

DISJOINT NON-FREE SUBGROUPS OF ABELIAN GROUPS

ANDREAS BLASS AND SAHARON SHELAH

ABSTRACT. Let G be an abelian group and let λ be the smallest rank of any group whose direct sum with a free group is isomorphic to G . If λ is uncountable, then G has λ pairwise disjoint, non-free subgroups. There is an example where λ is countably infinite and G does not have even two disjoint, non-free subgroups.

1. INTRODUCTION

In a discussion between the first author and John Irwin, the question arose whether every non-free, separable, torsion-free abelian group has two disjoint non-free subgroups. (Of course, in the context of subgroups, “disjoint” means that the intersection is $\{0\}$.) The main result of this paper is a strong affirmative answer. To state the result in appropriate generality, we need some terminology.

Convention 1. All groups in this paper are understood to be abelian and torsion-free. In particular, “free group” means “free abelian group”.

Definition 2. The *non-free rank* of a group G , written $\text{nfrk}(G)$, is the smallest cardinal κ such that G can be split as the direct sum of a group of rank $\leq \kappa$ and a free group.

Theorem 3. *If $\text{nfrk}(G)$ is uncountable, then G has $\text{nfrk}(G)$ pairwise disjoint, non-free subgroups.*

Recall that any countable, separable group is free. It follows that, if G is separable and not free, then $\text{nfrk}(G)$ is necessarily uncountable, so the theorem applies to G . It gives not only two disjoint non-free subgroups as in the original question but $\text{nfrk}(G) \geq \aleph_1$ of them.

The number of disjoint, non-free subgroups obtained in the theorem is the most one could hope for. Indeed, if $G \cong H \oplus F$ where F is free and H has rank and therefore cardinality equal to the infinite

Blass was partially supported by the United States National Science Foundation, grant DMS-0070723.

Shelah was partially supported by the US-Israel Binational Science Foundation, grant 2002323. This is paper 870 in his list of publications.

cardinal $\text{nfrk}(G)$, then any non-free subgroup S of G must have a non-zero intersection with H . Otherwise the projection to F would map S one-to-one into the free group F and it would follow that S is free. Therefore, disjoint non-free subgroups of G must intersect $H - \{0\}$ in disjoint non-empty sets. So there cannot be more such subgroups than $|H| = \text{nfrk}(G)$.

Although the theorem gives an optimal result for separable groups, the fact that it does not explicitly mention separability raises another question: Is the uncountability hypothesis really needed? We shall answer this question affirmatively by exhibiting a (necessarily non-separable) non-free group G such that $\text{nfrk}(G) = \aleph_0$ and G does not have two disjoint non-free subgroups, let alone \aleph_0 of them.

This paper contains, in addition to this introduction and a section of known preliminary results, three sections. The first two are devoted to the proof of Theorem 3. Section 3 contains the proof for the case that $\text{nfrk}(G)$ is (uncountable and) regular. Section 4 contains the additional arguments needed to extend the result to the case of singular $\text{nfrk}(G)$. Finally, Section 5 presents our counterexample for the case of countable non-free rank.

2. PRELIMINARIES

In this section, we collect for reference some conventions, definitions, and known results that will be needed in our proofs. The book [1] of Eklof and Mekler serves as a standard reference for this material.

Convention 4. Group operations will be written additively. For $n \in \mathbb{Z}$ and x a group element, the notation nx means the sum of n copies of x if $n > 0$, it means $(-n)(-x)$ if $n < 0$, and it means 0 if $n = 0$.

Definition 5. A subgroup H of G is *pure* in G if, whenever $x \in G$ and $nx \in H$ for some non-zero integer n , then $x \in H$. If H is an arbitrary subgroup of G , then we write H_* for the *purification* of H , the smallest pure subgroup of G that includes H .

If X is any subset of a group G , we write $\langle X \rangle$ for the subgroup of G generated by X . Its purification $\langle X \rangle_*$ is called the *pure subgroup generated by X* .

For a prime number p , the field of p -adic numbers will be denoted by \mathbb{Q}_p , and the subring of p -adic integers will be denoted by \mathbb{Z}_p .

Convention 6. When we refer to the numerators or denominators of rational numbers, we always mean that the rational numbers are regarded as fractions in reduced form.

Recall that the rational field \mathbb{Q} is a subfield of \mathbb{Q}_p and that the intersection $\mathbb{Q} \cap \mathbb{Z}_p$ consists of those rational numbers whose denominators are not divisible by p . A useful consequence is that a rational number is in \mathbb{Z}_p for all primes p if and only if it is an integer. Recall also that $p\mathbb{Z}_p$ is the unique maximal ideal of \mathbb{Z}_p and that the quotient $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to the p -element field $\mathbb{Z}/p\mathbb{Z}$, the isomorphism being induced by the inclusion of \mathbb{Z} in \mathbb{Z}_p .

Let λ be an uncountable regular cardinal and G a group of cardinality λ . A *filtration* of G is an increasing sequence $(G_\alpha : \alpha < \lambda)$ of subgroups of G , each of cardinality $< \lambda$, continuous at limit ordinals $\beta < \lambda$ (i.e., $\bigcup_{\alpha < \beta} G_\alpha = G_\beta$), and with $\bigcup_{\alpha < \lambda} G_\alpha = G$. Although there are many filtrations of G , any two of them, say (G_α) and (G'_α) , agree almost everywhere in the sense that the set of agreement $\{\alpha < \lambda : G_\alpha = G'_\alpha\}$ includes (in fact is) a closed unbounded set (*club*) in λ .

Recall that a subset of λ is *stationary* if it intersects every club and that the intersection of any fewer than λ clubs is again a club. It follows that a stationary set cannot be the union of fewer than λ non-stationary sets. We shall also need Fodor's theorem and a variant of it. Fodor's theorem says that, if S is a stationary subset of λ and $f : S \rightarrow \lambda$ is a regressive function (i.e., $f(\alpha) < \alpha$ for all $\alpha \in S$), then f is constant on a stationary subset of S . The variant that we shall need is the following.

Lemma 7. *Let (G_α) be a filtration of G , let $S \subseteq \lambda$ be stationary, and let $f : S \rightarrow G$ be a function such that $f(\alpha) \in G_\alpha$ for all $\alpha \in S$. Then f is constant on some stationary subset of S .*

Proof. Replacing S by its intersection with the club of all limit ordinals, we may assume that every $\alpha \in S$ is a limit ordinal. For such α , continuity of the filtration tells us that $f(\alpha) \in G_{g(\alpha)}$ for some $g(\alpha) < \alpha$. By Fodor's theorem, the regressive function g is constant, say with value β , on some stationary subset T of S . Since $|G_\beta| < \lambda$, we have a decomposition of T into fewer than λ sets $T_x = \{\alpha \in T : f(\alpha) = x\}$ for $x \in G_\beta$. So one of the pieces T_x must be stationary, and the lemma is established. \square

The connection between filtrations and freeness is given by the following definition and lemma.

Definition 8. The *Gamma invariant* of G is

$$\Gamma(G) = \{\alpha < \lambda : G/G_\alpha \text{ has a non-free subgroup of cardinality } < \lambda\}.$$

This definition seems to depend on the choice of filtration, but in fact it doesn't, modulo restriction to a club, because any two filtrations

agree on a club. In particular, the statement “ $\Gamma(G)$ is stationary” has the same truth value for all choices of the filtration.

Lemma 9. $\Gamma(G)$ is stationary if and only if $\text{nfrk}(G) = \lambda$.

Proof. Suppose first that $\Gamma(G)$ is not stationary and is therefore disjoint from some club C . Let $c : \lambda \rightarrow C$ be the function enumerating C in increasing order. We construct, by induction on α , a free basis B_α for $G_{c(\alpha)}/G_{c(0)}$ such that $B_\alpha \subseteq B_\beta$ whenever $\alpha < \beta$. This is trivial for $\alpha = 0$. Continuity of c (because C is closed) lets us simply take unions at limit stages. At a successor step, say from α to $\alpha + 1$, remember that $c(\alpha) \in C \subseteq \lambda - \Gamma(G)$, so $G_{c(\alpha+1)}/G_{c(\alpha)}$ has a free basis. Pick one representative in $G_{c(\alpha+1)}$ for each member of this basis, and adjoin the chosen representatives to B_α to get $B_{\alpha+1}$. At the end of the induction, $\bigcup_{\alpha < \lambda} B_\alpha$ is a free basis for $G/G_{c(0)}$. Since homomorphisms onto free groups split, G is isomorphic to the direct sum of $G_{c(0)}$ and a free group. So $\text{nfrk}(G) \leq |G_{c(0)}| < \lambda$.

For the converse, suppose $G = H \oplus F$ where $|H| < \lambda$ and F is free. Fix a free basis for F , necessarily of cardinality λ , and enumerate it in a sequence of order-type λ . Then G has a filtration whose α^{th} element G_α is the subgroup generated by H and the first α elements in the enumeration of the basis of F . Then each of the quotients G/G_α is free, with a basis represented by all but the first α elements of the basis of F . Therefore, all subgroups of G/G_α are also free. So $\Gamma(G)$ is empty for this filtration, and therefore non-stationary for all filtrations. \square

It will be convenient to use filtrations normalized as follows.

Lemma 10. G has a filtration $(G_\alpha : \alpha < \lambda)$ such that

- each G_α is a pure subgroup of G , and
- whenever G/G_α has a non-free subgroup of cardinality $< \lambda$ (i.e., whenever $\alpha \in \Gamma(G)$ as calculated with this filtration), then $G_{\alpha+1}/G_\alpha$ is such a subgroup.

Proof. Starting with any filtration (H_α) of G , we produce a new filtration with the desired additional properties by defining G_α inductively. Start with $G_0 = (H_0)_*$, and at limit ordinals take unions (as demanded by the definition of filtration). At a successor step from α to $\alpha + 1$, first choose a subgroup K of G , of cardinality $< \lambda$, such that $G_\alpha \subsetneq K$ and such that, if possible, K/G_α is not free. Then let $G_{\alpha+1} = (H_{\alpha+1} + K)_*$. \square

3. PROOF FOR REGULAR NON-FREE RANK

In this section, we shall establish Theorem 3 in the case that $\text{nfrk}(G)$ is an uncountable, regular cardinal λ . We may assume, without loss of generality, that $|G| = \lambda$. Indeed, if G were larger, we could use, in place of G , the summand H in the decomposition $G \cong H \oplus F$ given by the definition of $\text{nfrk}(G)$. This H has rank, cardinality, and non-free rank all equal to λ , and of course if we find λ pairwise disjoint non-free subgroups in H then these will serve in G as well. From now on, assume $|G| = \lambda$.

For the rest of this section, we fix a filtration $(G_\alpha : \alpha < \lambda)$ with the properties in Lemma 10. We define $\Gamma(G)$ using this filtration in Definition 8, and we note that, by Lemma 9, $\Gamma(G)$ is a stationary subset of λ .

For each $\alpha \in \Gamma(G)$, the properties of (G_α) in Lemma 10 imply that $G_{\alpha+1}/G_\alpha$ is a torsion-free, non-free group. Let Y_α be a maximal linearly independent subset of this group. By linear independence, the subgroup $\langle Y_\alpha \rangle$ generated by Y_α is free, and by maximality, its purification $\langle Y_\alpha \rangle_*$ is all of $G_{\alpha+1}/G_\alpha$. Choose, for each element of Y_α , a representative in $G_{\alpha+1}$, and let X_α be the set of these chosen representatives. Thus, the projection from $G_{\alpha+1}$ to its quotient modulo G_α maps X_α one-to-one onto Y_α .

Expressing the properties of Y_α in the quotient group as properties “modulo G_α ” of X_α in the group $G_{\alpha+1}$ we obtain the following.

Lemma 11.

- X_α is linearly independent modulo G_α . That is, if G_α contains a linear combination, with integer coefficients, of members of X_α , then all the coefficients are zero.
- $(\langle X_\alpha \rangle + G_\alpha)_* = G_{\alpha+1}$.

Proof. For the first assertion, notice that such a linear combination, when projected to the quotient modulo G_α , becomes a linear combination of members of Y_α that equals zero. So the linear independence of Y_α in the quotient group gives the required conclusion.

For the second assertion, consider an arbitrary element $a \in G_{\alpha+1}$. Its image \bar{a} in $G_{\alpha+1}/G_\alpha$ has a multiple $n\bar{a} \in \langle Y_\alpha \rangle$ for some non-zero $n \in \mathbb{Z}$. Since the projection maps X_α onto Y_α and thus maps $\langle X_\alpha \rangle$ onto $\langle Y_\alpha \rangle$, we have an element $z \in \langle X_\alpha \rangle$ projecting to $n\bar{a}$. So na and z project to the same element, which means $na = z + g$ for some $g \in G_\alpha$. This equation establishes that a is in the purification of $\langle X_\alpha \rangle + G_\alpha$, as required. \square

Temporarily fix an arbitrary stationary subset T of $\Gamma(G)$. Using T we define a subgroup L of G by

$$L = \left\langle \bigcup_{\beta \in T} X_\beta \right\rangle_*.$$

Our immediate objective, and indeed the main part of our argument for Theorem 3 when $\text{nfrk}(G)$ is uncountable, is to show that L is not free.

For this purpose, we shall use Lemma 9 and show that $\Gamma(L)$ is stationary. (So we shall have not only that L is not free but that $\text{nfrk}(L) = \lambda$.) In fact, we shall prove more, namely that $\Gamma(L)$ contains all of the stationary set T except for some non-stationary subset.

To begin the analysis of $\Gamma(L)$, we must choose a filtration of L to use in the definition of $\Gamma(L)$, and two natural choices present themselves. One is the restriction to L of the filtration we already have for G , i.e., $(G_\alpha \cap L : \alpha < \lambda)$. The other is obtained from the way L is generated by the X_α 's; this filtration is $(L_\alpha : \alpha < \lambda)$, where

$$L_\alpha = \left\langle \bigcup_{\substack{\beta \in T \\ \beta < \alpha}} X_\beta \right\rangle_*.$$

We shall want to use each of these occasionally. Fortunately, as mentioned earlier, these two filtrations (like any two filtrations of the same group) agree on a club. Let T_1 be the intersection of T with such a club. To show that $\Gamma(L)$ contains all but a non-stationary part of T , it suffices, since $T - T_1$ is non-stationary, to prove that $\Gamma(L)$ contains all but a non-stationary part of T_1 .

So our objective is now to show that the “exceptional” set

$$W = \{\alpha \in T_1 : \alpha \notin \Gamma(L)\}$$

is not stationary. Suppose, toward a contradiction, that W is stationary.

For each $\alpha \in W$, we have the following two facts:

- $L_{\alpha+1}/L_\alpha$ is free.
- $L_\alpha = L_{\alpha+1} \cap G_\alpha$.

The first of these is immediate because $\alpha \notin \Gamma(L)$. The second follows from

$$L_\alpha \subseteq L_{\alpha+1} \cap G_\alpha \subseteq L \cap G_\alpha = L_\alpha,$$

where the first inclusion uses the fact that L_α is the pure subgroup of G generated by a subset $\bigcup_{\beta \in T, \beta < \alpha} X_\beta$ of G_α and G_α is a pure subgroup of G . The second inclusion is trivial, and the final equality uses the fact that $\alpha \in T_1$ so the two filtrations agree at α .

Combining the two facts just established, we have, for each $\alpha \in W$, that the following group is free:

$$\frac{L_{\alpha+1}}{L_{\alpha+1} \cap G_\alpha} \cong \frac{L_{\alpha+1} + G_\alpha}{G_\alpha}.$$

Contrast this with the fact that, since $\alpha \in T \subseteq \Gamma(G)$, the following group is not free:

$$\frac{G_{\alpha+1}}{G_\alpha} = \frac{(\langle X_\alpha \rangle + G_\alpha)_*}{G_\alpha}.$$

(Here we use that our filtration was chosen to satisfy the second conclusion of Lemma 10.) Since subgroups of free groups are free, we conclude that

$$(\langle X_\alpha \rangle + G_\alpha)_* \not\subseteq L_{\alpha+1} + G_\alpha.$$

Choose, for each $\alpha \in W$, some element $g_\alpha \in (\langle X_\alpha \rangle + G_\alpha)_*$ that is not in $L_{\alpha+1} + G_\alpha$. By definition of purification, we have some $n_\alpha \in \mathbb{Z} - \{0\}$ such that $n_\alpha g_\alpha \in \langle X_\alpha \rangle + G_\alpha$. That is, $n_\alpha g_\alpha = c_\alpha + h_\alpha$ where c_α is a linear combination, with integer coefficients, of elements of X_α and where $h_\alpha \in G_\alpha$.

Because we assumed, toward a contradiction, that W is stationary, and because $h_\alpha \in G_\alpha$ for all $\alpha \in W$, Lemma 7 gives us a stationary set $W_1 \subseteq W$ such that h_α is the same element h for all $\alpha \in W_1$. Furthermore, since the values of n_α all lie in the countable set \mathbb{Z} , there is a stationary $W_2 \subseteq W_1$ such that n_α has the same value n for all $\alpha \in W_2$.

Let $\alpha < \beta$ be any two elements of W_2 . Then we have

$$\begin{aligned} n g_\alpha &= c_\alpha + h \\ n g_\beta &= c_\beta + h. \end{aligned}$$

Subtract to get

$$n(g_\beta - g_\alpha) = c_\beta - c_\alpha.$$

Both c_α and c_β are linear combinations of elements of $X_\alpha \cup X_\beta \subseteq L_{\beta+1}$. So $L_{\beta+1}$ contains the right side of the last equation. As $L_{\beta+1}$ is pure in G , it follows that $g_\beta - g_\alpha \in L_{\beta+1}$ and so $g_\beta \in g_\alpha + L_{\beta+1}$.

Notice that G_β includes G_α (because $\alpha < \beta$), and X_α (because $X_\alpha \subseteq G_{\alpha+1}$ and $\alpha + 1 \leq \beta$), and therefore $\langle X_\alpha \rangle + G_\alpha$, and therefore $(\langle X_\alpha \rangle + G_\alpha)_*$ (because, according to the first conclusion of Lemma 10, G_β is a pure subgroup of G). We chose g_α from this last group, so we have $g_\alpha \in G_\beta$. But then the result from the preceding paragraph gives us that $g_\beta \in G_\beta + L_{\beta+1}$, which contradicts our choice of g_β .

This contradiction completes the proof that W cannot be stationary and therefore L is not free.

Recall that L was defined in terms of an arbitrary but fixed stationary $T \subseteq \Gamma(G)$. We shall now need to vary T , so, to indicate the dependence of L on T , we write $L(T)$ for what was previously called simply L .

Lemma 12. *If T_1 and T_2 are disjoint stationary subsets of $\Gamma(G)$, then the subgroups $L(T_1)$ and $L(T_2)$ are disjoint.*

Proof. Notice first that, if two subgroups H_1 and H_2 are disjoint, then so are their purifications. Indeed, if the purifications had a common non-zero element x , then x would have non-zero multiples $n_1x \in H_1$ and $n_2x \in H_2$. But then n_1n_2x would be a non-zero element in $H_1 \cap H_2$ contrary to hypothesis.

So to prove the lemma, it suffices to prove that the subgroups generated by $\bigcup_{\beta \in T_1} X_\beta$ and $\bigcup_{\beta \in T_2} X_\beta$ are disjoint. Suppose, toward a contradiction, that they are not disjoint, choose a non-zero element in their intersection, and write it first as a linear combination c_1 of elements of $\bigcup_{\beta \in T_1} X_\beta$ and second as a linear combination c_2 of elements of $\bigcup_{\beta \in T_2} X_\beta$. Let β be the largest of the finitely many ordinals such that elements of X_β occur in these linear combinations with non-zero coefficients. Since T_1 and T_2 are disjoint, β is in only one of them, say T_1 . Split c_1 as $c'_1 + c''_1$ where $c'_1 \neq 0$ contains the terms from X_β and c''_1 contains the terms from X_α 's with $\alpha < \beta$. Then $c'_1 = c_2 - c''_1 \in G_\beta$. This contradicts the linear independence of X_β modulo G_β in Lemma 11. \square

Because of this lemma and the non-freeness of $L(T)$ for all stationary $T \subseteq \Gamma(G)$, we can get as many pairwise disjoint, non-free subgroups of G as we can get pairwise disjoint, stationary subsets of $\Gamma(G)$. It remains only to quote Solovay's famous theorem [6, Theorem 9] that, for any uncountable regular cardinal λ , every stationary subset of λ can be partitioned into λ pairwise disjoint, stationary subsets.

4. PROOF FOR SINGULAR NON-FREE RANK

In this section, we complete the proof of Theorem 3 by treating the case of singular $\text{nfrk}(G)$. An important ingredient of the proof is the following consequence of the second author's singular compactness theorem.

Lemma 13. *Assume that $\kappa < \lambda$ are infinite cardinals, that λ is singular, that G is a group of cardinality λ , and that every subgroup H of G with $|H| < \lambda$ has $\text{nfrk}(H) \leq \kappa$. Then $\text{nfrk}(G) \leq \kappa$.*

Proof. This follows from the singular compactness theorem of [5]. It is explicitly stated in [2, Theorem 2]. (To avoid possible confusion,

note that in [2] “ κ -generated” means generated by strictly fewer than κ elements.) For an easier proof, see [3]. \square

Our use of this lemma will be via the following consequence.

Lemma 14. *Let G be a group whose non-free rank is a singular cardinal λ . For any infinite cardinal $\kappa < \lambda$, there exists a subgroup H of G whose non-free rank is regular and satisfies $\kappa < \text{nfrk}(H) = |H| < \lambda$.*

Proof. Given G , λ , and κ as in the statement of the lemma, let H be a subgroup of G with $\text{nfrk}(H)$ as small as possible subject to the constraint that $\text{nfrk}(H) > \kappa$. Such an H certainly exists, since $\text{nfrk}(G) > \kappa$. By definition of $\text{nfrk}(H)$, we can split H as the direct sum of a free group and a group H_1 of cardinality $\text{nfrk}(H)$. Then $|H_1| = \text{nfrk}(H) = \text{nfrk}(H_1)$. Replacing H with H_1 if necessary, we assume from now on that $|H| = \text{nfrk}(H)$.

It remains only to show that $\text{nfrk}(H)$ is a regular cardinal and that $\text{nfrk}(H) < \lambda$. The latter follows from the former, because λ is singular, and so we need only prove the regularity of $\text{nfrk}(H)$. Suppose, toward a contradiction, that $\text{nfrk}(H)$ is a singular cardinal. Notice that, for every subgroup K of H with cardinality smaller than $|H|$, we have $\text{nfrk}(K) \leq |K| < |H| = \text{nfrk}(H)$, and so, by minimality of $\text{nfrk}(H)$, we must have $\text{nfrk}(K) \leq \kappa$. By Lemma 13, $\text{nfrk}(H) \leq \kappa$ also, but this contradicts the choice of H . \square

Using the lemma, we prove the singular case of Theorem 3 as follows. Let G be a group with $\text{nfrk}(G) = \lambda$ singular, let $\mu < \lambda$ be the cofinality of λ , and let $(\kappa_\xi : \xi < \mu)$ be a strictly increasing μ -sequence of cardinals with supremum λ . Inductively define a μ -sequence of subgroups G_ξ of G , with the following properties for all $\xi < \mu$:

- $|G_\xi| = \text{nfrk}(G_\xi)$.
- $\text{nfrk}(G_\xi)$ is a regular cardinal.
- $\kappa_\xi < \text{nfrk}(G_\xi) < \lambda$.
- $\sum_{\eta < \xi} |G_\eta| < \text{nfrk}(G_\xi)$.

Once G_η is defined and has the desired properties for all $\eta < \xi$, we obtain G_ξ by applying Lemma 14 with κ equal to the larger of κ_ξ and $\sum_{\eta < \xi} |G_\eta|$. Notice that this sum is strictly smaller than λ because $|G_\eta| < \lambda$ for all $\eta < \xi$ by induction hypothesis and because $\xi < \mu = \text{cf}(\lambda)$. Thus, our κ is smaller than λ , and the H provided by Lemma 14 serves as the required G_ξ .

The regular case of Theorem 3, proved in the previous section, applies to each G_ξ and provides a family \mathcal{D}_ξ of $\text{nfrk}(G_\xi)$ pairwise disjoint non-free subgroups of G_ξ .

Although the groups in any \mathcal{D}_ξ are pairwise disjoint, there may be a non-zero intersection between a group $H \in \mathcal{D}_\xi$ and a group $K \in \mathcal{D}_\eta$ for some $\eta < \xi$. But this does not happen too often. Specifically, for fixed η , K , and ξ , the number of such H 's is at most $|K| \leq |G_\eta|$ because different H 's from \mathcal{D}_ξ would meet $K - \{0\}$ in disjoint sets. Therefore, if we keep η and ξ fixed but let K vary through all elements of \mathcal{D}_η , then the number of $H \in \mathcal{D}_\xi$ that have non-zero intersection with some such K is at most $|G_\eta| \cdot |\mathcal{D}_\eta|$. Since \mathcal{D}_η is a family of disjoint subgroups of G_η , its cardinality is at most $|G_\eta|$, and so the product $|G_\eta| \cdot |\mathcal{D}_\eta|$ is also at most $|G_\eta|$. Now keeping ξ fixed but letting η vary through all ordinals $< \xi$, we find that the number of $H \in \mathcal{D}_\xi$ that have non-zero intersection with some K in some earlier \mathcal{D}_η is at most

$$\sum_{\eta < \xi} |G_\eta| < \text{nfrk}(G_\xi).$$

Therefore, if we discard these H 's from \mathcal{D}_ξ , what remains is a family \mathcal{D}'_ξ , still of cardinality $\text{nfrk}(G_\xi)$, still consisting of pairwise disjoint, non-free subgroups of G_ξ , but enjoying the additional property that all its members are disjoint from all members of earlier \mathcal{D}_η 's and, a fortiori, from all members of earlier \mathcal{D}'_η 's.

Therefore, $\mathcal{D}' = \bigcup_{\xi < \mu} \mathcal{D}'_\xi$ is a family of pairwise disjoint non-free subgroups of G . Its cardinality satisfies

$$\begin{aligned} |\mathcal{D}'| &= \sup\{|\mathcal{D}'_\xi| : \xi < \mu\} \\ &= \sup\{\text{nfrk}(G_\xi) : \xi < \mu\} \\ &\geq \sup\{\kappa_\xi : \xi < \mu\} \\ &= \lambda, \end{aligned}$$

so the theorem is proved.

5. COUNTEREXAMPLE FOR COUNTABLE NON-FREE RANK

In this section, we construct an example of a group G with $|G| = \text{nfrk}(G) = \aleph_0$ and with no two disjoint non-free subgroups. This shows that the uncountability assumption in Theorem 3 cannot be removed.

The group G will be a subgroup of the direct sum of \aleph_0 copies of the additive group \mathbb{Q} of rational numbers. G will be defined as the set of solutions, in this direct sum, of infinitely many p -adic conditions for all primes p . The construction of G will proceed in three phases. The first will set up some conventions and bookkeeping. The second will define, for each prime p , a certain set of vectors over \mathbb{Z}/p and a lifting of these vectors to \mathbb{Z}_p . The third will use these vectors to define the group G . After the construction is complete, we shall verify first that

$\text{nfrk}(G) = \aleph_0$ and second that G does not have two disjoint, non-free subgroups.

5.1. Conventions and bookkeeping. By a *vector* over a ring R , we shall usually mean an infinite sequence of elements of R , the sequence being indexed by the set \mathbb{N} of positive integers. (We do not include 0 in the index set, as we shall have a separate, special use for 0 later.) The rings relevant to our work will include \mathbb{Z} , \mathbb{Q} , \mathbb{Z}/p , \mathbb{Z}_p , and \mathbb{Q}_p for prime numbers p .

Occasionally, we shall need to refer to vectors of finite length l , with components indexed by $\{1, 2, \dots, l\}$, but this finiteness (and the value of l) will always be explicitly stated. Furthermore, it would do no harm to identify any finitely long vector with the infinite vector obtained by appending a sequence of zeros. We use the notation $\vec{x} \upharpoonright l$ for the vector of length l consisting of just the first l components of \vec{x} ; under the identification in the preceding sentence, $\vec{x} \upharpoonright l$ can also be considered as obtained from \vec{x} by replacing all components beyond the first l by 0.

We call a vector \vec{x} *finitely supported* if the set $\{i \in \mathbb{N} : x_i \neq 0\}$, called the *support* of \vec{x} , is finite. The inner product of two vectors is defined as

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i \in \mathbb{N}} x_i y_i,$$

provided at least one of the vectors is finitely supported, so that the infinite sum makes sense.

Partition the set P of prime numbers into infinitely many infinite pieces, and label the pieces as $P_{\vec{x}}$, where \vec{x} ranges over the non-zero, finitely supported, infinite vectors over \mathbb{Z} . (The number of such vectors is \aleph_0 , so the indexing makes sense.) Fix this indexed partition of P for the rest of the proof.

Also fix an enumeration, as $(\vec{\lambda}_i : i \in \mathbb{N})$, of all the finitely supported vectors over \mathbb{Q} .

Recall that \mathbb{Q} is canonically identified with a subfield of \mathbb{Q}_p , and that under this identification \mathbb{Z} becomes a subring of \mathbb{Z}_p . This inclusion of \mathbb{Z} in \mathbb{Z}_p induces an isomorphism between the quotients modulo p , $\mathbb{Z}/p \cong \mathbb{Z}_p/p\mathbb{Z}_p$. We write $[x]_p$, or just $[x]$ when p is clear from the context, for the equivalence class of x modulo p ; here x is in \mathbb{Z} or \mathbb{Z}_p , and $[x]$ is in \mathbb{Z}/p . We refer to x as a *representative* of $[x]$. We use the same notation for vectors; $[\vec{x}]_p$ is obtained from \vec{x} by reducing all components modulo p .

5.2. Useful sets of vectors. For this subsection, let p be a fixed prime. Later, the work we do here will be applied to all primes, but it

is notationally and conceptually easier to begin with just one p . Let \vec{x} be the unique vector such that $p \in P_{\vec{x}}$. Recall that this \vec{x} is a non-zero, finitely supported vector over \mathbb{Z} . Thus, $[\vec{x}]$ is a finitely supported (but possibly zero) vector over \mathbb{Z}/p .

We define a finite set M of vectors over \mathbb{Z}/p as follows. Choose an integer l larger than p and all elements of the support of \vec{x} . M will be described as a set of finite vectors, in $(\mathbb{Z}/p)^l$, but we really mean the infinite vectors obtained by appending a sequence of zeros.

If $[\vec{x}] = \vec{0}$, then M consists of all the vectors in $(\mathbb{Z}/p)^l$. If, on the other hand, $[\vec{x}] \neq \vec{0}$, then we proceed as follows. Call an index i or the corresponding vector $\vec{\lambda}_i$ (from the enumeration fixed above) *relevant* if $i < p-1$ and p does not divide the denominator of any component of $\vec{\lambda}_i$. For each relevant i , the components of $\vec{\lambda}_i$ are in \mathbb{Z}_p , so it makes sense to reduce them modulo p , obtaining a vector $[\vec{\lambda}_i]$ over \mathbb{Z}/p . Choose a non-zero $a \in \mathbb{Z}/p$ that is distinct from $\langle [-\vec{\lambda}_i], [\vec{x}] \rangle$ for all relevant i . The requirement, in the definition of relevance, that $i < p-1$ means that at most $p-2$ indices are relevant, so a suitable a exists. Now let M consist of all those $\vec{m} \in (\mathbb{Z}/p)^l$ such that $\langle \vec{m}, [\vec{x}] \rangle = a$.

Before proceeding further, we summarize the properties of M that we shall need later.

Lemma 15. *The set $M \subseteq (\mathbb{Z}/p)^l$ defined here has the following properties.*

- (1) *For each $k \leq l$, the truncations to k of the vectors in M span $(\mathbb{Z}/p)^k$.*
- (2) *The same holds for the truncations to k of the translates $[\vec{\lambda}_i] + M$ for each relevant $\vec{\lambda}_i$.*
- (3) *As \vec{m} varies through M , all the inner products $\langle \vec{m}, [\vec{x}] \rangle$ have the same value a .*
- (4) *This value a is non-zero and different from $\langle [-\vec{\lambda}_i], [\vec{x}] \rangle$ for all relevant i provided $[\vec{x}] \neq \vec{0}$.*

Proof. If $[\vec{x}] = \vec{0}$ then item (3) is obvious with $a = 0$ and item (4) doesn't apply. If $[\vec{x}] \neq \vec{0}$ then items (3) and (4) are explicitly in the definition of M .

As for items (1) and (2), it suffices to prove these for $k = l$, since truncation to smaller k 's commutes with linear combinations. So assume $k = l$. If $[\vec{x}] = \vec{0}$ then both (1) and (2) are obvious as M is all of $(\mathbb{Z}/p)^l$. So assume $[\vec{x}] \neq \vec{0}$. Then M is defined as a certain affine hyperplane in $(\mathbb{Z}/p)^l$. Since $a \neq 0$, this hyperplane does not pass through the origin, and therefore it spans $(\mathbb{Z}/p)^l$. This proves (1). For (2), observe

that $[\vec{\lambda}_i] + M$ is another affine hyperplane, a translate of M . It does not pass through the origin either, because $a \neq \langle [-\vec{\lambda}_i], [\vec{x}] \rangle$. So it, too, spans $(\mathbb{Z}/p)^l$, and (2) is proved. \square

The next step is to lift M to an infinite set of vectors over \mathbb{Z}_p . For this, we regard the vectors in M as having infinite length (rather than l), by appending zeros. Arbitrarily choose, for each $\vec{m} \in M$, a countable infinity of vectors $\vec{\psi}$ over \mathbb{Z}_p with $[\vec{\psi}] = \vec{m}$. Let Ψ be the set of all the resulting vectors, for all \vec{m} together.

Partition the set Ψ into infinitely many pieces Ψ_k ($k \in \mathbb{N}$) with $|\Psi_k| = k + 1$. For each k , we shall modify the $k + 1$ vectors in Ψ_k so that their truncations to k become affinely independent over \mathbb{Q}_p . Affine independence means that not only do these $k + 1$ vectors span \mathbb{Q}_p^k but they continue to span if any fixed vector is added to all of them. Any set of $k + 1$ vectors in a k -dimensional vector space over \mathbb{Q}_p (or over any valued field with a non-trivial valuation) can be made affinely independent by an arbitrarily small perturbation. For our modification of Ψ_k , we use a perturbation that is small in the p -adic norm, so that each vector $\vec{\psi}$ is modified by adding something divisible in \mathbb{Z}_p by p . Thus, the reduction modulo p , $[\vec{\psi}]$ is unaffected. Let the perturbed vectors constitute Φ_k , and let $\Phi = \bigcup_{k \in \mathbb{N}} \Phi_k$. Note that all the vectors in Φ have, like those in Ψ , all their components in \mathbb{Z}_p .

Lemma 16. *The set Φ of vectors over \mathbb{Z}_p constructed here has the following properties.*

- (1) *For each $k \leq p$, the vectors $[\vec{\varphi}] \upharpoonright k$ for $\vec{\varphi} \in \Phi$ span $(\mathbb{Z}/p)^k$.*
- (2) *The same holds for the translates $[\vec{\lambda}_i] + [\vec{\varphi}]$, truncated at k , for each relevant $\vec{\lambda}_i$.*
- (3) *As $\vec{\varphi}$ varies through Φ , all the inner products $\langle [\vec{\varphi}], [\vec{x}] \rangle$ have the same value a .*
- (4) *This value a is non-zero and different from $\langle [-\vec{\lambda}_i], [\vec{x}] \rangle$ for all relevant i provided $[\vec{x}] \neq \vec{0}$.*
- (5) *For every k , there are $k + 1$ vectors $\vec{\varphi}_0, \vec{\varphi}_1, \dots, \vec{\varphi}_k \in \Phi$ such that, for any vector λ over \mathbb{Q} , the truncated vectors $(\vec{\varphi}_i + \vec{\lambda}) \upharpoonright k$ span \mathbb{Q}_p^k .*

Proof. By construction, the reductions modulo p of the vectors in Φ are the same as those of the vectors in Ψ , namely the vectors in M (each repeated \aleph_0 times). Therefore, the first four conclusions of the present lemma follow immediately from the corresponding items in Lemma 15. (For the first item, remember that the l in Lemma 15 was $\geq p$.)

The final conclusion of the present lemma is the result of our modification of Ψ to obtain Φ . The $k+1$ vectors in Φ_k are affinely independent, and therefore their translates by any vector $\vec{\lambda}$ over \mathbb{Q}_p are linearly independent. This applies in particular to vectors $\vec{\lambda}$ over $\mathbb{Q} \subseteq \mathbb{Q}_p$. \square

5.3. The counterexample group. In the preceding subsection, we worked with a fixed prime p ; now we let p vary. The set Φ constructed above will now be called $\Phi(p)$.

Our group G will be a subgroup of the additive group $\mathbb{Q} \times (\mathbb{Q})^{(\mathbb{N})}$, which consists of pairs (x_0, \vec{x}) where x_0 is a rational number and \vec{x} is a finitely supported vector of rational numbers. Recall that the components of \vec{x} are indexed as x_i for positive integers i , so our use of the notation x_0 causes no conflict. We define

$$G = \{(x_0, \vec{x}) \in \mathbb{Q} \times (\mathbb{Q})^{(\mathbb{N})} : (\forall p \in P)(\forall \vec{\varphi} \in \Phi(p)) x_0 + \langle \vec{\varphi}, \vec{x} \rangle \in \mathbb{Z}_p\}.$$

Notice that the inner product in the definition makes sense because \vec{x} is finitely supported, the components of \vec{x} , being in \mathbb{Q} , can be regarded as elements of \mathbb{Q}_p , and the components of $\vec{\varphi}$, being in \mathbb{Z}_p , are also in \mathbb{Q}_p . Thus, the inner product is defined in \mathbb{Q}_p , and the requirement is that it be in \mathbb{Z}_p .

Lemma 17. $\mathbb{Z} \times (\mathbb{Z})^{(\mathbb{N})} \subseteq G$.

Proof. Remember that all components of all $\vec{\varphi}$ in $\Phi(p)$ are p -adic integers. So if the components of \vec{x} are integers, and thus also p -adic integers, then the inner product $\langle \vec{\varphi}, \vec{x} \rangle$ is a p -adic integer, and so is its sum with the integer x_0 . \square

This lemma and the fact that $\mathbb{Q} \times (\mathbb{Q})^{(\mathbb{N})}$ is countable show that the rank and the cardinality of G are \aleph_0 .

We close this subsection by introducing a subgroup of G that will play a central role in our proof that G has the desired properties. Let L be the subgroup of those elements of G for which at most the single component x_0 is non-zero. So

$$L = \{(x_0, \vec{0}) : (\forall p \in P)(\forall \vec{\varphi} \in \Phi(p)) x_0 \in \mathbb{Z}_p\}.$$

Recall that a rational number is in \mathbb{Z}_p for all primes p if and only if it is an integer. Therefore, $L = \mathbb{Z} \times \{0\}^{(\mathbb{N})}$. Thus, L is isomorphic to \mathbb{Z} , and it is clearly a pure subgroup of G .

5.4. The non-free rank of G . We now prove that $\text{nfrk}(G) = \aleph_0$. Since $|G| = \aleph_0$, we need only show that $\text{nfrk}(G)$ is infinite. For this purpose, it is useful to consider the quotient group G/L .

Lemma 18. *For each element $\xi \in G/L$, there are infinitely many primes p that divide ξ , i.e., such that $\xi \in p(G/L)$.*

Proof. Given ξ , choose a representative $(x_0, \vec{x}) \in G$ for it modulo L . The components x_i (both x_0 and the components of \vec{x}) are rational numbers and only finitely many are non-zero, so let d be a common denominator. Then $d\xi$ is represented by a vector $(dx_0, d\vec{x})$ of integers. If we prove the assertion of the lemma for $d\xi$ then it will follow for ξ , using the same primes except for the finitely many that divide d . (In detail, if $d\xi = p\eta$ and p doesn't divide d , then the Euclidean algorithm gives integers a, b with $ad + bp = 1$. Then $\xi = ad\xi + bp\xi = ap\eta + bp\xi$ is divisible by p .)

So, by renaming, we may assume that all the components x_i are integers. We may also assume that $\vec{x} \neq \vec{0}$, for otherwise we would have $\xi = 0$ in G/L and the conclusion of the lemma would be trivially correct. So $P_{\vec{x}}$ is an infinite set of primes. We shall show that every $p \in P_{\vec{x}}$ divides ξ , thereby completing the proof.

Since $\Phi(p)$ satisfies conclusion (3) of Lemma 16, we know that the inner products $\langle \vec{\varphi}, \vec{x} \rangle$ have the same value modulo p for all $\varphi \in \Phi(p)$. Write this value as $[a] \in \mathbb{Z}/p$, where $a \in \mathbb{Z}$, consider the element $(-a, \vec{x}) \in \mathbb{Z} \times (\mathbb{Z})^{(\mathbb{N})} \subseteq G$, and consider its quotient by p ,

$$z = \left(-\frac{a}{p}, \frac{1}{p}\vec{x} \right) \in \mathbb{Q} \times (\mathbb{Q})^{(\mathbb{N})}.$$

Is this quotient z in G ? Of the requirements for membership in G , the q -adic ones for primes q other than p are certainly satisfied, since division by p preserves the property of being a q -adic integer. There remain the p -adic requirements. But for each $\varphi \in \Phi(p)$, we have that $\langle \vec{\varphi}, \vec{x} \rangle \equiv a \pmod{p}$ and therefore $-a + \langle \vec{\varphi}, \vec{x} \rangle$ is divisible by p in \mathbb{Z}_p . This means that $-(a/p) + \langle \vec{\varphi}, (1/p)\vec{x} \rangle \in \mathbb{Z}_p$, i.e., that z satisfies the p -adic requirement arising from φ . Since this happens for every $\varphi \in \Phi(p)$, we conclude that $z \in G$. But pz differs from (x_0, \vec{x}) by an element of L , namely $(x_0 + a, \vec{0})$. So pz represents ξ ; that is, $\xi \in p(G/L)$ as required. \square

Corollary 19. *G/L has no non-zero, free, pure subgroup.*

Proof. Any pure subgroup of G/L would inherit from G/L the divisibility property in the lemma and therefore cannot be free unless it is the zero group. \square

Lemma 20. *The non-free rank of G is \aleph_0 .*

Proof. As remarked above, we need only prove that $\text{nfrk}(G)$ cannot be finite. Suppose, toward a contradiction, that G can be split as $H \oplus F$,

where H has finite rank and F is free. Choose a free basis B for F , and express the generator $(1, \vec{0})$ of L as a linear combination of a vector from H and a finite set B_0 of vectors from B . Then $L \subseteq H \oplus \langle B_0 \rangle$. Let $B_1 = B - B_0$ and notice that B_1 is infinite because G has infinite rank, H has finite rank, and B_0 is finite. Since $G = H \oplus \langle B_0 \rangle \oplus \langle B_1 \rangle$, we have

$$\frac{G}{L} \cong \left(\frac{H \oplus \langle B_0 \rangle}{L} \right) \oplus \langle B_1 \rangle.$$

This makes $\langle B_1 \rangle$ a non-zero, pure, free subgroup of G/L , contrary to Corollary 19. \square

5.5. Non-free subgroups of G . To complete the verification of our counterexample, we must show that G does not have two disjoint, non-free subgroups. The following lemma is the main ingredient in that proof.

Lemma 21. *If H is a subgroup of G of finite rank and $H \cap L = \{0\}$, then H is free.*

Proof. Let H be as in the hypothesis. Because it has finite rank and because every element of G has finite support, H is a subgroup of $\mathbb{Q} \times \mathbb{Q}^k \times \{\vec{0}\}$ for some k . Whenever convenient, we shall ignore the $\{\vec{0}\}$ factor and pretend that elements of H are of the form (x_0, \vec{x}) with $\vec{x} \in \mathbb{Q}^k$.

Recall (for example from the proof of Lemma 12) that purifications of disjoint subgroups are disjoint. Since H is disjoint from L , the purification H_* of H in $\mathbb{Q} \times \mathbb{Q}^k$ is disjoint from the purification $L_* = \mathbb{Q} \times \{0\}^k$ of L . These purifications are \mathbb{Q} -linear subspaces of $\mathbb{Q} \times \mathbb{Q}^k$, and so we know, by linear algebra, that there is a vector $\vec{\lambda} \in \mathbb{Q}^k$ such that all elements (x_0, \vec{x}) of H satisfy $x_0 = \langle \vec{\lambda}, \vec{x} \rangle$. Being in G , these elements of H also satisfy

$$(1) \quad \langle \vec{\lambda} + \vec{\varphi}, \vec{x} \rangle = x_0 + \langle \vec{\varphi}, \vec{x} \rangle \in \mathbb{Z}_p$$

for all primes p and all $\varphi \in \Phi(p)$.

By appending zeros, we regard $\vec{\lambda}$ as an infinite vector. Being a vector over \mathbb{Q} with finite support, it occurs in our enumeration as λ_i for a certain i , which we fix for the rest of this proof.

Call a prime p *good* if it has the following properties:

- $p \geq k$.
- $p - 1 > i$.
- p does not divide the denominator of any component of $\lambda = \lambda_i$.

We note for future reference that these conditions are satisfied by all but finitely many primes. We also note that the second and third of these conditions make λ relevant for p in the sense used in Lemmas 15 and 16.

Temporarily fix a good prime p . Assume, toward a contradiction, that p divides the denominator of some component of \vec{x} for some element (x_0, \vec{x}) of H . Fix such an (x_0, \vec{x}) , let p^m be the highest power of p that divides the denominator of a component of \vec{x} , and let $\vec{y} = p^m \vec{x}$. Then all components of \vec{y} have denominators prime to p , and at least one of these components also has its numerator prime to p . Since $(x_0, \vec{x}) \in H$ and since $m > 0$, formula (1) above gives us, for every $\vec{\varphi} \in \Phi(p)$,

$$\langle \vec{\lambda} + \vec{\varphi}, \vec{y} \rangle = p^m \langle \vec{\lambda} + \vec{\varphi}, \vec{x} \rangle \in p\mathbb{Z}_p,$$

and so $\langle [\vec{\lambda} + \vec{\varphi}], [\vec{y}] \rangle = 0$ in \mathbb{Z}/p . Since this holds for all $\vec{\varphi} \in \Phi(p)$ and since, by conclusion (2) of Lemma 16 the corresponding vectors $[\vec{\lambda} + \vec{\varphi}] \upharpoonright k$ span $(\mathbb{Z}/p)^k$, it follows that $[\vec{y}] \upharpoonright k$ is the zero vector in $(\mathbb{Z}/p)^k$. The components of \vec{y} beyond the first k all vanish by our choice of k . Therefore, all components of \vec{y} are divisible by p as p -adic integers, which means that, as rational numbers, they have numerators divisible by p . That contradicts our choice of m and \vec{y} .

This contradiction shows that good primes p cannot divide denominators of components of \vec{x} when $(x_0, \vec{x}) \in H$. Nor can they divide $x_0 = \langle \vec{\lambda}, \vec{x} \rangle$ because, being good, they do not divide denominators of components of $\vec{\lambda}$. Thus, good primes do not divide the denominators of any components of elements of H .

We now turn our attention to those finitely many primes p that are not good. Although it is possible for such a prime to divide the denominator of a component of an element of H , we intend to show that this divisibility cannot be with great multiplicity. That is, there is a bound $m \in \mathbb{N}$ such that no power of p higher than p^m divides the denominator of any element of H .

Fix one of these bad primes p , and choose $k + 1$ vectors $\vec{\varphi}_j \in \Phi(p)$ as in assertion (5) of Lemma 16. Applying that assertion with our present $\vec{\lambda}$, we obtain that the $k + 1$ vectors $(\vec{\varphi}_j + \vec{\lambda}) \upharpoonright k$ span \mathbb{Q}_p^k . As \mathbb{Q}_p^k is k -dimensional, we can select k of these vectors $\zeta_j = \vec{\varphi}_j + \vec{\lambda}$ whose truncations form a basis for \mathbb{Q}_p^k . This means that the matrix Z whose rows are these truncations is a non-singular $k \times k$ matrix over \mathbb{Q}_p . (We use here that the components of $\vec{\varphi}_j$ are in $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ and the components of $\vec{\lambda}$ are in $\mathbb{Q} \subseteq \mathbb{Q}_p$.) Recall that any p -adic number can be written as

a p -adic integer divided by a power of p . So we can choose an integer m so large that all entries of the matrix $p^m Z^{-1}$ are in \mathbb{Z}_p .

For any $(x_0, \vec{x}) \in H \subseteq G$, we have, by formula (1), that each $\langle \vec{\zeta}_j, \vec{x} \rangle \in \mathbb{Z}_p$, which means that, if we regard \vec{x} as a column vector, then $Z\vec{x}$ is a vector \vec{z} with components in \mathbb{Z}_p . Then $\vec{x} = Z^{-1}\vec{z}$ has all its components in $p^{-m}\mathbb{Z}_p$. That is, the denominator of any component of \vec{x} cannot be divisible by a higher power of p than p^m .

To get the same result for x_0 , we may need to increase m as follows. Since $\vec{\lambda}$ has finite support, let p^r be the highest power of p that divides the denominator of any component of $\vec{\lambda}$. Then, since $x_0 = \langle \vec{\lambda}, \vec{x} \rangle$, no power of p higher than p^{m+r} can divide the denominator of x_0 .

Summarizing, we have an upper bound for the powers of p that can divide the denominator of any component of any member of H .

Let D be the product of these powers p^{m+r} for all of the finitely many bad primes. What we have shown is that every component of every element of H has, as its denominator, a divisor of D . Indeed, such a denominator cannot have good prime factors, and the remaining primes, the bad ones, cannot divide such a denominator to a higher power than they divide D . This means that H is a subgroup of

$$\frac{1}{D} \left(\mathbb{Z} \times \mathbb{Z}^k \times \{\vec{0}\} \right).$$

This group, isomorphic to \mathbb{Z}^{k+1} , is free, and therefore so is H . \square

Corollary 22. *Every subgroup of G that is disjoint from L is free.*

Proof. If H is such a subgroup, then all its finite-rank subgroups are free, by the lemma. Also, H is countable, because G is. By Pontryagin's criterion ([4, Lemma 16] or [1, Theorem IV.2.3]), it follows that H is free. \square

Finally, we deduce that G does not have two disjoint non-free subgroups. Let two non-free subgroups of G be given. By the corollary just proved, each of them contains a non-zero element of L . As $L \cong \mathbb{Z}$, these elements have a common non-zero multiple, which has to be in both of the given subgroups.

REFERENCES

- [1] Paul Eklof and Alan Mekler, *Almost Free Modules. Set Theoretic Methods*, North-Holland (1990), second edition (2002).
- [2] Howard Hiller, Martin Huber, and Saharon Shelah, "The structure of $\text{Ext}(A, Z)$ and $V = L$," *Math. Z.* 162 (1978) 39–50.
- [3] Wilfrid Hodges, "In singular cardinality, locally free algebras are free," *Algebra Universalis* 12 (1981) 205–220.

- [4] Lev Pontryagin, “The theory of topological commutative groups,” *Ann. Math.* 35 (1934) 361–388.
- [5] Saharon Shelah, “A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals,” *Israel J. Math.* 21 (1975) 319–349.
- [6] Robert Solovay, “Real-valued measurable cardinals,” *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)*, Amer. Math. Soc. (1971) 397–428.

MATHEMATICS DEPARTMENT, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI
48109–1043, U.S.A.

Email address: `ablass@umich.edu`

MATHEMATICS DEPARTMENT, HEBREW UNIVERSITY, JERUSALEM 91904, IS-
RAEL, AND MATHEMATICS DEPARTMENT, RUTGERS UNIVERSITY, NEW BRUNSWICK,
NJ 08903, U.S.A.