beta_t \Vdash \underset{\sim}{\varepsilon}_{\xi_\beta} = \varepsilon^\beta_t$) and $q^\beta_t \in \mathbb{P}_{\xi_\beta}$.

Then, for $t \in T_\beta$, we let r^β_t be the answer given to Complete by $\mathbf{st}(\mathrm{rk}(t), p_t)$ in the appropriate partial play of $\partial^\lambda_0(\mathbb{P}_{\mathrm{rk}(t)}, p_t)$, where at stage β Incomplete put q^β_t (see (c), $(\oplus)_\beta$). It follows from 1.3(ii) that $\bar{r}^\beta = \langle r^\beta_t : t \in T_\beta\rangle$ is a tree of conditions. Plainly, $\mathcal{T}_\beta, \bar{q}^\beta, \bar{r}^\beta$ and $\bar{\varepsilon}^\beta$ satisfy all relevant (restrictions of the) demands (a)–(f).

Now suppose that β is a successor ordinal, say $\beta = \beta_0 + 1$. Let

$$T_\beta = T_{\beta_0} \cup \big\{t \cup \{\langle \xi_{\beta_0}, \varepsilon\rangle\} : t \in T_{\beta_0}\ \&\ \mathrm{rk}_{\beta_0}(t) = \gamma\ \&\ \varepsilon < \varepsilon^{\beta_0}_t\big\}$$

and for $t \in T_\beta$ define q_t as follows:

- if $t \in T_{\beta_0}$, then $q_t = r^{\beta_0}_t$,

- if $t \in T_\beta \setminus T_{\beta_0}$, then $q_t = r^{\beta_0}_{t\restriction\xi_{\beta_0}} {}^\frown p_t\restriction[\xi_{\beta_0}, \xi_\beta)$.

Then $\bar{q} = \langle q_t : t \in T_\beta\rangle$ is a tree of conditions, $r^{\beta_0}_t \leq q_t$ for $t \in T_{\beta_0}$. It follows from 1.5(1) that we may choose a tree of conditions $\bar{q}^\beta = \langle q^\beta_t : t \in T_\beta\rangle$ such that $\bar{q} \leq \bar{q}^\beta$ and

- if $\beta < \beta^*$, $t \in T_\beta$ and $\mathrm{rk}_\beta(t) = \gamma$, then the condition q^β_t decides $\underset{\sim}{\varepsilon}_{\xi_\beta}$ and, say, $q^\beta_t \Vdash \underset{\sim}{\varepsilon}_{\xi_\beta} = \varepsilon^\beta_t$.

Next, like in the limit case, $\bar{r}^\beta = \langle r^\beta_t : t \in T_\beta\rangle$ is obtained by applying the strategies $\mathbf{st}(\mathrm{rk}(t), p_t)$ suitably. Easily, $\mathcal{T}_\beta, \bar{q}^\beta, \bar{r}^\beta$ and $\bar{\varepsilon}^\beta$ satisfy the demands (a)–(f).

After the inductive construction is carried out look at T_{β^*}, $\bar{q}^{\beta^*}$ and $\langle \bar{\varepsilon}^\beta : \beta < \beta^*\rangle$. $\square$

2. ABC of reasonable completeness

Remark 2.1. Note that if a forcing notion $\mathbb{Q}$ is strategically $({<}\lambda)$–complete and $\mathcal{U}$ is a normal filter on λ, then the normal filter generated by $\mathcal{U}$ in $\mathbf{V}^{\mathbb{Q}}$ is proper. Abusing notation, we may denote the normal filter generated by $\mathcal{U}$ in $\mathbf{V}^{\mathbb{Q}}$ also by $\mathcal{U}$ or by $\mathcal{U}^{\mathbb{Q}}$. Thus if $\underset{\sim}{A}$ is a $\mathbb{Q}$–name for a subset of λ, then $p \Vdash_{\mathbb{Q}} \underset{\sim}{A} \in \mathcal{U}^{\mathbb{Q}}$ if and only if for some $\mathbb{Q}$–names $\underset{\sim}{A}_\alpha$ for elements of $\mathcal{U}^{\mathbf{V}}$ we have that $p \Vdash_{\mathbb{Q}} \underset{\alpha<\lambda}{\triangle} \underset{\sim}{A}_\alpha \subseteq \underset{\sim}{A}$ (where $\triangle$ denotes the operation of diagonal intersection).

Let us note that many of the arguments in this section would be much simpler if we restricted ourselves to $(<\lambda)$–complete forcing notions. Unfortunately, the forcing notions that we would like to cover tend to be only *strategically* $({<}\lambda)$–complete, see [RS07, §B.6].

Definition 2.2. Let $\mathbb{Q}$ be a strategically $({<}\lambda)$–complete forcing notion.

(1) For a condition $p \in \mathbb{Q}$ we define a game[1] $\Game^{\mathrm{rcA}}_{\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\mathrm{rcA}}_{\bar{\mu}}(p, \mathbb{Q})$ lasts λ steps and during a play a sequence
$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\Big\rangle$$
is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and a system $\langle p^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$,

$(\beth)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t \in I_\alpha)(p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play
$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\Big\rangle$$
of $\Game^{\mathrm{rcA}}_{\bar{\mu}}(p, \mathbb{Q})$ if and only if

$(\circledast)^{\mathrm{rc}}_{\mathrm{A}}$ there is a condition $p^* \in \mathbb{Q}$ stronger than p and such that[2]
$$p^* \Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha < \lambda : \big(\exists t \in I_\alpha\big)\big(q^\alpha_t \in \Gamma_{\mathbb{Q}}\big)\big\} = \lambda \text{ ”}.$$

(2) Games $\Game^{\mathrm{rcB}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q}), \Game^{\mathrm{rcC}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ are defined similarly, except that the winning criterion $(\circledast)^{\mathrm{rc}}_{\mathrm{A}}$ is replaced by

[1] rc stands for **r**easonable **c**ompleteness

[2] equivalently, for every $\alpha < \lambda$ the set $\big\{q^\alpha_t : t \in I_\alpha\big\}$ is pre-dense above p^*

$(\circledast)^{\rm rc}_{\rm B}$ there is a condition $p^*\in\mathbb{Q}$ stronger than p and such that

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda:\big(\exists t\in I_\alpha\big)\big(q^\alpha_t\in\Gamma_{\mathbb{Q}}\big)\big\}\in\mathcal{U}^{\mathbb{Q}} \text{ ”},$$

$(\circledast)^{\rm rc}_{\rm C}$ there is a condition $p^*\in\mathbb{Q}$ stronger than p and such that

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda:\big(\exists t\in I_\alpha\big)\big(q^\alpha_t\in\Gamma_{\mathbb{Q}}\big)\big\}\in\big(\mathcal{U}^{\mathbb{Q}}\big)^+ \text{ ”},$$

respectively.

(3) For a condition $p\in\mathbb{Q}$ we define a game $\Game^{\rm rc\mathbf{b}}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$ between Generic and Antigeneric as follows. A play of $\Game^{\rm rc\mathbf{b}}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$ lasts λ steps and during a play a sequence

$$\Big\langle\zeta_\alpha,\langle p^\alpha_\xi,q^\alpha_\xi:\xi<\zeta_\alpha\rangle:\alpha<\lambda\Big\rangle$$

is constructed. Suppose that the players have arrived to a stage $\alpha<\lambda$ of the game. Now, Generic chooses a non-zero ordinal $\zeta_\alpha<\mu_\alpha$ and then the two players play a subgame of length ζ_α alternately choosing successive terms of a sequence $\langle p^\alpha_\xi,q^\alpha_\xi:\xi<\zeta_\alpha\rangle$. At a stage $\xi<\zeta_\alpha$ of the subgame, first Generic picks a condition $p^\alpha_\xi\in\mathbb{Q}$ and then Antigeneric answers with a condition q^α_ξ stronger than p^α_ξ.

At the end, Generic wins the play

$$\Big\langle\zeta_\alpha,\langle p^\alpha_\xi,q^\alpha_\xi:\xi<\zeta_\alpha\rangle:\alpha<\lambda\Big\rangle$$

of $\Game^{\rm rc\mathbf{b}}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\mathbf{b}}$ there is a condition $p^*\in\mathbb{Q}$ stronger than p and such that

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda:\big(\exists\xi<\zeta_\alpha\big)\big(q^\alpha_\xi\in\Gamma_{\mathbb{Q}}\big)\big\}\in\mathcal{U}^{\mathbb{Q}} \text{ ”}.$$

(4) Games $\Game^{\rm rc\mathbf{a}}_{\bar\mu}(p,\mathbb{Q})$ and $\Game^{\rm rc\mathbf{c}}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$ are defined similarly except that the winning criterion $(\circledast)^{\rm rc}_{\mathbf{b}}$ is changed so that "$\in\mathcal{U}^{\mathbb{Q}}$" is replaced by "$=\lambda$" or "$\in\big(\mathcal{U}^{\mathbb{Q}}\big)^+$", respectively.

(5) We say that a forcing notion $\mathbb{Q}$ is *reasonably A–bounding over $\bar\mu$* if
   (a) $\mathbb{Q}$ is strategically $({<}\lambda)$–complete, and
   (b) for any $p\in\mathbb{Q}$, Generic has a winning strategy in the game $\Game^{\rm rcA}_{\bar\mu}(p,\mathbb{Q})$.

In an analogous manner we define when the forcing notion $\mathbb{Q}$ is *reasonably X–bounding over $\mathcal{U},\bar\mu$* (for ${\rm X}\in\{{\rm B},{\rm C},\mathbf{a},\mathbf{b},\mathbf{c}\}$) — just using the game $\Game^{\rm rcX}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$ appropriately.

If $\mu_\alpha=\lambda$ for each $\alpha<\lambda$, then we may omit $\bar\mu$ and say *reasonably B–bounding over $\mathcal{U}$* etc. If $\mathcal{U}$ is the filter generated by club subsets of λ, we may omit it as well.

(6) Let $\mathbf{st}$ be a strategy for Generic in the game $\Game^{\rm rcB}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$. We will say that a sequence $\Big\langle I_\alpha,\langle p^\alpha_t,q^\alpha_t:t\in I_\alpha\rangle:\delta<\alpha<\lambda\Big\rangle$ is *a δ–delayed play according to* $\mathbf{st}$ if it has an extension $\Big\langle I_\alpha,\langle p^\alpha_t,q^\alpha_t:t\in I_\alpha\rangle:\alpha<\lambda\Big\rangle$ which is a play agreeing with $\mathbf{st}$ and such that $p^\alpha_t=q^\alpha_t$ for $\alpha\le\delta$, $t\in I_\alpha$.

Remark 2.3. If $\mathbf{st}$ is a winning strategy for Generic in the game $\Game^{\rm rcB}_{\mathcal{U},\bar\mu}(p,\mathbb{Q})$, and $\bar\sigma=\Big\langle I_\alpha,\langle p^\alpha_t,q^\alpha_t:t\in I_\alpha\rangle:\delta\le\alpha<\lambda\Big\rangle$ is a δ–delayed play according to $\mathbf{st}$, then $\bar\sigma$ satisfies the condition $(\circledast)^{\rm rc}_{\rm B}$.

Observation 2.4. *For $\mathcal{U},\bar\mu$ as in 0.1, $X\in\{A,B,C,\mathbf{a},\mathbf{b},\mathbf{c}\}$ and a forcing notion $\mathbb{Q}$, let $\Phi(\mathbb{Q},X,\mathcal{U},\bar\mu)$ be the statement*

"$\mathbb{Q}$ is reasonably X–bounding over $\mathcal{U}, \bar{\mu}$".

Then the following implications hold

$$\begin{array}{ccccccc}
\Phi(\mathbb{Q}, A, \bar{\mu}) & \Rightarrow & \Phi(\mathbb{Q}, B, \mathcal{U}, \bar{\mu}) & \Rightarrow & \Phi(\mathbb{Q}, C, \mathcal{U}, \bar{\mu}) & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\Phi(\mathbb{Q}, \mathbf{a}, \bar{\mu}) & \Rightarrow & \Phi(\mathbb{Q}, \mathbf{b}, \mathcal{U}, \bar{\mu}) & \Rightarrow & \Phi(\mathbb{Q}, \mathbf{c}, \mathcal{U}, \bar{\mu}) & \Rightarrow & \mathbb{Q} \text{ is } \lambda\text{–proper.}
\end{array}$$

Theorem 2.5. *Assume that $\lambda, \mathcal{U}, \bar{\mu}$ are as in 0.1 and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ is a λ–support iteration such that for every $\xi < \gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{" } \underset{\sim}{\mathbb{Q}}_\xi \text{ is reasonably } B\text{–bounding over } \mathcal{U}, \bar{\mu} \text{ ".}$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is reasonably $\mathbf{b}$–bounding over $\mathcal{U}, \bar{\mu}$ (and so also λ–proper).

Proof. For each $\xi < \gamma$ pick a $\mathbb{P}_\xi$–name $\underset{\sim}{\mathbf{st}}^0_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{" } \underset{\sim}{\mathbf{st}}^0_\xi \text{ is a winning strategy for Complete in } \Game^\lambda_0\big(\underset{\sim}{\mathbb{Q}}_\xi, \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}\big) \text{ such that}$$
$$\text{if Incomplete plays } \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \text{ then Complete answers with } \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \text{ as well ".}$$

Also, for $\xi \leq \gamma$ and $r \in \mathbb{P}_\xi$, let $\mathbf{st}(\xi, r)$ be a winning strategy of Complete in $\Game^\lambda_0(\mathbb{P}_\xi, r)$ with the coherence properties given in 1.3.

We are going to describe a strategy $\mathbf{st}$ for Generic in the game $\Game^{\mathrm{rc}\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{P}_\gamma)$. In the course of the play, at a stage $\delta < \lambda$, Generic will be instructed to construct aside

$(\otimes)_\delta$ $\mathcal{T}_\delta, \bar{p}^\delta_*, \bar{q}^\delta_*, r^-_\delta, r_\delta, w_\delta, \langle \varepsilon_{\delta,\xi}, \bar{p}_{\delta,\xi}, \bar{q}_{\delta,\xi} : \xi \in w_\delta \rangle$, and $\underset{\sim}{\mathbf{st}}_\xi$ for $\xi \in w_{\delta+1} \setminus w_\delta$.

These objects will be chosen so that if

$$\big\langle \zeta_\delta, \langle p^\delta_\zeta, q^\delta_\zeta : \zeta < \zeta_\delta \rangle : \delta < \lambda \big\rangle$$

is a play of $\Game^{\mathrm{rc}\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{P}_\gamma)$ in which Generic follows $\mathbf{st}$, and the side objects constructed at stage $\delta < \lambda$ are listed in $(\otimes)_\delta$, then the following conditions are satisfied (for each $\delta < \lambda$).

$(*)_1$ $r^-_\delta, r_\delta \in \mathbb{P}_\gamma$, $r_0(0) = p(0)$, $w_\delta \subseteq \gamma$, $|w_\delta| = |\delta| + 1$, $\bigcup\limits_{\alpha<\lambda} \mathrm{Dom}(r_\alpha) = \bigcup\limits_{\alpha<\lambda} w_\alpha$, $w_0 = \{0\}$, $w_\delta \subseteq w_{\delta+1}$ and if δ is limit then $w_\delta = \bigcup\limits_{\alpha<\delta} w_\alpha$.

$(*)_2$ For each $\alpha < \delta < \lambda$ we have $(\forall \xi \in w_{\alpha+1})(r_\alpha(\xi) = r_\delta(\xi))$ and $p \leq r^-_\alpha \leq r_\alpha \leq r^-_\delta \leq r_\delta$.

$(*)_3$ If $\xi \in \gamma \setminus w_\delta$, then

$$r_\delta{\restriction}\xi \Vdash \text{" the sequence } \langle r^-_\alpha(\xi), r_\alpha(\xi) : \alpha \leq \delta \rangle \text{ is a legal partial play of}$$
$$\Game^\lambda_0\big(\underset{\sim}{\mathbb{Q}}_\xi, \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi}\big) \text{ in which Complete follows } \underset{\sim}{\mathbf{st}}^0_\xi \text{ "}$$

and if $\xi \in w_{\delta+1} \setminus w_\delta$, then $\underset{\sim}{\mathbf{st}}_\xi$ is a $\mathbb{P}_\xi$–name for a winning strategy of Generic in $\Game^{\mathrm{rcB}}_{\mathcal{U},\bar{\mu}}(r_\delta(\xi), \underset{\sim}{\mathbb{Q}}_\xi)$ such that if $\langle \underset{\sim}{p}^\alpha_t : t \in I_\alpha \rangle$ is given by that strategy to Generic at stage α, then I_α is an ordinal below μ_α. (And $\mathbf{st}_0$ is a suitable winning strategy of Generic in $\Game^{\mathrm{rcB}}_{\mathcal{U},\bar{\mu}}(p(0), \mathbb{Q}_0)$.)

$(*)_4$ $\mathcal{T}_\delta = (T_\delta, \mathrm{rk}_\delta)$ is a standard $(w_\delta, 1)^\gamma$–tree, $|T_\delta| < \mu_\delta$.

$(*)_5$ $\bar{p}^\delta_* = \langle p^\delta_{*,t} : t \in T_\delta \rangle$ and $\bar{q}^\delta_* = \langle q^\delta_{*,t} : t \in T_\delta \rangle$ are standard trees of conditions, $\bar{p}^\delta_* \leq \bar{q}^\delta_*$.

$(*)_6$ For $t\in T_\delta$ we have $\big(\mathrm{Dom}(p)\cup\bigcup\limits_{\alpha<\delta}\mathrm{Dom}(r_\alpha)\cup w_\delta\big)\cap \mathrm{rk}_\delta(t)\subseteq \mathrm{Dom}(p^\delta_{*,t})$ and for each $\xi\in\mathrm{Dom}(p^\delta_{*,t})\setminus w_\delta$:

$$p^\delta_{*,t}{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{`` if the set }\{r_\alpha(\xi):\alpha<\delta\}\cup\{p(\xi)\}\text{ has an upper bound in }\underaccent{\tilde}{\mathbb{Q}}_\xi,\text{ then }p^\delta_{*,t}(\xi)\text{ is such an upper bound ''.}$$

$(*)_7$ $\zeta_\delta=|\{t\in T_\delta:\mathrm{rk}_\delta(t)=\gamma\}|$ and for some enumeration $\{t\in T_\delta:\mathrm{rk}_\delta(t)=\gamma\}=\{t_\zeta:\zeta<\zeta_\delta\}$, for each $\zeta<\zeta_\delta$ we have

$$p^\delta_{*,t_\zeta}\le p^\delta_\zeta\le q^\delta_\zeta\le q^\delta_{*,t_\zeta}.$$

$(*)_8$ If $\xi\in w_\delta$, then $\underaccent{\tilde}{\varepsilon}_{\delta,\xi}$ is a $\mathbb{P}_\xi$–name for an ordinal below μ_δ, $\underaccent{\tilde}{\bar p}_{\delta,\xi},\underaccent{\tilde}{\bar q}_{\delta,\xi}$ are $\mathbb{P}_\xi$–names for sequences of conditions in $\underaccent{\tilde}{\mathbb{Q}}_\xi$ of length $\underaccent{\tilde}{\varepsilon}_{\delta,\xi}$.

$(*)_9$ If $\xi\in w_{\beta+1}\setminus w_\beta$, $\beta<\lambda$, then

$$\Vdash_{\mathbb{P}_\xi}\text{`` }\langle\underaccent{\tilde}{\varepsilon}_{\alpha,\xi},\underaccent{\tilde}{\bar p}_{\alpha,\xi},\underaccent{\tilde}{\bar q}_{\alpha,\xi}:\beta<\alpha<\lambda\rangle\text{ is a delayed play of }\Game^{\mathrm{rcB}}_{\mathcal{U},\bar\mu}(r_\beta(\xi),\underaccent{\tilde}{\mathbb{Q}}_\xi)\text{ in which Generic uses }\underaccent{\tilde}{\mathbf{st}}_\xi\text{ ''.}$$

$(*)_{10}$ If $t\in T_\delta$, $\mathrm{rk}_\delta(t)=\xi<\gamma$, then the condition $p^\delta_{*,t}$ decides the value of $\underaccent{\tilde}{\varepsilon}_{\delta,\xi}$, say $p^\delta_{*,t}\Vdash$"$\underaccent{\tilde}{\varepsilon}_{\delta,\xi}=\varepsilon^t_{\delta,\xi}$", and $\{(s)_\xi:t\vartriangleleft s\in T_\delta\}=\varepsilon^t_{\delta,\xi}$ and

$$q^\delta_{*,t}\Vdash_{\mathbb{P}_\xi}\text{`` }\underaccent{\tilde}{\bar p}_{\delta,\xi}(\varepsilon)\le p^\delta_{*,t^\frown\langle\varepsilon\rangle}(\xi)\text{ for }\varepsilon<\varepsilon^t_{\delta,\xi}\text{ and }\underaccent{\tilde}{\bar q}_{\delta,\xi}=\langle q^\delta_{*,s}(\xi):t\vartriangleleft s\in T_\delta\rangle\text{ ''.}$$

$(*)_{11}$ If $t_0,t_1\in T_\delta$, $\mathrm{rk}_\delta(t_0)=\mathrm{rk}_\delta(t_1)$ and $\xi\in w_\delta\cap\mathrm{rk}_\delta(t_0)$, $t_0{\restriction}\xi=t_1{\restriction}\xi$ but $\big(t_0\big)_\xi\neq\big(t_1\big)_\xi$, then

$$q^\delta_{*,t_0\restriction\xi}\Vdash_{\mathbb{P}_\xi}\text{`` the conditions }q^\delta_{*,t_0}(\xi),q^\delta_{*,t_1}(\xi)\text{ are incompatible ''.}$$

$(*)_{12}$ $\mathrm{Dom}(r_\delta)=\bigcup\limits_{t\in T_\delta}\mathrm{Dom}(q^\delta_{*,t})\cup\mathrm{Dom}(p)$ and if $t\in T_\delta$, $\xi\in\mathrm{Dom}(r_\delta)\cap\mathrm{rk}_\delta(t)\setminus w_\delta$, and $q^\delta_{*,t}{\restriction}\xi\le q\in\mathbb{P}_\xi$, $r_\delta{\restriction}\xi\le q$, then

$$q\Vdash_{\mathbb{P}_\xi}\text{`` if the set }\{r_\alpha(\xi):\alpha<\delta\}\cup\{q^\delta_{*,t}(\xi),p(\xi)\}\text{ has an upper bound in }\underaccent{\tilde}{\mathbb{Q}}_\xi,\text{ then }r_\delta(\xi)\text{ is such an upper bound ''.}$$

To describe the instructions given by $\mathbf{st}$ at stage $\delta<\lambda$ of a play of $\Game^{\mathrm{rc}\mathbf{b}}_{\mathcal{U},\bar\mu}(p,\mathbb{P}_\gamma)$ let us assume that

$$\big\langle\zeta_\alpha,\langle p^\alpha_\zeta,q^\alpha_\zeta:\zeta<\zeta_\alpha\rangle:\alpha<\delta\big\rangle$$

is the result of the play so far and that Generic constructed objects listed in $(\otimes)_\alpha$ (for $\alpha<\delta$) with properties $(*)_1$–$(*)_{12}$.

First, Generic uses her favourite bookkeeping device to determine w_δ such that the demands in $(*)_1$ are satisfied (and that at the end we will have $\bigcup\limits_{\alpha<\lambda}\mathrm{Dom}(r_\alpha)=\bigcup\limits_{\alpha<\lambda}w_\alpha$). Now Generic lets $\mathcal{T}'_\delta$ be a standard $(w_\delta,1)^\gamma$–tree such that for each $\xi\in w_\delta\cup\{\gamma\}$ we have $\{t\in T'_\delta:\mathrm{rk}'_\delta(t)=\xi\}=\prod\limits_{\varepsilon\in w_\delta\cap\xi}\mu_\delta$. Then for $\xi\in w_\delta$ she chooses $\mathbb{P}_\xi$–names $\underaccent{\tilde}{\varepsilon}_{\delta,\xi},\underaccent{\tilde}{\bar p}_{\delta,\xi}$ such that $\underaccent{\tilde}{\varepsilon}_{\delta,\xi}$ is a name for an ordinal below μ_δ and $\underaccent{\tilde}{\bar p}_{\delta,\xi}$ is a name for a sequence of conditions in $\underaccent{\tilde}{\mathbb{Q}}_\xi$ of length $\underaccent{\tilde}{\varepsilon}_{\delta,\xi}$ and

$$\Vdash_{\mathbb{P}_\xi}\text{`` }\underaccent{\tilde}{\varepsilon}_{\delta,\xi},\underaccent{\tilde}{\bar p}_{\delta,\xi}\text{ is the answer to the delayed play }\langle\underaccent{\tilde}{\varepsilon}_{\alpha,\xi},\underaccent{\tilde}{\bar p}_{\alpha,\xi},\underaccent{\tilde}{\bar q}_{\alpha,\xi}:\xi\in w_\alpha\ \&\ \alpha<\delta\rangle\text{ given to Generic by }\underaccent{\tilde}{\mathbf{st}}_\xi\text{ ''.}$$

She lets $\bar p^{\delta,0}_*=\langle p^{\delta,0}_{*,t}:t\in T'_\delta\rangle$ be a tree of conditions defined so that $\mathrm{Dom}(p^{\delta,0}_{*,t})=\big(\mathrm{Dom}(p)\cup\bigcup\limits_{\alpha<\delta}\mathrm{Dom}(r_\alpha)\cup w_\delta\big)\cap\mathrm{rk}'_\delta(t)$ and for each $\xi\in\mathrm{Dom}(p^{\delta,0}_{*,t})$

$(*)_{13}$ $p^{\delta,0}_{*,t}(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for an element of $\underset{\sim}{\mathbb{Q}}_\xi$ such that
- if $\xi \in w_\delta$, then

$$\Vdash_{\mathbb{P}_\xi} \text{“ if } (t)_\xi < \underset{\sim}{\varepsilon}_{\delta,\xi} \text{ then } p^{\delta,0}_{*,t}(\xi) = \underset{\sim}{\bar{p}}_{\delta,\xi}\big((t)_\xi\big), \text{ otherwise } p^{\delta,0}_{*,t}(\xi) = \underset{\sim}{\emptyset}_{\mathbb{Q}_\xi} \text{ ”},$$

- if $\xi \notin w_\delta$, then

$$\Vdash_{\mathbb{P}_\xi} \text{“ if the set } \{r_\alpha(\xi) : \alpha < \delta\} \cup \{p(\xi)\} \text{ has an upper bound in } \underset{\sim}{\mathbb{Q}}_\xi, \text{ then } p^{\delta,0}_{*,t}(\xi) \text{ is such an upper bound ”}.$$

Now Generic uses 1.5(3) and then 1.5(2) to choose a standard tree $(w_\delta, 1)^\gamma$–tree $\mathcal{T}_\delta = (T_\delta, \mathrm{rk}_\delta)$ and a tree of conditions $\bar{p}^\delta_* = \langle p^\delta_{*,t} : t \in T_\delta\rangle$ such that

$(*)^{\mathrm{a}}_{14}$ $T_\delta \subseteq T'_\delta$ and for every $t \in T_\delta$ such that $\mathrm{rk}_\delta(t) = \xi \in w_\delta$ the condition $p^\delta_{*,t}$ decides the value of $\underset{\sim}{\varepsilon}_{\delta,\xi}$, say $p^\delta_{*,t} \Vdash \underset{\sim}{\varepsilon}_{\delta,\xi} = \varepsilon^t_{\delta,\xi}$, and

$(*)^{\mathrm{b}}_{14}$ if $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \xi \in w_\delta$, then $\{\alpha < \lambda : t \cup \{\langle \xi, \alpha\rangle\} \in T_\delta\} = \varepsilon^t_{\delta,\xi}$, and

$(*)^{\mathrm{c}}_{14}$ $p^{\delta,0}_{*,t} \le p^\delta_{*,t}$ for all $t \in T_\delta$, and if $t_0, t_1 \in T_\delta$, $\mathrm{rk}_\delta(t_0) = \mathrm{rk}_\delta(t_1)$, $\xi \in \mathrm{Dom}(t_0)$, and $t_0{\restriction}\xi = t_1{\restriction}\xi$ but $(t_0)_\xi \neq (t_1)_\xi$, then

$$p^\delta_{*,t_0\restriction\xi} \Vdash_{\mathbb{P}_\xi} \text{“ the conditions } p^\delta_{*,t_0}(\xi), p^\delta_{*,t_1}(\xi) \text{ are incompatible in } \underset{\sim}{\mathbb{Q}}_\xi \text{ ”},$$

Thus Generic has written aside $\mathcal{T}_\delta$, $\bar{p}^\delta_*$, w_δ and $\langle \underset{\sim}{\varepsilon}_{\delta,\xi}, \underset{\sim}{\bar{p}}_{\delta,\xi} : \xi \in w_\delta\rangle$. (It should be clear that they satisfy the relevant demands in $(*)_1$, $(*)_4$–$(*)_6$, $(*)_8$ and $(*)_9$, $(*)_{10}$.) Now she turns to the play of $\eth^{\mathrm{rc}\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{P}_\gamma)$ and she puts

$$\zeta_\delta = |\{t \in T_\delta : \mathrm{rk}_\delta(t) = \gamma\}|$$

and she also picks an enumeration $\langle t_\zeta : \zeta < \zeta_\delta\rangle$ of $\{t \in T_\delta : \mathrm{rk}_\delta(t) = \gamma\}$. The two players start playing the subgame of level δ of length ζ_δ. During the subgame Generic constructs partial plays $\langle (r^\zeta_i, s^\zeta_i) : i \le \zeta_\delta\rangle$ of $\eth^\lambda_0(\mathbb{P}_\gamma, p^\delta_{*,t_\zeta})$ (for $\zeta < \zeta_\delta$) in which Complete uses the strategy $\mathbf{st}(\gamma, p^\delta_{*,t_\zeta})$ and such that

$(*)^{\mathrm{a}}_{15}$ if $\zeta, \xi < \zeta_\delta$, $t \in T_\delta$, $t \lhd t_\zeta$, $t \lhd t_\xi$, $i \le \zeta_\delta$, then $r^\zeta_i{\restriction}\mathrm{rk}_\delta(t) = r^\xi_i{\restriction}\mathrm{rk}_\delta(t)$ and $s^\zeta_i{\restriction}\mathrm{rk}_\delta(t) = s^\xi_i{\restriction}\mathrm{rk}_\delta(t)$;

$(*)^{\mathrm{b}}_{15}$ if $p^\delta_\zeta, q^\delta_\zeta$ are the conditions played at stage ζ of the subgame, then $p^\delta_{*,t_\zeta} \le r^\zeta_i \le p^\delta_\zeta \le q^\delta_\zeta = r^\zeta_\zeta$ for all $i < \zeta$.

So suppose that the two players have arrived at a stage $\zeta < \zeta_\delta$ of the subgame and $\big\langle \langle (r^\xi_i, s^\xi_i) : i < \zeta\rangle : \xi < \zeta_\delta\big\rangle$ has been defined. Generic looks at $\langle (r^\zeta_i, s^\zeta_i) : i < \zeta\rangle$ – it is a play of $\eth^\lambda_0(\mathbb{P}_\gamma, p^\delta_{*,t_\zeta})$ in which Complete uses $\mathbf{st}(\gamma, p^\delta_{*,t_\zeta})$, so we may find a condition $p^\delta_\zeta \in \mathbb{P}_\gamma$ stronger than all r^ζ_i, s^ζ_i for $i < \zeta$ (and $p^\delta_\zeta \ge p^\delta_{*,t_\zeta}$). She plays this condition as her move at stage ζ of the subgame and Antigeneric answers with $q^\delta_\zeta \ge p^\delta_\zeta$. Generic lets $r^\zeta_\zeta = q^\delta_\zeta$ and she defines r^ξ_ζ for $\xi < \zeta_\delta$, $\xi \neq \zeta$, as follows. Let $t \in T_\delta$ be such that $t \lhd t_\zeta$, $t \lhd t_\xi$ and $\mathrm{rk}_\delta(t)$ is the largest possible. Generic declares that

$$\mathrm{Dom}(r^\xi_\zeta) = \big(\mathrm{Dom}(r^\zeta_\zeta) \cap \mathrm{rk}_\delta(t)\big) \cup \bigcup_{i<\zeta} \mathrm{Dom}(s^\xi_i) \cup \mathrm{Dom}(p^\delta_{*,t_\xi}),$$

and $r^\xi_\zeta{\restriction}\mathrm{rk}_\delta(t) = r^\zeta_\zeta{\restriction}\mathrm{rk}_\delta(t)$, and for $\varepsilon \in \mathrm{Dom}(r^\xi_\zeta) \setminus \mathrm{rk}_\delta(t)$ she lets $r^\xi_\zeta(\varepsilon)$ be the $<^*_\chi$–first $\mathbb{P}_\varepsilon$–name for a member of $\underset{\sim}{\mathbb{Q}}_\varepsilon$ such that

$$r^\xi_\zeta{\restriction}\varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“ } r^\xi_\zeta(\varepsilon) \text{ is an upper bound to } \{p^\delta_{*,t_\xi}(\varepsilon)\} \cup \{s^\xi_i(\varepsilon) : i < \zeta\} \text{ ”}$$

(remember 1.3(iv)). Finally, s^ξ_ζ (for $\xi<\zeta_\delta$) is defined as the condition given to Complete by $\mathbf{st}(\gamma,p^\delta_{*,t_\xi})$ in answer to $\langle (r^\xi_i,s^\xi_i): i<\zeta\rangle^\frown\langle r^\xi_\zeta\rangle$. It follows from 1.3(ii) that $(*)^{\rm a}_{15}$ is still satisfied for the s^ξ_ζ.

After the subgame is completed and both $p^\delta_\zeta,q^\delta_\zeta$ and $\big\langle\langle (r^\xi_i,s^\xi_i): i<\zeta_\delta\rangle:\xi<\zeta_\delta\big\rangle$ have been determined, Generic chooses $r^0_{\zeta_\delta}$ as any upper bound to $\langle s^0_i: i<\zeta_\delta\rangle$ and then defines $r^\xi_{\zeta_\delta}$ for $\xi\in\zeta_\delta\setminus 1$ like r^ξ_ζ for $\xi\neq\zeta$ above. Also $s^\xi_{\zeta_\delta}$ (for $\xi<\zeta_\delta$) are chosen like earlier (as results of applying $\mathbf{st}(\gamma,p^\delta_{*,t_\xi})$). Finally, Generic picks a standard tree of conditions $\bar q^\delta_*=\langle q^\delta_{*,t}: t\in T_\delta\rangle$ such that $(\forall\zeta<\zeta_\delta)(q^\delta_{*,t_\zeta}=s^\zeta_{\zeta_\delta})$. (Note that $(*)_5$, $(*)_7$ hold.)

Now Generic defines $r^-_\delta,r_\delta\in\mathbb{P}_\gamma$ so that

$$\mathrm{Dom}(r^-_\delta)=\mathrm{Dom}(r_\delta)=\bigcup_{t\in T_\delta}\mathrm{Dom}(q^\delta_{*,t})\cup\mathrm{Dom}(p)$$

and

$(*)^{\rm a}_{16}$ if $\xi\in\mathrm{Dom}(r^-_\delta)\setminus w_\delta$, then:
$r^-_\delta(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for an element of $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$r^-_\delta{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{“ } r^-_\delta(\xi)\text{ is an upper bound of }\{r_\alpha(\xi):\alpha<\delta\}\cup\{p(\xi)\}\text{ and if } t\in T_\delta,\ \mathrm{rk}_\delta(t)>\xi,\text{ and } q^\delta_{*,t}{\restriction}\xi\in\Gamma_{\mathbb{P}_\xi}\text{ and the set }\{r_\alpha(\xi):\alpha<\delta\}\cup\{q^\delta_{*,t}(\xi),p(\xi)\}\text{ has an upper bound in }\underset{\sim}{\mathbb{Q}}_\xi,\text{ then } r^-_\delta(\xi)\text{ is such an upper bound ”},$$

and $r_\delta(\xi)$ is the $<^*_\chi$–first $\mathbb{P}_\xi$–name for an element of $\underset{\sim}{\mathbb{Q}}_\xi$ such that

$$r_\delta{\restriction}\xi\Vdash_{\mathbb{P}_\xi}\text{“ } r_\delta(\xi)\text{ is given to Complete by }\underset{\sim}{\mathbf{st}}^0_\xi\text{ as the answer to }\langle r^-_\alpha(\xi),r_\alpha(\xi):\alpha<\delta\rangle^\frown\langle r^-_\delta(\xi)\rangle\text{ ”}$$

$(*)^{\rm b}_{16}$ if $\xi\in w_{\alpha+1}$, $\alpha<\delta$, then $r^-_\delta(\xi)=r_\delta(\xi)=r_\alpha(\xi)$.

(Note that by a straightforward induction on $\xi\in\mathrm{Dom}(r_\delta)$ one easily applies $(*)_3$ from previous stages to show that r^-_δ,r_δ are well defined and $r_\delta\geq r^-_\delta\geq r_\alpha,p$ for $\alpha<\delta$. Remember also $(*)_{11}$ and/or $(*)^{\rm c}_{14}$.) If $\delta=0$ we also stipulate $r^-_0(0)=r_0(0)=p(0)$.

Finally, for each $\xi\in w_\delta$, Generic chooses a $\mathbb{P}_\xi$–name $\underset{\sim}{\bar q}_{\delta,\xi}$ for a sequence of conditions in $\underset{\sim}{\mathbb{Q}}_\xi$ of length $\underset{\sim}{\varepsilon}_{\delta,\xi}$ such that

$$\Vdash_{\mathbb{P}_\xi}\text{“ }(\forall\varepsilon<\underset{\sim}{\varepsilon}_{\delta,\xi})(\underset{\sim}{\bar p}_{\delta,\xi}(\varepsilon)\leq\underset{\sim}{\bar q}_{\delta,\xi}(\varepsilon))\text{ and if } t\in T_\delta,\ \mathrm{rk}_\delta(t)>\xi,\text{ and } q^\delta_{*,t}{\restriction}\xi\in\Gamma_{\mathbb{P}_\xi}\text{ then }\underset{\sim}{\bar q}_{\delta,\xi}\big((t)_\xi\big)=q^\delta_{*,t}(\xi)\text{ ”}.$$

Generic also picks $w_{\delta+1}$ by the bookkeeping device mentioned at the beginning and for $\xi\in w_{\delta+1}\setminus w_\delta$ she fixes $\underset{\sim}{\mathbf{st}}_\xi$ as in $(*)_3$.

This completes the description of the side objects constructed by Generic at stage δ. Verification that they satisfy our demands $(*)_1$–$(*)_{12}$ is straightforward, and thus the description of the strategy $\mathbf{st}$ is complete.

We are going to argue now that $\mathbf{st}$ is a winning strategy for Generic. To this end suppose that

$$\big\langle\zeta_\delta,\langle p^\delta_\zeta,q^\delta_\zeta:\zeta<\zeta_\delta\rangle:\delta<\lambda\big\rangle$$

is the result of a play of $\Game^{\rm rc\mathbf{b}}_{\mathcal{U},\bar\mu}(p,\mathbb{P}_\gamma)$ in which Generic followed $\mathbf{st}$ and constructed aside objects listed in $(\otimes)_\delta$ (for $\delta<\lambda$) so that $(*)_1$–$(*)_{12}$ hold.

We define a condition $r\in\mathbb{P}_\gamma$ as follows. Let $\mathrm{Dom}(r)=\bigcup\limits_{\delta<\lambda}\mathrm{Dom}(r_\delta)$ and for $\xi\in\mathrm{Dom}(r)$ let $r(\xi)$ be a $\mathbb{P}_\xi$–name for a condition in $\mathbb{Q}_\xi$ such that if $\xi\in w_{\alpha+1}\setminus w_\alpha$, $\alpha<\lambda$ (or $\xi=0=\alpha$), then

$$\Vdash_{\mathbb{P}_\xi} \text{“ } r(\xi)\geq r_\alpha(\xi) \text{ and } r(\xi)\Vdash_{\mathbb{Q}_\xi} \big\{\delta<\lambda\colon \big(\exists\varepsilon<\varepsilon_{\delta,\xi}\big)\big(\bar{q}_{\delta,\xi}(\varepsilon)\in\Gamma_{\mathbb{Q}_\xi}\big)\big\}\in(\mathcal{U}^{\mathbb{P}_\xi})^{\mathbb{Q}_\xi} \text{ ”}.$$

Clearly r is well defined (remember $(*)_9$) and $(\forall\delta<\lambda)(r_\delta\leq r)$ and $r\geq p$. For each $\xi\in\mathrm{Dom}(r)$ choose a sequence $\langle A^\xi_i : i<\lambda\rangle$ of $\mathbb{P}_{\xi+1}$–names for elements of $\mathcal{U}\cap\mathbf{V}$ such that

$(*)^\xi_{17}$ $\quad r{\restriction}(\xi+1)\Vdash_{\mathbb{P}_{\xi+1}} \big(\forall\delta\in\mathop{\triangle}\limits_{i<\lambda} A^\xi_i\big)\big(\exists\varepsilon<\varepsilon_{\delta,\xi}\big)\big(\bar{q}_{\delta,\xi}(\varepsilon)\in\Gamma_{\mathbb{Q}_\xi}\big)$.

Claim 2.5.1. *For each limit ordinal $\delta<\lambda$,*

$$r\Vdash_{\mathbb{P}_\gamma} \text{“ } \big(\forall\xi\in w_\delta\big)\big(\delta\in\mathop{\triangle}\limits_{i<\lambda} A^\xi_i\big)\ \Rightarrow\ \big(\exists t\in T_\delta\big)\big(\mathrm{rk}_\delta(t)=\gamma\ \&\ q^\delta_{*,t}\in\Gamma_{\mathbb{P}_\gamma}\big) \text{ ”}.$$

Proof of the Claim. Suppose that $r'\geq r$ and a limit ordinal $\delta<\lambda$ are such that

$(*)_{18}$ $\quad r'\Vdash_{\mathbb{P}_\gamma}$ “ $\big(\forall\xi\in w_\delta\big)\big(\delta\in\mathop{\triangle}\limits_{i<\lambda} A^\xi_i\big)$ ”.

We are going to show that there is $t\in T_\delta$ such that $\mathrm{rk}_\delta(t)=\gamma$ and the conditions $q^\delta_{*,t}$ and r' are compatible (and then the claim will readily follow). To this end let $\langle\varepsilon_\alpha:\alpha\leq\alpha^*\rangle=w_\delta\cup\{\gamma\}$ be the increasing enumeration. By induction on $\alpha\leq\alpha^*$ we will choose conditions $r^*_\alpha,r^{**}_\alpha\in\mathbb{P}_{\varepsilon_\alpha}$ and $t=\langle(t)_{\varepsilon_\alpha}:\alpha<\alpha^*\rangle\in T_\delta$ such that letting $t^\alpha_\circ=\langle(t)_{\varepsilon_\beta}:\beta<\alpha\rangle\in T_\delta$ we have

- $(*)^\alpha_{19}$ $q^\delta_{*,t^\alpha_\circ}\leq r^*_\alpha$ and $r'{\restriction}\varepsilon_\alpha\leq r^*_\alpha$,
- $(*)^\alpha_{20}$ $\langle r^*_\beta{}^\frown r'{\restriction}[\varepsilon_\beta,\gamma), r^{**}_\beta{}^\frown r'{\restriction}[\varepsilon_\beta,\gamma):\beta<\alpha\rangle$ is a partial legal play of $\Game^\lambda_0(\mathbb{P}_\gamma,r')$ in which Complete uses her winning strategy $\mathbf{st}(\gamma,r')$.

Suppose that $\alpha\leq\alpha^*$ is a limit ordinal and we have already defined $t^\alpha_\circ=\langle(t)_{\varepsilon_\beta}:\beta<\alpha\rangle$ and $\langle r^*_\beta,r^{**}_\beta:\beta<\alpha\rangle$. Let $\xi=\sup(\varepsilon_\beta:\beta<\alpha)$. It follows from $(*)^\beta_{20}$ (for $\beta<\alpha$) that we may find a condition $s\in\mathbb{P}_\xi$ stronger than all r^{**}_β (for $\beta<\alpha$). Let $r^*_\alpha\in\mathbb{P}_{\varepsilon_\alpha}$ be such that $r^*_\alpha{\restriction}\xi=s$ and $r^*_\alpha{\restriction}[\xi,\varepsilon_\alpha)=r'{\restriction}[\xi,\varepsilon_\alpha)$. It follows from $(*)^\beta_{19}$ that $q^\delta_{*,t^\alpha_\circ}{\restriction}\xi\leq s=r^*_\alpha{\restriction}\xi$ and $r'{\restriction}\xi\leq s=r^*_\alpha{\restriction}\xi$. Note also that $(\forall\beta<\alpha)(r^{**}_\beta\leq s{\restriction}\varepsilon_\beta=r^*_\alpha{\restriction}\varepsilon_\beta)$, so $(\forall\beta<\alpha)(r^{**}_\beta{}^\frown r'{\restriction}[\varepsilon_\beta,\gamma)\leq r^*_\alpha{}^\frown r'{\restriction}[\varepsilon_\alpha,\gamma))$. Now by induction on $\zeta\leq\varepsilon_\alpha$ we show that $q^\delta_{*,t^\alpha_\circ}{\restriction}\zeta\leq r^*_\alpha{\restriction}\zeta$ and $r'{\restriction}\zeta\leq r^*_\alpha{\restriction}\zeta$. For $\zeta\leq\xi$ we are already done, so assume that $\zeta\in[\xi,\varepsilon_\alpha)$ and we have shown $q^\delta_{*,t^\alpha_\circ}{\restriction}\zeta\leq r^*_\alpha{\restriction}\zeta$ and $r'{\restriction}\zeta\leq r^*_\alpha{\restriction}\zeta$. It follows from $(*)_6+(*)_3$ that $r^*_\alpha{\restriction}\zeta\Vdash(\forall i<\delta)(r_i(\zeta)\leq p^\delta_{*,t^\alpha_\circ}(\zeta))$ and therefore we may use $(*)_{12}$ to conclude that

$$r^*_\alpha{\restriction}\zeta\Vdash_{\mathbb{P}_\zeta} q^\delta_{*,t^\alpha_\circ}(\zeta)\leq r_\delta(\zeta)\leq r(\zeta)\leq r'(\zeta)=r^*_\alpha(\zeta).$$

The limit stages are trivial and we see that $(*)^\alpha_{19}$ and (a part of) $(*)^\alpha_{20}$ hold. Finally we let $r^{**}_\alpha\in\mathbb{P}_{\varepsilon_\alpha}$ be the condition given to Complete by $\mathbf{st}(\gamma,r')$ as the response to $\langle r^*_\beta{}^\frown r'{\restriction}[\varepsilon_\beta,\gamma), r^{**}_\beta{}^\frown r'{\restriction}[\varepsilon_\beta,\gamma):\beta<\alpha\rangle{}^\frown\langle r^*_\alpha\rangle$.

Now suppose that $\alpha=\beta+1\leq\alpha^*$ and we have already defined $r^*_\beta,r^{**}_\beta\in\mathbb{P}_{\varepsilon_\beta}$ and $t^\beta_\circ\in T_\delta$. It follows from $(*)^{\varepsilon_\beta}_{17}+(*)_{18}+(*)^\beta_{19}+(*)_{10}$ that

$$r^{**}_\beta\Vdash_{\mathbb{P}_{\varepsilon_\beta}} \text{“ } r'(\varepsilon_\beta)\Vdash_{\mathbb{Q}_{\varepsilon_\beta}} \big(\exists\varepsilon<\varepsilon^{t^\beta_\circ}_{\delta,\varepsilon_\beta}\big)\big(q^\delta_{*,t^\beta_\circ{}^\frown\langle\varepsilon\rangle}(\varepsilon_\beta)\in\Gamma_{\mathbb{Q}_{\varepsilon_\beta}}\big) \text{ ”}.$$

Therefore we may choose $\varepsilon = (t)_{\varepsilon_\beta} < \varepsilon^{t^\beta_\circ}_{\delta,\varepsilon_\beta}$ (thus defining $t^\alpha_\circ$) and a condition $s \in \mathbb{P}_{\varepsilon_\beta+1}$ such that $s{\restriction}\varepsilon_\beta \geq r^{**}_\beta \geq q^\delta_{*,t^\beta_\circ}$ and

$$s{\restriction}\varepsilon_\beta \Vdash \text{“ } s(\varepsilon_\beta) \geq r'(\varepsilon_\beta) \ \&\ s(\varepsilon_\beta) \geq q^\delta_{*,t^\alpha_\circ}(\varepsilon_\beta) \text{ ”}.$$

We let $r^*_\alpha \in \mathbb{P}_{\varepsilon_\alpha}$ be such that $r^*_\alpha{\restriction}(\varepsilon_\beta+1) = s$ and $r^*_\alpha{\restriction}(\varepsilon_\beta,\varepsilon_\alpha) = r'{\restriction}(\varepsilon_\beta,\varepsilon_\alpha)$. Exactly like in the limit case we argue that $(*)^\alpha_{19}$ and (a part of) $(*)^\alpha_{20}$ hold and then in the same manner as there we define r^{**}_α.

Finally note that $t \in T_\delta$, $\mathrm{rk}_\delta(t) = \gamma$, and the condition $r^*_{\alpha^*}$ witnesses that r' and $q^\delta_{*,t}$ are compatible. □

Now note that

$$\Vdash_{\mathbb{P}_\gamma} \text{“ } \big\{\delta < \lambda : \big(\forall \xi \in w_\delta\big)\big(\delta \in \mathop{\triangle}_{i<\lambda} \underset{\sim}{A}^\xi_i\big)\big\} \in \mathcal{U}^{\mathbb{P}_\gamma} \text{ ”},$$

and hence by 2.5.1 we have

$$r \Vdash_{\mathbb{P}_\gamma} \text{“ } \big\{\delta < \lambda : \big(\exists t \in T_\delta\big)\big(\mathrm{rk}_\delta(t) = \gamma \ \&\ q^\delta_{*,t} \in \Gamma_{\mathbb{P}_\gamma}\big)\big\} \in \mathcal{U}^{\mathbb{P}_\gamma} \text{ ”}.$$

Therefore, by $(*)_7$,

$$r \Vdash_{\mathbb{P}_\gamma} \text{“ } \big\{\delta < \lambda : \big(\exists \zeta < \zeta_\delta\big)\big(q^\delta_\zeta \in \Gamma_{\mathbb{P}_\gamma}\big)\big\} \in \mathcal{U}^{\mathbb{P}_\gamma} \text{ ”}$$

and the proof of the theorem is complete. □

Remark 2.6. The reason for the weaker “**b**–bounding” in the conclusion of 2.5 (and not “B–bounding”) is that in our description of the strategy **st**, we would have to make sure that the conditions played by Antigeneric form a tree of conditions. Playing a subgame and keeping the demands of $(*)_{15}$ are a convenient way to deal with this issue.

Similar work and arguments may be carried out for A/**a** –bounding. However, in a subsequent paper [RS11b] we find out that getting *reasonably* **a**–*bounding* for the limit of the iteration is not sufficient for the applications there. With these applications in mind we introduce a stronger property which more precisely captures what can be claimed on iterations of reasonably A–bounding forcing notions.

Definition 2.7. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ be a λ–support iteration.

(1) For a condition $p \in \mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ we define a game $\partial^{\mathrm{tree}\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ between two players, Generic and Antigeneric, as follows. A play of $\partial^{\mathrm{tree}\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ lasts λ steps and in the course of the play a sequence $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$ is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a standard $(w,1)^\gamma$–tree $\mathcal{T}_\alpha$ such that $|T_\alpha| < \mu_\alpha$ and a tree of conditions $\bar{p}^\alpha = \langle p^\alpha_t : t \in T_\alpha\rangle \subseteq \mathbb{P}_\gamma$,

$(\beth)_\alpha$ then Antigeneric answers by picking a tree of conditions $\bar{q}^\alpha = \langle q^\alpha_t : t \in T_\alpha\rangle \subseteq \mathbb{P}_\gamma$ such that $\bar{p}^\alpha \leq \bar{q}^\alpha$.

At the end, Generic wins the play $\langle \mathcal{T}_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$ of $\partial^{\mathrm{tree}\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ if and only if

$(\circledast)^{\mathrm{tree}}_{\mathbf{A}}$ there is a condition $p^* \in \mathbb{P}_\gamma$ stronger than p and such that

$$p^* \Vdash_{\mathbb{P}_\gamma} \text{“ } \big(\forall \alpha < \lambda\big)\big(\exists t \in T_\alpha\big)\big(\mathrm{rk}_\alpha(t) = \gamma \ \&\ q^\alpha_t \in \Gamma_{\mathbb{P}_\gamma}\big) \text{ ”}$$

(2) We say that $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is *reasonably** $A(\bar{\mathbb{Q}})$*–bounding over* $\bar\mu$ if Generic has a winning strategy in the game $\partial^{\mathrm{tree}\mathbf{A}}_{\bar\mu}(p,\bar{\mathbb{Q}})$ for every $p \in \mathbb{P}_\gamma$.

Theorem 2.8. *Assume that $\lambda,\bar{\mu}$ are as in 0.1 and $\bar{\mathbb{Q}}=\langle \mathbb{P}_\xi,\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi:\xi<\gamma\rangle$ is a λ–support iteration such that for every $\xi<\gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \mathop{\mathbb{Q}}\limits_{\sim}{}_\xi \text{ is reasonably } A\text{–bounding over } \bar{\mu} \text{ ”.}$$

Then $\mathbb{P}_\gamma=\lim(\bar{\mathbb{Q}})$ is reasonably $A(\bar{\mathbb{Q}})$–bounding over $\bar{\mu}$.*

Proof. This is a variation on the proof of Theorem 2.5, but let us sketch the proof of our present version. For each $\xi<\gamma$ pick a $\mathbb{P}_\xi$–name $\mathop{\mathbf{st}}\limits_{\sim}{}^0_\xi$ such that

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \mathop{\mathbf{st}}\limits_{\sim}{}^0_\xi \text{ is a winning strategy for Complete in } \Game^\lambda_0\big(\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi,\emptyset_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi}\big) \text{ such that}$$
$$\text{if Incomplete plays } \emptyset_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi} \text{ then Complete answers with } \emptyset_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi} \text{ as well ”.}$$

Let $p\in\mathbb{P}_\gamma$. We are going to describe a strategy $\mathbf{st}$ for Generic in the game $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p,\bar{\mathbb{Q}})$. In the course of the play, at a stage $\delta<\lambda$, Generic is instructed to construct aside

$(\otimes)_\delta$ $\quad r^-_\delta, r_\delta, w_\delta,\ \langle \mathop{\varepsilon}\limits_{\sim}{}_{\delta,\xi},\mathop{\bar{p}}\limits_{\sim}{}_{\delta,\xi},\mathop{\bar{q}}\limits_{\sim}{}_{\delta,\xi}:\xi\in w_\delta\rangle$, and $\mathop{\mathbf{st}}\limits_{\sim}{}_\xi$ for $\xi\in w_{\delta+1}\setminus w_\delta$.

These objects are to be chosen so that if $\big\langle \mathcal{T}_\delta,\bar{p}^\delta,\bar{q}^\delta:\delta<\lambda\big\rangle$ is a play of $\Game^{\rm tree\mathbf{A}}_{\bar{\mu}}(p,\bar{\mathbb{Q}})$ in which Generic follows $\mathbf{st}$, and the additional objects constructed at stage $\delta<\lambda$ are listed in $(\otimes)_\delta$, then the following conditions are satisfied (for each $\delta<\lambda$).

- $(*)_1$ $r^-_\delta,r_\delta\in\mathbb{P}_\gamma$, $r_0(0)=p(0)$, $w_\delta\subseteq\gamma$, $|w_\delta|=|\delta+1|$, $\bigcup\limits_{\alpha<\lambda}{\rm Dom}(r_\alpha)=\bigcup\limits_{\alpha<\lambda} w_\alpha$, $w_0=\{0\}$, $w_\delta\subseteq w_{\delta+1}$ and if δ is limit then $w_\delta=\bigcup\limits_{\alpha<\delta} w_\alpha$.
- $(*)_2$ For each $\alpha<\delta<\lambda$ we have $(\forall\xi\in w_{\alpha+1})(r_\alpha(\xi)=r_\delta(\xi))$ and $p\le r^-_\alpha\le r_\alpha\le r^-_\delta\le r_\delta$.
- $(*)_3$ If $\xi\in\gamma\setminus w_\delta$, then

$$r_\delta{\restriction}\xi\Vdash \text{“ the sequence } \langle r^-_\alpha(\xi),r_\alpha(\xi):\alpha\le\delta\rangle \text{ is a legal partial play of } \Game^\lambda_0\big(\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi,\emptyset_{\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi}\big) \text{ in which Complete follows } \mathop{\mathbf{st}}\limits_{\sim}{}^0_\xi \text{ ”}$$

  and if $\xi\in w_{\delta+1}\setminus w_\delta$, then $\mathop{\mathbf{st}}\limits_{\sim}{}_\xi$ is a $\mathbb{P}_\xi$–name for a winning strategy of Generic in $\Game^{\rm rcA}_{\bar{\mu}}(r_\delta(\xi),\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi)$ such that if $\langle p^\alpha_t:t\in I_\alpha\rangle$ is given by that strategy to Generic at stage α, then I_α is an ordinal below μ_α. Also $\mathbf{st}_0$ is a suitable winning strategy of Generic in $\Game^{\rm rcA}_{\bar{\mu}}(p(0),\mathbb{Q}_0)$.
- $(*)_4$ $\mathcal{T}_\delta=(T_\delta,{\rm rk}_\delta)$ is a standard $(w_\delta,1)^\gamma$–tree, $|T_\delta|<\mu_\delta$.
- $(*)_5$ $\bar{p}^\delta=\langle p^\delta_t:t\in T_\delta\rangle$ and $\bar{q}^\delta=\langle q^\delta_t:t\in T_\delta\rangle$ are standard trees of conditions in $\bar{\mathbb{Q}}$, $\bar{p}^\delta\le\bar{q}^\delta$ and $\mathcal{T}_\delta,\bar{p}^\delta$ and $\bar{q}^\delta$ are the innings of the two players at stage δ.
- $(*)_6$ For $t\in T_\delta$ we have $\big({\rm Dom}(p)\cup\bigcup\limits_{\alpha<\delta}{\rm Dom}(r_\alpha)\cup w_\delta\big)\cap{\rm rk}_\delta(t)\subseteq{\rm Dom}(p^\delta_t)$ and for each $\xi\in{\rm Dom}(p^\delta_t)\setminus w_\delta$:

$$p^\delta_t{\restriction}\xi\Vdash_{\mathbb{P}_\xi} \text{“ if the set } \{r_\alpha(\xi):\alpha<\delta\}\cup\{p(\xi)\} \text{ has an upper bound in } \mathop{\mathbb{Q}}\limits_{\sim}{}_\xi, \text{ then } p^\delta_t(\xi) \text{ is such an upper bound ”.}$$

- $(*)_7$ If $\xi\in w_\delta$, then $\mathop{\varepsilon}\limits_{\sim}{}_{\delta,\xi}$ is a $\mathbb{P}_\xi$–name for an ordinal below μ_δ, $\mathop{\bar{p}}\limits_{\sim}{}_{\delta,\xi},\mathop{\bar{q}}\limits_{\sim}{}_{\delta,\xi}$ are $\mathbb{P}_\xi$–names for $\mathop{\varepsilon}\limits_{\sim}{}_{\delta,\xi}$–sequences of conditions in $\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi$.
- $(*)_8$ If $\xi\in w_{\beta+1}\setminus w_\beta$, $\beta<\lambda$, then

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \langle \mathop{\varepsilon}\limits_{\sim}{}_{\alpha,\xi},\mathop{\bar{p}}\limits_{\sim}{}_{\alpha,\xi},\mathop{\bar{q}}\limits_{\sim}{}_{\alpha,\xi}:\beta<\alpha<\lambda\rangle \text{ is a delayed play of } \Game^{\rm rcA}_{\bar{\mu}}(r_\beta(\xi),\mathop{\mathbb{Q}}\limits_{\sim}{}_\xi) \text{ in which Generic uses } \mathop{\mathbf{st}}\limits_{\sim}{}_\xi \text{ ”.}$$

- $(*)_9$ If $t\in T_\delta$, ${\rm rk}_\delta(t)=\xi<\gamma$, then the condition p^δ_t decides the value of $\mathop{\varepsilon}\limits_{\sim}{}_{\delta,\xi}$, say $p^\delta_t\Vdash$“$\mathop{\varepsilon}\limits_{\sim}{}_{\delta,\xi}=\varepsilon^t_{\delta,\xi}$”, and $\{(s)_\xi:t\vartriangleleft s\in T_\delta\}=\varepsilon^t_{\delta,\xi}$ and

$$q^\delta_t\Vdash_{\mathbb{P}_\xi} \text{“ } \mathop{\bar{p}}\limits_{\sim}{}_{\delta,\xi}(\varepsilon)\le p^\delta_{t^\frown\langle\varepsilon\rangle}(\xi) \text{ and } \mathop{\bar{q}}\limits_{\sim}{}_{\delta,\xi}(\varepsilon)=q^\delta_{t^\frown\langle\varepsilon\rangle}(\xi) \text{ for } \varepsilon<\varepsilon^t_{\delta,\xi} \text{ ”.}$$

$(*)_{10}$ If $t_0, t_1 \in T_\delta$, $\mathrm{rk}_\delta(t_0) = \mathrm{rk}_\delta(t_1)$ and $\xi \in w_\delta \cap \mathrm{rk}_\delta(t_0)$, $t_0 \restriction \xi = t_1 \restriction \xi$ but $(t_0)_\xi \neq (t_1)_\xi$, then

$$p^\delta_{t_0\restriction\xi} \Vdash_{\mathbb{P}_\xi} \text{" the conditions } p^\delta_{t_0}(\xi), p^\delta_{t_1}(\xi) \text{ are incompatible ".}$$

$(*)_{11}$ $\mathrm{Dom}(r_\delta) = \bigcup\limits_{t\in T_\delta} \mathrm{Dom}(q^\delta_t) \cup \mathrm{Dom}(p)$ and if $t \in T_\delta$, $\xi \in \mathrm{Dom}(r_\delta) \cap \mathrm{rk}_\delta(t) \setminus w_\delta$, and $q^\delta_t \restriction \xi \le q \in \mathbb{P}_\xi$, $r_\delta \restriction \xi \le q$, then

$q \Vdash_{\mathbb{P}_\xi}$ " if the set $\{r_\alpha(\xi) : \alpha < \delta\} \cup \{q^\delta_t(\xi), p(\xi)\}$ has an upper bound in $\mathbb{Q}_\xi$, then $r^-_\delta(\xi)$ is such an upper bound ".

The detailed description of the strategy **st** closely follows the description of the strategy **st** in the proof of 2.5 (after the formulation of $(*)_1$–$(*)_{12}$ there). To argue that **st** is a winning strategy for Generic, suppose that $\langle \mathcal{T}_\delta, \bar{p}^\delta, \bar{q}^\delta : \delta < \lambda \rangle$ is the result of a play of $\Game^{\mathrm{tree}\mathbf{A}}_{\bar{\mu}}(p, \bar{\mathbb{Q}})$ in which Generic followed **st** and constructed objects listed in $(\otimes)_\delta$ (for $\delta < \lambda$) so that $(*)_1$–$(*)_{11}$ hold. Define a condition $r \in \mathbb{P}_\gamma$ as follows. Let $\mathrm{Dom}(r) = \bigcup\limits_{\delta<\lambda} \mathrm{Dom}(r_\delta)$ and for $\xi \in \mathrm{Dom}(r)$ let $r(\xi)$ be a $\mathbb{P}_\xi$–name for a condition in $\mathbb{Q}_\xi$ such that if $\xi \in w_{\alpha+1} \setminus w_\alpha$, $\alpha < \lambda$ (or $\xi = 0 = \alpha$), then

$$\Vdash_{\mathbb{P}_\xi} \text{" } r(\xi) \ge r_\alpha(\xi) \text{ and } r(\xi) \Vdash_{\mathbb{Q}_\xi} (\forall \delta < \lambda)(\exists \varepsilon < \varepsilon_{\delta,\xi})(\bar{q}_{\delta,\xi}(\varepsilon) \in \Gamma_{\mathbb{Q}_\xi}) \text{ ".}$$

Clearly r is well defined (remember $(*)_8$) and $(\forall \delta < \lambda)(r_\delta \le r)$ and $r \ge p$. An argument following the lines of the proof of Claim 2.5.1 shows that for each $\delta < \lambda$ the family $\{q^\delta_t : t \in T_\delta \ \&\ \mathrm{rk}_\delta(t) = \gamma\}$ is pre-dense above r. $\square$

We do not know if iterations of reasonably x–bounding forcing notions are reasonably x–bounding or even λ–proper (for $x \in \{\mathbf{a}, \mathbf{b}\}$). In a subsequent paper [RS11b] we introduce a property called *nice double x–bounding* and we show that it is preserved in λ–support iterations (see [RS11b, 2.9, 2.10]). This property is in some cases stronger than *being reasonably x–bounding,* but it puts some restrictions on $\bar{\mu}$. In this context the following problem is very natural.

Problem 2.9. (1) Do we have a result parallel to 2.5 for reasonably C–bounding forcings?
(2) Let $x \in \{\mathbf{a}, \mathbf{b}\}$. Are λ–support iterations of reasonably x–bounding forcing notions still reasonably x–bounding? At least λ–proper?

3. Consequences of reasonable ABC

Let us note that Theorem 2.8 improves [RS07, Theorem A.2.4]. Before we explain why, we should recall the following definition.

Definition 3.1 ([RS07, Def. A.2.1])**.** Let $\mathbb{P}$ be a forcing notion.

(1) A complete λ–tree of height $\alpha < \lambda$ is a set of sequences $s \subseteq {}^{\le\alpha}\lambda$ such that
- s has the $\lhd$–smallest element denoted root(s),
- s is closed under initial segments longer than lh(root(s)), and
- the union of any $\lhd$–increasing sequence of members of s is in s, and
- $(\forall \eta \in s)(\exists \nu \in s)(\eta \lhd \nu \ \&\ \mathrm{lh}(\nu) = \alpha)$.

(2) For a condition $p \in \mathbb{P}$ and an ordinal $i_0 < \lambda$ we define a game $\Game^{\mathrm{Sacks}}_{\bar{\mu}}(i_0, p, \mathbb{P})$ of two players, *Generic* and *Antigeneric* . A play lasts at most λ moves indexed by ordinals from the interval $[i_0, \lambda)$, and during it the players construct a sequence $\langle (s_i, \bar{p}^i, \bar{q}^i) : i_0 \le i < \lambda \rangle$ as follows. At stage i of the

play (where $i_0 \leq i < \lambda$), first Generic chooses $s_i \subseteq {}^{\leq i+1}\lambda$ and a system $\bar{p}^i = \langle p^i_\eta : \eta \in s_i \cap {}^{i+1}\lambda\rangle$ such that

(α) s_i is a complete λ–tree of height $i+1$ and $\mathrm{lh}(\mathrm{root}(s_i)) = i_0$,

(β) for all j such that $i_0 \leq j < i$ we have $s_j = s_i \cap {}^{\leq j+1}\lambda$,

(γ) $p^i_\eta \in \mathbb{P}$ for all $\eta \in s_i \cap {}^{i+1}\lambda$, and

(δ) if $i_0 \leq j < i$, $\nu \in s_i \cap {}^{j+1}\lambda$ and $\nu \vartriangleleft \eta \in s_i \cap {}^{i+1}\lambda$, then $q^j_\nu \leq p^i_\eta$ and $p \leq p^i_\eta$,

(ε) $|s_i \cap {}^{i+1}\lambda| < \mu_i$.

Then Antigeneric answers choosing a system $\bar{q}^i = \langle q^i_\eta : \eta \in s_i \cap {}^{i+1}\lambda\rangle$ of conditions in $\mathbb{P}$ such that $p^i_\eta \leq q^i_\eta$ for each $\eta \in s_i \cap {}^{i+1}\lambda$.

Generic wins a play if she always has legal moves (so the play really lasts λ steps) and there are a condition $q \geq p$ and a $\mathbb{P}$–name $\underset{\sim}{\rho}$ such that

($\circledast$) $$q \Vdash_{\mathbb{P}} \text{“ } \underset{\sim}{\rho} \in {}^\lambda\lambda \ \&\ \big(\forall i \in [i_0,\lambda)\big)\big(\underset{\sim}{\rho}{\restriction}(i+1) \in s_i \ \&\ q^i_{\underset{\sim}{\rho}\restriction(i+1)} \in \Gamma_{\mathbb{P}}\big) \text{ ”}.$$

(3) We say that $\mathbb{P}$ has the *strong $\bar{\mu}$–Sacks property* whenever
 (a) $\mathbb{P}$ is strategically $(<\lambda)$–complete, and
 (b) Generic has a winning strategy in the game $\Game^{\rm Sacks}_{\bar{\mu}}(i_0,p,\mathbb{P})$ for any $i_0 < \lambda$ and $p \in \mathbb{P}$.

The following proposition explains why 2.8 is stronger than [RS07, Theorem A.2.4].

Proposition 3.2. *Assume that $\lambda,\bar{\mu}$ are as in Context 0.1 and that additionally $(\forall i < j < \lambda)(\mu_i \leq \mu_j)$. Let $\mathbb{Q}$ be a forcing notion. Then*

$\mathbb{Q}$ is reasonably A–bounding over $\bar{\mu}$

if and only if

$\mathbb{Q}$ has the strong $\bar{\mu}$–Sacks property.

Proof. Suppose that $\mathbb{Q}$ is reasonably A–bounding over $\bar{\mu}$. Since the sequence $\bar{\mu}$ is non-decreasing, it is enough to show that Generic has a winning strategy in $\Game^{\rm Sacks}_{\bar{\mu}}(0,p,\mathbb{Q})$ for each $p \in \mathbb{Q}$ (as then almost the same strategy will be good in $\Game^{\rm Sacks}_{\bar{\mu}}(i,p,\mathbb{Q})$ for any $i < \lambda$).

Let $p \in \mathbb{Q}$. We are going to define a strategy $\mathbf{st}$ for Generic in the game $\Game^{\rm Sacks}_{\bar{\mu}}(0,p,\mathbb{Q})$. To describe it, let us fix a winning strategy $\mathbf{st}_0$ of Complete in $\Game^\lambda_0(\mathbb{Q},p)$ and a winning strategy $\mathbf{st}_1$ of Generic in $\Game^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q})$. Now, at a stage $\delta < \lambda$ of the play the strategy $\mathbf{st}$ will tell Generic to write aside

($\boxtimes$)$_\delta$ $\quad I_\delta$ and $\langle r^{0,\delta}_t, r^{1,\delta}_t : t \in I_\delta\rangle$ and $\langle r^\delta_\eta : \eta \in s_\delta \cap {}^{\delta+1}\lambda\rangle$

so that if $\langle (s_\delta,\bar{p}^\delta,\bar{q}^\delta) : \delta < \lambda\rangle$ is a play of $\Game^{\rm Sacks}_{\bar{\mu}}(0,p,\mathbb{Q})$ in which Generic follows $\mathbf{st}$, then the following conditions $(\odot)_1$–$(\odot)_4$ are satisfied (for each $\delta < \lambda$).

$(\odot)_1$ $\big\langle I_\alpha, \langle r^{0,\alpha}_t, r^{1,\alpha}_t : t \in I_\alpha\rangle : \alpha \leq \delta\big\rangle$ is a partial legal play of $\Game^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q})$ in which Generic uses $\mathbf{st}_1$.

$(\odot)_2$ For each $\eta \in s_\delta \cap {}^{\delta+1}\lambda$ the sequence $\langle q^\alpha_{\eta\restriction(\alpha+1)}, r^\alpha_{\eta\restriction(\alpha+1)} : \alpha \leq \delta\rangle$ is a partial legal play of $\Game^\lambda_0(\mathbb{Q},p)$ in which Complete uses $\mathbf{st}_0$.

$(\odot)_3$ If $t \in I_\delta$, $\alpha < \delta$, $\nu \in s_\alpha \cap {}^{\alpha+1}\lambda$, then either $r^\alpha_\nu, r^{1,\delta}_t$ are incompatible or $r^\alpha_\nu \leq r^{1,\delta}_t$.

$(\odot)_4$ $\langle p^\delta_\nu : \nu \in s_\delta \cap {}^{\delta+1}\lambda\rangle$ is an antichain in $\mathbb{Q}$.

So suppose that the two players arrived to a stage $\delta < \lambda$ of the game $\Game^{\rm Sacks}_{\bar{\mu}}(0,p,\mathbb{Q})$ and the objects listed in $(\boxtimes)_\alpha$ (for $\alpha < \delta$) as well as $\langle (s_\alpha,\bar{p}^\alpha,\bar{q}^\alpha) : \alpha < \delta\rangle$ have been constructed. First Generic uses $\mathbf{st}_1$ to pick the answer $\big(I_\delta, \langle r^{0,\delta}_t : t \in I_\delta\rangle\big)$ to $\big\langle I_\alpha, \langle r^{0,\alpha}_t, r^{1,\alpha}_t : t \in I_\alpha\rangle : \alpha < \delta\big\rangle$ in $\partial^{\rm rcA}_{\bar\mu}(p,\mathbb{Q})$. Then she uses the strategic completeness of $\mathbb{Q}$ and 1.2 to choose a system $\langle r^*_t : t \in I_\delta\rangle$ of conditions in $\mathbb{Q}$ such that

- $(\odot)_5$ if $t \in I_\delta$, then $r^{0,\delta}_t \le r^*_t$ and for every $\alpha < \delta$ and $\nu \in s_\alpha \cap {}^{\alpha+1}\lambda$, either r^α_ν, r^*_t are incompatible or $r^\alpha_\nu \le r^*_t$, and also either p, r^*_t are incompatible or $p \le r^*_t$,
- $(\odot)_6$ if $t_0, t_1 \in I_\delta$, $t_0 \ne t_1$, then the conditions $r^*_{t_0}, r^*_{t_1}$ are incompatible in $\mathbb{Q}$.

Now she lets $s^* = \{\eta \in {}^\delta\lambda : (\forall \alpha < \delta)(\eta{\restriction}(\alpha+1) \in s_\alpha)\}$ and

$$s^- = \{\eta \in s^* : (\exists t \in I_\delta)(\forall \alpha < \delta)(r^\alpha_{\eta\restriction(\alpha+1)} \le r^*_t \ \&\ p \le r^*_t)\},$$

and for each $\eta \in s^-$ she fixes an enumeration $\langle t^\eta_\xi : \xi < \xi_\eta\rangle$ of the set

$$\big\{t \in I_\delta : \big(\forall \alpha < \delta\big)\big(r^\alpha_{\eta\restriction(\alpha+1)} \le r^*_t \ \&\ p \le r^*_t\big)\big\}.$$

Now Generic defines

$$s^+_\delta = \big\{\nu \in {}^{\delta+1}\lambda : \big(\nu{\restriction}\delta \in s^* \setminus s^- \ \&\ \nu(\delta) = 0\big) \text{ or } \big(\nu{\restriction}\delta \in s^- \ \&\ \nu(\delta) < \xi_{\nu\restriction\delta}\big)\big\}$$

and she lets s_δ be a λ–tree of height $\delta+1$ such that $s_\delta \cap {}^{\delta+1}\lambda = s^+_\delta$. For $\nu \in s^+_\delta$ she also chooses p^δ_ν so that

- if $\nu{\restriction}\delta \notin s^-$, then $p^\delta_\nu \in \mathbb{Q}$ is an upper bound to $\{r^\alpha_{\nu\restriction(\alpha+1)} : \alpha < \delta\} \cup \{p\}$ (remember $(\odot)_2$),
- if $\nu{\restriction}\delta \in s^-$, then $p^\delta_\nu = r^*_{t^{\nu\restriction\delta}_{\nu(\delta)}}$.

And now, in the play of $\partial^{\rm Sacks}_{\bar\mu}(0,p,\mathbb{Q})$, Generic puts

$$s_\delta \quad \text{and} \quad \langle p^\delta_\nu : \nu \in s^+_\delta\rangle$$

and Antigeneric answers with $\langle q^\delta_\nu : \nu \in s^+_\delta\rangle$ (so that $q^\delta_\nu \ge p^\delta_\nu$). Conditions r^δ_ν (for $\nu \in s^+_\delta$) are determined using $\mathbf{st}_0$ (so that the demand in $(\odot)_2$ is satisfied). Finally, Generic defines also $r^{1,\delta}_t$ for $t \in I_\delta$ so that

- if $t = t^\eta_\xi$ for some $\eta \in s^-$ and $\xi < \xi_\eta$, then $r^{1,\delta}_t = r^\delta_{\eta\frown\langle\xi\rangle}$,
- otherwise $r^{1,\delta}_t = r^*_t$.

This completes the description of what Generic plays and what she writes aside — it should be clear that the requirements of $(\odot)_1$–$(\odot)_4$ are satisfied. Now, why is $\mathbf{st}$ a winning strategy? So suppose that $\langle (s_\delta, \bar p^\delta, \bar q^\delta) : \delta < \lambda\rangle$ is a play of $\partial^{\rm Sacks}_{\bar\mu}(0,p,\mathbb{Q})$ in which Generic follows $\mathbf{st}$, and I_δ, $\langle r^{0,\delta}_t, r^{1,\delta}_t : t \in I_\delta\rangle$ and $\langle r^\delta_\eta : \eta \in s_\delta \cap {}^{\delta+1}\lambda\rangle$ (for $\delta < \lambda$) are the objects constructed by Generic aside, so they satisfy $(\odot)_1$–$(\odot)_4$. It follows from $(\odot)_1$ and the choice of $\mathbf{st}_1$ that there is a condition $p^* \ge p$ such that

- $(\odot)_7$ for every $\delta < \lambda$ the set $\big\{r^{1,\delta}_t : t \in I_\delta\big\}$ is pre-dense above p^*.

We claim that then also

- $(\odot)_8$ for every $\delta < \lambda$ the set $\big\{r^\delta_\eta : \eta \in s_\delta \cap {}^{\delta+1}\lambda\big\}$ is pre-dense above p^*

(and this clearly implies that Generic won the play, remember $(\odot)_4$). Assume towards contradiction that $(\odot)_8$ fails and let $\delta < \lambda$ be the smallest ordinal for which we may find a condition $q \ge p^*$ such that q is incompatible with every r^δ_η for $\eta \in s_\delta \cap {}^{\delta+1}\lambda$. It follows from $(\odot)_7$ that we may pick $t \in I_\delta$ such that the conditions $r^{1,\delta}_t, q$ are compatible. By the previous sentence and by the definition of $r^{1,\delta}_t$ we get that $t \ne t^\eta_\xi$ for all $\xi < \xi_\eta$, $\eta \in s^-$ and thus $r^{1,\delta}_t = r^*_t$. Look at the condition r^*_t

(satisfying $(\odot)_5+(\odot)_6$) — it must be stronger than p and by the minimality of δ we have that $\big(\forall\alpha<\delta\big)\big(\exists\nu\in s_\alpha\cap{}^{\alpha+1}\lambda\big)\big(r^\alpha_\nu\leq r^*_t\big)$. It follows from $(\odot)_4$ from stages $\alpha<\delta$ that there is $\eta\in s^*$ such that $\big(\forall\alpha<\delta\big)\big(r^\alpha_{\eta\restriction(\alpha+1)}\leq r^*_t\big)$. Then $t\in s^-$ and hence $t=t^\eta_\xi$ for some $\xi<\xi_\eta$, contradicting what we already got.

The converse implication should be clear. □

The following easy proposition explains why the names of the properties defined in 2.2 include the adjective "bounding".

Proposition 3.3. *Let $\lambda,\mathcal{U}$ and $\bar{\mu}$ be as in 0.1. Assume that $\mathbb{Q}$ is a forcing notion, $p\in\mathbb{Q}$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}$–name for an element of ${}^\lambda\lambda$.*

1. *If $\mathbb{Q}$ is reasonably* **a***–bounding over $\bar{\mu}$, then there are a condition $q\geq p$ and a sequence $\bar{a}=\langle a_\alpha:\alpha<\lambda\rangle$ such that*
   - *$a_\alpha\subseteq\lambda$, $|a_\alpha|<\mu_\alpha$ for all $\alpha<\lambda$,*
   - *$q\Vdash_{\mathbb{Q}}$ " $(\forall\alpha<\lambda)(\underset{\sim}{\tau}(\alpha)\in a_\alpha)$ ".*
2. *If $\mathbb{Q}$ is reasonably* **b***–bounding over $\mathcal{U},\bar{\mu}$, then there are a condition $q\geq p$ and a sequence $\bar{a}=\langle a_\alpha:\alpha<\lambda\rangle$ such that*
   - *$a_\alpha\subseteq\lambda$, $|a_\alpha|<\mu_\alpha$ for all $\alpha<\lambda$,*
   - *$q\Vdash_{\mathbb{Q}}$ " $\{\alpha<\lambda:\underset{\sim}{\tau}(\alpha)\in a_\alpha\}\in\mathcal{U}^{\mathbb{Q}}$ ".*
3. *If $\mathbb{Q}$ is reasonably* **c***–bounding over $\mathcal{U},\bar{\mu}$, then there are a condition $q\geq p$ and a sequence $\bar{a}=\langle a_\alpha:\alpha<\lambda\rangle$ such that*
   - *$a_\alpha\subseteq\lambda$, $|a_\alpha|<\mu_\alpha$ for all $\alpha<\lambda$,*
   - *$q\Vdash_{\mathbb{Q}}$ " $\{\alpha<\lambda:\underset{\sim}{\tau}(\alpha)\in a_\alpha\}\in\big(\mathcal{U}^{\mathbb{Q}}\big)^+$ ".*

4. A model

In this section, in addition to the assumptions stated in 0.1 we will also assume that

Context 4.1.
(d) $S\subseteq\lambda$ is stationary and co-stationary, $S\in\mathcal{U}$,
(e) $\mathcal{V}$ is a normal filter on λ, $\lambda\setminus S\in\mathcal{V}$.

Definition 4.2.
1. Let $\alpha<\beta<\lambda$. *An (α,β)–extending function* is a mapping $c:\mathcal{P}(\alpha)\longrightarrow\mathcal{P}(\beta)\setminus\mathcal{P}(\alpha)$ such that $c(u)\cap\alpha=u$ for all $u\in\mathcal{P}(\alpha)$.
2. Let C be an unbounded subset of λ. *A C–extending sequence* is a sequence $\mathfrak{c}=\langle c_\alpha:\alpha\in C\rangle$ such that each c_α is an $(\alpha,\min(C\setminus(\alpha+1)))$–extending function.
3. Let $C\subseteq\lambda$, $|C|=\lambda$, $\beta\in C$, $w\subseteq\beta$ and let $\mathfrak{c}=\langle c_\alpha:\alpha\in C\rangle$ be a C–extending sequence. We define $\mathrm{pos}^+(w,\mathfrak{c},\beta)$ as the family of all subsets u of β such that
   (i) if $\alpha_0=\min\big(\{\alpha\in C:(\forall\xi\in w)(\xi<\alpha)\}\big)$, then $u\cap\alpha_0=w$ (so if $\alpha_0=\beta$, then $u=w$), and
   (ii) if $\alpha_0,\alpha_1\in C$, $w\subseteq\alpha_0<\alpha_1=\min(C\setminus(\alpha_0+1))\leq\beta$, then either $c_{\alpha_0}(u\cap\alpha_0)=u\cap\alpha_1$ or $u\cap\alpha_0=u\cap\alpha_1$,
   (iii) if $\sup(w)<\alpha_0=\sup(C\cap\alpha_0)\notin C$, $\alpha_1=\min\big(C\setminus(\alpha_0+1)\big)\leq\beta$, then $u\cap\alpha_1=u\cap\alpha_0$.
   For $\alpha_0\in\beta\cap C$ such that $w\subseteq\alpha_0$, the family $\mathrm{pos}(w,\mathfrak{c},\alpha_0,\beta)$ consists of all elements u of $\mathrm{pos}^+(w,\mathfrak{c},\beta)$ which satisfy also the following condition:
   (iv) if $\alpha_1=\min\big(C\setminus(\alpha_0+1)\big)\leq\beta$, then $u\cap\alpha_1=c_{\alpha_0}(w)$.
4. A C–extending sequence $\mathfrak{c}=\langle c_\alpha:\alpha\in C\rangle$ is *S–closed* provided that

(i) C is a club of λ, and
(ii) if $\alpha \in C$ and $u \subseteq \alpha$, then $\alpha \in c_\alpha(u)$, and
(iii) if $\xi \in S \setminus C$, $\alpha \in C \cap \xi$, $u \subseteq \alpha$ and $\xi = \sup\big(c_\alpha(u) \cap \xi\big)$, then $\xi \in c_\alpha(u)$.

(5) A set $w \subseteq \lambda$ is S–closed if $\xi = \sup\big(w \cap \xi\big) \in S$ implies $\xi \in w$.

(6) Let $\mathfrak{c} = \langle c_\alpha : \alpha \in C\rangle$ be an S–closed C–extending sequence, $\alpha_0, \beta \in C$, $w \subseteq \alpha_0 < \beta$. Assume also that $w \cup \{\alpha_0\}$ is S–closed. Then we let

$$\begin{aligned} \mathrm{pos}^+_S(w, \mathfrak{c}, \beta) = \quad & \big\{u \in \mathrm{pos}^+(w, \mathfrak{c}, \beta) : u \cup \{\beta\} \text{ is } S\text{–closed }\big\} \\ \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \beta) = \quad & \big\{u \in \mathrm{pos}(w, \mathfrak{c}, \alpha_0, \beta) : u \cup \{\beta\} \text{ is } S\text{–closed }\big\} \end{aligned}$$

Note that if C is a club (e.g. $\mathfrak{c}$ is S–closed), then clause 4.2(3)(iii) is satisfied vacuously.

Observation 4.3. (1) *Assume that $\mathfrak{c}$ is a C–extending sequence, $\alpha_0, \alpha_1, \beta \in C$, $\alpha_0 < \alpha_1 \le \beta$ and $w \subseteq \alpha_0$.*
(a) *If $u \in \mathrm{pos}(w, \mathfrak{c}, \alpha_0, \alpha_1)$ and $v \in \mathrm{pos}^+(u, \mathfrak{c}, \beta)$, then $v \in \mathrm{pos}(w, \mathfrak{c}, \alpha_0, \beta)$.*
(b) *If $v \in \mathrm{pos}(w, \mathfrak{c}, \alpha_0, \beta)$, then $v \cap \alpha_1 \in \mathrm{pos}(w, \mathfrak{c}, \alpha_0, \alpha_1)$ and $v \in \mathrm{pos}^+(v \cap \alpha_1, \mathfrak{c}, \beta)$.*
(c) *Similarly for* pos^+.

(2) *Assume that $\mathfrak{c}$ is an S–closed C–extending sequence, $\alpha_0, \alpha_1, \beta \in C$, $\alpha_0 < \alpha_1 \le \beta$, $w \subseteq \alpha_0$ and $w \cup \{\alpha_0\}$ is S–closed.*
(a) *If $u \in \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \alpha_1)$ and $v \in \mathrm{pos}^+_S(u, \mathfrak{c}, \beta)$, then $v \in \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \beta)$.*
(b) *If $v \in \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \beta)$, then $v \cap \alpha_1 \in \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \alpha_1)$ and $v \in \mathrm{pos}^+_S(v \cap \alpha_1, \mathfrak{c}, \beta)$.*
(c) *Similarly for* pos^+_S.
(d) $\emptyset \ne \mathrm{pos}_S(w, \mathfrak{c}, \alpha_0, \beta) = \{u \in \mathrm{pos}^+_S(w, \mathfrak{c}, \beta) : u \cap \alpha_0 = w \ \&\ \alpha_0 \in u\}$.

Definition 4.4. We define a forcing notion $\mathbb{Q}^1_S$ as follows.
A condition in $\mathbb{Q}^1_S$ is a triple $p = (w^p, C^p, \mathfrak{c}^p)$ such that

(i) $C^p \subseteq \lambda$ is a club of λ and $w^p \subseteq \min(C^p)$ is such that the set $w^p \cup \{\min(C^p)\}$ is S–closed,
(ii) $\mathfrak{c}^p = \langle c^p_\alpha : \alpha \in C^p\rangle$ is an S–closed C^p–extending sequence.

The order $\le_{\mathbb{Q}^1_S} = \le$ **of** $\mathbb{Q}^1_S$ is given by
$p \le_{\mathbb{Q}^1_S} q$ if and only if

(a) $C^q \subseteq C^p$ and $w^q \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \min(C^q))$ and
(b) if $\alpha_0 < \alpha_1$ are two successive members of C^q, $u \in \mathrm{pos}^+_S(w^q, \mathfrak{c}^q, \alpha_0)$, then $c^q_{\alpha_0}(u) \in \mathrm{pos}_S(u, \mathfrak{c}^p, \alpha_0, \alpha_1)$.

For $p \in \mathbb{Q}^1_S$, $\alpha \in C^p$ and $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$ we let $p{\restriction_\alpha} u \stackrel{\text{def}}{=} (u, C^p \setminus \alpha, \mathfrak{c}^p{\restriction}(C^p \setminus \alpha))$.

Remark 4.5. Note that in 4.4(b) we may replace $\mathrm{pos}_S(u, \mathfrak{c}^p, \alpha_0, \alpha_1)$ by $\mathrm{pos}^+_S(u, c^p, \alpha_1)$ (remember 4.2(4)(ii) and 4.3(2)(d)).

Proposition 4.6. (1) *$\mathbb{Q}^1_S$ is a $({<}\lambda)$–complete forcing notion of cardinality 2^λ.*

(2) *If $p \in \mathbb{Q}^1_S$ and $\alpha \in C^p$, then*
- *for each $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$, $p{\restriction_\alpha} u \in \mathbb{Q}^1_S$ is a condition stronger than p, and*
- *the family $\{p{\restriction_\alpha} u : u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)\}$ is pre-dense above p.*

(3) *Let $p \in \mathbb{Q}^1_S$ and $\alpha < \beta$ be two successive members of C^p. Suppose that for each $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$ we are given a condition $q_u \in \mathbb{Q}^1_S$ such that*

$p{\restriction}_\beta c^p_\alpha(u) \leq q_u$. Then there is a condition $q \in \mathbb{Q}^1_S$ such that letting $\alpha' = \min(C^q \setminus \beta)$ we have
(a) *$p \leq q$, $w^q = w^p$, $C^q \cap \beta = C^p \cap \beta$ and $c^q_\delta = c^p_\delta$ for $\delta \in C^q \cap \alpha$, and*
(b) *$\bigcup\big\{w^{q_u} : u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)\big\} \subseteq \alpha'$, and*
(c) *$q_u \leq q{\restriction}_{\alpha'} c^q_\alpha(u)$ for every $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$.*

(4) *Assume that $p \in \mathbb{Q}^1_S$, $\alpha \in C^p$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^1_S$–name such that $p \Vdash$ "$\underset{\sim}{\tau} \in \mathbf{V}$". Then there is a condition $q \in \mathbb{Q}^1_S$ stronger than p and such that*
(a) *$w^q = w^p$, $\alpha \in C^q$ and $C^q \cap \alpha = C^p \cap \alpha$, and*
(b) *if $u \in \mathrm{pos}^+_S(w^q, \mathfrak{c}^q, \alpha)$ and $\gamma = \min(C^q \setminus (\alpha+1))$, then the condition $q{\restriction}_\gamma c^q_\alpha(u)$ forces a value to $\underset{\sim}{\tau}$.*

Proof. (1) It should be clear that $\mathbb{Q}^1_S$ is a forcing notion of size 2^λ. To show that it is $({<}\lambda)$–complete suppose that $\gamma < \lambda$ is a limit ordinal and $\bar{p} = \langle p_\xi : \xi < \gamma\rangle \subseteq \mathbb{Q}^1_S$ is $\leq_{\mathbb{Q}^1_S}$–increasing. We put $w^q = \bigcup\limits_{\xi<\gamma} w^{p_\xi}$, $C^q = \bigcap\limits_{\xi<\gamma} C^{p_\xi}$ and for $\delta \in C^q$ we define $c^q_\delta : \mathcal{P}(\delta) \longrightarrow \mathcal{P}\big(\min(C^q \setminus (\delta+1))\big)$ so that

- if $u \in \bigcap\limits_{\xi<\gamma} \mathrm{pos}^+_S(w^{p_\xi}, \mathfrak{c}^{p_\xi}, \delta)$, then $c^q_\delta(u) = \bigcup\limits_{\xi<\gamma} c^{p_\xi}_\delta(u)$,
- if $u \subseteq \delta$ but it is not in $\bigcap\limits_{\xi<\gamma} \mathrm{pos}^+_S(w^{p_\xi}, \mathfrak{c}^{p_\xi}, \delta)$, then $c^q_\delta(u) = u \cup \{\delta\}$.

Finally we put $\mathfrak{c}^q = \langle c^q_\delta : \delta \in C^q\rangle$ and $q = (w^q, C^q, \mathfrak{c}^q)$. One easily checks that $q \in \mathbb{Q}^1_S$ is a condition stronger than all p_ξ's.

(2) Straightforward (remember 4.3(2)).

(3) Note that if $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$, $\alpha_u = \min(C^{q_u})$, then $w^{q_u} \in \mathrm{pos}(u, c^p, \alpha, \alpha_u)$. We let $w^q = w^p$ and $C^q = (C^p \cap \beta) \cup \bigcap\big\{C^{q_u} : u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)\big\}$ (plainly, C^q is a club of λ). Let $\alpha' = \min(C^q \setminus (\alpha+1)) = \min(C^q \setminus \beta)$. For $\delta \in C^q \cap \alpha = C^p \cap \alpha$ put $c^q_\delta = c^p_\delta$. Next, choose an (α, α')–extending function $c^q_\alpha : \mathcal{P}(\alpha) \longrightarrow \mathcal{P}(\alpha')$ such that $(\forall u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha))(c^q_\alpha(u) \in \mathrm{pos}^+_S(w^{q_u}, \mathfrak{c}^{q_u}, \alpha'))$ and $(c^q_\alpha(u) \setminus \alpha) \cup \{\alpha'\}$ is S–closed for each $u \subseteq \alpha$. (Remember 4.3(2d); note that, by the definition of C^q, $w^{q_u} \subseteq \alpha'$ for each $u \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$.) Finally, if $\delta_0 < \delta_1$ are two successive members of $C^q \setminus \alpha'$, then choose a (δ_0, δ_1)–extending function $c^q_{\delta_0} : \mathcal{P}(\delta_0) \longrightarrow \mathcal{P}(\delta_1)$ so that

(i) if $v \subseteq \delta_0$, $u = v \cap \alpha \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \alpha)$ and $v \in \mathrm{pos}^+_S(w^{q_u}, \mathfrak{c}^{q_u}, \delta_0)$, then $c^q_{\delta_0}(v) \in \mathrm{pos}_S(v, \mathfrak{c}^{q_u}, \delta_0, \delta_1)$;
(ii) if $v \subseteq \delta_0$, $v \in \mathrm{pos}^+_S(w^p, \mathfrak{c}^p, \delta_0)$ but we are not in a case covered by (i), then $c^q_{\delta_0}(v) \in \mathrm{pos}_S(v, \mathfrak{c}^p, \delta_0, \delta_1)$;
(iii) in all other cases we let $c^q_{\delta_0}(v) = v \cup \{\delta_0\}$.

Let $\mathfrak{c}^q = \langle c^q_\delta : \delta \in C^q\rangle$ and $q = (w^q, C^q, \mathfrak{c}^q)$. It should be clear that $q \in \mathbb{Q}^1_S$ is a condition as required.

(4) Easily follows from (3). $\square$

Definition 4.7. Suppose that $\gamma < \lambda$ is a limit ordinal and $\bar{p} = \langle p_\xi : \xi < \gamma\rangle \subseteq \mathbb{Q}^1_S$ is $\leq_{\mathbb{Q}^1_S}$–increasing. The condition q constructed as in the proof of 4.6(1) for $\bar{p}$ will be called *the natural limit of $\bar{p}$*.

Proposition 4.8. (1) *Suppose $\bar{p} = \langle p_\xi : \xi < \lambda\rangle$ is a $\leq_{\mathbb{Q}^1_S}$–increasing sequence of conditions from $\mathbb{Q}^1_S$ such that*
(a) *$w^{p_\xi} = w^{p_0}$ for all $\xi < \lambda$, and*
(b) *if $\gamma < \lambda$ is limit, then p_γ is the natural limit of $\bar{p}{\restriction}\gamma$, and*

(c) *for each* $\xi<\lambda$, *if* $\delta\in C^{p_\xi}$, $\mathrm{otp}(C^{p_\xi}\cap\delta)=\xi$, *then* $C^{p_{\xi+1}}\cap(\delta+1)=C^{p_\xi}\cap(\delta+1)$ *and for every* $\alpha\in C^{p_{\xi+1}}\cap\delta$ *we have* $c^{p_{\xi+1}}_\alpha=c^{p_\xi}_\alpha$.

Then the sequence $\bar{p}$ *has an upper bound in* $\mathbb{Q}^1_S$.

(2) *Suppose that* $p\in\mathbb{Q}^1_S$ *and* $\underset{\sim}{h}$ *is a* $\mathbb{Q}^1_S$*–name such that* $p\Vdash$ "$\underset{\sim}{h}:\lambda\longrightarrow\mathbf{V}$". *Then there is a condition* $q\in\mathbb{Q}^1_S$ *stronger than* p *and such that*

($\otimes$) *if* $\delta<\delta'$ *are two successive points of* C^q, $u\in\mathrm{pos}^+_S(w^q,\mathfrak{c}^q,\delta)$, *then the condition* $q{\restriction}_{\delta'}c^q_\delta(u)$ *decides the value of* $\underset{\sim}{h}{\restriction}(\delta+1)$.

Proof. (1) First let us note that if $\delta\in\underset{\xi<\lambda}{\triangle}C^{p_\xi}$ is a limit ordinal, then $\delta\in\bigcap\limits_{\xi<\lambda}C^{p_\xi}$ and $c^{p_{\delta+1}}_\delta=c^{p_\xi}_\delta$ for all $\xi\geq\delta+2$ (by assumptions (b) and (c)). Now, we put $w^q=w^{p_0}$ and $C^q=\{\delta\in\underset{\xi<\lambda}{\triangle}C^{p_\xi}:\delta$ is limit $\}$, and for $\delta\in C^q$ we let $c^q_\delta=c^{p_{\delta+1}}_\delta$ (thus defining $\mathfrak{c}^q=\langle c^q_\delta:\delta\in C^q\rangle$). It should be clear that $q=(w^q,C^q,\mathfrak{c}^q)\in\mathbb{Q}^1_S$ is an upper bound to $\bar{p}$.

(2) Follows from (1) above and 4.6(4). $\square$

Definition 4.9. We let $\underset{\sim}{W}$ and $\underset{\sim}{\eta},\underset{\sim}{\nu}$ be $\mathbb{Q}^1_S$–names such that

$$\Vdash_{\mathbb{Q}^1_S}\underset{\sim}{W}=\bigcup\{w^p:p\in\Gamma_{\mathbb{Q}^1_S}\}$$

and

$$\Vdash_{\mathbb{Q}^1_S}\text{“}\ \underset{\sim}{\eta},\underset{\sim}{\nu}\in{}^\lambda\lambda\text{ and if }\langle\delta_\xi:\xi<\lambda\rangle\text{ is the increasing enumeration of }\mathrm{cl}(\underset{\sim}{W}),\ \text{and }\delta_\xi\leq\alpha<\delta_{\xi+1},\ \xi<\lambda,\ \text{then }\underset{\sim}{\eta}(\alpha)=\xi\text{ and }\underset{\sim}{\nu}(\alpha)=\delta_{\xi+4}\text{”}.$$

Proposition 4.10. (1) $\Vdash_{\mathbb{Q}^1_S}$ "$\underset{\sim}{W}$ *is an unbounded S–closed subset of* λ ". *Consequently* $\Vdash_{\mathbb{Q}^1_S}$ " $\underset{\sim}{W}\in\mathcal{U}^{\mathbb{Q}^1_S}$ ".

(2) $\Vdash_{\mathbb{Q}^1_S}$ " $\underset{\sim}{W},\lambda\setminus\underset{\sim}{W}\in\big(\mathcal{V}^{\mathbb{Q}^1_S}\big)^+$ ".

(3) $\Vdash_{\mathbb{Q}^1_S}\big(\forall f\in{}^\lambda\lambda\cap\mathbf{V}\big)\big(\forall A\in\mathcal{V}^{\mathbb{Q}^1_S}\big)\big(\exists\alpha\in A\big)\big(f(\alpha)<\underset{\sim}{\nu}(\alpha)\big)$.

Proof. (2) Suppose that $p\in\mathbb{Q}^1_S$ and $\underset{\sim}{A}_i$ (for $i<\lambda$) are $\mathbb{Q}^1_S$–names for elements of $\mathcal{V}\cap\mathbf{V}$. Build inductively sequences $\langle p_i:i<\lambda\rangle\subseteq\mathbb{Q}^1_S$ and $\langle A_i:i<\lambda\rangle\subseteq\mathcal{V}$ such that

(a) $\big(\forall i<j<\lambda\big)\big(p\leq p_i\leq p_j\big)$,

(b) $p_{i+1}\Vdash_{\mathbb{Q}^1_S}\underset{\sim}{A}_i=A_i$ and $i\leq\sup(w^{p_i})$ for all $i<\lambda$,

(c) if $\gamma<\lambda$ is limit, then p_γ is the natural limit of $\langle p_i:i<\gamma\rangle$.

Pick a limit ordinal $\delta\in\underset{i<\lambda}{\triangle}A_i\setminus S$ such that $\delta=\sup\big(\bigcup\limits_{i<\delta}w^{p_i}\big)\in C^{p_\delta}$ (possible by the normality of $\mathcal{V}$; remember (b,c) above). Then $p_\delta\Vdash\delta\in\underset{i<\lambda}{\triangle}\underset{\sim}{A}_i$. Put $\beta=\min\big(C^{p_\delta}\setminus(\delta+1)\big)$.

Let $w=c^{p_\delta}_\delta(w^{p_\delta})$ and $p^*=p_\delta{\restriction}_\beta w$. Then $p^*\geq p_\delta$ and $p^*\Vdash\delta\in\underset{\sim}{W}$.

On the other hand, since $\delta=\sup(w^{p_\delta})\notin S$, we have $w^{p_\delta}\in\mathrm{pos}^+_S(w^{p_\delta},\mathfrak{c}^{p_\delta},\beta)$ so we may let $p^{**}=p_\delta{\restriction}_\beta w^{p_\delta}$. Then $p^{**}\geq p_\delta$ and $p^{**}\Vdash\delta\notin\underset{\sim}{W}$.

(3) Suppose that $p\in\mathbb{Q}^1_S$, $f\in{}^\lambda\lambda$ and $\langle\underset{\sim}{A}_\alpha:\alpha<\lambda\rangle$ is a sequence of $\mathbb{Q}^1_S$–names for members of $\mathcal{V}\cap\mathbf{V}$. By induction on $\alpha<\lambda$ construct a sequence $\langle p_\alpha,A_\alpha:\alpha<\lambda\rangle$ such that for each α:

(i) $p_\alpha\in\mathbb{Q}^1_S$, $A_\alpha\subseteq\lambda$, $A_\alpha\in\mathcal{V}$, $p_0=p$, $p_\alpha\leq_{\mathbb{Q}^1_S}p_{\alpha+1}$, $\min(C^{p_{\alpha+1}})>\alpha$ and

(ii) if α is a limit ordinal, then p_α is the natural limit of $\langle p_\beta:\beta<\alpha\rangle$, and

(iii) $p_{\alpha+1}\Vdash_{\mathbb{Q}^1_S}\underset{\sim}{A}_\alpha\cap(\lambda\setminus S)=A_\alpha$.

Next pick a limit ordinal $\delta\in\mathop{\triangle}\limits_{\alpha<\lambda}A_\alpha\cap(\lambda\setminus S)$ such that $(\forall\alpha<\delta)(\min(C^{p_\alpha})<\delta)$. Then $p_\delta\Vdash\delta\in\mathop{\triangle}\limits_{\alpha<\lambda}\underset{\sim}{A}_\alpha$ and $\delta=\min(C^{p_\delta})$ and $w^{p_\delta}\subseteq\delta$ is S–closed, so we may let $w^q=w^{p_\delta}$, $C^q=C^{p_\delta}\setminus\big(f(\delta)+1\big)$ and $\mathfrak{c}^q=\mathfrak{c}^{p_\delta}\restriction C^q$ to get a condition $q\in\mathbb{Q}^1_S$ stronger than p and such that

$$q\Vdash_{\mathbb{Q}^1_S}\text{“ }\delta\in\mathop{\triangle}\limits_{\alpha<\lambda}\underset{\sim}{A}_\alpha\text{ and }f(\delta)<\underset{\sim}{\nu}(\delta)\text{ ”.}$$

$\square$

Proposition 4.11. *The forcing notion $\mathbb{Q}^1_S$ is reasonably B–bounding over $\mathcal{U}$.*

Proof. By 4.6(1), $\mathbb{Q}^1_S$ is $({<}\lambda)$–complete, so we have to verify 2.2(5b) only. Let $p\in\mathbb{Q}^1_S$ and let $\bar{\mu}'=\langle\mu'_\alpha:\alpha<\lambda\rangle$, $\mu'_\alpha=\lambda$ for each $\alpha<\lambda$. We are going to describe a strategy $\mathbf{st}$ for Generic in $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$.

In the course of a play the strategy $\mathbf{st}$ instructs Generic to build aside an increasing sequence of conditions $\bar{p}^*=\langle p^*_\alpha:\alpha<\lambda\rangle\subseteq\mathbb{Q}^1_S$ such that

(a) $p^*_0=p$ and $w^{p^*_\alpha}=w^p$ for all $\alpha<\lambda$, and
(b) if $\gamma<\lambda$ is limit, then p^*_γ is the natural limit of $\bar{p}^*\restriction\gamma$, and
(c) for each $\alpha<\lambda$, if $\delta\in C^{p^*_\alpha}$, $\mathrm{otp}(C^{p^*_\alpha}\cap\delta)=\alpha$, then $C^{p^*_{\alpha+1}}\cap(\delta+1)=C^{p^*_\alpha}\cap(\delta+1)$ and for every $\xi\in C^{p^*_{\alpha+1}}\cap\delta$ we have $c^{p^*_{\alpha+1}}_\xi=c^{p^*_\alpha}_\xi$, and
(d) after stage $\alpha<\lambda$ of the play of $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$, the condition $p^*_{\alpha+1}$ is determined (conditions p^*_α for non-successor $\alpha<\lambda$ are determined by (a),(b) above).

So suppose that the players arrived to a stage $\alpha<\lambda$ of $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$, and Generic (playing according to $\mathbf{st}$ so far) has constructed aside an increasing sequence $\langle p^*_\xi:\xi\leq\alpha\rangle$ of conditions (satisfying (a)–(d)). Let $\delta\in C^{p^*_\alpha}$ be such that $\mathrm{otp}(C^{p^*_\alpha}\cap\delta)=\alpha$ and let $\gamma=\min(C^{p^*_\alpha}\setminus(\delta+1))$. Now Generic makes her move in $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$:

- $I_\alpha=\mathrm{pos}^+_S(w^{p^*_\alpha},\mathfrak{c}^{p^*_\alpha},\delta)$, and
- $p^\alpha_u=p^*_\alpha\restriction_\gamma c^{p^*_\alpha}_\delta(u)$ for $u\in I_\alpha$.

Let $\langle q^\alpha_u:u\in I_\alpha\rangle\subseteq\mathbb{Q}^1_S$ be the answer of Antigeneric, so $p^*_\alpha\restriction_\gamma c^{p^*_\alpha}_\delta(u)\leq q^\alpha_u$ for each $u\in\mathrm{pos}^+(w^{p^*_\alpha},\mathfrak{c}^{p^*_\alpha},\delta)$. Now Generic uses 4.6(3) (with $\delta,\gamma,p^*_\alpha,q^\alpha_u$ here standing for α,β,p,q_u there) to pick a condition $p^*_{\alpha+1}$ such that, letting $\alpha'=\min(C^{p^*_{\alpha+1}}\setminus\gamma)$, we have

(e) $p^*_\alpha\leq p^*_{\alpha+1}$, $w^{p^*_{\alpha+1}}=w^p$, $C^{p^*_{\alpha+1}}\cap\gamma=C^{p^*_\alpha}\cap\gamma$ and $c^{p^*_{\alpha+1}}_\xi=c^{p^*_\alpha}_\xi$ for $\xi\in C^{p^*_{\alpha+1}}\cap\delta$, and
(f) $\bigcup\big\{w^{q^\alpha_u}:u\in I_\alpha\big\}\subseteq\alpha'$, and
(g) $q^\alpha_u\leq p^*_{\alpha+1}\restriction_{\alpha'}c^{p^*_{\alpha+1}}_\delta(u)$ for every $u\in I_\alpha$.

We claim that $\mathbf{st}$ is a winning strategy for Generic in $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$. So suppose that

$$\Big\langle I_\alpha,\langle p^\alpha_u,q^\alpha_u:u\in I_\alpha\rangle:\alpha<\lambda\Big\rangle$$

is a play of $\Game^{\rm rcB}_{\mathcal{U},\bar{\mu}'}(p,\mathbb{Q}^1_S)$ in which Generic uses $\mathbf{st}$, and let $\bar{p}^*=\langle p^*_\alpha:\alpha<\lambda\rangle\subseteq\mathbb{Q}^1_S$ be the sequence constructed aside by Generic, so it satisfies (a)–(c) above, and thus also the assumptions of 4.8(1). Let p^* be an upper bound to $\bar{p}$ (which exists by 4.8(1)). Now note that

$$p^*\Vdash_{\mathbb{Q}^1_S}\text{“ if }\alpha\in C^{p^*}\cap\underset{\sim}{W}\text{ and }u=\underset{\sim}{W}\cap\alpha,\text{ then }q^\alpha_u\in\Gamma_{\mathbb{Q}^1_S}\text{ ”}$$

and therefore

$$p^* \Vdash_{\mathbb{Q}^1_S} \text{“ } \big(\forall \alpha \in C^{p^*} \cap \underset{\sim}{W}\big)\big(\exists u \in I_\alpha\big)\big(q^\alpha_u \in \Gamma_{\mathbb{Q}^1_S}\big) \text{ ”}.$$

Since $p^* \Vdash C^{p^*} \cap \underset{\sim}{W} \in \mathcal{U}^{\mathbb{Q}^1_S}$ (by 4.10) we may conclude that the condition p^* witnesses that Generic won the play. $\square$

Definition 4.12. Let $\mathcal{F}$ be a filter on λ including all co-bounded subsets of λ, $\emptyset \notin \mathcal{F}$.

(1) We say that a family $F \subseteq {}^\lambda\lambda$ is *$\mathcal{F}$–dominating* whenever
$$\big(\forall g \in {}^\lambda\lambda\big)\big(\exists f \in F\big)\big(\{\alpha < \lambda : g(\alpha) < f(\alpha)\} \in \mathcal{F}\big).$$
(2) The $\mathcal{F}$–dominating number $\mathfrak{d}_{\mathcal{F}}$ is the minimal size of an $\mathcal{F}$–dominating family in ${}^\lambda\lambda$.
(3) If $\mathcal{F}$ is the filter of co-bounded subsets of λ, then the corresponding dominating number is also denoted by $\mathfrak{d}_\lambda$. If $\mathcal{F}$ is the filter generated by club subsets of λ, then the corresponding dominating number is called $\mathfrak{d}_{\rm cl}$.

It was shown in Cummings and Shelah [CS95] that $\mathfrak{d}_\lambda = \mathfrak{d}_{\rm cl}$ (whenever $\lambda > \beth_\omega$ is regular). The following corollary is an interesting addition to that result.

Corollary 4.13. *It is consistent that λ is an inaccessible cardinal and there are two normal filters $\mathcal{U}', \mathcal{U}''$ on λ such that $\mathfrak{d}_{\mathcal{U}'} \neq \mathfrak{d}_{\mathcal{U}''}$.*

Proof. Start with the universe where $\lambda, \mathcal{U}, \mathcal{V}, S$ are as in 0.1 + 4.1 and $2^\lambda = \lambda^+$. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \lambda^{++}\rangle$ be a λ–support iteration such that for every $\xi < \lambda^{++}$, $\Vdash_{\mathbb{P}_\xi}$ “ $\underset{\sim}{\mathbb{Q}}_\xi = \mathbb{Q}^1_S$ ”.

It follows from 2.5 that $\mathbb{P}_{\lambda^{++}}$ is reasonably **b**–bounding over $\mathcal{U}$, and hence also λ–proper. Therefore using 4.6(1) and [RS07, Theorem A.1.10] (see also Eisworth [Eis03, §3]) one can easily argue that the limit $\mathbb{P}_{\lambda^{++}}$ of the iteration satisfies the λ^{++}–cc, $\Vdash_{\mathbb{P}_{\lambda^{++}}} 2^\lambda = \lambda^{++}$, $\mathbb{P}_{\lambda^{++}}$ is strategically ($<\lambda$)–complete and λ–proper. Thus, the forcing with $\mathbb{P}_{\lambda^{++}}$ does not collapse cardinals and it follows from 3.3 that

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{“ } {}^\lambda\lambda \cap \mathbf{V} \text{ is } \big(\mathcal{U}\big)^{\mathbb{P}_{\lambda^{++}}}\text{–dominating in } {}^\lambda\lambda \text{ ”}$$

and it follows from 4.10(3) that for each $\xi < \lambda^{++}$

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{“ } {}^\lambda\lambda \cap \mathbf{V}^{\mathbb{P}_\xi} \text{ is not } \big(\mathcal{V}\big)^{\mathbb{P}_{\lambda^{++}}}\text{–dominating in } {}^\lambda\lambda \text{ ”}$$

Therefore we may easily conclude that

$$\Vdash_{\mathbb{P}_{\lambda^{++}}} \text{“ if } \mathcal{U}' = \big(\mathcal{U}\big)^{\mathbb{P}_{\lambda^{++}}},\ \mathcal{U}'' = \big(\mathcal{V}\big)^{\mathbb{P}_{\lambda^{++}}} \text{ then } \mathfrak{b}_\lambda = \mathfrak{d}_{\mathcal{U}'} = \lambda^+ < 2^\lambda = \lambda^{++} = \mathfrak{d}_{\mathcal{U}''} = \mathfrak{d}_{\rm cl} = \mathfrak{d}_\lambda \text{ ”}.$$

$\square$

5. Two bad examples of forcing notions

In this section we give two more examples of forcing notions that have some of the properties studied in the paper - but not strong enough to allow us to quote results obtained earlier. They are test cases for our future research.

Definition 5.1. We define a forcing notion $\mathbb{P}^{\bar{\mu}}$ as follows.
A condition in $\mathbb{P}^{\bar{\mu}}$ is a pair $p = (f^p, C^p)$ such that

$$C^p \subseteq \lambda \text{ is a club of } \lambda \text{ and } f^p \in \prod\{\mu_\iota : \iota \in \lambda \setminus C^p\}.$$

The order $\leq_{\mathbb{P}^{\bar\mu}}=\leq$ of $\mathbb{P}^{\bar\mu}$ is given by:
$p \leq_{\mathbb{P}^{\bar\mu}} q$ if and only if $C^q \subseteq C^p$ and $f^p \subseteq f^q$.

Proposition 5.2. *Assume in addition to 0.1 that the sequence $\bar\mu$ is increasing unbounded in λ. Let $\mathcal{D}_\lambda$ be the club filter on λ. Then the forcing notion $\mathbb{P}^{\bar\mu}$ is reasonably* $\mathbf{b}$*–bounding over $\mathcal{D}_\lambda$ but it is not reasonably B–bounding over $\mathcal{D}_\lambda$.*

Proof. It should be clear that $\mathbb{P}^{\bar\mu}$ is a $(<\lambda)$–complete forcing notion.

Let $p\in\mathbb{P}^{\bar\mu}$. We are going to describe a winning strategy **st** for Generic in the game $\Game^{\rm rc\mathbf{b}}(p,\mathbb{P}^{\bar\mu})$. The strategy **st** instructs Generic to construct aside (in addition to the innings in the play)

- $(\odot)_1$ a closed increasing sequence $\langle\delta_\alpha:\alpha<\lambda\rangle\subseteq\lambda$ and an increasing sequence $\langle r_\alpha:\alpha<\lambda\rangle$ of conditions in $\mathbb{P}^{\bar\mu}$, so that
- $(\odot)_2$ $r_0=p$, $\delta_0=\min(C^p)$ and $\{\delta_\beta:\beta\leq\alpha\}=C^{r_\alpha}\cap(\delta_\alpha+1)$ for $\alpha<\lambda$,
- $(\odot)_3$ $\delta_{\alpha+1}$ and $r_{\alpha+1}$ are known right after the stage α of the play.

Suppose that the players have arrived at a stage $\alpha<\lambda$ of the play, and Generic playing according to **st** constructed aside $\langle\delta_{\beta+1},r_{\beta+1}:\beta<\alpha\rangle$. If α is limit, then $(\odot)_1$ determines δ_α and Generic lets $r_\alpha\in\mathbb{P}^{\bar\mu}$ be such that $C^{r_\alpha}=\bigcap\limits_{\beta<\alpha}C^{r_\beta}$ and $f^{r_\alpha}=\bigcup\limits_{\beta<\alpha}f^{r_\beta}$. (Clearly, relevant parts of $(\odot)_1+(\odot)_2$ are satisfied.) Now Generic picks an enumeration $\langle\eta_\xi:\xi<\zeta_\alpha\rangle$ of $\prod\big\{\mu_\iota:\iota\in\{\delta_\beta:\beta\leq\alpha\}\big\}$ (for some limit $\zeta_\alpha<\lambda$) and puts ζ_α as her inning in $\Game^{\rm rc\mathbf{b}}(p,\mathbb{P}^{\bar\mu})$. Now the two players start a subgame of length ζ_α. The innings of Generic in the subgame are essentially determined by the following demands:

- $(\odot)_4$ $p^\alpha_0=(\eta_0\cup f^{r_\alpha},C^{r_\alpha}\setminus\{\delta_\beta:\beta\leq\alpha\})$,

and for $\xi<\zeta<\zeta_\alpha$

- $(\odot)_5$ $\eta_\xi\subseteq f^{p^\alpha_\xi}$ and $(f^{q^\alpha_\xi}{\restriction}(\lambda\setminus\{\delta_\beta:\beta\leq\alpha\}),C^{q^\alpha_\xi}\cup\{\delta_\beta:\beta\leq\alpha\})\leq_{\mathbb{P}^{\bar\mu}}(f^{p^\alpha_\zeta}{\restriction}(\lambda\setminus\{\delta_\beta:\beta\leq\alpha\}),C^{p^\alpha_\zeta}\cup\{\delta_\beta:\beta\leq\alpha\})$.

After the subgame is over, Generic lets $r_{\alpha+1}\in\mathbb{P}^{\bar\mu}$ be such that

$$f^{r_{\alpha+1}}=\bigcup\big\{f^{q^\alpha_\zeta}{\restriction}(\lambda\setminus\{\delta_\beta:\beta\leq\alpha\}):\zeta<\zeta_\alpha\big\}\ \text{ and }\ C^{r_{\alpha+1}}=\bigcap_{\zeta<\zeta_\alpha}C^{q^\alpha_\zeta}\cup\{\delta_\beta:\beta\leq\alpha\}$$

and she takes $\delta_{\alpha+1}=\min\big(C^{r_{\alpha+1}}\setminus(\delta_\alpha+1)\big)$.

This completes the description of the strategy **st**. Suppose that

$$\Big\langle\zeta_\alpha,\langle p^\alpha_\xi,q^\alpha_\xi:\xi<\zeta_\alpha\rangle:\alpha<\lambda\Big\rangle$$

is a play of $\Game^{\rm rc\mathbf{b}}(p,\mathbb{P}^{\bar\mu})$ in which Generic followed **st** and she constructed aside $\langle\delta_\alpha,r_\alpha:\alpha<\lambda\rangle$. Then $(\bigcup\limits_{\alpha<\lambda}f^{r_\alpha},\bigcap\limits_{\alpha<\lambda}C^{r_\alpha})\in\mathbb{P}^{\bar\mu}$ is a condition witnessing that $(\circledast)^{\rm rc}_{\mathbf{b}}$ of 2.2(3) holds, so Generic wins this play.

To show that $\mathbb{P}^{\bar\mu}$ is not reasonably B–bounding over $\mathcal{D}_\lambda$, we will describe a winning strategy $\mathbf{st}^*$ for Antigeneric in $\Game^{\rm rcB}(\emptyset_{\mathbb{P}^{\bar\mu}},\mathbb{P}^{\bar\mu})$. In the course of the play Antigeneric will construct (in addition to his innings) an increasing sequence $\langle\xi_\delta:\delta<\lambda\rangle$ of ordinals below λ. Suppose that the two players have arrived at the stage $\delta<\lambda$ of the play and Generic has chosen a set I_δ of cardinality less than λ and a system $\langle p^\delta_t:t\in I_\delta\rangle$ of conditions in $\mathbb{P}^{\bar\mu}$. Now, Antigeneric lets $C=\bigcap\{C^{p^\delta_t}:t\in I_\delta\}$ (it is a club of λ) and then he picks $\xi_\delta\in C$ such that

- $(\boxdot)_1$ $\mu_{\xi_\delta}>|I_\delta|$ and $\xi_\delta>\sup(\xi_\alpha:\alpha<\delta)+\delta$.

Next for every $t \in I_\delta$, Antigeneric chooses a condition $q^\delta_t \in \mathbb{P}^{\bar{\mu}}$ such that

$(\boxdot)_2$ $C^{q^\delta_t} = C \setminus (\xi_\delta + 1)$, $p^\delta_t \leq q^\delta_t$ and

$(\boxdot)_3$ if $t, s \in I_\delta$ are distinct, then $f^{q^\delta_t}(\xi_\delta) \neq f^{q^\delta_s}(\xi_\delta)$.

This completes the description of $\mathbf{st}^*$.

Suppose that $\langle I_\delta, \langle p^\delta_t, q^\delta_t : t \in I_\delta \rangle : \delta < \lambda \rangle$ is a play of $\Game^{\rm rcB}(\emptyset_{\mathbb{P}^{\bar{\mu}}}, \mathbb{P}^{\bar{\mu}})$ in which Antigeneric plays according to $\mathbf{st}^*$ and $\langle \xi_\delta : \delta < \lambda \rangle$ is the sequence constructed aside. Let $p^* \in \mathbb{P}^{\bar{\mu}}$. We claim that for every $\delta \in C^{p^*}$ the family $\{q^\delta_t : t \in I_\delta\}$ is not predense above p^*. So suppose that $\delta \in C^{p^*}$. It follows from $(\boxdot)_1 + (\boxdot)_2$ that $\delta, \xi_\delta \in \bigcap \{\lambda \setminus C^{q^\delta_t} : t \in I_\delta\}$.

Case 1 $\quad \xi_\delta \in C^{p^*}$.

Then we may find a condition $r \geq p^*$ such that $f^r(\xi_\delta) \notin \{f^{q^\delta_t}(\xi_\delta) : t \in I_\delta\}$ (remember $(\boxdot)_1$). The condition r is incompatible with all q^δ_t (for $t \in I_\delta$).

Case 2 $\quad \xi_\delta \notin C^{p^*}$.

If $f^{p^*}(\xi_\delta) \notin \{f^{q^\delta_t}(\xi_\delta) : t \in I_\delta\}$, then p^* is incompatible with all q^δ_t for $t \in I_\delta$. Otherwise there is a unique $s \in I_\delta$ such that $f^{p^*}(\xi_\delta) = f^{q^\delta_s}(\xi_\delta)$ and the condition p^* is incompatible with all q^δ_t for $t \in I_\delta \setminus \{s\}$. Since $\delta \in C^{p^*} \setminus C^{q^\delta_s}$, we may pick a condition $r \geq p^*$ such that $\delta \notin C^r$ and $f^r(\delta) \neq f^{q^\delta_s}(\delta)$. Then r is incompatible with all q^δ_t for $t \in I_\delta$.

Now we may easily argue that Generic lost the play. $\square$

An iterable property which will be introduced in the subsequent paper [RS11b] will capture also $\mathbb{P}^{\bar{\mu}}$. The second example is a very close relative of the forcing notion $\mathbb{Q}^1_S$ from the previous section. Yet at the moment we do not know if we can iterate it.

Definition 5.3. We define a forcing notion $\mathbb{Q}^2_{\mathcal{U}}$ as follows.
A condition in $\mathbb{Q}^2_{\mathcal{U}}$ is a triple $p = (w^p, C^p, \mathfrak{c}^p)$ such that

(i) $C^p \in \mathcal{U}$, $w^p \subseteq \min(C^p)$,
(ii) $\mathfrak{c}^p = \langle c^p_\alpha : \alpha \in C^p \rangle$ is a C^p–extending sequence.

The order $\leq_{\mathbb{Q}^2_{\mathcal{U}}} = \leq$ **of** $\mathbb{Q}^2_{\mathcal{U}}$ is given by
$p \leq_{\mathbb{Q}^2_{\mathcal{U}}} q$ if and only if

(a) $C^q \subseteq C^p$ and $w^q \in {\rm pos}^+(w^p, \mathfrak{c}^p, \min(C^q))$ and
(b) if $\alpha_0, \alpha_1 \in C^q$, $\alpha_0 < \alpha_1 = \min(C^q \setminus (\alpha_0+1))$ and $u \in {\rm pos}^+(w^q, \mathfrak{c}^q, \alpha_0)$, then $c^q_{\alpha_0}(u) \in {\rm pos}(u, \mathfrak{c}^p, \alpha_0, \alpha_1)$.

For $p \in \mathbb{Q}^2_{\mathcal{U}}$, $\alpha \in C^p$ and $u \in {\rm pos}^+(w^p, \mathfrak{c}^p, \alpha)$ we let $p{\restriction}_\alpha u \stackrel{\rm def}{=} (u, C^p \setminus \alpha, \mathfrak{c}^p{\restriction}(C^p \setminus \alpha))$.

Proposition 5.4. (1) *$\mathbb{Q}^2_{\mathcal{U}}$ is a $({<}\lambda)$–complete forcing notion of cardinality 2^λ.*

(2) *If $p \in \mathbb{Q}^2_{\mathcal{U}}$ and $\alpha \in C^p$, then*
- *for each $u \in {\rm pos}^+(w^p, \mathfrak{c}^p, \alpha)$, $p{\restriction}_\alpha u \in \mathbb{Q}^2_{\mathcal{U}}$ is a condition stronger than p, and*
- *the family $\{p{\restriction}_\alpha u : u \in {\rm pos}^+(w^p, \mathfrak{c}^p, \alpha)\}$ is pre-dense above p.*

(3) *Let $p \in \mathbb{Q}^2_{\mathcal{U}}$ and $\alpha < \beta$ be two successive members of C^p. Suppose that for each $u \in {\rm pos}^+(w^p, \mathfrak{c}^p, \alpha)$ we are given a condition $q_u \in \mathbb{Q}^2_{\mathcal{U}}$ such that $p{\restriction}_\beta c^p_\alpha(u) \leq q_u$. Then there is a condition $q \in \mathbb{Q}^2_{\mathcal{U}}$ such that letting $\alpha' = \min(C^q \setminus \beta)$ we have*

(a) *$p \leq q$, $w^q = w^p$, $C^q \cap \beta = C^p \cap \beta$ and $c^q_\delta = c^p_\delta$ for $\delta \in C^q \cap \alpha$, and*

(b) $\bigcup\{w^{q_u} : u \in \mathrm{pos}^+(w^p, \mathfrak{c}^p, \alpha)\} \subseteq \alpha'$, *and*
(c) $q_u \leq q{\restriction}_{\alpha'} c^q_\alpha(u)$ *for every* $u \in \mathrm{pos}^+(w^p, \mathfrak{c}^p, \alpha)$.

(4) *Assume that* $p \in \mathbb{Q}^2_{\mathcal{U}}$, $\alpha \in C^p$ *and* $\underset{\sim}{\tau}$ *is a* $\mathbb{Q}^2_{\mathcal{U}}$*–name such that* $p \Vdash$ "$\underset{\sim}{\tau} \in \mathbf{V}$". *Then there is a condition* $q \in \mathbb{Q}^2_{\mathcal{U}}$ *stronger than* p *and such that*
(a) $w^q = w^p$, $\alpha \in C^q$ *and* $C^q \cap \alpha = C^p \cap \alpha$, *and*
(b) *if* $u \in \mathrm{pos}^+(w^q, \mathfrak{c}^q, \alpha)$ *and* $\gamma = \min(C^q \setminus (\alpha+1))$, *then the condition* $q{\restriction}_\gamma c^q(u)$ *forces a value to* $\underset{\sim}{\tau}$.

Proof. Fully parallel to 4.6. $\square$

Definition 5.5. *The natural limit* of an $\leq_{\mathbb{Q}^2_{\mathcal{U}}}$–increasing sequence $\bar{p} = \langle p_\xi : \xi < \gamma\rangle \subseteq \mathbb{Q}^2_{\mathcal{U}}$ (where $\gamma < \lambda$ is a limit ordinal) is the condition $q = (w^q, C^q, \mathfrak{c}^q)$ defined as follows:

- $w^q = \bigcup\limits_{\xi<\gamma} w^{p_\xi}$, $C^q = \bigcap\limits_{\xi<\gamma} C^{p_\xi}$ and
- $\mathfrak{c}^q = \langle c^q_\delta : \delta \in C^q\rangle$ is such that for $\delta \in C^q$ and $u \subseteq \delta$ we have $c^q_\delta(u) = \bigcup\limits_{\xi<\gamma} c^{p_\xi}_\delta(u)$.

Proposition 5.6. (1) *Suppose* $\bar{p} = \langle p_\xi : \xi < \lambda\rangle$ *is a* $\leq_{\mathbb{Q}^2_{\mathcal{U}}}$*–increasing sequence of conditions from* $\mathbb{Q}^2_{\mathcal{U}}$ *such that*
(a) $w^{p_\xi} = w^{p_0}$ *for all* $\xi < \lambda$, *and*
(b) *if* $\gamma < \lambda$ *is limit, then* p_γ *is the natural limit of* $\bar{p}{\restriction}\gamma$, *and*
(c) *for each* $\xi < \lambda$, *if* $\delta \in C^{p_\xi}$, $\mathrm{otp}(C^{p_\xi} \cap \delta) = \xi$, *then* $C^{p_{\xi+1}} \cap (\delta+1) = C^{p_\xi} \cap (\delta+1)$ *and for every* $\alpha \in C^{p_{\xi+1}} \cap \delta$ *we have* $c^{p_{\xi+1}}_\alpha = c^{p_\xi}_\alpha$.

Then the sequence $\bar{p}$ *has an upper bound in* $\mathbb{Q}^2_{\mathcal{U}}$.

(2) *Suppose that* $p \in \mathbb{Q}^2_{\mathcal{U}}$ *and* $\underset{\sim}{h}$ *is a* $\mathbb{Q}^2_{\mathcal{U}}$*–name such that* $p \Vdash$ "$\underset{\sim}{h} : \lambda \longrightarrow \mathbf{V}$". *Then there is a condition* $q \in \mathbb{Q}^2_{\mathcal{U}}$ *stronger than* p *and such that*
($\otimes$) *if* $\delta < \delta'$ *are two successive points of* C^q, $u \in \mathrm{pos}(w^q, \mathfrak{c}^q, \delta)$, *then the condition* $q{\restriction}_{\delta'} c^q_\delta(u)$ *decides the value of* $\underset{\sim}{h}{\restriction}(\delta+1)$.

Proof. Fully parallel to 4.8. $\square$

Definition 5.7. We let $\underset{\sim}{W}$ and $\underset{\sim}{\eta}, \underset{\sim}{\nu}$ be $\mathbb{Q}^2_{\mathcal{U}}$–names such that

$$\Vdash_{\mathbb{Q}^2_{\mathcal{U}}} \underset{\sim}{W} = \bigcup\{w^p : p \in \Gamma_{\mathbb{Q}^2_{\mathcal{U}}}\}$$

and

$$\Vdash_{\mathbb{Q}^2_{\mathcal{U}}} \text{" } \underset{\sim}{\eta}, \underset{\sim}{\nu} \in {}^\lambda\lambda \text{ and if } \langle \delta_\xi : \xi < \lambda\rangle \text{ is the increasing enumeration of } \mathrm{cl}(\underset{\sim}{W}), \text{ and } \delta_\xi \leq \alpha < \delta_{\xi+1},\ \xi < \lambda, \text{ then } \underset{\sim}{\eta}(\alpha) = \xi \text{ and } \underset{\sim}{\nu}(\alpha) = \delta_{\xi+4} \text{ ".}$$

Note that if $p \in \mathbb{Q}^2_{\mathcal{U}}$, then

$$p \Vdash_{\mathbb{Q}^2_{\mathcal{U}}} \text{" } \underset{\sim}{W} \subseteq \bigcup\{[\alpha_0, \alpha_1) : \alpha_0, \alpha_1 \in C^p \ \&\ \alpha_1 = \min\big(C^p \setminus (\alpha_0+1)\big)\} \text{ "}$$

and

$$p \Vdash_{\mathbb{Q}^2_{\mathcal{U}}} \text{" } \{\alpha \in C^p : \big[\alpha, \min(C^p \setminus (\alpha+1))\big) \cap \underset{\sim}{W} \neq \emptyset\}, \{\alpha \in C^p : \big[\alpha, \min(C^p \setminus (\alpha+1))\big) \cap \underset{\sim}{W} = \emptyset\} \in \big(\mathcal{U}^{\mathbb{Q}^2_{\mathcal{U}}}\big)^+ \text{ ".}$$

Proposition 5.8. $\Vdash_{\mathbb{Q}^2_{\mathcal{U}}} \big(\forall f \in {}^\lambda\lambda \cap \mathbf{V}\big)\big(\forall A \in \mathcal{U}^{\mathbb{Q}^2_{\mathcal{U}}}\big)\big(\exists \alpha \in A\big)\big(f(\alpha) < \underset{\sim}{\nu}(\alpha)\big)$.

Proof. Fully parallel to 4.10. $\square$

Proposition 5.9. *The forcing notion* $\mathbb{Q}^2_{\mathcal{U}}$ *is reasonably* C*–bounding over* $\mathcal{U}$.

Proof. Fully parallel to 4.11. □

The following problem is a particular case of 2.9(1).

Problem 5.10. Are λ–support iterations of $\mathbb{Q}^2_{\mathcal{U}}$ λ–proper?

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