DECISIVE CREATURES AND LARGE CONTINUUM

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ABSTRACT. For $f > g \in \omega^\omega$ let $c^\forall_{f,g}$ be the minimal number of uniform trees with g-splitting needed to $\forall^\infty$-cover the uniform tree with f-splitting. $c^\exists_{f,g}$ is the dual notion for the $\exists^\infty$-cover.

Assuming CH and given $\aleph_1$ many (sufficiently different) pairs (f_ϵ, g_ϵ) and cardinals κ_ϵ such that $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$, we construct a partial order forcing that $c^\exists_{f_\epsilon,g_\epsilon} = c^\forall_{f_\epsilon,g_\epsilon} = \kappa_\epsilon$.

For this, we introduce a countable support semiproduct of decisive creatures with bigness and halving. This semiproduct satisfies fusion, pure decision and continuous reading of names.

1. INTRODUCTION

While there is extensive literature on separating various cardinal characteristics with forcing, much less is known about forcing different values to many cardinal characteristics *simultaneously*. In the paper *Many simple cardinal invariants* [GS93], Goldstern and the second author construct a partial order P that forces pairwise different values to $\aleph_1$ many instances of the cardinal characteristic $c^\forall_{f,g}$, defined as follows:

Let $f, g \in \omega^\omega$ (usually we have $f(n) > g(n)$ for all n). An (f, g)-slalom is a sequence $S = (S(n))_{n\in\omega}$ such that $S(n) \subseteq f(n)$ and $|S(n)| \leq g(n)$. A family $\mathcal{S}$ of (f, g)-slaloms is a $(\forall, f, g)$-cover, if for all $r \in \prod_{n\in\omega} f(n)$ there is an $S \in \mathcal{S}$ such that $r(n) \in S(n)$ for all but finitely many $n \in \omega$. $c^\forall_{f,g}$ is the smallest size of a $(\forall, f, g)$-cover.

In [GS93], plans to investigate the dual notion were announced as well (this investigation was promised in a paper called 448a): A Family $\mathcal{S}$ of (f, g)-slaloms is an $(\exists, f, g)$-cover, if for all $r \in \prod f$ there is an $S \in \mathcal{S}$ such that $r(n) \in S(n)$ for infinitely many $n \in \omega$. $c^\exists_{f,g}$ is the smallest size of an $(\exists, f, g)$-cover.

In this paper, we assume that we have a sequence of $\aleph_1$ many (sufficiently different) pairs $(f_\alpha, g_\alpha)_{\alpha\in\omega_1}$ and cardinals $(\kappa_\alpha)_{\alpha\in\omega_1}$ such that $\kappa_\alpha^{\aleph_0} = \kappa_\alpha$. We also assume CH. We then construct a partial order preserving cardinals that forces $c^\exists_{f_\alpha,g_\alpha} = c^\forall_{f_\alpha,g_\alpha} = \kappa_\alpha$ for all $\alpha \in \omega_1$.

So in particular, in the extension there are ω_1 many different cardinals below the continuum, i.e. $2^{\aleph_0} \geq \aleph_{\omega_1}$. Therefore we cannot use countable support iterations. When we try to keep any $c^\forall_{f,g}$ small during the iteration, we also have to avoid Cohen reals, so we cannot use finite support iterations either. Instead, we use a subset of the countable support product of lim-inf creature forcings $\mathbb{Q}^*_\infty$.

Creature forcing in general is introduced in the monograph *Norms on possibilities I: forcing with trees and creatures* [RS99] by Rosłanovski and the second author. We use the same notation, but we do not assume that the reader is familiar with creature forcing and

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recall all the definitions that we use. The forcing of the proof in [GS93] can be interpreted as creature forcing as well, more specifically as a lim-sup creature forcing.

The constructions for lim-inf creature forcing are generally more complicated than for lim-sup, and [RS99] shows that such forcings can collapse ω_1. We introduce the notion of decisiveness. This additional assumption (in connection with bigness and halving) makes the lim-inf forcing similar to $\mathbb{Q}^*_f$ of [RS99], and in particular proper and ω^ω-bounding. It also allows us to construct a countable support semiproduct of such lim-inf forcings, satisfying fusion and pure decision and therefore properness, continuous reading of names and the ω^ω-bounding property.

The theorems in this paper are due to the second author. The first authors contribution was to fill in some details, to ask the second author to fill in other details, and to write the paper.

2. LIM-INF CREATURE FORCINGS

We will use lim-inf forcings made up of simple, forgetful, decisive creatures with bigness and halving.

Such forcings are defined by a parameter, the creating pair $(\mathbf{K},\mathbf{\Sigma})$. (We will usually only write $\mathbf{K}$ and assume that $\mathbf{\Sigma}$ is clear from the context.)

We have the following:

- a function $\mathbf{H} : \omega \to \omega \setminus \{0\}$,
- a strictly increasing function $\mathbf{F} : \omega \to \omega$ such that $\mathbf{F}(0) = 0$,
- for every $n \in \omega$ a set $\mathbf{K}(n)$ such that each $\mathfrak{c} \in \mathbf{K}(n)$ has the form $(\mathrm{val}(\mathfrak{c}), \mathrm{nor}(\mathfrak{c}), \mathrm{dist}(\mathfrak{c}))$, where
  - $\mathrm{val}(\mathfrak{c})$ is a nonempty subset of $\prod_{\mathbf{F}(n)\leq i<\mathbf{F}(n+1)} \mathbf{H}(i)$,
  - $\mathrm{nor}(\mathfrak{c})$ is a non-negative real number, and
  - $\mathrm{dist}(\mathfrak{c}) \in \omega$.
- If $|\mathrm{val}(\mathfrak{c})| = 1$, then $\mathrm{nor}(\mathfrak{c}) = 0$.

A $\mathfrak{c} \in \mathbf{K}(n)$ is called n-creature. The intended meaning of the n-creature $\mathfrak{c}$ is the following: the set of possible values for the generic object $\underaccent{\tilde}{\eta} \in \prod_{i\in\omega} \mathbf{H}(i)$ restricted to the interval $[\mathbf{F}(n), \mathbf{F}(n+1) - 1]$ is the set $\mathrm{val}(\mathfrak{c})$. $\mathrm{nor}(\mathfrak{c})$ measures the amount of "freedom" the creature $\mathfrak{c}$ leaves on its interval. If $\mathfrak{c}$ determines its part of the generic real (i.e. if $\mathrm{val}(\mathfrak{c})$ is a singleton) then $\mathrm{nor}(\mathfrak{c}) = 0$ (i.e. $\mathfrak{c}$ leaves no freedom).

We set $\mathbf{K} := \bigcup_{n\in\omega} \mathbf{K}(n)$.

By some simple coding we could assume without much loss of generality that either $\mathbf{H}(n) = 2$ for all $n \in \omega$ or that $\mathbf{F}(n) = n$ for all n. In our application $\mathbf{F}(n) = n$, i.e. an n-creature lives on the singleton $\{n\}$.

We also have a function $\mathbf{\Sigma} : \mathbf{K} \to \mathcal{P}(\mathbf{K})$ satisfying:

- If $\mathfrak{c} \in \mathbf{K}(n)$ and $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ then $\mathfrak{d} \in \mathbf{K}(n)$.
- $\mathbf{\Sigma}$ is reflexive, i.e. $\mathfrak{c} \in \mathbf{\Sigma}(\mathfrak{c})$.
- $\mathbf{\Sigma}$ is transitive, i.e. $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ and $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{d})$ implies $\mathfrak{d}' \in \mathbf{\Sigma}(\mathfrak{c})$.
- If $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ then $\mathrm{val}(\mathfrak{d}) \subseteq \mathrm{val}(\mathfrak{c})$ and $\mathrm{nor}(\mathfrak{d}) \leq \mathrm{nor}(\mathfrak{c})$.
- $\mathbf{\Sigma}(\mathfrak{c})$ is finite.

The intended meaning is that $\mathbf{\Sigma}(\mathfrak{c})$ is the set of creatures that are stronger than $\mathfrak{c}$.

We say "$\mathbf{K}$ contains trunk-creatures" if for all $\mathfrak{c} \in \mathbf{K}$ and $\nu \in \mathrm{val}(\mathfrak{c})$ there is a $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ such that $\mathrm{val}(\mathfrak{d}) = \{\nu\}$. It doesn't make a real difference whether we assume this or not. Even if we do not have trunk-creatures, we extend the definitions of nor, val and $\mathbf{\Sigma}$ to

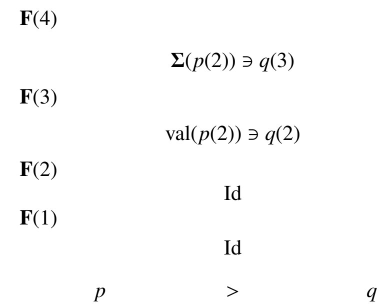

FIGURE 1. $q \leq p$, $\mathrm{trnklg}(p) = 2$, $\mathrm{trnklg}(q) = 3$.

sequences $t \in \prod_{\mathbf{F}(n) \leq i < \mathbf{F}(n+1)} \mathbf{H}(n)$ (to simplify notation later on): We set

$$\mathrm{nor}(t) := 0, \qquad \mathrm{val}(t) := \{t\}, \qquad t \in \mathbf{\Sigma}(\mathfrak{c}) \text{ iff } t \in \mathrm{val}(\mathfrak{c}), \qquad s \in \mathbf{\Sigma}(t) \text{ iff } s = t.$$

A pair $(\mathbf{K}, \mathbf{\Sigma})$ as above is a creating pair as defined in [RS99, 1.2]. It satisfies the following additional properties:

- finitary [RS99, 1.1.3]: $\mathbf{H}(n)$ and $\mathbf{\Sigma}(\mathfrak{c})$ are always finite.
- simple [RS99, 2.1.7]: $\mathbf{\Sigma}$ is defined on single creatures only.[1]
- forgetful [RS99, 1.2.5]: $\mathrm{val}(\mathfrak{c})$ does not depend on values of the generic real outside of the interval of $\mathfrak{c}$.[2]
- nice and smooth [RS99, 1.2.5]: A technical requirement that is trivial in the case of forgetful simple creating pairs.

In [RS99] two main cases of forcings are examined: creature forcings [RS99, 1.2.6] (defined by a creating pair [RS99, 1.2.2]) and tree creature forcings [RS99, 1.3.5] (defined via a tree-creating pair [RS99, 1.3.3]). So in this paper we deal with creature forcings.[3]

We now define the lim-inf forcing $\mathbb{Q}^*_\infty(\mathbf{K}, \mathbf{\Sigma})$:

Definition 2.1.
- A condition $p \in \mathbb{Q}^*_\infty(\mathbf{K}, \mathbf{\Sigma})$ consists of a trunk $t \in \prod_{i<\mathbf{F}(n)} \mathbf{H}(i)$ for some n and a sequence $(\mathfrak{c}_i)_{i \geq n}$ such that $\mathfrak{c}_i \in \mathbf{K}(i)$ and $\lim(\mathrm{nor}(\mathfrak{c}_i)) = \infty$.

  We set $\mathrm{trunk}(p) := t$, $\mathrm{trnklg}(p) := n$, and

$$p(i) := \begin{cases} \mathfrak{c}_i & \text{if } i \geq n, \\ t \restriction [\mathbf{F}(i), \mathbf{F}(i+1) - 1] & \text{otherwise.} \end{cases}$$

- $q \leq p$ if $\mathrm{trnklg}(q) \geq \mathrm{trnklg}(p)$ and $q(i) \in \mathbf{\Sigma}(p(i))$ for all i.

So in particular $q \leq p$ implies that $\mathrm{trunk}(q)$ extends $\mathrm{trunk}(p)$, see Figure 2.

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[1]In non-simple creating pairs we can have something like $\mathfrak{d} \in \mathbf{\Sigma}(\{\mathfrak{c}_1, \mathfrak{c}_2\})$, e.g. $\mathfrak{c}_1$ could live on the interval I_1, $\mathfrak{c}_2$ on I_2, and $\mathfrak{d}$ is $\mathfrak{c}_1$ and $\mathfrak{c}_2$ "glued together".

[2]In the general case, $\mathrm{val}(\mathfrak{c})$ is defined as a set of pairs (u, v) where $v \in \prod_{i<\mathbf{F}(n+1)} \mathbf{H}(i)$ and $u = v \restriction \mathbf{F}(n)$. The intended meaning is that $\mathfrak{c}$ implies: If the generic object $\underset{\sim}{\eta}$ restricted to $\mathbf{F}(n)$ is u, then the possible values v for $\underset{\sim}{\eta} \restriction \mathbf{F}(n+1)$ are those v such that $(u, v) \in \mathrm{val}(\mathfrak{c})$.
Then "$\mathfrak{c}$ is forgetful" is defined as: If $(u, v) \in \mathrm{val}(\mathfrak{c})$ and $u' \in \prod_{i<\mathbf{F}(n)} \mathbf{H}(i)$ then $(u', v) \in \mathrm{val}(\mathfrak{c})$.
In the forgetful case $\mathrm{val}(\mathfrak{c})$ and $\{v : (\exists u)\,(u, v) \in \mathrm{val}(\mathfrak{c})\}$ carry the same information. So for simplicity of notation in this paper we call the latter set $\mathrm{val}(\mathfrak{c})$.

[3]Actually every simple forgetful creating pair can be interpreted as tree-creating pair as well. The resulting tree-forcing however is very different to the creature forcing: the creature forcing corresponds to the "homogeneous" trees only.

Remarks.
- The requirement that $|\mathrm{val}(\mathfrak{c})|=1$ implies $\mathrm{nor}(\mathfrak{c})=0$ is not really needed, we do however need $\mathrm{nor}(\mathfrak{c})<M$ for some M (i.e. a uniform bound for all n-creatures).
- If $\mathbf{K}$ contains trunk-creatures, then we could omit the trunks in the definitions of $\mathbb{Q}^*_\infty$. In contrast, in the definition of the product in section 5, the distinction between the trunk and just a very small creature is essential.
- In [GS93] lim-sup creature forcings are used, defined as $\mathbb{Q}^*_{w\infty}$ in [RS99, 1.2.6]. The lim-inf case $\mathbb{Q}^*_\infty$ is generally harder to handle, and [RS99] does not deal with them in a general way.[4] We will introduce additional assumptions to guarantee that $\mathbb{Q}^*_\infty$ is proper and ω^ω-bounding. (These assumptions we use will actually make $\mathbb{Q}^*_\infty$ similar to $\mathbb{Q}^*_f$ of [RS99].)

Note that in general the generic filter G is *not* determined by the generic real $\underset{\sim}{\eta} := \bigcup_{p\in G}\mathrm{trunk}(p)$. This would be true in some special cases, e.g. if $\mathfrak{c}$ is determined by $\mathrm{val}(\mathfrak{c})$.[5] But we will be interested in creating pairs with halving. In this case, $\mathrm{dist}(\mathfrak{c})$ is relevant and G is not determined by $\underset{\sim}{\eta}$.

3. Bigness and halving, properness of $\mathbb{Q}^*_\infty$

We will now introduce the properties that guarantee that $\mathbb{Q}^*_\infty$ is proper.

Definition 3.1. Let $0<r\leq 1$, $k\in\omega$.
- $\mathfrak{c}$ is (k,r)-big if for all $F:\mathrm{val}(\mathfrak{c})\to k$ there is a $\mathfrak{d}\in\mathbf{\Sigma}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d})\geq\mathrm{nor}(\mathfrak{c})-r$ and $F\restriction\mathrm{val}(\mathfrak{d})$ is constant.[6]
- $\mathbf{K}(n)$ is (k,r)-big if every $\mathfrak{c}\in\mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c})>1$ is (k,r)-big.
- $\mathfrak{c}$ is r-halving,[7] if there is a $\mathrm{half}(\mathfrak{c})\in\mathbf{\Sigma}(\mathfrak{c})$ such that
  - $\mathrm{nor}(\mathrm{half}(\mathfrak{c}))\geq\mathrm{nor}(\mathfrak{c})-r$, and
  - if $\mathfrak{d}\in\mathbf{\Sigma}(\mathrm{half}(\mathfrak{c}))$ and $\mathrm{nor}(\mathfrak{d})>0$, then there is a $\mathfrak{c}'\in\mathbf{\Sigma}(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{c}')\geq\mathrm{nor}(\mathfrak{c})-r$ and $\mathrm{val}(\mathfrak{c}')\subseteq\mathrm{val}(\mathfrak{d})$.

  (Note: this c' generally is not in $\mathbf{\Sigma}(\mathrm{half}(\mathfrak{c}))$.)
- $\mathbf{K}(n)$ is r-halving, if all $\mathfrak{c}\in\mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c})>1$ have r-halving.

Facts.
- *If r' is smaller than r, then (k,r')-bigness implies (k,r)-bigness, and r'-halving implies r-halving.*
- $\mathfrak{c}$ *cannot be* $(|\mathrm{val}(\mathfrak{c})|,r)$*-big for any* $0<r<\mathrm{nor}(\mathfrak{c})$.

Theorem 3.2. *Set $\varphi(<n):=|\prod_{i<\mathbf{F}(n)}\mathbf{H}(i)|$ and $r(n):=1/(n\varphi(<n))$. If $\mathbf{K}(n)$ has $(2,r(n))$–bigness and $r(n)$-halving, then $\mathbb{Q}^*_\infty(\mathbf{K},\mathbf{\Sigma})$ is ω^ω-bounding and proper.*

The proof is similar to (but simpler than) the proof of Theorem 5.4, so we will not give it here. We just mention the basic concept:

Definition 3.3.
- $q\leq_n p$ means: $q\leq p$, and there is an $h\geq n$ such that $q\restriction h=p\restriction h$ and $\mathrm{nor}(q(i))\geq n$ for all $i\geq h$.

[4] There are good reasons for this: [RS99, 1.4.5] proofs that generally $\mathbb{Q}^*_\infty$ can collapse ω_1. In the rest of [RS99] $\mathbb{Q}^*_\infty$ is only considered in a special case (incompatible with simple) where $\mathbb{Q}^*_\infty$ is actually equivalent to other forcings that are better behaved (cf. [RS99, p23 and 2.1.3]).

[5] To be more exact: If $\mathrm{dist}(\mathfrak{c})$ is 0 for all $\mathfrak{c}$ and $\mathrm{nor}(\mathfrak{c})$ is a function of $\mathrm{val}(\mathfrak{c})$ and $\mathrm{val}(\mathfrak{d})\subseteq\mathrm{val}(\mathfrak{c})$ implies $\mathfrak{d}\in\mathbf{\Sigma}(\mathfrak{c})$, then the generic filter is determined by $\underset{\sim}{\eta}$.

[6] This is a variant of, but technically not quite the same as, [RS99, 2.2.1].

[7] Cf. [RS99, 2.2.7]. The original definition used $\mathrm{nor}(\mathrm{half}(\mathfrak{c}))\geq\mathrm{nor}(\mathfrak{c})/2$ instead of $\mathrm{nor}(\mathfrak{c})-r$, therefore the name halving.

- If $s \in \prod_{i<n} \mathbf{F}(p(i))$, then $q = s \wedge q$ is defined by $\operatorname{trnklg}(q) = \max(n, \operatorname{trnklg}(p))$ and $q(i) := \begin{cases} s(i) & \text{if } i < n \\ p(i) & \text{otherwise.} \end{cases}$
- Let $\underset{\sim}{\tau}$ be a name of an ordinal. q essentially decides $\underset{\sim}{\tau}$, if there is an n such that $s \wedge q$ decides $\underset{\sim}{\tau}$ for all $s \in \prod_{i<n} \operatorname{val}(q(i))$.

It is clear that $\mathbb{Q}^*_\infty$ satisfies fusion: If $(p_n)_{n\in\omega}$ satisfy $p_{n+1} \leq_n p_n$, then there is a p_ω such that $p_\omega \leq_n p_n$.

It also can be shown (as in the proof of Lemma 5.7) that $\mathbb{Q}^*_\infty$ satisfies pure decision: If $p \in \mathbb{Q}^*_\infty$, $n \in \omega$ and $\underset{\sim}{\tau}$ is a $\mathbb{Q}^*_\infty$-name for an ordinal, then there is a $q \leq_n p$ essentially deciding $\underset{\sim}{\tau}$.

Then properness and ω^ω-bounding follows as in the proof of Theorem 5.4 on page 5.

Remarks. The notions bigness and halving are not canonical in the sense that we could modify many parameters and still get very similar properties (which still imply properness). For example instead of subtracting r from the norm, we could multiply the norm with a factor $r < 1$. We could always use $\leq$ instead of $<$. Instead of requiring halving (or bigness) for all creatures with norm > 1, we could require it for all creatures with norm $> r$. (If $\mathbf{K}$ contains trunk-creatures, we can require it for all creatures, since then creatures with norm $\leq r$ always have bigness and halving). Instead of requiring that we can un-halve every $\mathfrak{d}$ that has norm > 0, we could require it for creatures with norm at least 1.

4. DECISIVENESS, PROPERNESS OF FINITE PRODUCTS

In this section, we fix a *finite* set I and for every $i \in I$ a creating pair $(\mathbf{K}_i, \mathbf{\Sigma}_i)$.

The product forcing $\prod_{i\in I} \mathbb{Q}^*_\infty(\mathbf{K}_i, \mathbf{\Sigma}_i)$ is equivalent to $\mathbb{Q}^*_\infty(\mathbf{K}_I, \mathbf{\Sigma}_I)$, where the creating pair is defined as follows: An n-creature $\mathfrak{c} \in \mathbf{K}_I(n)$ corresponds to a sequence $(\mathfrak{c}_i)_{i\in I}$, where $\mathfrak{c}_i \in \mathbf{K}_i(n)$. $\operatorname{val}(\mathfrak{c}) := \prod_{i\in I} \operatorname{val}(\mathfrak{c}_i)$, $\operatorname{nor}(\mathfrak{c}) := \min(\{\operatorname{nor}(\mathfrak{c}_i) : i \in I\})$, $\operatorname{dist}(\mathfrak{c}) := (\operatorname{dist}(\mathfrak{c}_i))_{i\in I}$, and $\mathbf{\Sigma}(\mathfrak{c}) := \{(\mathfrak{d}_i)_{i\in I} : \mathfrak{d}_i \in \mathbf{\Sigma}(\mathfrak{c}_i)\}$.[8]

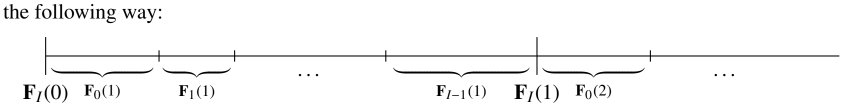
If each $\mathbf{K}_i(n)$ is r-halving, then $\mathbf{K}_I(n)$ is r-halving as well: Set $\operatorname{half}(\mathfrak{c}) := (\operatorname{half}(\mathfrak{c}_i))_{i\in I}$. (If $\mathfrak{d} \in \mathbf{\Sigma}(\operatorname{half}(\mathfrak{c}))$ and $\operatorname{nor}(\mathfrak{d}) > 0$, then $\mathfrak{d} = (\mathfrak{d}_i)_{i\in I}))$ and $\operatorname{nor}(\mathfrak{d}_i > 0)$, so we can un-halve every $\mathfrak{d}_i$.)

However, to handle bigness we have to introduce a new notion:

Definition 4.1. Let $0 < r \leq 1$, $k, K, n > 0$.

- $\mathfrak{c}$ is hereditarily (k, r)-big, if every $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ with $\operatorname{nor}(\mathfrak{c}) > 1$ is (k, r)-big.
- $\mathfrak{c}$ is (K, n, r)-decisive, if there are $\mathfrak{d}^-, \mathfrak{d}^+ \in \mathbf{\Sigma}(\mathfrak{c})$ such that $\operatorname{nor}(\mathfrak{d}^-), \operatorname{nor}(\mathfrak{d}^+) \geq \operatorname{nor}(\mathfrak{c}) - r$, $|\operatorname{val}(\mathfrak{d}^-)| \leq K$ and $\mathfrak{d}^+$ is hereditarily $(2^{K^n}, r)$-big. $\mathfrak{d}^-$ is called a "small successor", $\mathfrak{d}^+$ a "big successor" of $\mathfrak{c}$.
- $\mathfrak{c}$ is (n, r)-decisive if $\mathfrak{c}$ is (K, n, r)-decisive for some K.
- $\mathfrak{c}$ is hereditarily (n, r)-decisive if every $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ with $\operatorname{nor}(\mathfrak{c}) > 1$ is (n, r)-decisive.
- $\mathbf{K}(n)$ is (n, r)-decisive if every $\mathfrak{c} \in \mathbf{K}(n)$ with $\operatorname{nor}(\mathfrak{c}) > 1$ is (n, r)-decisive.

[8] So an n-creature "lives" on the product $\prod_{i\in I}[\mathbf{F}_i(n), \mathbf{F}_i(n+1) - 1]$. This does not fit our restrictive framework, so we should just "linearize" the product. Assume $I \in \omega$, i.e. $I = \{0, \dots, I-1\}$. Set $\mathbf{F}_I(n) := \sum_{i\in I} \mathbf{F}_i(n)$ and write it in the following way:



Now it should be clear how to formally define $\mathbf{H}_I$, $\mathbf{K}_I$, $\mathbf{\Sigma}_I$ etc. If we use the more general framework of [RS99], this linearization is not necessary.

Facts.
- *If* $\mathfrak{c}$ *is* (K,n,r)*-decisive, hereditarily* (k,r)*-big and* $\mathrm{nor}(\mathfrak{c}) > 1+r$*, then* $k<K$.
- *If* $\mathfrak{c}$ *is* (n,r)*-decisive (i.e.* $\mathfrak{c}$ *is* (K,n,r)*-decisive for* some K*), then for* every $K\in\omega$ *there is a* $\mathfrak{d}\in\mathbf{\Sigma}(\mathfrak{c})$ *such that* $\mathrm{nor}(\mathfrak{d})\geq\mathrm{nor}(\mathfrak{d})-r$ *and* either $|\mathrm{val}(\mathfrak{d})|\leq K$ or $\mathfrak{d}$ *is hereditarily* $(2^{K^n},r)$*-big.*
- *If* $\mathfrak{c}$ *is hereditarily* (n,r)*-decisive and* $\mathrm{nor}(\mathfrak{c})>1+k\cdot r$*, then there are* $\mathfrak{d}^-,\mathfrak{d}^+\in\mathbf{\Sigma}(\mathfrak{c})$ *and* K *such that* $\mathrm{nor}(\mathfrak{d}^-)\geq\mathrm{nor}(\mathfrak{c})-r$*,* $\mathrm{nor}(\mathfrak{d}^+)\geq\mathrm{nor}(\mathfrak{c})-k\cdot r$*,* $|\mathrm{val}(\mathfrak{d}^-)|\leq K$ *and* $\mathfrak{d}^+$ *is hereditarily* $(L(k),r)$*-big, where* $L(1)=K$ *and* $L(m+1)=2^{L(m)^n}$.

The last fact implies that decisiveness can be used to increase bigness:

Lemma 4.2. *Assume* $\mathfrak{c}$ *is hereditarily* (n,r)*-decisive and hereditarily* (B,r)*-big, and* $\mathrm{nor}(\mathfrak{c})>1+k\cdot r$*. Then there is a hereditarily* $(L(k),r)$*-big* $\mathfrak{d}\in\mathbf{\Sigma}(\mathfrak{c})$ *such that* $\mathrm{nor}(\mathfrak{d})\geq\mathrm{nor}(\mathfrak{c})-k\cdot r$*, where* $L(0)=B$ *and* $L(m+1)=2^{L(m)^n}$.

Lemma 4.3. *Assume that* $k,m,t\geq 1$*,* $0<r\leq 1$*,* $\mathfrak{c}_0,\dots,\mathfrak{c}_{k-1}\in\mathbf{K}$*, and* F *satisfy the following:*
- $\mathrm{nor}(\mathfrak{c}_i)>1+r\cdot(k-1)$*,*
- $\mathfrak{c}_i$ *is hereditarily* (k,r)*-decisive and hereditarily* $(2^{m^t},r)$*-big, and*
- F *is a function from* $\prod_{i\in k}\mathrm{val}(\mathfrak{c}_i)$ *to* 2^{m^t}.

Then there are $\mathfrak{d}_0,\dots,\mathfrak{d}_{k-1}\in\mathbf{K}$ *such that*
- $\mathfrak{d}_i\in\mathbf{\Sigma}(\mathfrak{c}_i)$*,*
- $\mathrm{nor}(\mathfrak{d}_i)\geq\mathrm{nor}(\mathfrak{c}_i)-r\cdot k$*, and*
- $F\restriction\prod_{i\in k}\mathrm{val}(\mathfrak{d}_i)$ *is constant.*

Proof. The case $k=1$ is just the definition of $(2^{m^t},r)$-big (decisive is not needed). So assume the lemma holds for k, and let us investigate the case $k+1$.

$\mathfrak{c}_k$ is $(k+1,r)$-decisive, i.e. there is an M such that $\mathfrak{c}_k$ is $(M,k+1,r)$-decisive. This implies that $M>2^{m^t}$. We call a creature $\mathfrak{d}$ "small" if $|\mathrm{val}(\mathfrak{d})|\leq M$, and "big" if $\mathfrak{d}$ is hereditarily $(2^{M^{(k+1)}},r)$-big ($\mathfrak{d}$ cannot be both if $\mathrm{nor}(\mathfrak{d})>1$).

We choose a sequence $(\mathfrak{d}_i)_{i\in k+1}$ satisfying $\mathfrak{d}_i\in\mathbf{\Sigma}(\mathfrak{c}_i)$ and $\mathrm{nor}(\mathfrak{d}_i)\geq\mathrm{nor}(\mathfrak{c}_i)-r$ the following way: For $i<k$ let $\mathfrak{d}_i$ be either small or big. If $\mathfrak{d}_0$ is small, then choose $\mathfrak{d}_k$ to be big, otherwise choose $\mathfrak{d}_k$ to be small. Set $S:=\{i\in k+1:\ \mathfrak{d}_i \text{ is small}\}$, and $L:=(k+1)\setminus S$. (We have the freedom to choose, since $\mathfrak{c}_k$ is $(M,k+1,r)$-decisive.) So $\{L,S\}$ is a non-trivial partition of $k+1$.

Set $Y:=\prod_{i\in S}\mathrm{val}(\mathfrak{d}_i)$. $|Y|\leq M^{|S|}$. So we can write Y as $\{\bar y_1,\dots,\bar y_{M^{|S|}}\}$.

Define F^* on $\prod_{i\in L}\mathrm{val}(\mathfrak{d}_i)$ by
$$F^*(\bar x):=(F(\bar x^\frown\bar y_1),\dots,F(\bar x^\frown\bar y_{M^{|S|}})).$$
So
$$|\mathrm{image}(F^*)|\leq|\mathrm{image}(F)|^{M^{|S|}}\leq 2^{m^tM^{|S|}}<2^{M^{|S|+1}}.$$

So we can apply the induction hypothesis to $k':=|L|<k+1$, $m':=M$, $t':=|S|+1\leq k+1$, $F':=F^*$ and $\mathfrak{c}'_i:=\mathfrak{d}_i$ for $i\in L$. This gives us $(\mathfrak{d}'_i)_{i\in L}$ such that
- $\mathfrak{d}'_i\in\mathbf{\Sigma}(\mathfrak{d}_i)\subseteq\mathbf{\Sigma}(\mathfrak{c}_i)$,
- $\mathrm{nor}(\mathfrak{d}'_i)\geq\mathrm{nor}(\mathfrak{d}_i)-r\cdot k'\geq\mathrm{nor}(\mathfrak{c}_i)-r(k+1)$, and
- $F^*\restriction\prod_{i\in L}\mathrm{val}(\mathfrak{d}'_i)$ is constant, say $(F^{**}(\bar y_1),\dots,F^{**}(\bar y_{M^{|S|}}))$.

So F^{**} is a function from $Y=\prod_{i\in S}\mathrm{val}(\mathfrak{d}_i)$ into 2^{m^t}. Now we apply the induction hypothesis again, this time to $k'':=|S|<k+1$, $m'':=m$, $t''=t$, $F'':=F^{**}$, and $\mathfrak{c}''_i:=\mathfrak{d}_i$ for $i\in S$. This gives us $(\mathfrak{d}'_i)_{i\in S}$ such that
- $\mathfrak{d}'_i\in\mathbf{\Sigma}(\mathfrak{d}_i)\subseteq\mathbf{\Sigma}(\mathfrak{c}_i)$,

- $\mathrm{nor}(\mathfrak{d}'_i) \geq \mathrm{nor}(\mathfrak{d}_i) - r \cdot k'' \geq \mathrm{nor}(\mathfrak{c}_i) - r(k+1)$, and
- $F^{**} \restriction \prod_{i \in S} \mathrm{val}(\mathfrak{d}'_i)$ is constant.

Then $(\mathfrak{d}'_i)_{i \leq k}$ is as required. □

Corollary 4.4. *If every $\mathbf{K}_i(n)$ is $(2, r)$-big and $(|I|, r)$-decisive for all $i \in I$, then every $\mathfrak{c} \in \mathbf{K}_I(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1 + r \cdot |I|$ is $(2, r \cdot |I|)$-big.*

Proof. Assume $F : \mathrm{val}(\mathfrak{c}) \to 2$. $\mathrm{val}(\mathfrak{c}) = \prod_{i \in I} \mathrm{val}(\mathfrak{c}_i)$. Apply Lemma 4.3 for $k = I$, $m = 1$ and $t = 1$. □

Corollary 4.5. *Set $\varphi(<n) := \max(|\prod_{m<\mathbf{F}_i(n)} \mathbf{H}_i(m)| : i \in I)$, and $r(n) := 1/(n\varphi(<n))$. Assume that for all $i \in I$ and $n \in \omega$, $\mathbf{K}_i(n)$ is $(2, r(n)/|I|)$-big, $(|I|, r(n)/|I|)$-decisive and $r(n)$-halving. Then $\prod_{i \in I} \mathbb{Q}^*_\infty(\mathbf{K}_i, \mathbf{\Sigma}_i)$ is ω^ω-bounding and proper.*

Proof. $\prod_{i \in I} \mathbb{Q}^*_\infty(\mathbf{K}_i, \mathbf{\Sigma}_i) = \mathbb{Q}^*_\infty(\mathbf{K}_I, \mathbf{\Sigma}_I)$. $\mathbf{K}_I(n)$ is $r(n)$-halving and $(2, r(n))$-big according to the last corollary. Now use Theorem 3.2. (Actually we have bigness only for creatures with norm least $1 + r \leq 2$ and not 1, but this does not make any difference.) □

Remarks.
- As usual, these notions are not canonical. One version of decisive (for which Lemma 4.3 holds as well) requires $\mathfrak{d}^+$ to be $(2^{K^{\lceil \mathrm{nor}(\mathfrak{c})/r \rceil}}, r)$-big (this notion does not use the parameter n). And of course we can e.g. substitute > 1 by > 0, or $\leq$ by $<$ etc.
- Decisiveness is quite costly: The n-th level will generally have to be very large compared to the levels before to achieve decisiveness as in the last corollary, i.e. $|\prod_{i<F_i(n+1)} \mathbf{H}(i)| \gg |\prod_{i<F_i(n)} \mathbf{H}(i)|$. In our application this will have the effect that we can separate (f, g) and (f', g') only if their growth rate is considerably different. It is very likely that with a more careful and technically more complicated analysis one can construct forcings that can separate cardinal invariants for pairs that are not so far apart, but this will have to use other concepts than decisiveness.

5. A semicountable support product

Fix a set I and for every $\alpha \in I$ a creating pair $(\mathbf{K}_\alpha, \mathbf{\Sigma}_\alpha)$. Also, fix a natural number $n^{\mathrm{dist}}(\alpha,\beta)$ for every $\alpha \neq \beta \in I$. We assume that $\varphi(=n)$ is an upper bound for all $|\prod_{\mathbf{F}_\alpha(n) \leq m < \mathbf{F}_\alpha(n+1)} \mathbf{H}_\alpha(n)|$, and set $\varphi(\leq n) := \prod_{m \leq n} \varphi(=m)$ and $\varphi(<n) := \prod_{m<n} \varphi(=m)$.

We define a forcing P which is between the finite and the countable support product:

Definition 5.1.
- P consists of conditions of the form $p = (p_\alpha)_{\alpha \in u}$ for some countable $u \subseteq I$ such that
  - $p_\alpha \in \mathbb{Q}^*_\infty(\mathbf{K}_\alpha)$,
  - if $n > \mathrm{trnklg}(p_\alpha)$, then $\mathrm{nor}(p_\alpha(n)) > 0$,
  - $|\mathrm{supp}(p, n)| < n$ for all $n > 0$, where we set
  $$\mathrm{supp}(p, n) := \{\alpha \in \mathrm{dom}(p) : \mathrm{trnklg}(p_\alpha) < n\},$$
  - moreover, $\lim_{n\to\infty}(|\mathrm{supp}(p, n)|/n) = 0$,
  - $\lim_{n\to\infty}(\mathrm{nor}^{\mathrm{min}}(p, n)) = \infty$, where we set
  $$\mathrm{nor}^{\mathrm{min}}(p, n) := \min(\{\mathrm{nor}(p_\alpha(n)) : \alpha \in \mathrm{supp}(p, n)\}),$$
  - $\alpha \neq \beta \in \mathrm{supp}(p, m)$ implies $m \geq n^{\mathrm{dist}}(\alpha, \beta)$.
- $q \leq p$ if
  - $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$,
  - if $\alpha \in \mathrm{dom}(p)$, then $\mathrm{trnklg}(q_\alpha) \geq \mathrm{trnklg}(p_\alpha)$,
  - $\mathrm{trnklg}(q_\alpha) = \mathrm{trnklg}(p_\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(p)$,

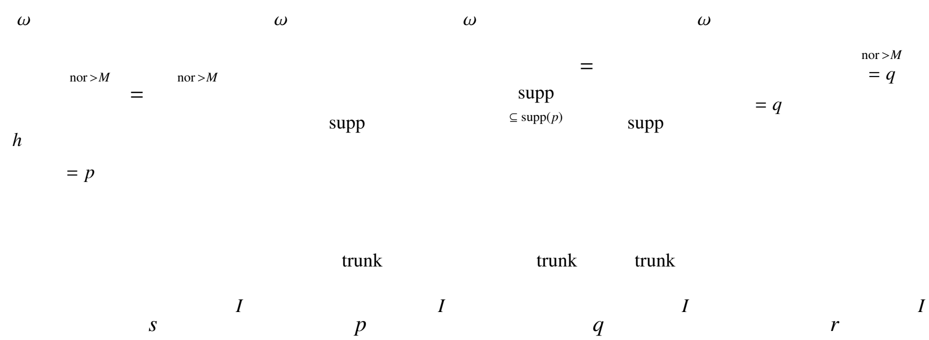


FIGURE 2. $q \leq p$, $s \leq^+_M p$, $r \leq^{\text{new}}_M p$ as in 5.6(1)

– if $\alpha \in \text{dom}(p)$, then $q_\alpha(n) \in \mathbf{\Sigma}(p_\alpha(n))$ for all n.

Note that $q \leq p$ implies $\text{nor}(q_\alpha(n)) \leq \text{nor}(p_\alpha(n))$ but not $\text{nor}^{\min}(q,n) \leq \text{nor}^{\min}(p,n)$. Figure 5 shows one way to visualize $q \leq p$.

If I is finite and $n^{\text{dist}} \equiv 0$, then P is just the product of $\mathbb{Q}^*_\infty(\mathbf{K}_\alpha)$. If I is countable, then P can be partitioned according to the function $\alpha \mapsto \text{trnklg}(p_\alpha)$ (modulo finite).

If $n^{\text{dist}} \not\equiv 0$, then conditions p and q with disjoint domains do not have to be compatible. If $n^{\text{dist}} \equiv 0$, then such conditions are compatible (take the union and enlarge some stems so that the $|\text{supp}(p \cup q, n)| < n$ requirement is satisfied).

Lemma 5.2. *If $J \subseteq I$, then $P_J = \{p \in P : \text{dom}(p) \subseteq J\}$ is a complete subforcing of P.*

Proof. If $p \in P$, then $p \restriction J \in P_J$. So if $p \perp q \in P_J$, then $p \perp q \in P$. Also, $p \restriction J$ is a reduction of p: If $q \leq p$, then q and $p \restriction J$ are compatible. □

Definition 5.3.
- $\text{val}^\Pi(p, \leq n) := \prod_{\alpha \in \text{dom}(p)} \prod_{m \leq n} \text{val}(p_\alpha(m))$, a set of size at most $\varphi(\leq n)^n$.
- If $w \subseteq \text{dom}(p)$ and $t \in \prod_{\alpha \in w} \prod_{0 \leq i < \mathbf{F}_\alpha(m+1)} \mathbf{H}_\alpha(i)$, then $t \wedge p$ is defined by
$$(t \wedge p)_\alpha(m) = \begin{cases} t_\alpha & \text{if } m \leq n \text{ and } \alpha \in w, \\ p_\alpha(m) & \text{otherwise.} \end{cases}$$
So $t \wedge p \in P$, and if $t \in \text{val}^\Pi(p, \leq n)$, then $t \wedge p \leq p$.
- If $\underset{\sim}{\tau}$ is a name of an ordinal, then p essentially decides $\underset{\sim}{\tau}$, if for some m and every $t \in \text{val}^\Pi(p, \leq m)$ the condition $t \wedge p$ decides $\underset{\sim}{\tau}$.

The following should be clear:

Facts.
- *If $p \in P$ and $n \in \omega$, then $\{t \wedge p : t \in \text{val}^\Pi(p, \leq n)\}$ is predense under p.*
- *If $q, q' \in P$ such that $\text{dom}(q') = \text{dom}(q)$, $\text{val}(q'_\alpha(h)) \subseteq \text{val}(q_\alpha(h))$ for all α and h, and $q'_\alpha(h) = q_\alpha(h)$ for all α and all h larger than some n, then $q' \leq^* q$ (i.e. q' forces that $q \in G$. Note that in general $q' \nleq q$.) If q essentially decides $\underset{\sim}{\tau}$, then q' essentially decides $\underset{\sim}{\tau}$.*

Theorem 5.4. *If $\mathbf{K}(n)$ is $(n, r(n))$-decisive, $(2, r(n))$-big and $r(n)$-halving, where*
$$r(n) = \frac{1}{n \cdot \varphi(<n)^n},$$
then P is proper and ω^ω-bounding.

The following notions and simple facts are used in the proof:

Definition 5.5.
- $r \leq^{\text{old}}_M p$, if there is an $h \geq M$ such that
  - $r \leq p$,
  - if $\alpha \in \text{dom}(p)$, then $\text{trnklg}(r_\alpha) = \text{trnklg}(p_\alpha)$,
  - if $\alpha \in \text{dom}(p)$ and $n < h$, then $p_\alpha(n) = r_\alpha(n)$,
  - if $\alpha \in \text{supp}(p, n)$ and $n \geq h$, then $\text{nor}(r_\alpha(n)) \geq M$.

  (We chose the name "old" because this property refers to $\text{dom}(p)$ only.)
- $r \leq^{\text{new}}_M p$, if
  - $r \leq p$,
  - if $n \in \omega$ and $\alpha \in \text{supp}(r, n) \setminus \text{dom}(p)$, then $\text{nor}(r_\alpha(n)) > M$ and $|\text{supp}(r, m)|/m \leq 1/(M + 1)$.
- $r \leq^{+}_M p$, if $r \leq^{\text{new}}_M p$ and $r \leq^{\text{old}}_M p$.

The following shows that the "new" part of $\leq^+$ is not really important.

Lemma 5.6. (1) *If $M \in \omega$ and $q \leq p$, then there is an $r \leq q$ such that $r \leq^{new}_M p$ and $r_\alpha(n) = q_\alpha(n)$ for $\alpha \in \text{dom}(p)$.*
(2) *If $q \leq^{old}_M p$, then there is an $r \leq q$ such that $r \leq^{+}_M p$.*
(3) *(Fusion) If $(p^m)_{m\in\omega}$ satisfies $p^{m+1} <^{+}_m p^m$ then there is a (canonical) p^ω such that $p^\omega \leq^{+}_m p_m$ for all m.*

Proof. (1) r is constructed from q by extending the trunks at positions $\alpha \notin \text{dom}(p)$ to sufficient length, see Figure 5.

(2) follows from (1).

(3) Set $\text{dom}(p^\omega) = \bigcup \text{dom}(p^n)$. $p^\omega_\alpha(n) = p^M_\alpha(n)$, where $M \geq n$ is minimal such that $\alpha \in \text{dom}(p^M)$. $p^\omega \in P$: Fix some k. Since $p^k \in P$, there is an l such that $\text{nor}^{\min}(p^k, n) > k$ and $|\text{supp}(p^k, n)|/n < 1/k$ for all $n > l$. Since $p^{k+1} \leq^{+}_k p^k$, the same holds for p^{k+1}, etc. So $\text{nor}^{\min}(p^\omega, n) > k$ and $|\text{supp}(p^\omega, n)|/n < 1/k$ for all $n > l$. The rest of the requirements for being a condition of P are "local" and therefore satisfied by p^ω as well. □

All we need to prove Theorem 5.4 is

Lemma 5.7. *If $\underset{\sim}{\tau}$ is a name of an ordinal, $p \in P$ and $M \in \mathbb{N}$, then there is a $q \leq^{+}_M p$ essentially deciding $\underset{\sim}{\tau}$.*

Then the rest is the following standard argument:

Proof of Theorem 5.4. *ω^ω-bounding:* Let $\underset{\sim}{f}$ be the name for a function from ω into ordinals and $p = p^0 \in P$. If p^n is already constructed, choose $p^{n+1} \leq^{+}_{n+1} p^n$ essentially deciding $\underset{\sim}{f}(n)$. In particular there is a finite set F_n of possibilities of $\underset{\sim}{f}(n)$. So the fusion of the sequence, p^ω, forces that $\underset{\sim}{f}(n) \in F_n$ for all $n \in \omega$.

This phenomenon is called "continuous reading of names for ω-sequences."

In particular, if the image of f is a subset of ω, then g defined by $g(n) = \sup(F_n)$ is an upper bound of $\underset{\sim}{f}$.

proper: Let $N \prec H(\chi)$ contain p^0, P. Let $(\underset{\sim}{\tau}_n)_{n\in\omega}$ be a list of the P-names of ordinals that are in N. Choose (in N) $p^{n+1} \leq^{+}_n p^n$ such that p^{n+1} essentially decides $\underset{\sim}{\tau}_n$. Then $p^\omega \leq p^0$ is N-generic. (More specifically P satisfies Axiom A, which implies proper.) □

Proof of Lemma 5.7. Let $\underset{\sim}{\tau}$ be a name of an ordinal, $p \in P$, $m \in \omega$. We have to show that there is a $q \leq^{+}_m p$ essentially deciding $\underset{\sim}{\tau}$.

Step 1: Assume we have a $p \in P$ such that $\text{nor}(p_\alpha(m)) > 2$ for all $m, \alpha \in \text{supp}(p, m)$. In this step we will define $S(p, M) \leq p$ for such a p and any $M \in \omega$.

Set $n := \min(\{\mathrm{trnklg}(p_\alpha) : \alpha \in \mathrm{dom}(p)\})$. Find the minimal h such that $\mathrm{nor}^{\min}(p, m) > M + 2$ for all $m \geq h$.

Enumerate $\mathrm{val}^\Pi(p, \leq n)$ as $s^1, \dots, s^N$. Note that $N \leq \varphi(=n)^n$. Set $p^0 := p$ and $M^0 := M + 2$. Given p^{l-1} and M^{l-1}, we set $M^l := M^{l-1} - r(h)$ and find a $p^l \leq p^{l-1}$ satisfying the following for all $\alpha \in \mathrm{dom}(p)$:

- $\mathrm{trnklg}(p^l_\alpha) = \mathrm{trnklg}(p_\alpha)$.
- If $m \leq n$ then $p^l_\alpha(m) = p_\alpha(m)$.
- If $n < m < h$, then $\mathrm{nor}(p^l_\alpha(m)) \geq \mathrm{nor}(p^{l-1}_\alpha(m)) - r(m)$.
- If $m \geq h$, then $\mathrm{nor}(p^l_\alpha(m)) \geq M^l$.
- One of the following two cases hold:
  dec: $s^l \wedge p^l$ essentially decides $\underset{\sim}{\tau}$, or
  half: it is no possible to satisfy dec, then $p^l_\alpha(m) = \mathrm{half}(p^{l-1}_\alpha(m))$ for all $m > n$ and $\alpha \in \mathrm{supp}(p^{l-1}, m)$.

So we first try to find a p^l satisfying dec, only if this is not possible we halve p^{l-1}_α.

We can construct such p^l for all $l \leq N$, since $2 - r(m) \cdot N > 1$ for all $m > n$. (Since $1 = r(m) \cdot \varphi(<m)^m \cdot m > r(m) \cdot N$.)

In case "half" there is is no $q \leq s^l \wedge p^l$ essentially deciding $\underset{\sim}{\tau}$ such that $\mathrm{trnklg}(q_\alpha) = \min(n + 1, \mathrm{trnklg}(p))$ for all $\alpha \in \mathrm{dom}(p)$:

Otherwise, pick an h' such that $\mathrm{nor}^{\min}(q, m) > M^l$ for $m \geq h'$. We can "un-halve" $q_\alpha(m)$ for $\alpha \in \mathrm{supp}(p, m)$, $n < m < h'$ to get $q'_\alpha(m) \in \mathbf{\Sigma}(p^{l-1}_\alpha(m))$ such that $\mathrm{nor}(q'_\alpha(m)) > \mathrm{nor}(p^{l-1}_\alpha(m)) - r(m)$ and $\mathrm{val}(q'_\alpha(m)) \subseteq \mathrm{val}(q_\alpha(m))$. Define q'' by

$$q''_\alpha(m) := \begin{cases} q'_\alpha(m) & \text{if } \alpha \in \mathrm{dom}(p) \text{ and } n < m < h', \\ q_\alpha(m) & \text{otherwise.} \end{cases}$$

This q'' essentially decides $\underset{\sim}{\tau}$, which is a contradiction to the fact that we chose case "half".

This construction defines an $F : \prod_{\alpha \in \mathrm{supp}(p,n)} \mathrm{val}(p_\alpha(n)) \to \{\mathrm{dec}, \mathrm{half}\}$. Each $\mathbf{K}_\alpha(n)$ is $(n, r(n))$-decisive and $(2, r(n))$-big, and $|\mathrm{supp}(p, n)| \leq n$. According to Lemma 4.3 there are $\mathfrak{d}_\alpha \in \mathbf{\Sigma}(p_\alpha(n))$ such that $F \restriction \prod \mathrm{val}(\mathfrak{d}_\alpha)$ is constant and $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(p_\alpha(n)) - n \cdot r(n)$.

Let $h' \geq h + M$ be such that $\mathrm{nor}(p^N_\alpha(m)) > M + 1$ and $|\mathrm{supp}(p^N, m)| < 1/M$ for all $m \geq h'$. Define $S(p, M)$ by $\mathrm{dom}(S(p, M)) = \mathrm{dom}(p^N)$ and for $\alpha \in \mathrm{dom}(S(p, N))$

$$S(p, M)_\alpha(m) = \begin{cases} p_\alpha(m) & \text{if } m < n \text{ and } \alpha \in \mathrm{dom}(p), \\ \mathfrak{d}_\alpha & \text{if } m = n \text{ and } \alpha \in \mathrm{dom}(p), \\ p^N_\alpha(m) & \text{if } \alpha \in \mathrm{dom}(p) \text{ or } m > h', \\ \text{some } s \in p^N_\alpha(m) & \text{otherwise.} \end{cases}$$

If the constant value of F is "dec", then we call $S(p, M)$ deciding, otherwise we call $S(p, M)$ halving. We get

- If $\alpha \in \mathrm{supp}(S(p, M), m) \setminus \mathrm{supp}(p)$, then $m > h$, $\mathrm{nor}(S(p, M)_\alpha(m)) > M + 1$, and $|\mathrm{supp}(S(p, M), m)| < 1/M$.
- If $\alpha \in \mathrm{dom}(p)$, then $\mathrm{trnklg}(S(p, M)_\alpha) = \mathrm{trnklg}(p_\alpha)$.
- If $\alpha \in \mathrm{dom}(S(p, M))$, then

$$\mathrm{nor}(S(p, M)_\alpha(m)) \geq \begin{cases} \mathrm{nor}(p_\alpha(n)) - n \cdot r(n) & \text{if } n = m \text{ and } \alpha \in \mathrm{dom}(p), \\ \mathrm{nor}(p_\alpha(m)) - \varphi(=n)^n r(m) & \text{if } \alpha \in \mathrm{dom}(p) \text{ and } n < m < h, \\ M + 1 & \text{otherwise.} \end{cases}$$

  (For the last line, note that $M^N = M + 2 - \varphi(=n)^n \cdot r(h) > M + 1$.)
- If $S(p, M)$ is deciding then $S(p, M)$ essentially decides τ.

- If $S(p, M)$ is halving, then there is no $q \leq S(p, M)$ essentially deciding $\underset{\sim}{\tau}$ such that $\mathrm{trnklg}(q_\alpha) \leq \min(n + 1, \mathrm{trnklg}(p_\alpha))$ for all $\alpha \in \mathrm{dom}(p)$.

To see the last item, assume q is a counterexample. Since $q \leq S(p, M)$, there is an $l \leq N$ such that $s^l \in \mathrm{val}^\Pi(q, \leq n)$. So $s^l \wedge q \leq s^l \wedge p^l$ essentially decides $\underset{\sim}{\tau}$, a contradiction.

Step 2: Now we show that $S(p, M)$ is deciding (for any M). Assume towards a contradiction that $S(p, M)$ is halving.

Recall that we set $n = \min(\mathrm{trnklg}(p_\alpha))$. Set $p^n = S(p, M)$. Assume that for some $l \geq n$ we already defined p^l such that there is no $q \leq p^l$ essentially deciding $\underset{\sim}{\tau}$ such that $\mathrm{trnklg}(q_\alpha) \leq \min(l + 1, \mathrm{trnklg}(p_\alpha))$ for all $\alpha \in \mathrm{dom}(p)$.

We define p^{l+1} the following way: Enumerate $\mathrm{val}^\Pi(p^l, \leq l)$ as $s^1, \dots, s^N$. Note that $N \leq \varphi(\leq l)^l$. Set $p^{l,0} = p^l$. Given $p^{l,k}$, set $p^{l,k+1} := S(s^k \wedge p^{l,k}, l)$. $p^{l,k+1}$ is halving (otherwise $p^{l,k+1} \leq p^l$ would essentially decide τ). Define p^{l+1} the following way: $\mathrm{dom}(p^{l+1}) = \mathrm{dom}(p^{l,N})$, and

$$p^{l+1}_\alpha(m) = \begin{cases} p^l_\alpha(m) & \text{if } m \leq l \text{ and } \alpha \in \mathrm{dom}(p^l) \\ p^{l,N}_\alpha(m) & \text{otherwise.} \end{cases}$$

Note that on level $m > l$ we decrease the norms at most N times by $\varphi(=l + 1)^{l+1} r(m)$ or to some value $\geq l + 1$, so for $\alpha \in \mathrm{supp}(p^{l+1}, m)$ and some k we get:

$$\mathrm{nor}(p^{l+1}_\alpha(m)) \geq \begin{cases} \mathrm{nor}(p^l_\alpha(m)) & \text{if } m \leq l \\ \mathrm{nor}(p^l_\alpha(m)) - \varphi(\leq l)^l r(m) & \text{if } m = l + 1 \\ \mathrm{nor}(p^l_\alpha(m)) - \varphi(\leq l + 1)^{l+1} r(m) & \text{if } l + 1 < m < k \\ l & \text{if } m \geq k. \end{cases}$$

(The last line holds because $\varphi(\leq l + 1)^{l+1} \cdot r(k) < 1$.)

So no $q \leq p^{l+1}$ with $\mathrm{trnklg}(q_\alpha) \geq \min(l + 2, \mathrm{trnklg}(p_\alpha))$ for $\alpha \in \mathrm{dom}(p)$ can essentially decide $\underset{\sim}{\tau}$.

Define q by $\mathrm{dom}(q) := \bigcup \mathrm{dom}(p^l)$ and $q_\alpha(l) = p^m_\alpha(l)$ for $\alpha \in \mathrm{dom}(p^m)$ and $l \leq m$. Let $q' \leq q \leq p$ decide $\underset{\sim}{\tau}$. Pick m such that $\mathrm{trnklg}(q'_\alpha) = \mathrm{trnklg}(p_\alpha)$ for all $\alpha \in \mathrm{dom}(p)$ with $\mathrm{trnklg}(p_\alpha) \geq m$. Then $q' \leq p^m$ is a contradiction to the property of p^m.

Step 3: Given p and M, we find an $h > M$ such that $\mathrm{nor}^{\min}(p, m) > 2 + M$ for all $m \geq h$. Enumerate $\mathrm{val}^\Pi(p, \leq h - 1)$ as $\{s^1, \dots, s^N\}$. Set $p^0 = p$, $p^{m+1} := S(s^m \wedge p^m, n)$, and define q by $q(i) = p(i)$ for $i < h$ and $q(i) = p^N(i)$ otherwise. Then $q \leq^+_M p$ essentially decides $\underset{\sim}{\tau}$. □

Theorem 5.8. *If $g : \omega \to \omega$ is monotone and $\mathbf{K}(n)$ is $(n, r(n))$-decisive, $(g(n) + 1, r(n))$-big and $r(n)$-halving (again, we set $r(n) := 1/(n \cdot \varphi(<n)^n)$), if $p \in P$, and $\underset{\sim}{r}$ is a name of a real such that $\underset{\sim}{r}(n) < g(n)$, then there is a $q \leq p$ rapidly deciding $\underset{\sim}{r}$, i.e.: $t \wedge q$ decides $\underset{\sim}{r}(n)$ for every $t \in \mathrm{val}^\Pi(q, \leq n - 1)$.*

Proof. The same proof as above works, with the following modification: Instead of $q \leq p^i$ essentially deciding $\underset{\sim}{\tau}$, we look for $q \leq p^i$ deciding $\underset{\sim}{s}(n)$, so we get $g(n) + 1$ colors: $g(n)$ many possible decisions and half if a decision is not possible.

This gives a $q \leq^+_M p$ such that $t \wedge q$ decides $\underset{\sim}{s}(n)$ for every $t \in \mathrm{val}^\Pi(q, \leq n - 1)$. □

Actually, each $g(n) + 1$ should be of the form 2^{m^t} to be able to apply Lemma 4.3. However, this is not important: On the contrary, we can decide even more rapidly. Recall that according to 4.2 we can increase bigness from e.g. $B(n)$ to $2^{B(n)^n}$, assuming that the norm of the creatures is not just > 1 but $> 1 + r(n)$. For any m we can strengthen a condition (by enlarging finitely many stems) such that $\mathrm{nor}(p_\alpha(n)) > m$ for all $\alpha \in \mathrm{supp}(p, n)$.

Therefore we get:

Corollary 5.9. *Set $L(m) = m^m$ and $L^k(m) = L(L^{k-1}(m))$. Fix $k \in \omega$. If $\mathbf{K}(n)$ is $(n, r(n))$-decisive, $(g(n), r(n))$-big and $r(n)$-halving , if $p \in P$, and $\underset{\sim}{r}$ is a name of a real such that $\underset{\sim}{r}(n) < L^k(g(n))$, then there is a $q \leq p$ rapidly deciding $\underset{\sim}{r}$, i.e.: $t \wedge q$ decides $\underset{\sim}{r}(n)$ for every $t \in \mathrm{val}^\Pi(q, \leq n-1)$.*

Remarks.

- Of course the assumptions here are once again not canonical. Instead of $|\operatorname{supp}(p,n)| < n$ we can require that $|\operatorname{supp}(p,n)| < k(n)$ for some function $k : \mathbb{N} \to \mathbb{N}$ (usually we would want $\limsup(k(n)) = \infty$, otherwise we get "semi-atoms").
- If this If k is a slow growing function, then $r(n)$ can be chosen accordingly bigger (so the bigness etc. is easier to satisfy). But such modifications would not help us much to separate $c^{\exists}_{f,g}$ for f, g that are closer together: As already mentioned, the really wasteful part is decisiveness, which cannot be eliminated from this kind of construction.
- Instead of $\lim(|\operatorname{supp}(p,n)|/n) = 0$ we could require e.g. $\lim(|\operatorname{supp}(p,n)|^2/n) = 0$; or $\lim(n - |\operatorname{supp}(p,n)|) = \infty$ (however then we cannot take the union of two conditions with disjoint domains); or just have no additional condition to $|\operatorname{supp}(p,n)| < n$, then we again get "semi-atoms".
- However, the proof for properness does not work any more if we change the requirement that there is a bound on $|\operatorname{supp}(p,m)|$ or that $q \leq p$ implies that $\mathrm{trnklg}(q_\alpha) = \mathrm{trnklg}(p_\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(p)$.
- Also, the parameters in the definition of halving and P have to be compatible in the sense that we have to be able to un-halve everything not in the trunk.

6. A decisive creature with bigness and halving

Let us construct a concrete example of a suitable $\mathbf{K}(n)$ and $\mathbf{\Sigma}$. We set $\mathbf{F}(n) := n$ for all n, i.e. the n-creatures live on the singleton $\{n\}$. We fix natural numbers n, B and $\varphi(<n)$.

Lemma 6.1. *Set $r := 1/(n\varphi(<n)^n)$. There are $\mathbf{K}(n)$ and $\mathbf{\Sigma}$ which are r-halving, (B, r)-big and (n, r)-decisive such that* $\mathrm{nor}(\mathfrak{c}) > n$ *for some* $n \in \mathbf{K}(n)$.

Without the last requirement the Lemma is trivial, just assume that $\mathrm{nor}(\mathfrak{c}) = 0$ for all $\mathfrak{c} \in \mathbf{K}$, and read the definitions of halving, big and decisive. On the other hand, the last requirement guarantees that we get a nontrivial $\mathbb{Q}^*_\infty$ when we put together $K(n)$ obtained inductively by the Lemma.

Proof. Set $\Psi(m) = 2^{2^{m^2}}$ and $a := 2^{\frac{1}{r}} = 2^{n\cdot\varphi(<n)^n}$. So $\log_a(2) = r$.

The pre-pre-norm:
There is an $J \in \omega$ and a function preprenor on the powerset of J such that the following holds:

1. preprenor is monotone, i.e. if $u_1 \subseteq u_2$ then $\mathrm{preprenor}(u_1) \leq \mathrm{preprenor}(u_2)$.
2. $\mathrm{preprenor}(\emptyset) = 0$, and $\mathrm{preprenor}(J) = a^{n+1}$.
3. If $\mathrm{preprenor}(u) = k + 1$ then there is an $M \in \mathbb{N}$ and a sequence $0 = j_0 < j_1 < \cdots < j_M$ such that $M \geq \min(B, \Psi(j_1 + n))$ and $\mathrm{preprenor}(u \cap [j_i, j_{i+1} - 1]) \geq k$ for all $i \in M$.

Proof: For finite subsets u of ω define $\mathrm{preprenor}(u) \geq k$ by induction on k: For all u set $\mathrm{preprenor}(u) \geq 0$, and $\mathrm{preprenor}(u) \geq 1$ iff u is nonempty. $\mathrm{preprenor}(u) \geq k + 1$ iff (3) as above holds. Then for every k there is an m such that $\mathrm{preprenor}(m) > k$. Pick J such that $\mathrm{preprenor}(J) = a^{n+1}$.

The pre-norm:
For a subset c of 2^J, we set

$$\mathrm{prenor}(c) := \max\{\mathrm{preprenor}(u) : u \subseteq J, c \restriction u = 2^u\}.$$

(where $c \restriction u$ is $\{b \restriction u : b \in c\}$). So $d \subseteq c$ implies $\mathrm{prenor}(d) \leq \mathrm{prenor}(c)$. We will use the following simple

Fact: Assume that $M \in \mathbb{N}$, J a set, $u \subseteq J$, $c \subseteq 2^J$, $c \restriction u = 2^u$, $c = \bigcup_{i \in M} c_i$, and that u_i ($i \in M$) are pairwise disjoint subsets of u. Then $2^{u_i} = c_i \restriction u_i$ for some $i \in M$.

Proof: Otherwise, for all $i \in M$ there is an $a_i \in 2^{u_i} \setminus (c_i \restriction u_i)$. Let $b \in 2^u$ contain the concatenation of these a_i. Then $b \in c \restriction u$, so $b \in c_i \restriction u$ for some $i \in M$, and $a_i \in c_i \restriction u_i$, a contradiction.

The creating pair:
We set $\mathbf{H}(m) = 2^J$. An n-creature $\mathfrak{c}$ is a pair (c, k) such that $c \subseteq 2^J$, $k \in \omega$ and $k \leq \mathrm{prenor}(c) - 1$. (More formally, we set $\mathrm{val}(\mathfrak{c}) = c$ and $\mathrm{dist}(\mathfrak{c}) = k$.) $\mathrm{nor}(\mathfrak{c})$ is determined from (c, k) by $\mathrm{nor}(\mathfrak{c}) := \log_a(\mathrm{prenor}(c) - k)$. For n-creatures $\mathfrak{c} \cong (c, k)$ and $\mathfrak{d} \cong (d, k')$ we define $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ by $d \subseteq c$ and $k' \geq k$, or in other notation:

$$\mathbf{\Sigma}(\mathfrak{c}) = \{\mathfrak{d} \in \mathbf{K}(n) : \mathrm{val}(\mathfrak{d}) \subseteq \mathrm{val}(\mathfrak{c}) \text{ and } \mathrm{dist}(\mathfrak{d}) \geq \mathrm{dist}(\mathfrak{c})\}.$$

(So this $\mathbf{K}(n)$ does not contain trunk creatures: for every n-creature (c, k), $\mathrm{prenor}(c) \geq k + 1 \geq 1$, so $|c| > 1$.)

Halving:
Assume $\mathrm{nor}(\mathfrak{c}) > 1$, i.e. $\mathrm{prenor}(c) - k > a > 2$. We define

$$\mathrm{half}(c, k) := (c, k + \lfloor(\mathrm{prenor}(c) - k)/2\rfloor).$$

Note that $\log_a(\lceil(\mathrm{prenor}(c) - k)/2\rceil) \geq \mathrm{nor}(c, k) - \log_a(2) = \mathrm{nor}(c, k) - r$. Then

$$\mathrm{nor}(\mathrm{half}(c, k)) = \log_a(\mathrm{prenor}(c) - k - \lfloor(\mathrm{prenor}(c) - k)/2\rfloor) \geq \mathrm{nor}(c, k) - r.$$

If $(d, k') \in \mathbf{\Sigma}(\mathrm{half}(c, k))$ and $\mathrm{nor}(d, k') > 0$, then

$$\mathrm{prenor}(d) \geq k' + 1 \geq k + \lfloor(\mathrm{prenor}(c) - k)/2\rfloor + 1,$$

and we can un-halve (d, k') to $(d, k) \in \mathbf{\Sigma}(c, k)$:

$$\mathrm{nor}(d, k) = \log_a(\mathrm{prenor}(d) - k) \geq \log_a(\lfloor(\mathrm{prenor}(c) - k)/2\rfloor + 1) \geq \mathrm{nor}(c, k) - r,$$

and $\mathrm{val}(d, k) = \mathrm{val}(d, k') = d$.

Bigness:
Let (c, l) be an n-creature and $\mathrm{nor}(c, l) = x + r \geq r$. Let $u \subseteq J$ witness $\mathrm{prenor}(c) = a^{x+r} + l = 2a^x + l$. So there is an increasing sequence $(j_i)_{i \in M+1}$ such that $M \geq \min(B, \Psi(j_1 + n))$ and

$$\mathrm{preprenor}(u \cap [j_i, j_{i+1} - 1]) \geq 2a^x + l - 1 \geq a^x + l$$

for all $i \in M$ (if $x > 0$, the last inequality is strict).

Take any $F : c \to M$. Then $c = \bigcup_{i \in M} F^{-1}\{i\}$. According to the Fact above there is an $i \in M$ such that $F^{-1}\{i\} \restriction u_i = 2^{u_i}$ for $u_i := u \cap [j_i, j_{i+1} - 1]$. We set $d := F^{-1}\{i\} \subseteq c$. Then $\mathrm{nor}(d, l) \geq \log_a(a^x) = x = \mathrm{nor}(c, l) - r$ (and if $x > 0$, then $\mathrm{nor}(d, l) > \mathrm{nor}(c, l) - r$). This shows that (c, l) is (M, r)-big.

Decisiveness:
Pick $(c, l) \in \mathbf{K}(n)$ such that $\mathrm{nor}(c, l) = x + r \geq r$. As above there is a witness $u \subseteq J$, M and $(j_i)_{i \in M+1}$. Set $u^- := u \cap [j_0, j_1 - 1]$. Let $d^- \subseteq c$ contain for every $a \in 2^{u^-}$ exactly one $b \in c$ such that $b \restriction u^- = a$. Then $|d| \leq 2^{j_1} =: K$ and (as above) $\mathrm{nor}(d^-, l) \geq \mathrm{nor}(c, l) - r$ (and $>$, if $x > 0$).

It is enough to show that there is a hereditarily $(2^{K^n}, r)$-big $\mathfrak{d}^+ \in \mathbf{\Sigma}(\mathfrak{c})$.

Let $F : c \to 2^{j_1} < M$ map b to $b \restriction j_1$. So as above there is an $i < M$ such that $F^{-1}\{i\} \restriction u_i = 2^{u_i}$ for $u' := u \cap [j_i, j_{i+1} - 1]$. Obviously $i \neq 0$. Set $d^+ := F^{-1}\{i\}$, and $\mathfrak{d}^+ = (d^+, l)$. Pick any $(d', l') \in \mathbf{\Sigma}(d^+, l)$. Let prenor$(d')$ be witnessed by $u', M', (j'_i)_{i \leq M'}$. Then $u' \cap j_1 = \emptyset$ (since every $b \in d'$ has the same $b \restriction j_1$). So $j'_1 > j_1$, and (by the same argument as above) $\mathfrak{d}'$ is $(\Psi(j_1 + n), r)$-big. This finishes the proof, since

$$\Psi(j_1 + n) = 2^{2^{(j_1+n)^2}} \geq 2^{2^{j_1 \cdot n}} = 2^{(2^{j_1})^n} = 2^{K^n}. \qquad \square$$

If we use this construction to iteratively build a creating pair $(\mathbf{K}(n))_{n \in \omega}$ with bigness etc., then $\mathbf{H}$ will be a very fast growing function: The J of this construction is a quite large number (in dependence on the triple $(n, \varphi({<}n), B)$. (An upper bound can be given as nested iterations of Ψ.) $\mathbf{H}(n) = 2^J$.

Of course we can find creatures $\mathbf{K}(n)$ with the same bigness etc. for any *larger* $\mathbf{H}(n)$ as well: Just map 2^J into $\mathbf{H}(n)$, and ignore all the elements of val$(\mathfrak{c})$ that are not in the image.

7. Many cardinal invariants

Definition 7.1.
- Let $f, g \in (\omega \setminus 1)^\omega$. A function S with domain ω is an (f, g)-slalom if $S(n) \subseteq f(n)$ and $|S(n)| \leq g(n)$ for all $n \in \omega$.
- Let $\mathcal{S}$ be a family of (f, g)-slaloms.

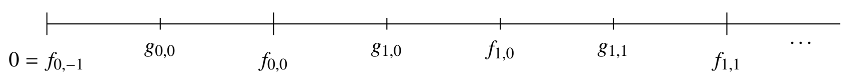
  $\mathcal{S}$ is an $(\exists, f, g)$-cover, if for all $\eta \in \prod_{n \in \omega} f(n)$ there is a $S \in \mathcal{S}$ such that $\eta(n) \in S(n)$ for infinitely many n.

  $\mathcal{S}$ is a $(\forall, f, g)$-cover, when the same is true when "infinitely many" is replaced by "all but finitely many".
- $c^{\exists}_{f,g} := \min\{|\mathcal{S}| : \mathcal{S}$ is an $(\exists, f, g)$-cover$\}$;

  $c^{\forall}_{f,g} := \min\{|\mathcal{S}| : \mathcal{S}$ is a $(\forall, f, g)$-cover$\}$.

For example, if $f \equiv 2$ and $g \equiv 1$, then $c^{\forall}_{f,g} = 2^{\aleph_0}$ and $c^{\exists}_{f,g} = 2$. However, if $(\forall n)(\exists m) f(m) > n \cdot g(m)$, then simple diagonalization shows that $2^{\aleph_0} \geq c^{\forall}_{f,g} \geq c^{\exists}_{f,g} \geq \aleph_1$.

Let $(f_{n,l})_{n \in \omega, -1 \leq l \leq n}$, and $(g_{n,l})_{n \in \omega, 0 \leq l \leq n}$ be natural numbers such that $0 = f_{0,-1}$ and (for $n \in \omega$) $f_{n+1,-1} = f_{n,n}$ and $f_{n,l-1} < g_{n,l} < f_{n,l}$. I.e. we have:



Moreover we require that $g_{n,l+1} \geq f^{3n}_{n,l}$, and that $f_{n,l}$ is larger than 2^J, where J is calculated from $(n, f^{n-1}_{n-1}(n-1), g_{n,l})$ as described in the last paragraph of the last section. (So $f_{n,l}$ is *much* larger than $g_{n,l}$.)

Let $\mathcal{J}$ be of size $\aleph_1$, and choose $(h_\epsilon)_{\epsilon \in \mathcal{J}}$ such that $h_\epsilon(m) \leq m$ and for all $\epsilon \neq \epsilon'$ there is an n such that $h_\epsilon(m) \neq h_{\epsilon'}(m)$ for all $m \geq n$.[9]

For $\epsilon \in \mathcal{J}$, set $f_\epsilon(n) := f_{n, h_\epsilon(n)}$ and $g_\epsilon(n) := g_{n, h_\epsilon(n)}$. We set $f_{\max}(m) := f_{m,m}$.

Theorem 7.2. *Assume CH. Let $\mathcal{J}$ be of size $\aleph_1$ and choose for all $\epsilon \in \mathcal{J}$ a cardinal κ_ϵ such that $\kappa_\epsilon = \kappa_\epsilon^{\aleph_0}$. Let (f_ϵ, g_ϵ) be as above. Then there is a proper, $\aleph_2$-cc partial order P which forces that $c^{\exists}_{f_\epsilon, g_\epsilon} = c^{\forall}_{f_\epsilon, g_\epsilon} = \kappa_\epsilon$ for all $\epsilon \in \mathcal{J}$.*

Set $I = \sum_{\epsilon \in \mathcal{J}} \kappa_\epsilon$, and call the κ_ϵ-part I_ϵ (i.e. I is the disjoint union $\bigcup I_\epsilon$ and $|I_\epsilon| = \kappa_\epsilon$). We will always use $\epsilon, \epsilon', \epsilon_1, \dots$ for the elements of $\mathcal{J}$ (corresponding to the $\aleph_1$ many cardinal invariants), and $\alpha, \beta, \dots$ for elements of I (the index set of the product).

[9] This is possible, since for all $k : \omega \to \omega$ with $\lim_{n \to \infty} k(\omega) = \infty$ and all countable $\mathcal{F} \subseteq \prod_{n \in \omega} k(n)$ there is a $g \in \prod_{n \in \omega} k(n)$ such that for all $f \in \mathcal{F}$ there is an $m \in \omega$ such that $f(n) \neq g(n)$ for all $n > m$.

Definition 7.3.
- $\epsilon(\alpha)$ is the (unique) $\epsilon \in \mathcal{J}$ such that $\alpha \in I_\epsilon$.
- For $\epsilon_1 \neq \epsilon_2 \in \mathcal{J}$ we set $n^{\text{dist}}(\epsilon_1, \epsilon_2)$ to be the least n such that $f_{\epsilon_1}(m) \neq f_{\epsilon_2}(m)$ for all $m \geq n$; and we set $n^{\text{dist}}(\epsilon, \epsilon) = 0$ for all $\epsilon \in \mathcal{J}$.
  For $\alpha, \beta \in I$ we set $n^{\text{dist}}(\alpha, \beta) := n^{\text{dist}}(\epsilon(\alpha), \epsilon(\beta))$.
  Similarly for $\epsilon' \in \mathcal{J}, \alpha \in I$ we set $n^{\text{dist}}(\alpha, \epsilon') = n^{\text{dist}}(\epsilon', \alpha) = n^{\text{dist}}(\epsilon(\alpha), \epsilon')$.

By induction on first $n \in \omega$ and then the finitely many values $f_\epsilon(n)$ ($\epsilon \in \mathcal{J}$) we choose creating pairs $(\mathbf{K}_\epsilon(n), \mathbf{\Sigma}_\epsilon)$ satisfying the following:

Lemma 7.4. *(1) $\mathbf{F}_\epsilon(n) = n$,*
(2) $\mathbf{H}_\epsilon(n) = f_\epsilon(n)$,
(3) $\mathbf{K}_\epsilon(n)$ is $(g_\epsilon(n), r(n))$-big, $r(n)$-halving and $(n, r(n))$-decisive, where

$$r(n) := \frac{1}{n \cdot (\varphi(<n)^n} \quad \varphi(<n) := \prod_{m<n} \varphi(=m) \quad \textit{and} \quad \varphi(=m) := \{\max f_\epsilon(m) : \epsilon \in \mathcal{J}\}.$$

For every $\alpha \in I$, we set $\mathbf{K}_\alpha := \mathbf{K}_{\epsilon(\alpha)}$, $f_\alpha := f_{\epsilon(\alpha)}$ and $g_\alpha := g_{\epsilon(\alpha)}$. So $\varphi(<n)$ and $r(n)$ satisfy the definitions of Theorems 5.4 and 5.8.

Lemma 7.5. *(CH)*
(1) P is ω^ω-bounding, proper and has continuous reading of names.
(2) P is $\aleph_2$-cc.

So in particular (1) and (2) imply that P does not change any cardinality or cofinality.

Proof. (1) is just Theorem 5.4. For (2) assume towards a contradiction that A is an antichain of size $\aleph_2$. Note that when we fix enumerations $\{\alpha_0, \alpha_1, \dots\}$ of all $\text{dom}(p)$, then there are only $\aleph_1$ many possibilities for the sequence $(\epsilon(\alpha_n), \text{trnklg}(p_{\alpha_n}))_{n \in \omega}$ (lets call this sequence "type of p" for now). Without loss of generality $\{\text{dom}(p) : p \in A\}$ forms a Δ-system with root u (by the Δ-system lemma and CH). Also we can assume that all $p \restriction u$ are the same and that the types of all $p \restriction (I \setminus u)$ are the same. So pick $p \neq q \in A$. Let n be such that $|\text{supp}(p, m) \cup \text{supp}(q, m)| < m$ for all $m \geq n$. Extend the stems in p up to n and call the result p'. Then $p' \cup q \leq p, q$: $\lim(|\text{supp}(p' \cup q, n)|/n) \leq \lim(|\text{supp}(p, n)|/n) + \lim(|\text{supp}(q, n)|/n) = 0$. Assume that $\alpha \neq \beta \in \text{supp}(p' \cup q, n)$, and that $\alpha \in \text{dom}(p) \setminus u$ and $\beta \in \text{dom}(q) \setminus u$. Then there is a $\beta' \in \text{supp}(p, n)$ such that $\epsilon(\beta') = \epsilon(\beta)$ and $\text{trnklg}(p_{\beta'}) = \text{trnklg}(q_\beta)$ (since the type of $p \restriction (I \setminus u)$ is the same as of $q \restriction (I \setminus u)$). So $n^{\text{dist}}(\alpha, \beta) = n^{\text{dist}}(\alpha, \beta') < n$. $\square$

We note some trivial consequences of Lemma 7.4:

Corollary 7.6. *(1) If $f_{\epsilon'}(n) > f_\epsilon(n)$, then $\mathbf{K}_{\epsilon'}(n)$ is $(f_\epsilon(n), r(n))$-big.*
(2) $\mathbf{K}_{\epsilon'}(n+1)$ is $(f_\epsilon(n), r(n))$-big for all $\epsilon, \epsilon' \in \mathcal{J}$.
(3) If $f_{\epsilon'}(n) > f_\epsilon(n)$, then $g_{\epsilon'}(n) \geq \varphi(<n)^n \cdot f_\epsilon^n(n)$.
(4) $f_\epsilon(n+1) \geq \varphi(=n)$.
(5) If $f_{\epsilon'}(n) > f_\epsilon(n)$, and $\mathfrak{c} \in \mathbf{K}_{\epsilon'}(n)$ such that $\text{nor}(\mathfrak{c}) > 1 + 3 \cdot r(n)$, then there is a hereditarily $(f_\epsilon(n)^{\varphi(<n)^n f_\epsilon(n)}, r(n))$-big $\mathfrak{d} \in \mathbf{\Sigma}(\mathfrak{c})$ with $\text{nor}(\mathfrak{d}) > \text{nor}(\mathfrak{c}) - 3 \cdot r(n)$.

Proof. (1) and (2) $\mathbf{K}_{\epsilon'}(n)$ is $g_{\epsilon'}(n)$-big, and $g_{\epsilon'}(n) > f_\epsilon(n)$ and $g_{\epsilon'}(n) > f_{\epsilon''}(n-1)$.
(3) and (4) follow from the initial construction of the $f_{i,j}$ and the fact that $g_{i,l+1} \geq f_{i,l}^{3i}$.
For (5), use 4.2. $\square$

Corollary 7.7. *We can rapidly read (f_ϵ, g_ϵ)-slaloms $\underset{\sim}{S}$: If $\epsilon \in \mathcal{J}$, if $\underset{\sim}{S}$ is a P-name for an (f_ϵ, g_ϵ)-slalom, and if $p \in P$, then there is a $q \leq p$ such that $s \wedge q$ decides $\underset{\sim}{S}(n)$ for all $s \in \text{val}^\Pi(q, \leq n)$. The analog holds for names $\underset{\sim}{r}$ such that $\underset{\sim}{r}(n) \leq f_\epsilon(n)$.*

Proof. This follows from 5.9, noting that there are less than $f_\epsilon(n)^{f_\epsilon(n)}$ many possibilities for $\underset{\sim}{S}(n)$, and that each $\mathbf{K}_\alpha(n+1)$ is $(f_\epsilon(n), r(n))$-big. □

Now we will show that P forces

$$\kappa_\epsilon \leq c^\exists_{f_\epsilon,g_\epsilon} \leq c^\forall_{f_\epsilon,g_\epsilon} \leq \kappa_\epsilon.$$

Lemma 7.8. *P forces $c^\forall_{f_\epsilon,g_\epsilon} \leq \kappa_\epsilon$.*

Proof. Recall that $I_\epsilon := \{\alpha \in I : \epsilon(\alpha) = \epsilon\}$ and set $P_\epsilon := \{p \in P : \mathrm{dom}(p) \subset J_\epsilon\}$. P_ϵ is a complete subforcing of P. Since every P_ϵ-name of a real can be read continuously, every real in the P_ϵ-extension corresponds to a condition in P_ϵ. There at most $\kappa_\epsilon^{\aleph_0} = \kappa_\epsilon$ many such conditions.

We will show that in a P-extension $V[G]$, the set of slaloms that are in the P_ϵ-extension $V[G \cap P_\epsilon]$ are a $(\forall, f_\epsilon, g_\epsilon)$-cover.

Let $p_0 \in P$ and $\underset{\sim}{r}$ be a P-name for a real such that $\underset{\sim}{r}(n) < f_\epsilon(n)$. So there is an $p \leq p_0$ rapidly reading $\underset{\sim}{r}$, i.e. $c \wedge p$ decides $\underset{\sim}{r}(n)$ for all $s \in \mathrm{val}^\Pi(p, {\leq}n)$. We can assume that $\mathrm{nor}(p_\alpha(n)) > 3$ for all $\alpha \in \mathrm{supp}(p, n)$. It is sufficient to find a $q \leq p$ and to define a P_ϵ-name $\underset{\sim}{S}$ of an (f_ϵ, g_ϵ)-slalom such that q forces: $\underset{\sim}{r}(n) \in \underset{\sim}{S}(n)$ for all but finitely many $n \in \omega$.

Fix $m \in \omega$. Set $M := \mathrm{supp}(p, m) \cap I_\epsilon$. ($M$ stands for "medium".) If $\alpha \in \mathrm{supp}(p, m) \setminus M$, then $m > n^{\mathrm{dist}}(\alpha, \epsilon)$, i.e. either $f_\alpha(m) < f_\epsilon(m)$ (in this case we set $\alpha \in S$ for "small") or $f_\alpha(m) > f_\epsilon(m)$ (then we set $\alpha \in L$ for "large"). So $\mathrm{supp}(p, m)$ is partitioned into S, M and L.

The name $\underset{\sim}{r}$ defines a function

$$F : \mathrm{val}^\Pi(p, {<}m) \times \left(\prod_{\alpha \in S \cup M \cup L} \mathrm{val}(p_\alpha(m)) \right) \to f_\epsilon(m).$$

Assume L is nonempty. $|\prod_{\alpha \in S \cup M} \mathrm{val}(p_\alpha(m))| \leq \mathbf{H}_\epsilon(m)^m = f_\epsilon(m)^m$. So we can rewrite F as

$$F' : \prod_{\alpha \in L} \mathrm{val}(p_\alpha(m)) \to f_\epsilon(m)^{\varphi({<}m)^m f_\epsilon(m)^m}.$$

According to 7.6(5), there are sufficiently large $q_\alpha(m) \in \mathbf{\Sigma}(p_\alpha)(m)$ such that F' restricted to $\prod_{\alpha \in L} \mathrm{val}(q_\alpha(m))$ is constant. This defines a $q \leq p$ (set $q_\alpha(m) = p_\alpha(m)$ if $\alpha \notin L$).

So given $\bar{t} \in \prod_{\alpha \in M} \mathrm{val}(q_\alpha(m))$, $t \wedge q$ has at most

$$\varphi({<}m)^m \cdot \prod_{\alpha \in S} \mathrm{val}(p_\alpha(m)) \leq \varphi({<}m)^m \cdot f^m_{\epsilon'} < f_\epsilon(m)$$

possibilities for $\underset{\sim}{r}$ (for a suitable ϵ'). We call this set of possibilities $\underset{\sim}{S}(m)$. Since $M \subseteq I_\epsilon$, $\underset{\sim}{S}$ can be interpreted as P_ϵ-name of an (f_ϵ, g_ϵ)-slalom, and q forces that $\underset{\sim}{r} \in \underset{\sim}{S}(n)$. □

Lemma 7.9. $\kappa_\epsilon \leq c^\exists_{f_\epsilon,g_\epsilon}$.

Proof. Assume towards a contradiction that p forces that $\underset{\sim}{S}$ is an $(\exists, f_\epsilon, g_\epsilon)$-cover, $\aleph_1 \leq \lambda < \kappa_\epsilon$ and $\underset{\sim}{S} = \{\underset{\sim}{S}_\alpha : \alpha \in \lambda\}$.

For every α, the set of $p' \leq p$ which continuously (and even rapidly) reads $\underset{\sim}{S}_\alpha$ is predense under p. Because of $\aleph_2$-cc, we can find a set D_α of such p' which is predense under p and has size $\aleph_1$. So the union of all D_α for $\alpha \in \lambda$ has size λ, and $J = \bigcup_{\alpha \in \lambda, p' \in D_\alpha} \mathrm{dom}(p')$ has size λ as well. Since $|I_\epsilon| = \kappa_\epsilon > \lambda$, there is a $\beta \in I_\epsilon \setminus J$. Fix this β.

Let $q_0 \leq p$ decide the α such that $\underset{\sim}{\eta}_\beta(n) \in \underset{\sim}{S}_\alpha(n)$ for infinitely many n. We set $\underset{\sim}{S} := \underset{\sim}{S}_\alpha$. Let $q \leq q_0$ be stronger than some $p' \in D_\alpha$, and let $\mathrm{nor}(q_\alpha(m)) > 10$ for all $\alpha \in \mathrm{supp}(q, m)$.

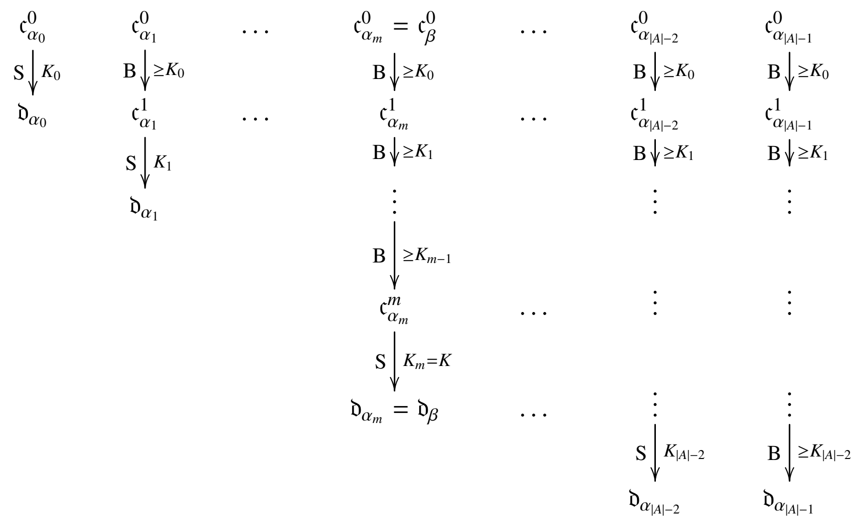


FIGURE 3.

We will show that we can strengthen q to a q' such that for all $n > \mathrm{trnklg}(q_\beta)$ the generic $\eta_\beta(n) < f_\eta(n)$ (determined by $q'_\beta(n)$) runs away from every possibility of $\underset{\sim}{S}(n)$ (determined by $q'_\alpha(m)$ for $m \leq n$ and $\alpha \neq \beta$). So we get a contradiction.

Fix an $n > \mathrm{trnklg}(q_\beta)$. Set $A := \mathrm{supp}(q, n)$. Without loss of generality, $\beta \in A$ and $|A| \geq 2$. According to the definition of P, $|A| \leq n$.

Set $\mathfrak{c}^0_\alpha := q_\alpha(n)$ for $\alpha \in A$. Assume we already have $(\mathfrak{c}^l_\alpha)_{\alpha\in A}$ and a list $(\alpha_k)_{k<l}$ of elements of A. $\mathfrak{c}^l_\alpha$ is $(K^l_\alpha, n, r(n))$-decisive for some K^l_α. Set $K_l := \min(\{K^l_\alpha : \alpha \in A \setminus \{\alpha_0, \dots, \alpha_{l-1}\}\})$, and choose α_l such that $K^l_{\alpha_l} = K_l$. Pick a K_l-small successor of $\mathfrak{c}^l_{\alpha_l}$ and call it $\mathfrak{d}_\alpha$. For $\alpha \in A \setminus \{\alpha_0, \dots, \alpha_l\}$, pick a K_l-big successor of $\mathfrak{c}^l_\alpha$ and call it $\mathfrak{c}^{l+1}_\alpha$. Cf. Figure 7. Iterate this construction $|A| - 1$ times. So there remains one α that has not been listed as an α_l, set $\alpha = \alpha_{|A|-1}$ and $\mathfrak{d}_\alpha = \mathfrak{c}^{|A|-1}_\alpha$.

Let m be such that $\beta = \alpha_m$, and set $K := K_m$, $S := \{\alpha_l : l < m\}$ and $L := \{\alpha_l : l > m\}$. So A is partitioned into the three parts $\{\beta\}$, S and L. (S or L could be empty), and we get:

- $\mathfrak{d}_\alpha \in \mathbf{\Sigma}(q_\alpha(n))$.
- $\mathrm{nor}(\mathfrak{d}_\alpha) \geq \mathrm{nor}(q_\alpha(n)) - n \cdot r(n)$.
- $K_{l+1} > 2^{{K_l}^n}$, since $\mathfrak{c}^{l+1}_{\alpha_{l+1}}$ is hereditarily $2^{{K_l}^n}$-big and $|\mathrm{val}(\mathfrak{d}_{\alpha_{l+1}})| \leq K_{l+1}$.
- $\mathfrak{d}_\beta$ is $2^{{K_{m-1}}^n}$-big (and $g_\epsilon(n)$-big), and $|\mathrm{val}(\mathfrak{d})| \leq K$.
- $\prod_{\alpha\in S} |\mathrm{val}(\mathfrak{d}_\alpha)| \leq {K_{m-1}}^n \leq \log_2(K)$.
- If $\alpha \in L$, then $\mathfrak{d}_\alpha$ is 2^{K^n}-big.

$\underset{\sim}{S}(n)$ is determined by $\mathrm{val}^\Pi(q, {<}n) \prod_{\alpha\in S\cup L} \mathrm{val}(q_\alpha(l))$, since $q \leq p' \in D_\alpha$ and $\beta \notin \mathrm{dom}(p')$. We actually are only interested in the part of $\underset{\sim}{S}(n)$ that is compatible with q', i.e. $\underset{\sim}{S}(n) \cap \mathfrak{d}_\beta$. This part has size $\leq K$. So we get a function

$$F : \mathrm{val}^\Pi(q, {<}n) \times \left(\prod_{\alpha\in S} \mathrm{val}(\mathfrak{d}_\alpha)\right) \times \left(\prod_{\alpha\in L} \mathrm{val}(\mathfrak{d}_\alpha)\right) \to \binom{K}{g_\epsilon(n)}.$$

Step 1: Assume L is non-empty (otherwise continue at step 2). Note that $\binom{K}{g_\epsilon(n)} \leq K^{g_\epsilon(n)}$ and $\varphi(<n) < g_\epsilon(n) < K$. So we can rewrite F as

$$F' : \prod_{\alpha\in L} \mathrm{val}(\mathfrak{d}_\alpha) \to K^{K\cdot\log_2 K}$$

Since $\mathfrak{d}_\alpha$ is decisive and (hereditarily) 2^{K^n}-big for $\alpha \in L$, we can assume without loss of generality that F' is constant (cf. 4.2).

Step 2: So we can rewrite F as

$$F'' : \mathrm{val}^{\Pi}(q, <n) \times \left(\prod_{\alpha\in S} \mathrm{val}(\mathfrak{d}_\alpha)\right) \to \binom{K}{g_\epsilon(n)}$$

The image of F'' has size at most $\varphi(<n)^n \cdot {K_{m-1}}^n$. Since $\mathfrak{d}$ is $2^{{K_{m-1}}^n}$-big as well as $g_\epsilon(n)$-big, and since $g_\epsilon(n) \geq \varphi(<n)^n$ we can assume that val $\mathfrak{d}_\beta$ is disjoint to the image of F'' (again, use 4.2). This gives a contradiction. □

Remarks.
- The proofs above show the simultaneous independence of $c^{\exists}_{f,g}$ for a certain families of f, g with very different growth behavior. On the other hand, simple ZFC results show that $c^{\exists}_{f,g}$ cannot be separated from $c^{\exists}_{f',g'}$ if (f, g) and (f', g') are too similar, cf. e.g. [GS93].
- Our proofs leave some freedom and could be "optimized" to get the independence result for slightly more families. However, the methods of this paper will not yield "sharp" results like the following: Either (f, g) and (f', g') are in a relation provable in ZFC, or can be separated. The reason is that decisiveness is wasteful, i.e. we have to assume that $f(n)$ is *much* bigger than $g(n)$. There are other methods that seem promising for obtaining sharper results. Good candidates seem "products with memory" (i.e. mixtures between products and iterations).

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