On the p-rank of $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ in certain models of ZFC

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ABSTRACT. We show that if the existence of a supercompact cardinal is consistent with ZFC, then it is consistent with ZFC that the p-rank of $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ is as large as possible for every prime p and any torsion-free abelian group G. Moreover, given an uncountable strong limit cardinal μ of countable cofinality and a partition of Π (the set of primes) into two disjoint subsets Π_0 and Π_1, we show that in some model which is very close to ZFC there is an almost-free abelian group G of size $2^\mu = \mu^+$ such that the p-rank of $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ equals $2^\mu = \mu^+$ for every $p \in \Pi_0$ and 0 otherwise, i.e. for $p \in \Pi_1$.

1. Introduction

In 1977 the first named author solved the well-known Whitehead problem by showing that it is undecidable in ordinary set-theory ZFC wether or not every abelian group G satisfying $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z}) = \{0\}$ has to be free (see [**Sh2**], [**Sh3**]). However, this did not clarify the structure of $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ for torsion-free abelian groups - a problem which has received much attention since then. Easy arguments show that $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ is always a divisible group for every torsion-free group G. Hence it is of the form

$$\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z}) = \bigoplus_{p\in\Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 ($p \in \Pi$) which are uniquely determined. The obvious question that arises is which sequences $(\nu_0, \nu_p : p \in \Pi)$ of cardinals can appear as the cardinal invariants of $\text{Ext}_{\mathbb{Z}}(G,\mathbb{Z})$ for some (which) torsion-free abelian group? Obviously, the trivial sequence consisting of zero entries only can be realized by any free abelian group. However, the solution of the Whitehead problem shows that it is undecidable in ZFC if these are the only ones. There are a few results about possible sequences $(\nu_0, \nu_p : p \in \Pi)$ provable in ZFC. On the other hand, assuming

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Gödel's constructible universe ($V = L$) plus there is no weakly compact cardinal a complete characterization of the cardinal invariants of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for torsion-free abelian groups G has recently been completed by the authors (see [**EkHu**], [**EkSh1**], [**GrSh1**], [**GrSh2**], [**HiHuSh**], [**MeRoSh**], [**SaSh1**], [**SaSh2**], [**ShSt**] and [**Sh4**] for references). In fact, it turned out that almost all divisible groups D may be realized as $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for some torsion-free abelian group G of almost any given size.

In this paper we shall take the opposite point of view. It is a theorem of ZFC that every sequence $(\nu_0, \nu_p : p \in \Pi)$ of cardinals such that $\nu_0 = 2^{\lambda_0}$ for some infinite λ_0 and $\nu_p \leq \nu_0$ is either finite or of the form 2^{λ_p} for some infinite λ_p can arise as the cardinal invariants of $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ for some torsion-free G. The first purpose of this paper is to show that this result is as best as possible by constructing a model of ZFC in which the only realizable sequences are of this kind. We shall assume therefore the consistency of the existence of a supercompact cardinal (see [**MeSh**]). This is a strong additional set-theoretic assumption which makes the model we are working in be far from ZFC.

On the other hand, we will also work in models very close to ZFC assuming only the existence of certain ladder systems on successor of strong limit cardinals of cofinality $\aleph_0$. Although this model is close to ZFC it allows us to construct almost-free torsion-free abelian groups G such that for instance $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ is torsion-free, i.e. G is coseparable. Also this can be considered as a result at the borderline of what is provable in models close to ZFC since the existence of non-free coseparable groups is independent of ZFC (see [**EkMe**, Chapter XII] and [**MeSh**]).

Our notation is standard and we write maps from the right. All groups under consideration are abelian and written additively. We shall abbreviate $\mathrm{Ext}_{\mathbb{Z}}(-,-)$ by $\mathrm{Ext}(-,-)$ and Π will denote the set of all primes. A Whitehead group is a torsion-free group G such that $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z}) = 0$. If H is a pure subgroup of the abelian group G, then we shall write $H \subseteq_* G$. We shall assume sufficient knowledge about forcing, large cardinals and prediction principles like weak diamond etc. as for example in [**EkMe**], [**Ku**] or [**Sh5**]. Also reasonable knowledge about abelian groups as for instance in [**Fu**] is assumed. However, the authors have tried to make the paper as accessible as possible to both algebraists and set theorists.

2. The structure of $\mathrm{Ext}(G, \mathbb{Z})$

In this section we recall the basic results on the structure of $\mathrm{Ext}(G, \mathbb{Z})$ for torsion-free groups G. Therefore let G be a torsion-free abelian group. It is easy to see that $\mathrm{Ext}(G, \mathbb{Z})$ is divisible, hence it is of the form

$$\mathrm{Ext}(G, \mathbb{Z}) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for certain cardinals ν_p, ν_0 $(p \in \Pi)$. Since the cardinals ν_p $(p \in \Pi)$ and ν_0 completely determine the structure of $\mathrm{Ext}(G,\mathbb{Z})$ we introduce the following terminology. Let $\mathrm{Ext}_p(G,\mathbb{Z})$ be the p-torsion part of $\mathrm{Ext}(G,\mathbb{Z})$ for $p \in \Pi$. We denote by $r_0^e(G)$ the *torsion-free rank* ν_0 of $\mathrm{Ext}(G,Z)$ which is the dimension of $\mathbb{Q} \otimes \mathrm{Ext}(G,\mathbb{Z})$ and by $r_p^e(G)$ the *p-rank* ν_p of $\mathrm{Ext}(G,\mathbb{Z})$ which is the dimension of $\mathrm{Ext}(G,\mathbb{Z})[p]$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$ for any prime number $p \in \Pi$. There are only a few results provable in ZFC when G is uncountable, but assuming additional set-theoretic assumptions a better understanding of the structure of $\mathrm{Ext}(G,\mathbb{Z})$ is obtained. For instance, in Gödel's universe and assuming that there is no weakly compact cardinal a complete characterization is known. The aim of this paper is to go to the borderline of the characterization. On the one hand we shall show that one can make the p-ranks of $\mathrm{Ext}(G,\mathbb{Z})$ as large as possible for every torsion-free abelian group G by working in a model of ZFC which assumes strong additional axioms (the existence of large cardinals). On the other hand we shall work in a model which is very close to ZFC but still allows to construct uncountable torsion-free groups G such that $\mathrm{Ext}(G,\mathbb{Z})$ is torsion-free.

We first justify our restriction to torsion-free G. Let A be any abelian group and $t(A)$ its torsion subgroup. Then $\mathrm{Hom}(t(A),\mathbb{Z}) = 0$ and hence we obtain the short exact sequence

$$0 \to \mathrm{Ext}(A/t(A),\mathbb{Z}) \to \mathrm{Ext}(A,\mathbb{Z}) \to \mathrm{Ext}(t(A),\mathbb{Z}) \to 0$$

which must split since $\mathrm{Ext}(A/t(A),\mathbb{Z})$ is divisible. Thus

$$\mathrm{Ext}(A,\mathbb{Z}) \cong \mathrm{Ext}(A/t(A),\mathbb{Z}) \oplus \mathrm{Ext}(t(A),\mathbb{Z}).$$

Since the structure of $\mathrm{Ext}(t(A),\mathbb{Z}) \cong \prod_{p\in\Pi} \mathrm{Hom}(A,\mathbb{Z}(p^\infty))$ is well-known in ZFC (see [**Fu**]) it is reasonable to assume that A is torsion-free and, of course, non-free. Using Pontryagin's theorem one proves

Lemma 2.1. *Suppose G is a countable torsion-free group which is not free. Then $r_0^e(G) = 2^{\aleph_0}$.*

Proof. See [**EkMe**, Theorem XII 4.1]. □

Similarly, we have for the p-ranks of G the following

Lemma 2.2. *If G is a countable torsion-free group, then for any prime p, either $r_p^e(G)$ is finite or $2^{\aleph_0}$.*

Proof. See [**EkMe**, Theorem XII 4.7]. □

This clarifies the structure of $\mathrm{Ext}(G,\mathbb{Z})$ for countable torsion-free groups G in ZFC. We now turn our attention to uncountable groups. There is a useful characterization of $r_p^e(G)$ using the exact sequence

$$0 \to \mathbb{Z} \overset{p}{\to} \mathbb{Z} \to \mathbb{Z}/pZ \to 0.$$

The induced sequence

$$\mathrm{Hom}(G,\mathbb{Z}) \overset{\varphi^p}{\to} \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Ext}(G,\mathbb{Z}) \overset{p_*}{\to} \mathrm{Ext}(G,\mathbb{Z})$$

shows that the dimension of

$$\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})/\,\mathrm{Hom}(G,\mathbb{Z})\varphi^p$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$ is exactly $r_p^e(G)$.

The following result due to Hiller, Huber and Shelah deals with the case when $\mathrm{Hom}(G,\mathbb{Z}) = 0$.

Lemma 2.3. *For any cardinal ν_0 of the form $\nu_0 = 2^{\mu_0}$ for some infinite μ_0 and any sequence of cardinals $(\nu_p : p \in \Pi)$ less than or equal to ν_0 such that each ν_p is either finite or of the form 2^{μ_p} for some infinite μ_p there is a torsion-free group G of cardinality μ_0 such that $\mathrm{Hom}(G,\mathbb{Z}) = 0$ and $r_0^e(G) = \nu_0$, $r_p^e(G) = \nu_p$ for all primes $p \in \Pi$.*

Proof. See [**HiHuSh**, Theorem 3(b)]. □

Together with the following lemma we have reached the borderline of what is provable in ZFC.

Lemma 2.4. *If G is torsion-free such that $\mathrm{Hom}(G,\mathbb{Z}) = 0$, then for all primes p, $r_p^e(G)$ is either finite or of the form 2^{μ_p} for some infinite $\mu_p \leq |G|$.*

Proof. See [**EkMe**, Lemma XII 5.2]. □

Assuming Gödel's axiom of constructibility one even knows a complete characterization in the case when $\mathrm{Hom}(G,\mathbb{Z}) = 0$.

Lemma 2.5 (V=L). *Suppose G is a torsion-free non-free group and let B be a subgroup of A of minimum cardinality ν such that A/B is free. Then $r_0^e(G) = 2^\nu$. In particular, $r_0^e(G)$is uncountable and $r_0^e(G) = 2^{|G|}$ if $\mathrm{Hom}(G,\mathbb{Z}) = 0$.*

Proof. See [**EkMe**, Theorem XII 4.4, Corollary XII 4.5]. □

Note that the above lemma is not true in ZFC since for any countable divisible group D it is consistent that there exists an uncountable torsion-free group G with $\mathrm{Ext}(G,\mathbb{Z}) \cong D$, hence $r_0^e(G) = 1$ is possible taking $D = \mathbb{Q}$ (see [**Sh4**]).

The following result is a collection of theorems due to Grossberg, Mekler, Roslanowski, Sageev and the authors. It shows that under the assumption of $(V = L)$ almost all possibilities for $r_p^e(G)$ can appear if the group is not of weakly compact cardinality or singular cardinality of cofinality $\aleph_0$.

LEMMA 2.6 (V=L). *Let ν be an uncountable cardinal and suppose that $(\nu_p : p \in \Pi)$ is a sequence of cardinals such that for each p, $0 \leq \nu_p \leq 2^\nu$. Moreover, let H be a torsion-free group of cardinality ν. Then the following hold.*

(i) *If ν is regular and less than the first weakly compact cardinal, then there is an almost-free group G of cardinality ν such that $r_0^e(G) = 2^\nu$ and for all primes p, $r_p^e(G) = \nu_p$;*
(ii) *If ν is a singular strong limit cardinal of cofinality ω, then there is no torsion-free group G of cardinality ν such that $r_p^e(G) = \nu$ for any prime p;*
(iii) *If ν is weakly compact and $r_p^e(H) \geq \nu$ for some prime p, then $r_p^e(H) = 2^\nu$;*
(iv) *If ν is singular less than the first weakly compact cardinal and of cofinality $\mathrm{cf}(\nu) > \aleph_0$, then there is a torsion-free group G of cardinality ν such that $r_0^e(G) = 2^\nu$ and for all primes p, $r_p^e(G) = \nu_p$.*

PROOF. For (i) see [**MeRoSh**, Theorem 3.7], for (ii) we refer to [**GrSh1**, Theorem 1.0], for (iii) see [**SaSh1**, Main Theorem] and (iv) is contained in [**ShSt**]. $\square$

The above results show that under the assumption of $(V = L)$ and the non-existence of weakly compact cardinals, the structure of $\mathrm{Ext}(G,\mathbb{Z})$ for torsion-free groups G of cardinality ν is clarified for all cardinals ν and almost all sequences $(\nu_0, \nu_p : p \in \Pi)$ can be realized as the cardinal invariants of some torsion-free abelian group in almost every cardinality. However, if we weaken the set-theoretic assumptions to GCH (the generalized continuum hypothesis), then even more is possible which was excluded by $(V = L)$ before (see Lemma 2.5).

LEMMA 2.7. *The following hold.*

(i) *Assume GCH. For any torsion-free group A of uncountable cardinality ν, if $\mathrm{Hom}(A,\mathbb{Z}) = 0$ and $r_0^e(A) < 2^\nu$, then for each prime p, $r_p^e(A) = 2^\nu$;*
(ii) *It is consisitent with ZFC and GCH that for any cardinal $\rho \leq \aleph_1$, there is a torsion-free group G_ρ such that $\mathrm{Hom}(G_\rho,\mathbb{Z}) = 0$, $r_0^e(G_\rho) = \rho$ and for all primes p, $r_p^e(G_\rho) = 2^{\aleph_1}$.*

PROOF. See [**EkMe**, Theorem XII 5.3] and [**EkMe**, Theorem XII 5.49]. $\square$

It is our aim in the next section to show that this rich structure of $\mathrm{Ext}(G,\mathbb{Z})$ (G torsion-free) which exists in $(V = L)$ does not appear in other models of ZFC. As a motivation we state two results from [**MeSh**] which show that using Cohen forcing we may enlarge the p-rank of $\mathrm{Ext}(G,\mathbb{Z})$ for torsion-free groups G.

Lemma 2.8. *Suppose G is contained in the p-adic completion of a free group F and $|G| > |F|$. Then, if $\lambda \geq |F|$ and λ Cohen reals are added to the universe, $|\operatorname{Ext}_p(G,\mathbb{Z})| \geq \lambda$. In particular, adding $2^{\aleph_0}$ Cohen reals to the universe implies that for every torsion-free reduced non-free abelian group G of cardinality less than the continuum, there is a prime p such that $r_p^e(G) > 0$.*

Proof. See [**MeSh**, Theorem 8]. □

Assuming the consistency of large cardinals we even get more. Recall that a cardinal κ is *compact* if it is uncountable regular and satisfies the condition that for every set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S. This is equivalent to saying that for any set A such that $|A| \geq \kappa$, there exists a fine measure on $P_\kappa(A)$ (the set of all subsets of A of size less than or equal to κ). If we require the measure to satisfy a normality condition, then we get a stronger notion. A fine measure U on $P_\kappa(A)$ is called *normal* if whenever $f : P_\kappa(A) \to A$ is such that $f(P) \in P$ for almost all $P \in P_\kappa(A)$, then f is constant on a set in U. A cardinal κ is called *supercompact* if for every set A such that $|A| \geq \kappa$, there exists a normal measure on $P_\kappa(A)$ (see [**EkMe**, Chapter II.2] or [**Je**, Chapter 6, 33. Compact cardinals] for further details on supercompact cardinals).

Lemma 2.9. *Suppose that it is consistent that a supercompact cardinal exists. Then it is consistent with either $2^{\aleph_0} = 2^{\aleph_1}$ or $2^{\aleph_0} < 2^{\aleph_1}$ that for any group G either $\operatorname{Ext}(G,\mathbb{Z})$ is finite or $r_0^e(G) \geq 2^{\aleph_0}$.*

Proof. See [**MeSh**, Theorem 11]. □

3. The free (p-)rank

In this section we introduce the *free (p-)rank* of a torsion-free group G (p a prime) which will induce upper bounds for the cardinal invariants of $\operatorname{Ext}(G,\mathbb{Z})$.

Definition 3.1. *For a prime $p \in \Pi$ let K_p be the class of all torsion-free groups G such that $G/p^\omega G$ is free. Moreover, let K_0 be the class of all free groups.*

Note, that for $G \in K_p$ (p a prime) we have $G = p^\omega G \oplus F$ for some free group F and hence $\operatorname{Ext}(G,\mathbb{Z})[p] = 0$ since $p^\omega G$ is p-divisible. Note that $p^\omega G$ is a pure subgroup of G. Thus $r_p^e(G) = 0$ for $G \in K_p$ and any prime p. Clearly, also $r_0^e(G) = 0$ for all $G \in K_0$.

Definition 3.2. *Let G be a torsion-free group. We call*

$$\mathrm{fr}-\mathrm{rk}_0(G) = \min\{\mathrm{rk}(H) : H \subseteq_* G \textit{ such that } G/H \in K_0\}$$

the free rank of G *and similarly we call*

$$\mathrm{fr}-\mathrm{rk}_p(G) = \min\{\mathrm{fr}-\mathrm{rk}(H) : H \subseteq_* G/p^\omega G \textit{ such that } (G/p^\omega G)/H \in K_p\}$$

the free p-rank of G *for any prime* $p \in \Pi$.

We have a first easy lemma.

LEMMA 3.3. *Let G be a torsion-free group and $p \in \Pi$ a prime. Then the following hold.*

(i) $r_p^e(G) = r_p^e(G/p^\omega G)$;
(ii) *If H is a pure subgroup of G, then* $r_p^e(H) \le r_p^e(G)$ *and* $r_0^e(H) \le r_0^e(G)$;
(iii) $\mathrm{fr}-\mathrm{rk}_0(G) \ge \mathrm{fr}-\mathrm{rk}_0(G/p^\omega G)$;
(iv) $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{fr}-\mathrm{rk}_p(G/p^\omega G)$;
(v) $\mathrm{fr}-\mathrm{rk}_p(G) \le \mathrm{fr}-\mathrm{rk}_0(G)$.

PROOF. We first show (i) and let p be a prime. Since $p^\omega G$ is pure in G we have that $p^\omega G$ is p-divisible, hence

$$0 \to p^\omega G \to G \to G/p^\omega G \to 0$$

induces the exact sequence

$$0 \to \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Hom}(p^\omega G, \mathbb{Z}/p\mathbb{Z}) = 0$$

the latter being trivial because $p^\omega G$ is p-divisible. Thus we have

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z})$$

and it follows easily that

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G,\mathbb{Z})\varphi^p \cong \mathrm{Hom}(G/p^\omega G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G/p^\omega G,\mathbb{Z})\varphi^p.$$

Therefore, $r_p^e(G) = r_p^e(G/p^\omega G)$.
In order to show (ii) we consider the exact sequence

$$0 \to H \to G \to G/H \to 0$$

which implies the exact sequence

$$\cdots \to \mathrm{Ext}(G/H,\mathbb{Z}) \xrightarrow{\alpha} \mathrm{Ext}(G,\mathbb{Z}) \to \mathrm{Ext}(H,\mathbb{Z}) \to 0.$$

Since G/H is torsion-free we conclude that $\mathrm{Ext}(G/H,\mathbb{Z})$ is divisible and hence $\mathrm{Im}(\alpha)$ is divisible. Thus $\mathrm{Ext}(G,\mathbb{Z}) = \mathrm{Ext}(H,\mathbb{Z}) \oplus \mathrm{Im}(\alpha)$ and therefore $r_0^e(H) \le r_0^e(G)$ and $r_p^e(H) \le r_p^e(G)$ for every prime p.
Claim (iii) is easily proved noting that, whenever $G = H \oplus F$ for some free group F, then $p^\omega G \subseteq H$ for every prime p, hence $G/p^\omega G = H/p^\omega G \oplus F$.
To show (iv) note that $p^\omega(G/p^\omega G) = \{0\}$ and hence $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{fr}-\mathrm{rk}_p(G/p^\omega G)$ easily follows.
Finally, (v) follows from (iii), (iv) and the definition of $\mathrm{fr}-\mathrm{rk}_p(G)$ and $\mathrm{fr}-\mathrm{rk}_0(G)$ since the class K_0 is contained in the class K_p for every prime p. $\square$

REMARK 3.4. *If G is a torsion-free group and p a prime, then Lemma 3.3 (i) and (iv) imply that, regarding the free p-rank of G, we may assume without loss of generality that G is p-reduced. This is also justified by the fact that*

$$\mathrm{fr}-\mathrm{rk}_p(G) = \{H \subseteq_* G \text{ such that } G/H \in K_p\}$$

is easily proven.

To simplify notations we let $\Pi_0 = \Pi \cup \{0\}$ in the sequel.

LEMMA 3.5. *Let G be a torsion-free group. Then the following hold.*

(i) $r_0^e(G) \leq 2^\lambda$ *where* $\lambda = \max\{\aleph_0, \mathrm{fr}-\mathrm{rk}_0(G)\}$*; In particular,* $r_0^e(G) \leq 2^{\mathrm{fr}-\mathrm{rk}_0(G)}$ *if* $\mathrm{fr}-\mathrm{rk}_0(G)$ *is infinite;*

(ii) $r_p^e(G) \leq p^{\mathrm{fr}-\mathrm{rk}_p(G)}$ *for all* $p \in \Pi$.

PROOF. In order to prove (i) choose a subgroup $H \subseteq G$ such that $\mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_0(G)$ and $G/H \in K_0$. Hence $G = H \oplus F$ for some free group F and so $\mathrm{Ext}(G,\mathbb{Z}) = \mathrm{Ext}(H,\mathbb{Z})$ which implies that $r_0^e(G) = r_0^e(H) \leq 2^\lambda$ where $\lambda = \max\{\aleph_0, \mathrm{rk}(H)\} = \max\{\aleph_0, \mathrm{fr}-\mathrm{rk}_0(G)\}$.
We now prove (ii). Let p be a prime, then $r_p^e(G) = r_p^e(G/p^\omega G)$ and $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{fr}-\mathrm{rk}_p(G/p^\omega H)$ by Lemma 3.3 (i) and (iv). Hence we may assume that $p^\omega G = \{0\}$ without loss of generality. Let $H \subseteq G$ be such that $\mathrm{fr}-\mathrm{rk}_0(H) = \mathrm{fr}-\mathrm{rk}_p(G)$ and $G/H \in K_p$. Then $G/H = D \oplus F$ for some free group F and some p-divisible group D. As in the proof of Lemma 3.3 (i) it follows that $r_p^e(G) = r_p^e(H)$. Now, we let $H = H' \oplus F'$ for some free group F' such that $\mathrm{rk}(H') = \mathrm{fr}-\mathrm{rk}_0(H) = \mathrm{fr}-\mathrm{rk}_p(G)$. Hence $\mathrm{Ext}(H,\mathbb{Z}) = \mathrm{Ext}(H',\mathbb{Z})$ and therefore $r_p^e(G) = r_p^e(H) = r_p^e(H')$. Consequently, $r_p^e(G) = r_p^e(H') \leq p^{\mathrm{rk}(H')} = p^{\mathrm{fr}-\mathrm{rk}_p(G)}$. □

Note, that for instance in $(V = L)$ for any torsion-free group G, $2^{\mathrm{fr}-\mathrm{rk}_0(G)}$ is the actual value of $r_0^e(G)$ by Lemma 2.5. The following lemma justifies that, as far as it concerns the free p-rank of a torsion-free group, one may also assume without loss of generality that $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{rk}(G)$ if $p \in \Pi_0$.

LEMMA 3.6. *Let G be a torsion-free group, $p \in \Pi_0$ and $H \subseteq_* G$ such that*

(i) $G/(H \oplus F) \in K_p$ *for some free group F;*

(ii) $\mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_p(G)$.

Then $\mathrm{fr}-\mathrm{rk}_p(H) = \mathrm{rk}(H)$ *and* $r_p^e(G) = r_p^e(H)$.

PROOF. Let G, H and p be given. If $p = 0$, then the claim is trivially true. Hence assume that $p \in \Pi$ and that $\mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_p(G)$. Then there is a free group F such that $H' = H \oplus F$ is a pure subgroup of G satisfying $G/H' \in K_p$. Thus $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_0(H')$. Without loss of generality we may assume that G/H' is p-divisible by splitting of the free part. By way of contradiction assume that $\mathrm{fr}-\mathrm{rk}_p(H) < \mathrm{rk}(H)$. Let $H_1 \subseteq_* H$ such that $H/H_1 \in K_p$ and

$\mathrm{fr}-\mathrm{rk}_0(H_1) = \mathrm{fr}-\mathrm{rk}_p(H) < \mathrm{rk}(H)$. Then there are a free group F_1 and a p-divisible group D such

$$H/H_1 = D \oplus F_1.$$

Choose a pure subgroup $H_1 \subseteq_* H_2 \subseteq_* H$ such that $H_2/H_1 \cong D$. Thus $H/H_2 \cong F_1$ and so $H \cong H_2 \oplus F_1$ and without loss of generality $H = H_2 \oplus F_1$. Consequently, $\mathrm{rk}(H_2) = \mathrm{rk}(H)$ since $\mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_p(G)$. Let $H_3 = H_1 \oplus F_1 \oplus F$. Then

$$\mathrm{fr}-\mathrm{rk}_0(H_3) = \mathrm{fr}-\mathrm{rk}_0(H_1) < \mathrm{rk}(H) = \mathrm{fr}-\mathrm{rk}_p(G).$$

Moreover,

$$G/H' \cong (G/H_3)\,/\,(H'/H_3)$$

is p-divisible. Since also $H'/H_3 \cong H_2/H_1$ is p-divisible and all groups under consideration are torsion-free we conclude that G/H_3 is p-divisible. Hence $\mathrm{fr}-\mathrm{rk}_p(G) \leq \mathrm{fr}-\mathrm{rk}_0(H_3) < \mathrm{fr}-\mathrm{rk}_p(G)$ - a contradiction. Finally, $r_p^e(G) = r_p^e(H)$ follows as in the proof of Lemma 3.5. □

We now show how to calculate explicitly $\mathrm{fr}-\mathrm{rk}_p(G)$ for torsion-free groups G of finite rank and $p \in \Pi$ (note that $\mathrm{fr}-\mathrm{rk}_0(G)$ can be easily calculated). Recall that a torsion-free group G of finite rank is *almost-free* if every subgroup H of G of smaller rank than the rank of G is free.

LEMMA 3.7. *Let G be a non-free torsion-free group of finite rank n and $p \in \Pi$. Then we can calculate* $\mathrm{fr}-\mathrm{rk}_p(G)$ *as follows.*

(i) *If G is almost-free, then let $H \subseteq \mathbb{Q}$ be the outer type of G. Then*
  (a) $\mathrm{fr}-\mathrm{rk}_p(G) = n$ *if H is not p-divisible;*
  (b) $\mathrm{fr}-\mathrm{rk}_p(G) = 0$ *if H is p-divisible.*

(ii) *If G is not almost-free, then choose a filtration $\{0\} = G_0 \subseteq_* G_1 \subseteq_* \cdots \subseteq_* G_m \subseteq_* G$ with G_{k+1}/G_k almost-free. Then*

$$\mathrm{fr}-\mathrm{rk}_p(G) = \sum_{k<m} \mathrm{fr}-\mathrm{rk}_p(G_{k+1}/G_k).$$

PROOF. Left to the reader. □

In order to prove our main Theorem 4.3 of Section 4 we need a further result on the class K_p for $p \in \Pi$.

LEMMA 3.8. *Let p be a prime and G a torsion-free group of infinite rank. Then the following hold.*

(i) *If G is of singular cardinality, then $G \in K_p$ if and only if every pure subgroup H of G of smaller cardinality than G satisfies $H \in K_p$;*

(ii) *$G \notin K_p$ if and only if $r_p^e(G) > 0$ whenever we add $|G|$ Cohen reals to the universe;*

(iii) *If* $\mathrm{rk}(G) \geq \aleph_0$*, then adding* $|G|$ *Cohen reals to the universe adds a new member to* $\mathrm{Ext}_p(G, \mathbb{Z})$ *preserving the old ones.*

PROOF. Let G and p be as stated. Part (i) is an easy application of the first author's Singular Compactness Theorem from [**Sh1**].
One implication of (ii) is trivial, hence assume that $G \notin K_p$. By Lemma 3.3 (ii) we may assume that G does not have any pure subgroup of smaller rank than G satisfying (ii). It is easily seen that the rank $\delta = \mathrm{rk}(G)$ of G must be uncountable. Thus $\delta > \aleph_0$ must be regular by (i). Let $G = \bigcup_{\alpha<\delta} G_\alpha$ be a filtration of G by pure subgroups G_α of G ($\alpha < \delta$). The claim now follows as in [**MeSh**] repeating [**MeSh**, Theorem 9 and Theorem 10] (compare also Lemma 2.8). The only difference is that in our situation the group G is not almost free, hence we require in [**MeSh**, Theorem 10] that for every α in the stationary set E there exists an element $a \in G_{\alpha+1} \backslash (G_\alpha + p^\omega G_{\alpha+1})$ which belongs to the p-adic closure of $G_\alpha + p^\omega G_{\alpha+1}$. This makes only a minor change in the proof of [**MeSh**, Theorem 10].
Finally, (iii) follows similar to (ii) from the proof of [**MeSh**, Theorem 11]. The proof is therefore left to the reader. □

Finally, we consider the *p-closure* of a pure subgroup H of some torsion-free abelian group G which shall be needed in the proof of Theorem 4.3.

DEFINITION 3.9. *Let* G *be torsion-free and* H *a pure subgroup of* G*. For every prime* $p \in \Pi$ *the set*

$$\mathrm{cl}_p(G, H) = \{x \in G : \text{ for all } n \in \mathbb{N} \text{ there is } y_n \in H \text{ such that } x - y_n \in p^n G\}$$

is called the p-closure of H*.*

We have a first easy lemma.

LEMMA 3.10. *Let* G *be torsion-free and* H *a pure subgroup of* G*. Then the following hold for all primes* $p \in \Pi$*.*

(i) $H \subseteq \mathrm{cl}_p(G, H)$*;*
(ii) $\mathrm{cl}_p(G, H)$ *is a pure subgroup of* G*;*
(iii) $\mathrm{cl}_p(G, H)/H$ *is p-divisible.*

PROOF. We fix a prime $p \in \Pi$. The first statement is trivial. In order to prove (ii) assume that $mx \in \mathrm{cl}_p(G, H)$ for some $m \in \mathbb{N}$ and $x \in G$. Then, for every $n \in \mathbb{N}$, there is $y_n \in H$ such that $mx - y_n \in p^n G$, say $y_n = p^n g_n$ for some $g_n \in G$. Without loss of generality we may assume that $(m, p) = 1$. Hence $1 = km + lp^n$ for some $k, l \in \mathbb{Z}$. Thus

$$x = kmx + lp^n x = kp^n g_n + ky_n + lp^n x$$

and hence $x - ky_n \in p^n G$ with $ky_n \in H$. Therefore $x \in \mathrm{cl}_p(G,H)$ and (ii) holds. Finally, (iii) follows easily from (ii). □

4. Supercompact cardinals and large p-ranks

In this section we shall assume that the existence of a supercompact cardinal is consistent with ZFC. We shall then determine the cardinal invariants $(r_0^e(G), r_p^e(G) : p \in \Pi)$ of $\mathrm{Ext}(G,\mathbb{Z})$ for every torsion-free abelian group in this model and show that they are as large as possible. We start with a theorem from [**MeSh**] (see also [**Da**]). Recall that for cardinals μ, γ and δ we can define a partially ordered set $Fn(\mu,\gamma,\delta)$ by putting

$$Fn(\mu,\gamma,\delta) = \{f : dom(f) \to \gamma : dom(f) \subseteq \mu, |dom(f)| < \delta\}.$$

The partrial order is given by $f \le g$ if and only if $g \subseteq f$ as functions.

Lemma 4.1. *Suppose κ is a supercompact cardinal, V is a model of ZFC which satisfies $2^{\aleph_0} = \aleph_1$ and $\mathcal{P} = Fn(\mu, 2, \aleph_1) \times Fn(\rho, 2, \aleph_0)$, where $\mu, \rho > \aleph_1$. Then $\mathcal{P}$ forces that every κ-free group is free.*

Proof. See [**MeSh**, Theorem 19]. □

As a corollary one obtains

Lemma 4.2. *If it is consistent with ZFC that a supercompact cardinal exists then both of the statements* every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} < 2^{\aleph_1}$ *and* every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} = 2^{\aleph_1}$ *are consistent with ZFC. Furthermore, if it is consistent that there is a supercompact cardinal then it is consistent that there is a cardinal $\kappa < 2^{\aleph_1}$ so that if κ Cohen reals are added to the universe then every κ-free group is free.*

Proof. See [**MeSh**, Corollary 20]. □

We are now ready to prove our main theorem of this section working in the model from Lemma 4.2. Assume that the existence of a supercompact cardinal is consistent with ZFC. Let V be any model in which there exists a supercompact cardinal κ such that the weak diamond principle $\diamondsuit^*_{\lambda^+}$ holds for all regular cardinals $\lambda \ge \kappa$. Now, we use Cohen forcing to add κ Cohen reals to V to obtain a new model $\mathcal{V}$. Thus, in $\mathcal{V}$ we still have $\diamondsuit^*_{\lambda^+}$ for all regular $\lambda \ge \kappa$ and also $\diamondsuit_\kappa$ holds. Moreover, we have $2^{\aleph_0} = \kappa$ and every κ-free group (of arbitrary cardinality) is free by [**MeSh**].

Theorem 4.3. *In any model $\mathcal{V}$ as described above, the following is true for every non-free torsion-free abelian group G and prime $p \in \Pi$.*

(i) $r_0^e(G) = 2^{\max\{\aleph_0, \mathrm{fr-rk}_0(G)\}}$;

(ii) *If* $\mathrm{fr}-\mathrm{rk}_p(G)$ *is finite, then* $r_p^e(G) = \mathrm{fr}-\mathrm{rk}_p(G)$*;*
(iii) *If* $\mathrm{fr}-\mathrm{rk}_p(G)$ *is infinite, then* $r_p^e(G) = 2^{\mathrm{fr}-\mathrm{rk}_p(G)}$.

We would like to remark first that the above theorem shows that $r_p^e(G)$ $(p \in \Pi_0)$ is as large as possible for every torsion-free abelian group G in the model described above. Moreover, Lemma 2.3 shows that every sequence of cardinals $(\nu_p : p \in \Pi_0)$ not excluded by Theorem 4.3 may be realized as the cardinal invariants of $\mathrm{Ext}(H, \mathbb{Z})$ for some torsion-free group H.

Proof. Let $p \in \Pi_0$ be fixed. By Lemma 3.3 we may assume that $p^\omega G = 0$ if $p \in \Pi$. Moreover, Lemma 3.6 shows that also $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{rk}(G)$ holds without loss of generality. We now prove the claim by induction on the rank $\lambda = \mathrm{rk}(G) = \mathrm{fr}-\mathrm{rk}_p(G)$.

Case A: λ is finite.
In this case we may assume without loss of generality that $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$ since G is of finite rank. If $p = 0$, then $r_0^e(G) = 2^{\aleph_0} = \kappa$ follows from Lemma 2.1 since G is not free. Thus assume that $p > 0$. Then $r_p^e(G)$ is the dimension of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ as a vectorspace over $\mathbb{Z}/p\mathbb{Z}$. Since $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ is the vectorspace dual of G/pG it follows that $r_p^e(G) = \mathrm{rk}(G) = \mathrm{fr}-\mathrm{rk}_p(G)$. Note that G is p-reduced by assumption.

Case B: $\lambda = \aleph_0$.
Since G is of countable rank it is well-known that there exists a decomposition $G = G' \oplus F$ of G where F is a free group and G' satisfies $\mathrm{Hom}(G', \mathbb{Z}) = \{0\}$. Moreover, by assumption $\mathrm{fr}-\mathrm{rk}_p(G) = \lambda = \aleph_0$, hence we obtain that $\mathrm{rk}(G') = \mathrm{fr}-\mathrm{rk}_p(G') = \aleph_0$. Therefore, $r_0^e(G) = r_0^e(G') = 2^{\aleph_0} = \kappa$ follows from Lemma 2.1. If $p > 0$ we conclude that $\mathrm{Hom}(G', \mathbb{Z}/p\mathbb{Z})$ has cardinality $2^{\aleph_0}$. Since $\mathrm{Hom}(G', \mathbb{Z}) = \{0\}$ it follows by Lemma 3.5 (ii) that

$$2^{\aleph_0} \leq r_p^e(G') = r_p^e(G) \leq 2^{\aleph_0}$$

and thus $r_p^e(G) = 2^{\aleph_0} = \kappa$ for $p \in \Pi_0$.

Case C: $\aleph_0 < \lambda < \kappa$.
Note that $\kappa = 2^{\aleph_0} = 2^\lambda$, hence $r_0^e(G) = 2^{\aleph_0} = \kappa$ follows from Lemma 2.5. In fact, by induction hypothesis every Whitehead group H of size less than λ has to be free (because $r_p^e(H) = \mathrm{fr}-\mathrm{rk}_0(H)$) which suffices for Lemma 2.5. Now, assume that $p \in \Pi$. By Lemma 3.5 (ii) we deduce

$$r_p^e(G) \leq 2^{\mathrm{fr}-\mathrm{rk}_p(G)} \leq 2^\lambda = 2^{\aleph_0}$$

and hence it remains to prove that $r_p^e(G) \geq 2^{\aleph_0}$. The proof is very similar to the proof of [**MeSh**, Theorem 8] (see also Lemma 2.8), hence we shall recall it only briefly. Let V be the ground model and P be the Cohen forcing, i.e. $P = P(\kappa \times \omega, 2, \omega) = \{h : \mathrm{Dom}(h) \to \{0,1\} : \mathrm{Dom}(h)$ *is a finite subset of* $\kappa \times \omega\}$. If $\mathbb{G}$

is a P-generic filter over V let $\tilde{h} = \bigcup_{g\in\mathbb{G}} g$. For notational reasons we may also write $\mathcal{V} = V[\mathbb{G}] = V[\tilde{h}]$ for the extension model determined by the generic filter $\mathbb{G}$. Let $A \subseteq [\kappa]^{\leq\lambda}$ such that G belongs to $V[\tilde{h}\restriction_{A\times\omega}]$. Without loss of generality we may assume that $\alpha \in A$ if and only if $\beta \in A$ whenever $\alpha + \lambda = \beta + \lambda$. We shall prove the claim by splitting the forcing. For each α such that $\lambda\alpha \in \kappa\backslash A$ let

$$\tilde{f}_\alpha \in V[A \times \omega \cup [\lambda\alpha, \lambda\alpha + \lambda) \times \omega]$$

be a member of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ computed by $\tilde{h}\restriction_{[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}$. Note that $\tilde{f}_\alpha$ exists by Proposition 3.8 (iii). Then $\tilde{f}_\alpha$ is also computed from $\tilde{h}\restriction_{[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}$ over $V[\tilde{h}\restriction_{\kappa\backslash[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}]$ and hence $\tilde{f}_\alpha$ is not equivalent to any $f' \in V[\tilde{h}\restriction_{\kappa\backslash[\lambda\alpha,\lambda\alpha+\lambda)\times\omega}]$ modulo $\mathrm{Hom}(G,\mathbb{Z})\varphi^p$. Thus the set of homomorphisms $\{\tilde{f}_\alpha : \lambda\alpha < \kappa, \lambda\alpha \notin A\}$ exemplifies that $r_p^e(G) \geq \kappa = 2^{\aleph_0}$ and hence

$$2^\lambda = 2^{\aleph_0} = \kappa \leq r_p^e(G) \leq 2^\lambda$$

which shows $r_p^e(G) = 2^\lambda = 2^{\mathrm{fr-rk}_p(G)}$.

<u>Case D:</u> $\lambda \geq \kappa$.
Let $p \in \Pi_0$. We distinguish two subcases. Note that for $p \in \Pi$ our assumption $\mathrm{fr}-\mathrm{rk}_p(G) = \mathrm{rk}(G)$ also implies that $\mathrm{fr}-\mathrm{rk}_0(G) = \mathrm{rk}(G)$ by Lemma 3.3 (v).

<u>Case D1:</u> $\lambda \geq \kappa$ is regular.
The case $p = 0$ follows as in [**EkMe**, Theorem XII 4.4]. Let $\langle G_\alpha : \alpha < \lambda\rangle$ be a filtration of G into pure subgroups G_α ($\alpha < \lambda$) so that if G/G_α is not λ-free, then $G_{\alpha+1}/G_\alpha$ is not free. Choose by [**EkHu**, Lemma 2.4] an associate free resolution of G, i.e. a free resolution

$$0 \to K \stackrel{\Phi}{\to} F \to G \to 0$$

of G such that $F = \bigoplus_{\alpha<\lambda} F_\alpha$ and $K = \bigoplus_{\alpha<\lambda} K_\alpha$ are free groups such that $|F_\alpha| < \lambda$ and $|K_\alpha| < \lambda$ for all $\alpha < \lambda$ and the induced sequences

$$0 \to \bigoplus_{\beta<\alpha} K_\beta \to \bigoplus_{\beta<\alpha} F_\beta \to G_\alpha \to 0$$

are exact for every $\alpha < \lambda$. Since $\mathrm{fr}-\mathrm{rk}_0(G) = \lambda$, the set $E = \{\alpha < \lambda : G_{\alpha+1}/G_\alpha$ is not free $\}$ is stationary. For any subset $E' \subseteq E$ let $K(E') = \bigoplus_{\alpha\in E'} K_\alpha$ and $G(E') = F/\Phi(K(E'))$. Then $\Gamma(G(E')) \geq \tilde{E}'$ where $\Gamma(G(E'))$ is the Γ-invariant of $G(E')$ (see [**EkMe**] for details on the Γ-invariant). Now, by assumption we have $\diamondsuit^*_\lambda$, hence we may decompose E into λ disjoint stationary sets E'_α, each of which is non-small, i.e. $\diamondsuit^*_\lambda(E'_\alpha)$ holds. Hence $G(E'_\alpha)$ is not free since $\tilde{E'_\alpha} \leq \Gamma(G(E'_\alpha))$ for every $\alpha < \lambda$. By [**MeSh**] we conclude that $G(E'_\alpha)$ is not κ-free and therefore has a non-free pure subgroup H_α of rank less than κ. By induction hypothesis it follows that $\mathrm{Ext}(H_\alpha, \mathbb{Z}) \neq 0$ and hence also $\mathrm{Ext}(G(E'_\alpha), \mathbb{Z}) \neq 0$. As in [**EkHu**, Lemma 1.1] (see also [**EkMe**, Lemma XII 4.2]) there is an epimorphism

$$\mathrm{Ext}(G, \mathbb{Z}) \to \prod_{\alpha<\lambda} \mathrm{Ext}(G(E'_\alpha), \mathbb{Z}) \to 0$$

and it easily follows that $r_0^e(G) \geq 2^\lambda$ and hence $r_0^e(G) = 2^\lambda$ (compare [**EkMe**, Lemma 4.3]).

Now assume that $p > 0$. Again, let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration of G into pure subgroups G_α ($\alpha < \lambda$) such that $\mathrm{Ext}_p(G_{\alpha+1}/G_\alpha, \mathbb{Z}) \neq 0$ if and only if $\mathrm{Ext}_p(G_\beta/G_\alpha, \mathbb{Z}) \neq 0$ for some $\beta > \alpha$. Fix $\alpha < \lambda$. We claim that $G/\operatorname{cl}_p(G, G_\alpha)$ is not free. By way of contradiction assume that $G/\operatorname{cl}_p(G, G_\alpha)$ is free. Hence $G = \operatorname{cl}_p(G, G_\alpha) \oplus F$ for some free group F. Therefore, $G/G_\alpha = (\operatorname{cl}_p(G, G_\alpha) \oplus F)/G_\alpha = \operatorname{cl}_p(G, G_\alpha)/G_\alpha \oplus F$ is a direct sum of a p-divisible group and a free group by Lemma 3.10 (iii). It follows that $\mathrm{fr}-\mathrm{rk}_p(G) \leq \mathrm{rk}(G_\alpha) < \lambda$ contradicting the fact that $\mathrm{fr}-\mathrm{rk}_p(G) = \lambda$. By [**MeSh**] we conclude that $G/\operatorname{cl}_p(G, G_\alpha)$ is not κ-free since we are working in the model $\mathcal{V}$. Let $G'/\operatorname{cl}_p(G, G_\alpha) \subseteq_* G/\operatorname{cl}_p(G, G_\alpha)$ be a non-free pure subgroup of $G/\operatorname{cl}_p(G, G_\alpha)$ of size less than κ. Then there exists $\alpha \leq \beta < \lambda$ such that $G' \subseteq_* G_\beta$. By purity it follows that $\operatorname{cl}_p(G, G_\alpha) \cap G_\beta = \operatorname{cl}_p(G_\beta, G_\alpha)$. Hence

$$G_\beta/\operatorname{cl}_p(G_\beta, G_\alpha) = (G_\beta + \operatorname{cl}_p(G, G_\alpha))/\operatorname{cl}_p(G, G_\alpha)$$

is torsion-free but not free. Without loss of generality we may assume that $\beta = \alpha+1$. Hence we may assume that for all $\alpha < \lambda$ the quotient $G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)$ is a torsion-free non-free group. Note that $G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)$ is also p-reduced since $\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)$ is the p-closure of G_α inside $G_{\alpha+1}$.
Since the cardinality of $G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)$ is less than λ the induction hypothesis applies. Hence $r_p^e(G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)) = 2^{\mathrm{fr}-\mathrm{rk}_p(G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha))}$. We claim that $\mathrm{Ext}_p(G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha), \mathbb{Z}) \neq \{0\}$ stationarily often. If not, then there is a cub $C \subseteq \lambda$ such that for all $\alpha < \beta \in C$ we have $\mathrm{Ext}_p(G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha), \mathbb{Z}) = \{0\}$ and equivalently $\mathrm{Ext}_p(G_\beta/G_\alpha, \mathbb{Z}) = 0$, hence $\mathrm{fr}-\mathrm{rk}_p(G_\beta) = \mathrm{fr}-\mathrm{rk}_p(G_\alpha)$ by induction hypothesis. As in [**EkMe**, Proposition XII 1.5] it follows that $\mathrm{Ext}_p(G/G_\alpha, \mathbb{Z}) = 0$ for all $\alpha \in C$. This easily contradicts the fact that $\mathrm{fr}-\mathrm{rk}_p(G) = \lambda$. It follows that without loss of generality for every $\alpha < \lambda$ there exist homomorphisms $h_\alpha^0, h_\alpha^1 \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$ such that

(1) $h_\alpha^0 \restriction_{\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)} = h_\alpha^1 \restriction_{\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)}$;
(2) There are no homomorphisms $g_\alpha^0, g_\alpha^1 \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z})$ such that
  (a) $g_\alpha^0 \restriction_{\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)} = g_\alpha^1 \restriction_{\operatorname{cl}_p(G_{\alpha+1}, G_\alpha)}$ (or equivalently $g_\alpha^0 \restriction_{G_\alpha} = g_\alpha^1 \restriction_{G_\alpha}$);
  (b) $h_\alpha^0 = g_\alpha^0 \varphi^p$ and $h_\alpha^1 = g_\alpha^1 \varphi^p$.

To see this note that there is a homomorphism $\varphi : G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha) \to \mathbb{Z}/p\mathbb{Z}$ which can not be factored by φ_p since $\mathrm{Ext}_p(G_{\alpha+1}/\operatorname{cl}_p(G_{\alpha+1}, G_\alpha), \mathbb{Z}) \neq \{0\}$. Let

$h^0_\alpha = 0$ and h^1_α be given by

$$h^1_\alpha : G_{\alpha+1} \to G_{\alpha+1}/\mathrm{cl}_p(G_{\alpha+1}, G_\alpha) \overset{\varphi}{\to} \mathbb{Z}/p\mathbb{Z}.$$

Then it is easy to check that h^0_α and h^1_α are as required. In particular we may assume that $h^0_\alpha = 0$ for every $\alpha < \lambda$. An immediate consequence is the following property (U).

(U) Let $f : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ and $g : G_\alpha \to \mathbb{Z}$ such that $g\varphi_p = f$. Then there exists $\tilde{f} : G_{\alpha+1} \to \mathbb{Z}/p\mathbb{Z}$ such that $\tilde{f} \restriction_{G_\alpha} = f$ and there is no homomorphism $\tilde{g} : G_{\alpha+1} \to \mathbb{Z}$ satisfying both $\tilde{g} \restriction_{G_\alpha} = g$ and $\tilde{g}\varphi_p = \tilde{f}$.

To see this, let $\hat{f} : G_{\alpha+1} \to \mathbb{Z}/p\mathbb{Z}$ be any extension of f which exists by the pure injectivity of $\mathbb{Z}/p\mathbb{Z}$. If $\hat{f}$ is as required let $\tilde{f} = \hat{f}$. Otherwise let $\hat{g} : G_{\alpha+1} \to \mathbb{Z}$ be such that $\hat{g} \restriction_{G_\alpha} = g$ and $\hat{g}\varphi_p = \hat{f}$. Put $\tilde{f} = \hat{f} - h^1_\alpha$. Then $\tilde{f} \restriction_{G_\alpha} = f$. Assume that there exists $\tilde{g} : G_{\alpha+1} \to \mathbb{Z}$ such that $\tilde{g} \restriction_{G_\alpha} = g$ and $\tilde{g}\varphi_p = \tilde{f}$. Choosing $g^1_\alpha = \tilde{g} - \hat{g}$ we conclude $-g^1_\alpha\varphi_p = h^1_\alpha$ contradicting (2). Note that $h^0_\alpha = 0$.
We now proceed exactly as in [**HiHuSh**, Proposition 1] to show that $r^e_p(G) = 2^{\mathrm{fr-rk}_p(G)} = 2^\lambda$. We therefore recall the proof only briefly and for simplicity we even shall assume that $\diamondsuit_\lambda$ holds. It is an easy exercise (and therefore left to the reader) to prove the result assuming the weak diamond principle only. Assume that $r^e_p(G) = \sigma < 2^\lambda$ and let $L = \{f^\alpha : \alpha < \sigma\}$ be a complete list of representatives of elements in $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G,\mathbb{Z})\varphi_p$. Without loss of generality let $\{g_\alpha : G_\alpha \to \mathbb{Z} : \alpha < \lambda\}$ be the Jensen functions given by $\diamondsuit_\lambda$, hence for every homomorphism $g : G \to \mathbb{Z}$ there exists α such that $g \restriction_{G_\alpha} = g_\alpha$. We now define a sequence of homomorphisms $\{f^*_\alpha : G_\alpha \to \mathbb{Z}/p\mathbb{Z} : \alpha < \lambda\}$ such that the following hold.

(1) $f^*_0 = f^0$;
(2) $f^*_\alpha \restriction_{G_\beta} = f^*_\beta$ for all $\beta < \alpha$;
(3) If $f^* = \bigcup_{\alpha<\lambda} f^*_\alpha$, then $f^* - f^\alpha$ is an element of $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ but not of $\mathrm{Hom}(G,\mathbb{Z})\varphi_p$.

Suppose that f^*_β has been defined for all $\beta < \alpha$. If α is a limit ordinal, then we let $f^*_\alpha = \bigcup_{\beta<\alpha} f^*_\beta$ which is a well-defined homomorphism by (2). If $\alpha = \beta + 1$ is a successor ordinal, then we distinguish two cases. If $f^*_\beta - f^\beta \restriction_{G_\beta} \neq g_\beta\varphi_p$, let $f^*_\alpha : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ be any extension of f^*_β which exists since $\mathbb{Z}/p\mathbb{Z}$ is pure injective and $G_\beta \subseteq_* G_\alpha$. If $f^*_\beta - f^\beta \restriction_{G_\beta} = g_\beta\varphi_p$, then (U) shows that there is a homomorphism $\tilde{f} : G_\alpha \to \mathbb{Z}/p\mathbb{Z}$ extending $f^*_\beta - f^\beta \restriction_{G_\beta}$ such that there is no $\tilde{g} : G_{\beta+1} \to \mathbb{Z}$ with both extending g_β and $\tilde{g}\varphi_p = \tilde{f}$. Finally, put $f^*_\alpha = \tilde{f} + f^\alpha \restriction_{G_\alpha}$ and $f^* = \bigcup\limits_{\alpha<\lambda} f^*_\alpha$.
It is now straightforward to see that f^* satisfies (3) and hence f^* contradicts the maximality of the list L.

Case D2: $\lambda \geq \kappa$ is singular.
First note that $\mathrm{fr}-\mathrm{rk}_p(G) > \kappa$ since $\kappa = 2^{\aleph_0}$ is regular. By induction on $\alpha < \lambda$ we choose subgroups K_α of G such that the following hold.

(1) K_α is a pure non-free subgroup of G;
(2) $|K_\alpha| < \kappa$;
(3) $K_\alpha \cap \sum_{\beta<\alpha} K_\beta = \{0\}$;
(4) $\sum_{\beta<\alpha} K_\beta$ is a pure subgroup of G.

Assume that we have succeeded in constructing the groups K_α $(\alpha < \lambda)$. Then

$$K = \sum_{\beta<\lambda} K_\beta = \bigoplus_{\beta<\lambda} K_\beta$$

is a pure subgroup of G and hence $r_p^e(G) \geq r_p^e(K)$ by Lemma 3.3 (ii). If $\mathrm{Ext}_p(K_\alpha, \mathbb{Z}) = 0$, then $K_\alpha \in K_p$ follows by induction. Since G is p-reduced we obtain $K_\alpha \in K_0$ contradicting (1). Thus, $\mathrm{Ext}_p(K_\alpha, \mathbb{Z}) \neq \{0\}$ for every $\alpha < \lambda$ which implies that $r_p^e(K) \geq 2^\lambda$ since $\mathrm{Ext}(K, \mathbb{Z}) \cong \prod_{\alpha<\lambda} \mathrm{Ext}(K_\alpha, \mathbb{Z})$. It therefore suffices to complete the construction of the groups K_α $(\alpha < \lambda)$. Assume that K_β for $\beta < \alpha$ has been constructed. Let $\mu = (\kappa + |\alpha|)^{<\kappa}$ which is a cardinal less than λ. Let H_α be such that

(i) $H_\alpha \subseteq_* G$;
(ii) $\sum_{\beta<\alpha} K_\beta \subseteq H_\alpha$;
(iii) $|H_\alpha| = \mu$;
(iv) If $K \subseteq G$ is of cardinality less than κ, then there is a subgroup $K' \subseteq_* G$ such that $H_\alpha \cap K \subseteq K'$ and K and K' are isomorphic over $K \cap H_\alpha$, i.e. there exists an isomorphism $\psi : K \to K'$ which is the identity if restricted to $K \cap H_\alpha$.

It is easy to see that H_α exists. Now, G/H_α is a non-free group since $\mathrm{fr}-\mathrm{rk}_0(G) = \lambda$ and $p^\omega(G/H_\alpha) = \{0\}$. Hence [**MeSh**] implies that there is $K'_\alpha \subseteq G$ such that $(K'_\alpha + H_\alpha)/H_\alpha$ is not free and $|K'_\alpha| < \kappa$. Let $K^0_\alpha \subseteq_* H_\alpha$ be as in (5), i.e. $K'_\alpha \cap H_\alpha \subseteq K^0_\alpha$ and there is an isomorphism $\psi_\alpha : K'_\alpha \to K^0_\alpha$ which is the identity on $K'_\alpha \cap H_\alpha$. Let $K_\alpha = \{x - x\psi_\alpha : x \in K'_\alpha\}$. Then K_α is as required. For instance

$$K_\alpha \cong K^0_\alpha / (K'_\alpha \cap H_\alpha) \cong K'_\alpha / (K'_\alpha \cap H_\alpha)$$

shows that K_α is not free. □

Corollary 4.4. *In the model $\mathcal{V}$ let $\langle \mu_p : p \in \Pi_0 \rangle$ be a sequence of cardinals. Then there exists a torsion-free non-free abelian group G such that $r_p^e(G) = \mu_p$ for all $p \in \Pi_0$ if and only if*

(i) $\mu_0 = 2^{\lambda_0}$ *for some infinite cardinal* λ_0;
(ii) $\mu_p \leq \mu_0$ *for all* $p \in \Pi$;

(iii) *μ_p is either finite or of the form 2^{λ_p} for some infinite cardinal λ_p.*

Proof. The proof follows easily from Lemma 2.3 and Theorem 4.3. □

Corollary 4.5. *In the model $\mathcal{V}$ let $\langle \mu_p : p \in \Pi_0\rangle$ be a sequence of cardinals. Then there exists a non-free $\aleph_1$-free abelian group G such that $r_p^e(G) = \mu_p$ for all $p \in \Pi_0$ if and only if $\mu_p \leq \mu_0$ and $\mu_p = 2^{\lambda_p}$ for some infinite cardinal λ_p for every $p \in \Pi_0$.*

Proof. By Theorem 4.3 we only have to prove the existence claim of the corollary. It suffices to construct $\aleph_1$-free groups G_p for $p \in \Pi_0$ such that $r_p^e(G) = r_0^e(G) = 2^{\aleph_0} = \kappa$ and $r_q^e(G) = 0$ for all $p \neq q \in \Pi$. Then $B = G_0^{(\lambda_0)} \oplus \bigoplus_{p\in\Pi} G_p^{(\lambda_p)}$ will be as required (see for instance the proof of [**HiHuSh**, Theorem 3(b)]). Fix $p \in \Pi_0$. From [**EkMe**, Theorem XII 4.10] or [**SaSh2**] it follows that there exists an $\aleph_1$-free non-free group G_p of size $2^{\aleph_0}$ such that $r_p^e(G_p) = 2^{\aleph_0} = \kappa$ if $p \in \Pi$. In [**EkMe**, Theorem XII 4.10] it is then assumed that $2^{\aleph_0} = \aleph_1$ to show that also $r_0^e(G_p) = \kappa$. However, since we work in the model $\mathcal{V}$ and G_p is not free it follows from Theorem 4.3 that $\mathrm{fr}-\mathrm{rk}_0(G_p) \geq \aleph_1$ and hence $r_p^0(G_p) = 2^{\mathrm{fr}-\mathrm{rk}_0(G_p)} = \kappa$. □

Recall that a reduced torsion-free group G is called *coseparable* if $\mathrm{Ext}(G,\mathbb{Z})$ is torsion-free. By [**MeSh**] it is consistent that all coseparable groups are free. However, by [**EkMe**, Theorem XII 4.10] there exist non-free coseparable groups assuming $2^{\aleph_0} = \aleph_1$. Note that the groups constructed in Lemma 2.3 are not reduced, hence do not provide examples of coseparable groups.

Corollary 4.6. *In the model $\mathcal{V}$ there exist non-free coseparable groups.*

Proof. Follows from Corollary 4.5 letting $\mu_0 = 2^{\aleph_0}$ and $\mu_p = 0$ for all $p \in \Pi$. □

5. A model close to ZFC

In this section we shall construct a coseparable group which is not free in a model of ZFC which is very close to ZFC. As mentioned in the previous Section 4 it is undecidable in ZFC if all coseparable groups are free.
Let $\aleph_0 < \lambda$ be a regular cardinal and S a stationary subset of λ consisting of limit ordinals of cofinality ω. We recall the definition of a ladder system on S (see for instance [**EkMe**, page 405]).

Definition 5.1. A *ladder system $\bar\eta$ on S* is a family of functions $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ such that $\eta_\delta : \omega \to \delta$ is strictly increasing with $\sup(\mathrm{rg}(\eta_\delta)) = \delta$, where $\mathrm{rg}(\eta_\delta)$ denotes the range of η_δ. We call the ladder system *tree-like* if for all $\delta, \nu \in S$ and every $n, n \in \omega$, $\eta_\delta(n) = \eta_\nu(m)$ implies $n = m$ and $\eta_\delta(k) = \eta_\nu(k)$ for all $k \leq n$.

In order to construct almost-free groups one method is to use κ-free ladder systems.

Definition 5.2. *Let κ be an uncountable regular cardinal. The ladder system $\bar{\eta}$ is called* κ-free *if for every subset $X \subseteq S$ of cardinality less than κ there is a sequence of natural numbers $\langle n_\delta : \delta \in X\rangle$ such that*

$$\langle\{\eta_\delta \restriction_l : n_\delta < l < \omega\} : \delta \in X\rangle$$

is a sequence of pairwise disjoint sets.

Finally, recall that a stationary set $S \subseteq \lambda$ with λ uncountable regular is called *non-reflecting* if $S \cap \kappa$ is not stationary in κ for every $\kappa < \lambda$ with $\mathrm{cf}(\kappa) > \aleph_0$.

Theorem 5.3. *Let μ be an uncountable strong limit cardinal such that $\mathrm{cf}(\mu) = \omega$ and $2^\mu = \mu^+$. Put $\lambda = \mu^+$ and assume that there exists a λ-free tree-like ladder system on a non-reflecting stationary subset $S \subseteq \lambda$. If $\Pi = \Pi_0 \cup \Pi_1$ is a partition of Π into disjoint subsets Π_0 and Π_1, then there exists an almost-free group G of size λ such that*

(i) $r_0^e(G) = 2^\lambda$;
(ii) $r_p^e(G) = 2^\lambda$ *if* $p \in \Pi_0$;
(iii) $r_p^e(G) = 0$ *if* $p \in \Pi_1$.

Proof. Let $\bar{\eta} = \langle\eta_\delta : \delta \in S\rangle$ be the λ-free ladder system where S is a stationary non-reflecting subset of λ consisting of ordinals less than λ of cofinality ω. Without loss of generality we may assume that $S = \lambda$. Let $\mathrm{pr} : \mu^2 \to \mu$ be a pairing function, hence pr is bijective and if $\alpha \in \mu$ then we shall denote by $(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha))$ the unique pair $(\beta, \gamma) \in \mu^2$ such that $\mathrm{pr}(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha)) = \alpha$. Let L be the free abelian group

$$L = \bigoplus_{\alpha<\mu} \mathbb{Z}x_\alpha$$

generated by the independent elements x_α $(\alpha < \mu)$. For notational simplicity we may assume that $\Pi_1 \neq \emptyset$ and let $\langle(p_\beta, f_\beta) : \beta < \lambda\rangle$ be a listing of all pairs (p, f) with $p \in \Pi_1$ and $f \in \mathrm{Hom}(L, \mathbb{Z}/p\mathbb{Z})$. Recall that $\lambda = 2^\mu$. By induction on $\beta < \lambda$ we shall choose triples $(g_\beta, \nu_\beta, \rho_\beta)$ such that the following conditions hold.

(1) $g_\beta \in \mathrm{Hom}(L, \mathbb{Z})$;
(2) $f_\beta = g_\beta\varphi_p$ where $\varphi_p : \mathrm{Hom}(L, \mathbb{Z}) \to \mathrm{Hom}(L, \mathbb{Z}/p\mathbb{Z})$ is the canonical map;
(3) $\nu_\beta, \rho_\beta : \omega \to \mu$ such that $\eta_\beta(n) = \mathrm{pr}_1(\nu_\beta(n)) = \mathrm{pr}_1(\rho_\beta(n))$;
(4) For all $\delta \leq \beta$ there exists $n = n(\delta, \beta) \in \omega$ such that for all $m \geq n$ we have $g_\delta(x_{\nu_\beta(m)}) = g_\delta(x_{\rho_\beta(m)})$;
(5) For all $\delta < \beta$ there exists $n = n(\delta, \beta) \in \omega$ such that for some sequence $\langle b_m^{\delta,\beta} : m \in [n, \omega)\rangle$ of natural numbers we have $\left(\prod_{p\in\Pi_1\cap m} p\right) b_{m+1}^{\delta,\beta} = b_m^{\delta,\beta} + g_\beta(x_{\nu_\delta(m)}) - g_\beta(x_{\rho_\delta(m)})$ for all $m \geq n$;
(6) $\nu_\beta(m) \neq \rho_\beta(m)$ for all $m \in \omega$.

Fix $\beta < \lambda$ and assume that we have constructed $(g_\delta, \nu_\delta, \rho_\delta)$ for all $\delta < \beta$. Choose a function $h_\beta : \beta \to \omega$ such that $h_\beta(\delta) > p_\delta$ for all $\delta < \beta$ and

$$\langle \{\eta_\delta \restriction_l : l \in [h_\beta(\delta), \omega)\} : \delta < \beta \rangle \tag{5.1}$$

is a sequence of pairwise disjoint sets. Note that such a choice is possible since the ladder system $\bar{\eta}$ is λ-free by assumption. Moreover, by (3) the pairing function pr implies that also

$$\langle \{\nu_\delta \restriction_l, \rho \restriction_l : l \in [h_\beta(\delta), \omega)\} : \delta < \beta \rangle \tag{5.2}$$

is a sequence of pairwise disjoint sets. Now, we choose the function g_β such that (2) and (5) hold. For $\delta < \beta$ let $n = n(\delta, \beta) = h_\beta(\delta)$. Since L is free we may choose first $g_\beta(x_\alpha)$ satisfying $g_\beta(x_\alpha) + p_\beta\mathbb{Z} = f_\beta(x_\alpha)$ for every α such that $\mathrm{pr}_1(\alpha) \neq \eta_\delta(l)$ for all $\delta < \beta$ and $l \geq n(\delta, \beta)$, that is to say for those α such that x_α does not appear in (5). Secondly, for $\delta < \beta$, we choose by induction on $m \geq n(\delta, \beta)$ integers $b^{\delta,\beta}_{m+1}$ such that

$$0 + p_\beta\mathbb{Z} = b^{\delta,\beta}_{m+1} + f_\beta(x_{\nu_\delta(m)}) - f_\beta(x_{\rho_\delta(m)}) + p_\beta\mathbb{Z}$$

and then choose $g_\beta(x_{\nu_\delta(m)})$ and $g_\beta(x_{\rho_\delta(m)})$ such that (5) holds for δ. Note that this inductive process is possible by the choice of h_β and condition (5.1).

Finally, let $\beta = \bigcup_{n\in\omega} A_n$ be the union of an increasing chain of sets A_n such that $|A_n| < \mu$ (recall that we have assumed without loss of generality that $S = \lambda$, so β is of cofinality ω). By induction on $n < \omega$ we now may choose $\rho_\beta(n)$ and $\nu_\beta(n)$ as distinct ordinals such that

- $\rho_\beta(n), \nu_\beta(n) \in \mu$
- $\rho_\beta(n), \nu_\beta(n) \notin \{\nu_\beta(m), \rho_\beta(m) : m < n\}$
- $\mathrm{pr}_1(\rho_\beta(n)) = \mathrm{pr}_1(\nu_\beta(n)) = \eta_\beta(n)$;
- $\langle g_\delta(x_{\nu_\beta(n)}) : \delta \in A_n \rangle = \langle g_\delta(x_{\rho_\beta(n)}) : \delta \in A_n \rangle$.

Hence (3), (4) and (6) hold and we have carried on the induction. Now, let G be freely generated by L and $\{y_{\beta,n} : \beta < \lambda, n \in \omega\}$ subject to the following relations for $\beta < \lambda$ and $n \in \omega$.

$$\left(\prod_{p \in \Pi_1 \cap n} p \right) y_{\beta,n+1} = y_{\beta,n} + x_{\nu_\beta(n)} - x_{\rho_\beta(n)}.$$

Then G is a torsion-free abelian group of size λ. Moreover, since the ladder system $\bar{\eta}$ is λ-free and S is stationary but not reflecting it follows by standard calculations using (5.2) that G is almost-free but not free (see for instance [**EkSh2**]). It remains to prove that (i), (ii) and (iii) of the Theorem hold. For $\beta < \lambda$ let

$$G_\beta = \langle L, y_{\delta,n} : \delta < \beta, n \in \omega \rangle_* \subseteq_* G$$

so that $G = \bigcup_{\beta<\lambda} G_\beta$ is the union of the continuous increasing sequence of pure subgroups G_β $(\beta < \lambda)$. We start by proving (iii). Thus let $p \in \Pi_1$ and choose

$f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$. By assumption there is $\beta < \lambda$ such that $(p, f \restriction_L) = (p_\beta, f_\beta)$. Inductively we shall define an increasing sequence of homomorphisms $g_{\beta,\gamma} : G_\gamma \to \mathbb{Z}$ for $\gamma \geq \beta$ such that $g_{\beta,\gamma}\varphi_p = f \restriction_{G_\gamma}$. For $\gamma = \beta$ we choose $n(\delta, \beta)$ and $\langle b_m^{\delta,\beta} : m \in [n(\delta,\beta), \omega) \rangle$ as in (5) for $\delta < \beta$. We let $g_{\beta,\beta} \restriction_L = g_\beta$ where g_β is chosen as in (1). Moreover, put $g_{\beta,\beta}(y_{\delta,m}) = b_m^{\delta,\beta}$ for $m \in [n(\delta,\beta), \omega)$ and $\delta < \beta$. By downwards induction we chose $g_{\beta,\beta}(y_{\delta,m})$ for $m < n(\delta\beta)$, $\delta < \beta$. It is easily seen that $g_{\beta,\beta}$ is as required, i.e. satisfies $g_{\beta,\beta}\varphi_p = f \restriction_{G_\beta}$. Now, assume that $\gamma > \beta$. If γ is a limit ordinal, then let $g_{\beta,\gamma} = \bigcup\limits_{\beta \leq \epsilon < \gamma} g_{\beta,\epsilon}$. If $\gamma = \epsilon + 1$, then (4) implies that there is $n(\beta, \epsilon) < \omega$ such that $g_\beta(x_{\nu_\epsilon(m)}) = g_\beta(x_{\rho_\epsilon(m)})$ for all $m \in [n(\beta,\epsilon), \omega)$. Therefore, putting $g_{\beta,\gamma} \restriction_{G_\epsilon} = g_{\beta,\epsilon}$ and $g_{\beta,\gamma}(y_{\epsilon,m}) = 0$ for $m \in [n(\beta,\epsilon), \omega)$ and determing $g_{\beta,\gamma}(y_{\epsilon,m})$ by downward induction for $m < n(\beta,\epsilon)$ we obtain $g_{\beta,\gamma}$ as required. Finally, let $g = \bigcup\limits_{\gamma \geq \beta} g_{\beta,\gamma}$ which satisfies $g\varphi_p = f$. Since f was chosen arbitrary it follows that $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z})\varphi_p$ for all $p \in \Pi_1$ and hence $r_p^e(G) = 0$ for $p \in \Pi_1$.

We now turn to $p \in \Pi_0$. By definition of G it follows that every homomorphism $\psi : L \to \mathbb{Z}$ has at most one extension to a homomorphism $\psi' : G \to \mathbb{Z}$. Thus $|\mathrm{Hom}(G, \mathbb{Z})| \leq 2^\mu$. However, for every $\beta < \lambda$, any homomorphism $\psi : G_\beta \to \mathbb{Z}/p\mathbb{Z}$ has more than one extension to a homomorphism $\psi' : G_{\beta+1} \to \mathbb{Z}/p\mathbb{Z}$ and hence $|\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})| = 2^\lambda > 2^\mu$. Consequently, $r_p^e(G) = 2^\lambda$. Similarly, it follows that $r_0^e(G) = 2^\lambda$ which finishes the proof. □

Corollary 5.4. *Let μ be an uncountable strong limit cardinal such that* $\mathrm{cf}(\mu) = \omega$ *and $2^\mu = \mu^+$. Put $\lambda = \mu^+$ and assume that there exists a λ-free ladder system on a stationary subset $S \subseteq \lambda$. Then there exists an almost-free non-free coseparable group of size λ.*

Proof. Follows from Theorem 5.3 letting $\Pi_0 = \emptyset$ and $\Pi_1 = \Pi$. □

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