TEST GROUPS FOR WHITEHEAD GROUPS

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Abstract. We consider the question of when the dual of a Whitehead group is a test group for Whitehead groups. This turns out to be equivalent to the question of when the tensor product of two Whitehead groups is Whitehead. We investigate what happens in different models of set theory.

1. Introduction

All groups in this note are abelian. A *Whitehead group*, or *W-group* for short, is defined to be an abelian group A such that $\mathrm{Ext}(A, \mathbb{Z}) = 0$. We are looking for groups C other than $\mathbb{Z}$ such that a group A is a W-group if and only if $\mathrm{Ext}(A, C) = 0$; such a C will be called *a test group for Whitehead groups*, or a *W-test group* for short. Notice that if C is a non-zero separable torsion-free group, then $\mathrm{Ext}(A, C) = 0$ implies A is a W-group (since $\mathbb{Z}$ is a summand of C), but the converse may not hold, that is, $\mathrm{Ext}(A, C)$ may be non-zero for some W-group A.

Among the separable torsion-free groups are the dual groups, where by a dual group we mean one of the form $\mathrm{Hom}(B, \mathbb{Z})$ for some group B. We call $\mathrm{Hom}(B, \mathbb{Z})$ the ($\mathbb{Z}$-)*dual* of B and denote it by B^*. We shall call a group a *W^*-group* if it is the dual of a W-group. The principal question we will consider is whether every W^*-group is a W-test group. This turns out to be equivalent to a question about tensor products of W-groups. (See Corollary 2.4.)

As is almost always the case with problems related to Whitehead groups, the answer depends on the chosen model of set theory. We have an easy affirmative answer if every W-group is free (for example in a model of V = L); therefore we will focus on models where there are non-free W-groups. We will exhibit models with differing results about whether W^*-groups are W-test groups, including information about some of the "classical" models where there are non-free W-groups. In particular, we will show that the answer to the question is independent of ZFC + GCH.

2. Theorems of ZFC

We begin with some theorems of ZFC.

Theorem 2.1. *A group A is a W^*-group if and only if it is the kernel of an epimorphism $\mathbb{Z}^\kappa \to \mathbb{Z}^\lambda$ for some cardinals κ and λ.*

Date: October 5, 2020.

1991 *Mathematics Subject Classification.* Primary 20K20; Secondary 03E35, 20A15, 20K35, 20K40.

Key words and phrases. Whitehead group, dual group, tensor product.

The third author would like to thank the United States-Israel Binational Science Foundation for their support. Publication 879.

Proof. If $A = B^*$ where B is a W-group, choose a free resolution

$$0 \to K \to F \to B \to 0$$

where F and K are free groups. Taking the dual yields

$$0 \to B^* = A \to F^* \to K^* \to \mathrm{Ext}(B, \mathbb{Z}) = 0.$$

Since F^* and K^* are products, this proves the claim in one direction.

Conversely, if H is the kernel of an epimorphism $\phi : \mathbb{Z}^\kappa \to \mathbb{Z}^\lambda$, then we have an exact sequence

$$0 \to H \to F^* \to K^* \to 0$$

for certain free groups F and K. Dualizing, we obtain an exact sequence

$$0 \to K \to F \to B \to 0$$

for a subgroup B of H^*. When we take the dual of the last sequence we obtain the original epimorphism ϕ, and conclude that its kernel is B^* and $\mathrm{Ext}(B, \mathbb{Z}) = 0$. $\square$

Theorem 2.2. *For W-groups A and B, there are natural isomorphisms*

$$\mathrm{Ext}(A, B^*) \cong \mathrm{Ext}(B, A^*)$$

and

$$\mathrm{Ext}(A, B^*) \cong \mathrm{Ext}(A \otimes B, \mathbb{Z}).$$

Proof. Recall that, for any pair of groups A and B, there are natural isomorphisms

$$\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(A \otimes B, \mathbb{Z}) \cong \mathrm{Hom}(B \otimes A, \mathbb{Z}) \cong \mathrm{Hom}(B, A^*).$$

Now let A and B be W-groups. Using these isomorphisms as vertical maps and a free resolution $0 \to K \to F \to A \to 0$, we can form a commutative diagram with exact rows as follows:

$$\begin{array}{cccccc}
0 \to & \mathrm{Hom}(A, B^*) \to & \mathrm{Hom}(F, B^*) \to & \mathrm{Hom}(K, B^*) \to & \mathrm{Ext}(A, B^*) & \to 0 \\
& \downarrow & \downarrow & \downarrow & & \\
0 \to & \mathrm{Hom}(A \otimes B, \mathbb{Z}) \to & \mathrm{Hom}(F \otimes B, \mathbb{Z}) \to & \mathrm{Hom}(K \otimes B, \mathbb{Z}) \to & \mathrm{Ext}(A \otimes B, \mathbb{Z}) & \to 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to & \mathrm{Hom}(B \otimes A, \mathbb{Z}) \to & \mathrm{Hom}(B \otimes F, \mathbb{Z}) \to & \mathrm{Hom}(B \otimes K, \mathbb{Z}) \to & \mathrm{Ext}(B \otimes A, \mathbb{Z}) & \to 0 \\
& \downarrow & \downarrow & \downarrow & & \\
0 \to & \mathrm{Hom}(B, A^*) \to & \mathrm{Hom}(B, F^*) \to & \mathrm{Hom}(B, K^*) \to & \mathrm{Ext}(B, A^*) & \to 0
\end{array}$$

The 0's at the end of the four rows are justified by the freeness of F and by the fact that $F \otimes B$, $B \otimes F$, and B are W-groups, respectively. All the horizontal and vertical maps are natural isomorphisms. Consequently, the diagram can be completed—preserving commutativity—by natural maps between the two top Exts and the two bottom Exts. These maps provide the desired natural isomorphisms. $\square$

As an immediate corollary we obtain:

Corollary 2.3. *Let A and B be W-groups. The following are equivalent:*
(i) $A \otimes B$ *is a W-group;*
(ii) $\mathrm{Ext}(A, B^*) = 0$*;*
(iii) $\mathrm{Ext}(B, A^*) = 0$. $\square$

Hence we have

Corollary 2.4. *In any model of ZFC, the following are equivalent:*
(i) the tensor product of any two W-groups is a W-group;
(ii) every W^*-group is a W-test group.

The following Proposition demonstrates that the hypothesis that A and B are W-groups is necessary for the equivalent conditions of Corollary 2.3 to hold.

Proposition 2.5. *For any abelian groups A and B, if either*
(i) $A \otimes B$ is a non-zero W-group, or
(ii) $\mathrm{Ext}(A, B^*) = 0$ and B^* is non-zero,
then A is a W-group.

Proof. For (ii), the conclusion follows since B^* is separable. For (i), we use the fact ([2, p.116]) that there is a (non-natural) isomorphism

$$\mathrm{Ext}(A, \mathrm{Hom}(B, \mathbb{Z})) \oplus \mathrm{Hom}(A, \mathrm{Ext}(B, \mathbb{Z})) \cong \mathrm{Ext}(A \otimes B, \mathbb{Z}) \oplus \mathrm{Hom}(\mathrm{Tor}(A, B), \mathbb{Z})$$

In our case the right-hand side reduces to $\mathrm{Ext}(A \otimes B, \mathbb{Z})$, so the hypothesis implies that $\mathrm{Ext}(A, B^*) = 0$. It suffices then to show that $B^* \neq 0$, but this follows from the fact that $A \otimes B$ is a W-group, hence is separable, and is also assumed to be non-zero. □

3. An Independence Result

In this section we will show that there are: (1) a model of ZFC where the equivalent conditions of Corollary 2.4 fail; and (2) a model of ZFC where the conditions of Corollary 2.4 hold. In particular, we will exhibit

(1) a model of ZFC + GCH such that there is a W-group B of cardinality $\aleph_1$ such that B^* is not a test group for W–groups of cardinality $\aleph_1$, and

(2) another model of ZFC + GCH in which there are non-free W-groups and for every W group B (of arbitrary cardinality, B^* is a test group for W-groups (of arbitrary cardinality).

Model (1). For the first model, we use the model in [6, Theorem 0.5] in which there exists a non-reflexive W-group. In fact, the following theorem is proved there:

Theorem 3.1. *It is consistent with ZFC + GCH that there is a non-free W-group B of cardinality $\aleph_1$ such that B^* is free.*

Let B be as in the theorem. Note that B^* must be of infinite rank, since B is not free, but there is a monomorphism : $B \to B^{**}$. Since B is a non-reflexive W-group, a result of Huber (see [8] or [5, XI.2.7]) implies that B is not $\aleph_1$-coseparable, i.e., $\mathrm{Ext}(B, \mathbb{Z}^{(\omega)}) \neq 0$. Thus $\mathrm{Ext}(B, B^*) \neq 0$.

Model (2). We shall work in a model of $\mathrm{Ax}(S)$ + $\diamondsuit^*(\omega_1 \setminus S)$ plus

(‡) $\diamondsuit_\kappa(E)$ holds for every regular cardinal $\kappa > \aleph_1$ and every stationary subset E of κ.

For the definition and implications of $\mathrm{Ax}(S)$ + $\diamondsuit^*(\omega_1 \setminus S)$, see [5, pp. 178-179 and Thm. XII.2.1]. This is the "classical" model of ZFC + GCH in which there are non-free W-groups of cardinality $\aleph_1$; in fact, whether an $\aleph_1$-free group of cardinality $\aleph_1$ is a W-group is determined by its Γ-invariant.

Theorem 3.2. *Assuming $\mathrm{Ax}(S)$ $+\diamondsuit^*(\omega_1 \setminus S)$, if A and B are W-groups of cardinality $\leq \aleph_1$, then $A \otimes B$ is a W-group.*

Proof. The case when either A or B is countable (hence free) is trivial so we can assume that A and B have cardinality $\aleph_1$. By Theorem XII.2.1 of [5], $\Gamma(A) \leq \tilde{S}$ and $\Gamma(B) \leq \tilde{S}$

since A and B are W-groups. It suffices to prove that $\Gamma(A \otimes B) \leq \tilde{S}$. Fix ω_1-filtrations $\{A_\nu : \nu < \omega_1\}$ and $\{B_\nu : \nu < \omega_1\}$ of A and B, respectively, such that

$$\{\nu < \omega_1 : \exists \mu > \nu \text{ s.t. } A_\mu / A_\nu \text{ is not free}\} \subseteq S$$

and similarly for B. Then $\{A_\nu \otimes B_\nu : \nu < \omega_1\}$ is an ω_1-filtration of $A \otimes B$. For each $\mu > \nu$ there is an exact sequence

$$0 \to A_\nu \otimes B_\nu \to A_\mu \otimes B_\mu \to (A_\mu \otimes (B_\mu / B_\nu)) \oplus ((A_\mu / A_\nu) \otimes B_\mu)$$

(cf. [2, Prop. 4.3a(c), p. 25]). If $\nu \notin S$, then the two summands on the right are free, and hence the quotient $A_\mu \otimes B_\mu / A_\nu \otimes B_\nu$ is free. Thus we have proved that $\Gamma(A \otimes B) \leq \tilde{S}$. □

The following is proved by the methods of proof of Theorem 3.1 in [1] (see also Corollary XII.1.13 of [5]).

Lemma 3.3. *In any model of (‡), every W-group A is the union of a continuous chain of subgroups $\{A_\nu : \nu < \sigma\}$ such that $A_0 = 0$ and for all $\nu < \sigma$, A_ν is a W-group and $A_{\nu+1}/A_\nu$ has cardinality $\leq \aleph_1$ and is a W-group.* □

The hypothesis $\mathrm{Ax}(S) + \diamondsuit^*(\omega_1 \backslash S)$ implies that the second assumption of the following theorem holds. Thus, we will show that our model has the desired property if we prove the following theorem.

Theorem 3.4. *Assume that (‡) holds and that the tensor product of two W-groups of cardinality $\leq \aleph_1$ is again a W-group. Then the tensor product of any two W-groups (of arbitrary cardinality) is again a W-group.*

Proof. Let A and B be W-groups. It suffices to prove that $A \otimes B$ is a W-group when at least one of the groups, say B, has cardinality $> \aleph_1$. Let $\{B_\nu : \nu < \sigma\}$ be a continuous chain as in the conclusion of Lemma 3.3. We shall first prove that if $|A| = \aleph_1$, then $A \otimes B$ is a W-group. Now $\{A \otimes B_\nu : \nu < \sigma\}$ is a continuous filtration of $A \otimes B$. To show that $A \otimes B$ is a W-group, it suffices to show that for all $\nu < \sigma$, $A \otimes B_{\nu+1}/A \otimes B_\nu$ is a W-group (cf. [5, XII.1.5]). Now, as above, there is an exact sequence

$$0 \to A \otimes B_\nu \to A \otimes B_{\nu+1} \to A \otimes (B_{\nu+1}/B_\nu) \to 0$$

since A is torsion-free. By hypothesis, the right-hand term is a W-group (of cardinality $\aleph_1$); hence $A \otimes B_{\nu+1}/A \otimes B_\nu$ is a W-group.

Next suppose that $|A| \gtrdot \aleph_1$. Again we use the continuous filtration $\{A \otimes B_\nu : \nu < \sigma\}$ and the displayed exact sequence above. Now the right-hand term $A \otimes (B_{\nu+1}/B_\nu)$ is a W-group by the first case (and the symmetry of the tensor product) because it is a tensor product of two W-groups one of which has cardinality $\aleph_1$. □

Remark. As mentioned above, Martin Huber has shown that every $\aleph_1$-coseparable group is reflexive. In Model (2) above and also in the model of the next section, all W-groups are $\aleph_1$-coseparable. This raises the question of whether (provably in ZFC) every $\aleph_1$-coseparable group, or every reflexive group, B satisfies $\mathrm{Ext}(B, B^*) = 0$, or even satisfies: B^* is a W-test group.

4. Martin's Axiom

A model of Martin's Axiom (MA) was the first model in which it was proved (in [9]) that there are non-free W-groups. So it is of interest to see what happens in that model. In fact, the conclusion is the same as that of Theorem 3.2. But we prove it by proving the following theorem:

Theorem 4.1. *(MA + ¬CH) If A and B are W-groups of cardinality $\leq \aleph_1$, then* $\mathrm{Ext}(A, B^*) = 0$.

Proof. The proof does not make use directly of the definition of Whitehead groups, but rather of the identification of W-groups of cardinality $\aleph_1$ as Shelah groups, which holds under the set-theoretic hypotheses here (but not in all models of ZFC). See [10] or [5, XII.2.5, XIII.3.6]. We fix a short exact sequence

$$0 \to B^* \overset{\iota}{\hookrightarrow} N \overset{\pi}{\to} A \to 0$$

and proceed to prove that it splits. We may assume that ι is the inclusion map. We also fix a set function $\gamma : A \to N$ such that for all $a \in A$, $\pi(\gamma(a)) = a$. We will show that the short exact sequence splits by proving the existence of a function $h : A \to B^*$ such that the function

$$\gamma - h : A \to N : a \mapsto \gamma(a) - h(a)$$

is a homomorphism. The function h will be obtained via a directed subset $\mathcal{G}$ of a c.c.c. poset P; the directed subset (which will be required to intersect certain dense subsets of P) will exist as a consequence of MA.

We define P to consist of all triples $p = (A_p, B_p, h_p)$ where A_p (resp. B_p) is a finitely generated summand of A (resp. B) and h_p is a function from A_p to B_p^*. Moreover, we require that the function which takes $a \in A_p$ to $\gamma(a) - h_p(a)$ is a homomorphism from A_p into $N/\{f \in B^* : f \restriction B_p \equiv 0\}$. (Strictly speaking, this is an abuse of notation: by $\gamma(a) - h_p(a)$ we mean the coset of $\gamma(a) - \eta_a$ where η_a is any element of B^* such that $\eta_a \restriction B_p = h_p(a)$.)

The partial ordering on P is defined as follows: $p = (A_p, B_p, h_p) \leq p' = (A'_p, B'_p, h'_p)$ if and only if $A_p \subseteq A'_p$, $B_p \subseteq B'_p$, and $h'_p(a) \restriction B_p = h_p(a)$ for all $a \in A_p$. The dense subsets that we use are:

$$D^1_a = \{p \in P : a \in A_p\}$$

for all $a \in A$ and

$$D^2_b = \{p \in P : b \in B_p\}$$

for all $b \in B$. Assuming that these sets are dense and that P is c.c.c., the axiom MA + ¬CH yields a directed subset $\mathcal{G}$ which has non-empty intersection with each of these dense subsets. We can then define h by: $h(a)(b) = h_p(a)(b)$ for some (all) $p \in \mathcal{G}$ such that $a \in A_p$ and $b \in B_p$. It is easy to check that h is well-defined and has the desired properties.

For use in proving both density and the c.c.c. property, we state the following claim, whose proof we defer to the end.

> (1) *Given a basis $\{x_i : i = 1, ..., n\}$ of a finitely generated pure subgroup A_0 of A, a basis $\{y_j : j = 1, ..., m\}$ of a finitely generated pure subgroup B_0 of B, and an indexed set $\{e_{ij} : i = 1, ..., n, j = 1, ..., m\}$ of elements of $\mathbb{Z}$, there is one and only one $p \in P$ such that $A_p = A_0$, $B_p = B_0$ and $h_p(x_i)(y_j) = e_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$.*

Assuming this claim, we proceed to prove the density of D^1_a. Given p, there is a finitely generated pure subgroup A' of A which contains A_p and a. Now A_p is a summand of A' so we can choose a basis $\{x_i : i = 1, ..., n\}$ of A' which includes a basis $\{x_i : i = 1, ..., k\}$ of A_p; choose a basis $\{y_j : j = 1, ..., m\}$ of B_p. Then by the claim there is an element p' of P such that $A_{p'} = A'$, $B_{p'} = B_p$, and $h_{p'}(x_i)(y_j) = h_p(x_i)(y_j)$ for all $i = 1, ..., k$. Clearly p' extends p and belongs to D^1_a. The proof of the density of D^2_b is similar.

Next we prove that P is c.c.c. We will make use of the following fact, which is proved in [3], Lemma 7.5. (It is proved there for strongly $\aleph_1$-free groups, there called groups with Chase's condition, but the proof may be adapted for Shelah groups; cf. [4, Theorem 7.1].)

> (2) *If G is a Shelah group of cardinality $\aleph_1$ and $\{S_\alpha : \alpha \in \omega_1\}$ is a family of finitely generated pure subgroups of G, then there is an uncountable subset I of ω_1 and a pure free subgroup G' of G such that $S_\alpha \subseteq G'$ for all $\alpha \in I$.*

Suppose that $\{p_\nu = (A_\nu, B_\nu, h_\nu) : \nu \in \omega_1\}$ is a subset of P. We must prove that there are indices $\mu \neq \nu$ such that p_μ and p_ν are compatible. By claim (2), passing to a subset, we can assume that there is a pure free subgroup A' of A and a pure free subgroup B' of B such that $A_\nu \subseteq A'$ and $B_\nu \subseteq B'$ for all $\nu \in \omega_1$. (Now we follow the argument in [3] for property (7.1.3).) Choose a basis X of A' and a basis Y of B'. By density we can assume that each A_ν is generated by a finite subset X_ν of X and each B_ν is generated by a finite subset, Y_ν, of Y. Moreover we can assume that there is a (finite) subset T of X (resp. W of Y) which is contained in each X_ν (resp. each Y_ν) and is maximal with respect to the property that it is contained in uncountably many X_ν (resp. Y_ν). Passing to a subset, we can assume that $h_\nu(x)(y)$ has a value independent of ν for each $x \in T$ and $y \in W$. By a counting argument we can find $\nu > 0$ such that $X_\nu \cap X_0 = T$ and $Y_\nu \cap Y_0 = W$. We define a member $q \in P$ which extends p_0 and p_ν, as follows. Let $A_q = \langle X_0 \cup X_\nu \rangle$ and $B_q = \langle Y_0 \cup Y_\nu \rangle$. Clearly these are pure subgroups of A (resp. B). Then by claim (1) there is a homomorphism $h_q : A_q \to B^*_q$ such that $q \geq p_0$ and $q \geq p_\nu$. In particular, for $x \in T$

$$h_q(x)(y) = \begin{cases} \text{the common value} & \text{if } y \in W \\ h_0(x)(y) & \text{if } y \in Y_0 - W \\ h_\nu(x)(y) & \text{if } y \in Y_\nu - W \end{cases}$$

and for $x \in X_0 - T$

$$h_q(x)(y) = \begin{cases} h_0(x)(y) & \text{if } y \in Y_0 \\ \text{arbitrary} & \text{if } y \in Y_\nu - W \end{cases}$$

and similarly for $x \in X_\nu - T$.

Thus we have shown the existence of p_μ and p_ν which are compatible, and it remains to prove claim (1). We will prove uniqueness first. Suppose that, with the notation of (1), there are p_1 and p_2 in P such that for $\ell = 1, 2$, $h_{p_\ell}(x_i)(y_j) = e_{ij}$ for all $i = 1, ..., n$ $j = 1, ..., m$. It suffices to prove that for all $a \in A_0$,

$$h_{p_1}(a)(y_j) = h_{p_2}(a)(y_j)$$

for all $j = 1, ..., m$. Write a as a linear combination of the basis: $a = \sum_{i=1}^n d_i x_i$. By the definition of P

$$\gamma(a) - \sum_{i=1}^n d_i \gamma(x_i) = h_{p_\ell}(a) - \sum_{i=1}^n d_i h_{p_\ell}(x_i).$$

for $\ell = 1, 2$. Note that the right-hand side is, by definition, a function on B_0. Applying both sides to $y_j \in B_0$, we obtain that

$$h_{p_\ell}(a)(y_j) = \sum_{i=1}^{n} d_i e_{ij} + (\gamma(a) - \sum_{i=1}^{n} d_i \gamma(x_i))(y_j)$$

for $\ell = 1, 2$. Since the right-hand side is independent of ℓ, we can conclude the desired identity. For existence, we use the last displayed equation to define h_p and easily check that it gives an element of P. $\square$

By starting with a base model of MA + $2^{\aleph_0} = \aleph_2$ and doing the standard iterated forcing to force all instances of diamond above $\aleph_1$ one obtains

Model (3). A model of MA + $2^{\aleph_0} = \aleph_2$ plus

> (‡) $\diamondsuit_\kappa(E)$ holds for every regular cardinal $\kappa > \aleph_1$ and every stationary subset E of κ.

Just as for Model (2), in this model the tensor product of any two Whitehead groups of arbitrary cardinality is again a W-group.

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