A SACKS REAL OUT OF NOWHERE

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Abstract. There is a proper countable support iteration of length ω adding no new reals at finite stages and adding a Sacks real in the limit.

1. Introduction

Preservation theorems are a central tool in forcing theory:

> Let $(P_\alpha, \underset{\sim}{Q}_\alpha)_{\alpha<\epsilon}$ be a forcing iteration. Assume that Q_α is (forced to be) nice for all $\alpha < \epsilon$. Then P_ϵ is nice.[1]

A niceness (or: preservation) property usually implies that the forcing does not change the universe too much. Among the most important preservation theorems are:

The finite support iteration of ccc forcings is ccc. [8]

and

The countable support iteration of proper forcings is proper. [6]

In this paper we investigate proper countable support iterations. So the limits are always proper. Many additional preservation properties are preserved as well, for example ω^ω-bounding (i.e. not adding an unbounded real). This is a special instance of a general preservation theorem by the second author ("Case A" of [7, XVIII §3]) which is also known as "first preservation theorem" (see [1]) or "tools-preservation" (see [4] and [5]) in the literature. Many additional preservation theorems for proper countable support iterations can be found in [7], or, from the point of view of large cardinals, in [9].

We investigate iterations where all iterands are NNR, i.e. do not add new reals. So the iterands (and therefore the limit as well) satisfy all instances of tools-preservation. However, it turns out that the limit can add a new real r. The first example was given by Jensen [3], and the phenomenon was further investigated in [7, V]. So what do we know about the real r? We know that it has to be bounded by an old real (i.e. a real in the ground model), corresponding to the iterable preservation property "ω^ω-bounding". r will even satisfy the stronger Sacks property. In particular r cannot be e.g. a Cohen, random, Laver or Matthias real. The examples mentioned have very indirect proofs that a real r is added, and do not give much "positive" information about r. So it is natural to ask which kind of reals *can* appear in proper NNR limits. Todd Eisworth asked this question for the simplest and best understood real that satisfies the Sacks property, the Sacks real. In this paper, we show that Sacks reals indeed can appear in this way:

Date: 2007-03-11.

2000 *Mathematics Subject Classification.* 03E40.

† supported by a European Union Marie Curie EIF fellowship, contract MEIF-CT-2006-024483.

‡ supported by the United States-Israel Binational Science Foundation (Grant no. 2002323), publication 905.

[1] Often preservation theorems also have the (weaker) form "If ϵ is a limit, and all P_α are nice for $\alpha < \epsilon$, then P_ϵ is nice."

Theorem 1. *There is a proper countable support iteration $(P_n, \underset{\sim}{Q}_n)_{n<\omega}$ such that no P_n adds a new real, and P_ω adds a Sacks real. Moreover, P_ω is equivalent to $S * \underset{\sim}{P}'$, where S is Sacks forcing and P' is NNR.*[2]

This can be interpreted in two ways:

On the one hand, it indicates limitations of possible preservation theorems: "Not adding a Sacks real" is obviously not iterable (even with rather strong additional assumptions).

On the other hand, it shows that Sacks forcing is exceptionally "harmless": It satisfies every usual iterable preservation property.[3] So the Sacks model (the model one gets starting with CH and iterating ω_2 Sacks forcings in a countable support iteration) has all the corresponding properties as well.[4]

In a continuation of this work we will say more about the kind of reals that can be added in limits of NNR iterations (e.g. generics for other finite splitting lim sup tree forcings). It turns out, that many of these reals can appear at limit stages, but some of them not at stage ω, but only at later stages, e.g. ω^2.

2. A simple case

In this section, we construct a proper, NNR countable support iteration and argue that the limit adds a real that it is in some way similar to a Sacks real.

In the rest of the paper, we deal with an analog (more general and more complicated) construction that adds a Sacks real. We do not give any proofs in this section, but refer to the proofs of the more general statements. We use the same symbols for the simpler objects in this section and for the analog constructions in the rest of the paper.

So the purpose of this section is to give some flavor of the constructions we use to prove Theorem 1, using a somewhat simplified notation.[5] The reader who does not feel the need of such an introduction can safely continue with the next section.

The forcing iteration will start with a preparatory forcing $\tilde{P}$, followed by $\underset{\sim}{Q}_0, \underset{\sim}{Q}_1, \dots$. $\tilde{P} * P_n$ stands for $\tilde{P} * \underset{\sim}{Q}_0 * \dots * \underset{\sim}{Q}_{n-1}$. δ always denotes a countable limit ordinal.

The preparatory forcing adds disjoint cofinal subsets $\nu_{\delta,n,m} \subseteq \delta$ of of order type ω for every limit ordinal $\delta < \omega_1$. In more detail:

Definition 2.1. A condition $\tilde{p}$ in $\tilde{P}$ consists of a limit ordinal $\mathrm{ht}(\tilde{p}) \in \omega_1$ and a sequence $(\nu_{\delta,n,m})_{\delta<\mathrm{ht}(\tilde{p}),n,m\in\omega}$, such that $\nu_{\delta,n,m} \subseteq \delta$ is cofinal and has order type ω, and ν_{δ,n,m_1} and ν_{δ,n,m_2} are disjoint for $m_1 \neq m_2$. $\tilde{P}$ is ordered by extension.

So $\tilde{P}$ is σ-closed.

Definition 2.2. $Q_0 = 2^{<\omega_1}$ ordered by extension.

Q_0 is also σ closed, and adds the generic sequence $\eta_0 \in 2^{\omega_1}$.

Given $\tilde{P} * P_n = \tilde{P} * Q_0 * \dots * Q_{n-1}$ such that Q_{n-1} adds the generic sequence $\eta_{n-1} \in 2^{\omega_1}$, we define the $\tilde{P} * P_n$-name Q_n:

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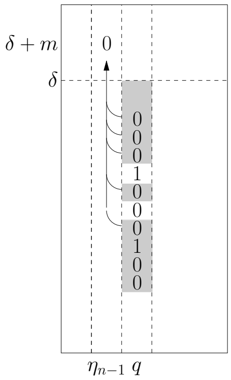
[2]We do not claim that P' is proper.

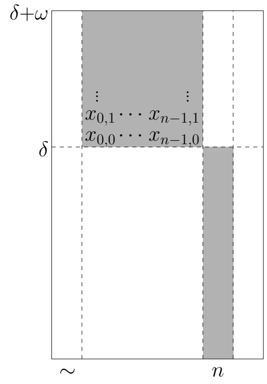
[3]More exactly: Sacks forcing satisfies every property X such that: Every proper NNR forcing satisfies X, X is preserved under proper countable support iterations, and if P does *not* satisfy X, then $P * \underset{\sim}{Q}$ does not satisfy X either for any NNR Q.

[4]Of course this already known for many of the popular preservation properties, cf. [2] or [9], which shows that in some respect Sacks forcing is the "most tame" forcing possible. This corresponds to the fact that all of the usual cardinal characteristics (apart from the continuum) are $\aleph_1$ in the Sacks model.

[5]Of course this section also shows that not only Sacks reals can appear at limits of NNR iterations, but other simply definable reals as well.



(a)



(b)

FIGURE 1. (a) η_{n-1} and q cohere at $\delta+m$. $\nu_{\delta,n-1,m}$ is indicated by the gray area. (b) An element of R: $r_{n,\delta+m}=x_{n,m}$. If $\alpha<\delta$, then $r_{n,\alpha}$ is a term using only variables $x_{i,j}$ with $i<n$.

Definition 2.3. Let q be a partial function from ω_1 to 2, $\delta\subseteq\mathrm{dom}(q)$. q and η_{n-1} cohere at $\delta+m$, if $\eta_{n-1}(\delta+m)=q(\alpha)$ for all but finitely many $\alpha\in\nu_{\delta,n-1,m}$. $q\in Q_n$, if $q\in 2^{<\omega_1}$ and q coheres with η_{n-1} at all $\delta+m$ for $\delta\leq\mathrm{dom}(q), m\in\omega$.

Cf. Figure 1(a).

Lemma 2.4. *The following is forced by P_n:*

- *If $q\in Q_n$ and $q'\in 2^{\mathrm{dom}(q)}$, $q'(\alpha)=q(\alpha)$ for all but finitely many $\alpha\in\mathrm{dom}(q)$, then $q'\in Q_n$.*
- *Each Q_n is separative, proper, and NNR, i.e. adds no new reals.*
- *Q_n adds a generic sequence $\eta_n\in 2^{\omega_1}$ defined by $\bigcup_{q\in G(n)}q$.*
- *If $n>0$, then Q_n is not σ-closed.*
- *$\tilde{P}*P_\omega$ adds a new real. In particular, $(\eta_n(m))_{n,m\in\omega}$ determines the generic filter.*

For a proof, see Lemmata 3.7, 3.8 and 3.9.

More generally, for any $f:\omega\to\omega$ and $\delta\in\omega$, no condition of $\tilde{P}*P_\omega$ can determine $\eta_n(\alpha)$ for all $n\geq n_0$ and $\delta+f(n)<\alpha<\delta+\omega$, cf. Figure 2(c).

However, for all $n_0\in\omega$ and $f:\omega\to\omega$, there are $p\in\tilde{P}*P_\omega$ that determine all $\eta_n(m)$ for $n<n_0$ or $m<f(n)$, cf. Figure 2(b). But these conditions are *not* dense. (See page 7 for an explanation.)

We now define a dense subforcing R of $\tilde{P}*P_\omega$.

Definition 2.5. A variable is one of the symbols $x_{i,j}$ for $i,j\in\omega$. (We use it as variable for the value of some $\eta_{i,\alpha}$, i.e. 0 or 1.) A term has the form $f(v_0,\dots,v_{l-1})$, where $l\geq 0$, each v_i is a variable and $f:2^l\to 2$ a function. An assignment a maps every variable (and therefore every term) to 0 or 1. A substitution ϕ maps every variable (and therefore every term) to a term. We write $t\circ a$ (or $t\circ\phi$) for the result of applying the assignment a (or substitution ϕ) to the term or variable t.

Definition 2.6. $R=\bigcup_{\delta<\omega_1}R_{\delta+\omega}$. $R_{\delta+\omega}$ consists of $(\tilde{p},\bar{r})$ such that

- $\tilde{p}\in\tilde{P}$, $\mathrm{ht}(\tilde{p})=\delta+\omega$.
- $\bar{r}=(r_{n,\alpha})_{n\in\omega,\alpha\in\delta+\omega}$.
- $r_{n,\delta+m}$ is the term $x_{n,m}$.
- For $\alpha<\delta$, $r_{n,\alpha}$ is a term using only variables $x_{i,j}$ with $i<n$.
- For $\alpha\leq\delta$ limit and $n,m\in\omega$, $r_{n,\alpha+m}=r_{n+1,\zeta}$ for all but finitely many $\zeta\in\nu_{\alpha,n,m}$.

We can interpret $(\tilde{p}, \bar{r}) \in R$ as a condition $(\tilde{p}, p(0), p(1), \dots)$ in $\tilde{P} * P_\omega$: After forcing with $\tilde{P} * P_n$, we have the generic sequence $(\eta_i)_{i<n}$. This defines a canonical assignment of $x_{i,j}$ for $i < n$, namely $x_{i,j} := \eta_i(\delta + j)$. This assignment evaluates $(r_{n,\alpha})_{\alpha<\delta}$ to a condition in Q_n, and we define $p(n)$ to be that condition. Using this identification, we get:

Lemma 2.7.
- *R is a dense subset of $\tilde{P} * P_\omega$.*
- *$(\tilde{q}, \bar{s}) \in R_{\delta'+\omega}$ is stronger than $(\tilde{p}, \bar{r}) \in R_{\delta+\omega}$ iff $\tilde{q} \leq \tilde{p}$ (in $\tilde{P}$) and $s_{n,\alpha} = r_{n,\alpha} \circ \phi$ for $\alpha < \delta, n \in \omega$ and the substitution ϕ that maps $x_{i,j}$ to $s_{i,\delta+j}$.*

For a proof, see Definition 4.6 and Lemma 4.8.

A condition $(\tilde{p}, \bar{r}) \in R_{\delta+\omega}$ can be made into a stronger condition $(\tilde{q}, \bar{s}) \in R_{\delta+\delta'+\omega}$ by stacking some $(\tilde{q}', \bar{s}') \in R_{\delta'+\omega}$ on top of $(\tilde{p}, \bar{r})$. On the other hand, if $(\tilde{q}, \bar{s})$ is stronger than $(\tilde{p}, \bar{r})$, then $(\tilde{q}, \bar{s})$ is the result of stacking some $(\tilde{q}, \bar{s})$ on top of $(\tilde{p}, \bar{r})$, cf Figure 3(b).

We now define the forcing Q_*:

Definition 2.8. Q_* is the set of all sequences $\bar{t} = (t_{i,j})_{i,j\in\omega}$ of terms.[6]
$\bar{t} \leq \bar{s}$, if there is a substitution ϕ such that:
- $t_{i,j} = s_{i,j} \circ \phi$ for all $i, j \in \omega$.
- ϕ maps $x_{n,m}$ to a term only depending on $x_{i,j}$ with $i \leq n$.
- For every (n, m) there is an (i, j) such that ϕ maps $x_{i,j}$ to $x_{n,m}$.

$\leq$ is transitive: If the substitution ϕ witnesses that $\bar{r}_2 \leq \bar{r}_1$, and ψ witnesses $\bar{r}_3 \leq \bar{r}_2$, then $\phi \circ \psi$ is a suitable substitution witnessing $\bar{r}_3 \leq \bar{r}_1$.

So Q_* is a nicely definable forcing. We now show that R adds a Q_*-generic filter:

Lemma 2.9. *We define the function σ with domain R by $(\tilde{p}, \bar{r}) \mapsto (r_{i,j})_{i,j\in\omega}$. Then $\sigma : R \to Q_*$ and σ preserves $\leq$. If G is R-generic over V, then*

$$G_* = \{\bar{t} \in Q_* : \exists (\tilde{p}, \bar{r}) \in G : \sigma(\bar{r}) \leq \bar{t}\}$$

is Q_*-generic over V.

Proof. Assume that $(\tilde{p}, \bar{r}) \in R_{\delta+\omega}$, and $(\tilde{q}, \bar{s}) \leq (\tilde{p}, \bar{r})$. Then the substitution ϕ that maps $x_{n,m}$ to $s_{n,\delta+m}$ shows that $\sigma(\bar{s}) \leq \sigma(\bar{r})$:

We have to show that for all (n, m) there is an (i, j) such that $s_{i,\delta+j} = x_{n,m}$. If $(\tilde{q}, \bar{s}) \in R_{\delta'+\omega}$, then $s_{n,\delta'+m} = x_{n,m}$. So there is a minimal limit $\alpha \geq \delta$ such that for some (i, j), $s_{i,\alpha+j} = x_{n,m}$. Assume towards a contradiction that $\alpha > \delta$. By the definition of R, $s_{i+1,\beta} = s_{i,\alpha+j}$ for some $\delta < \beta < \alpha$, a contradiction.

Now assume towards a contradiction that $(\tilde{p}, \bar{r}) \in R$ forces that G_* does not meet the dense set $D \subseteq Q_*$. We first define $\bar{r}' \in Q_*$ stronger than $\sigma(\bar{r})$ by the following substitution ϕ:

For each n, let $(t^n_m)_{m\in\omega}$ enumerate (with infinite repetitions) the constant term 0 and all variables $x_{i,j}$ with $i < n$. Let ϕ map $x_{n,m}$ to t^n_m.

Now pick some $\bar{s}$ in D such that $\bar{s} \leq \bar{r}'$, witnessed by some substitution ψ. We claim that there is a $(\tilde{q}, \bar{s}') \leq (\tilde{p}, \bar{r})$ such that $\sigma(\bar{s}') = \bar{s} \in D$, a contradiction.

Fix (n, m). By the definition of $\leq$, there is an (i, j) such that ψ maps $x_{i,j}$ to $x_{n,m}$. So there is an $i_{n,m}$ such that for all $i \geq i_{n,m}$ there are infinitely many j such that $x_{i,j} \circ \phi \circ \psi = x_{n,m}$.

Fix an injective function from $\omega^{<\omega}$ to ω, $(a_1, \dots, a_l) \mapsto \ulcorner a_1, \dots, a_l \urcorner$ with coinfinite range.

We now define $(\tilde{p}^\Delta, \bar{r}^\Delta) \in R_{\omega\cdot\omega+\omega}$, such that $r^\Delta_{i,j} = x_{i,j} \circ \phi \circ \psi$. Then we can stack $(\tilde{p}^\Delta, \bar{r}^\Delta)$ on top of $(\tilde{p}, \bar{r})$ to get $(\tilde{q}, \bar{s}')$ as required. So we have to define $\bar{\nu}$

[6] We could also just use the (open) subset of Q_* defined by $(\forall n, m)(\exists^\infty i)(\exists^\infty j) t_{i,j} = x_{n,m}$. This would not make a difference, since $\sigma''G$ is in this subset anyway.

and $r^\Delta_{n,\alpha}$ for $\omega \leq \alpha < \omega \cdot \omega$. (We already know the values required for $r^\Delta_{n,m}$, and $r^\Delta_{n,\omega\cdot\omega+m}$ has to be $x_{n,m}$.)

- $\nu_{\omega\cdot\omega,n_0,m_0} = \{\omega \cdot k + \ulcorner n_0, m_0 \urcorner : k \geq i_{n_0,m_0} + 1\}$.
  If $\alpha \in \nu_{\omega\cdot\omega,n_0,m_0}$, then $r^\Delta_{n_0+1,\alpha} = x_{n_0,m_0}$.
- If $k > 1$, and $m = \ulcorner n_0, m_0, a_2, \dots, a_l \urcorner$ is a code of a sequence of length $l+1$ for some $l \geq 1$ and $n_0, m_0, a_2, \dots, a_{l+1} \in \omega$, if $n = n_0 + l$, and if $k \geq i_{n_0,m_0} + 1 - l$, then we define
  $\nu^\Delta_{\omega\cdot(k+1),n,m} = \{\omega \cdot k + \ulcorner n_0, m_0, a_2, \dots, a_l, j \urcorner : j \in \omega\}$.
  For $\alpha \in \nu_{\omega\cdot(k+1),n,m}$, we set $r^\Delta_{n+1,\alpha} = x_{n_0,m_0}$.
- If the same assumptions hold for $k = 1$, we let $\nu^\Delta_{\omega,n,m}$ consist of l such that $x_{n+1,l} \circ \phi \circ \psi = x_{n_0,m_0}$. (We also make sure that $\nu^\Delta_{\omega,n,m}$ and $\nu^\Delta_{\omega,n,m'}$ are disjoint for $m \neq m'$.) This is possible since $n + 1 \geq i_{n_0,m_0}$.
- If $r^\Delta_{n,\alpha}$ is not yet defined, we set it to be (the constant term) 0. If $\nu^\Delta_{\alpha,n,m}$ is not defined for $\alpha > \omega$, we choose it outside of the domain of $\ulcorner \cdot \urcorner$. If $\nu^\Delta_{\omega,n,m}$ is not defined, we let it consist of l such that $x_{n+1,l} \circ \phi \circ \psi = 0$. $\square$

3. The NNR iteration

In the rest of the paper, δ always denotes a countable limit ordinal.

First we define a σ-closed preparatory forcing $\tilde{P}$, which gives us for ever $\alpha \in \omega_1$ a subset of α of order type ω and some simple coding sequences.

Definition 3.1. $\tilde{p} \in \tilde{P}$, if for some $\mathrm{ht}(\tilde{p}) < \omega_1$, $\tilde{p}$ consists of sequences

$$\nu_{\delta,n,m},\ j_{\delta,n,m},\ \text{and}\ f_{\delta,n,m,k}\ \text{for}\ \delta < \mathrm{ht}(\tilde{p}),\ m \in \omega,\ \text{and}\ n \in \omega$$

such that

- each $\nu_{\delta,n,m}$ is a cofinal, unbounded subset of δ of order type ω.
- $m_1 \neq m_2$ implies that ν_{δ,n,m_1} and ν_{δ,n,m_2} are disjoint.
- $j_{\delta,n,m}$ is an increasing function from ω to ω.
- $f_{\delta,n,m,k}$ is a surjective function from $2^{[j_{\delta,n,m}(k), j_{\delta,n,m}(k+1)-1]}$ to 2.

$\tilde{P}$ is ordered by extension.

Lemma 3.2. *$\tilde{P}$ is σ-closed, and forces that $2^{\aleph_0} = \omega_1$.*

Proof. Given $\bar{\nu}$, define A_δ by $l \in A_\delta$ iff $\delta + 2l \in \nu_{\delta+\omega,0,0}$. Then $\{A_\delta : \delta < \omega_1\}$ contains all old reals and therefore all reals. $\square$

Fixing $\bar{\nu}$, $\bar{j}$ and $\bar{f}$, δ and $n > 0$, we define functions $g_{\delta,n-1,m} : \omega \times 2^\delta \to \{0,1\}$ the following way, cf. Figure 2(a):

Fix $\eta_n \in 2^\delta$. For $i \in \omega$, let ζ_i be the i-th element of $\nu_{\delta,n-1,m}$. Set $b_i = \eta_n(\zeta_i)$. So $\bar{b} = (b_i)_{i\in\omega} \in 2^\omega$. Fix $k \in \omega$. Look at $\bar{b} \restriction [j_{\delta,n-1,m}(k), j_{\delta,n-1,m}(k+1) - 1]$. This is a 0-1-sequence of appropriate length, so we can apply $f_{\delta,n-1,m,k}$. We call the result $g_{\delta,n-1,m}(k, \eta_n)$. To summarize:

Definition 3.3. Let ζ_i be the i-th element of $\nu_{\delta,n-1,m}$.

$$g_{\delta,n-1,m}(k, \eta) = f_{\delta,n-1,m,k}\big((\eta(\zeta_i))_{j_{\delta,n-1,m}(k) \leq i < j_{\delta,n-1,m}(k+1)}\big).$$

We will be interested in sequences $(\eta_n)_{n\in\omega}$ that cohere with respect to $g_{\delta,n,m}$. Again, cf. Figure 2(a):

Definition 3.4. Let η_{n-1} and η_n be partial functions from ω_1 to 2, $\delta + m \in \mathrm{dom}(\eta_{n-1})$, $\delta \subseteq \mathrm{dom}(\eta_n)$, $k_0 \in \omega$. η_{n-1} and η_n cohere at $\delta + m$ above k_0 (with respect to $g_{\delta,n-1,m}$), if

$$\eta_{n-1}(\delta + m) = g_{\delta,n-1,m}(k, \eta_n)$$

for all $k \geq k_0$. η_{n-1} and η_n cohere at $\delta + m$, if they cohere at some k_0.

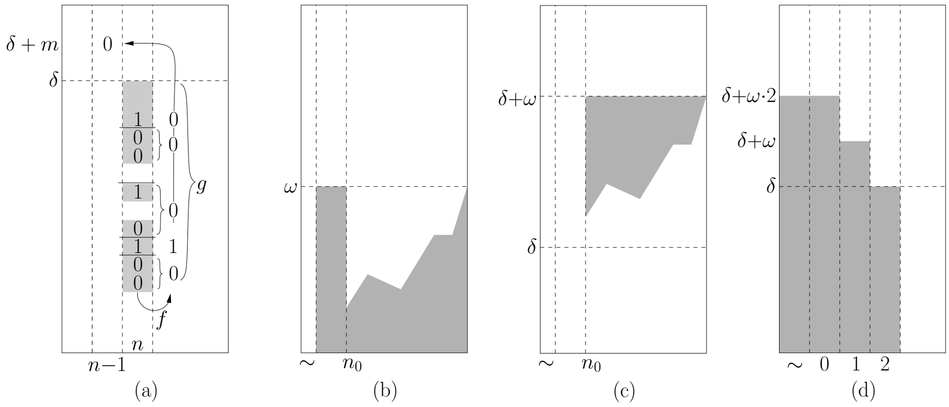


FIGURE 2. (a) η_{n-1} and η_n cohere at $\delta+m$ above $k_0=2$. $\nu_{\delta,n-1,m}$ is indicated by the gray area, $j_{\delta,n-1,m}$ corresponds to the partition of this area, $f_{\delta,n-1,m}$ calculates $g_{\delta,n-1,m}$. (b) There are conditions which determine the gray values, but these conditions are not dense. (c) No condition can determine all gray values. (d) A typical condition in $\tilde{P}*P_3$.

Let G_0 be $\tilde{P}$-generic over V, and define in $V[G_0]$ the forcing notion Q_0:

Definition 3.5. $p\in Q_0$ iff $p\in 2^{\mathrm{ht}(p)}$ for some $\mathrm{ht}(p)<\omega_1$. Q_0 is ordered by extension.

So Q_0 is σ-closed and adds the generic $\eta_0\in 2^{\omega_1}$. Assume that $n\geq 1$, $\tilde{P}*P_n=\tilde{P}*Q_0*\cdots*Q_{n-1}$, and Q_{n-1} adds the generic sequence $\eta_{n-1}\in 2^{\omega_1}$. Let G_n be $\tilde{P}*P_n$-generic over V. In $V[G_n]$, we define Q_n the following way:

Definition 3.6. $p\in Q_n$ iff $p\in 2^{\mathrm{ht}(p)}$ for some $\mathrm{ht}(p)<\omega_1$ limit and η_{n-1} and p cohere everywhere, i.e. at all $\delta+m<\mathrm{ht}(p)+\omega$. Q_n is ordered by extension.

Lemma 3.7. *The following is forced by $\tilde{P}*P_n$:*
1. *If $q\in Q_n$, $q'\in 2^{\mathrm{dom}(q)}$ and $q'(\alpha)=q(\alpha)$ for all but finitely many $\alpha\in\mathrm{dom}(q)$, then $q'\in Q_n$.*
2. *If $p\in Q_n$ and $\beta>\mathrm{ht}(p)$, then there is a $q\leq p$ with $\mathrm{ht}(q)=\beta$, i.e. Q_n adds the generic $\eta_n=\bigcup G(n)\in 2^{\omega_1}$.*
3. *Q_n is separative (and in particular nontrivial).*
4. *If $n>0$, then Q_n is not σ-closed.*

Proof. (1) is clear and implies (4).

(2) Let $(\delta_i,m_i)_{i\in\omega}$ enumerate all pairs (δ,m) such that $\mathrm{ht}(p)<\delta\leq\beta$ and $m\in\omega$. Define an increasing sequence p_i of partial functions from β to $\{0,1\}$:

For $i=0$, set $p_i=p$. For $i>0$, assume that $\mathrm{dom}(p_{i-1})=\mathrm{ht}(p)\cup\bigcup_{j<i}\nu_{\delta_j,n-1,m_j}$. Then $\nu_{\delta_i,n-1,m_i}\cap\mathrm{dom}(p_{i-1})$ is finite:

- If $j<i$ and $\delta_j=\delta_i$, then $m_j\neq m_i$ and $\nu_{\delta_i,n-1,m_i}$ and $\nu_{\delta_j,n-1,m_j}$ are disjoint.
- If $j<i$ and $\delta_j\neq\delta_i$, then $\nu_{\delta_i,n-1,m_i}\cap\nu_{\delta_j,n-1,m_j}$ are finite (since $\nu_{\delta,n-1,m}$ is a cofinal subset of δ of order type ω).
- For the same reason, $\nu_{\delta_i,n-1,m_i}\cap\mathrm{ht}(p)$ is finite.

Therefore we can extend p_{i-1} to some p_i by adding values at $\nu_{\delta_i,n,m_i}\setminus\mathrm{dom}(p_i)$ that cohere with η_{n-1} at δ_i+m_i (recall that $f_{\delta_i,n-1,m_i,k}$ is onto). Set $p_\omega=\bigcup p_n$, and fill in arbitrary values (e.g. 0) at $\beta\setminus\mathrm{dom}(p_\omega)$. This gives a $q\leq p$ with $\mathrm{ht}(q)=\beta$.

(3) follows from (1) and (2). $\square$

Remark: For this proof, as well as for most of the following, the preparatory forcing $\tilde{P}$ is not necessary: The definition of Q_n works for any reasonably defined sequences $\bar{\nu}$, $\bar{j}$, $\bar{f}$. Only in the proof of Lemma 6.5, in (2), we need that these sequences are generic.

Lemma 3.8. *$(\eta_n)_{n\in\omega}$ is determined by G_0 and $(\eta_n{\restriction}\omega)_{n\in\omega}$. In particular, $\tilde{P} * P_\omega$ adds a new real.*
More generally, if $\delta < \omega_1$, $n_0 \in \omega$ and $f : \omega \to \omega$, then G_0 together with the values $\eta_n(\alpha)$ for $n \geq n_0$, $\delta + f(n) < \alpha < \delta + \omega$ determine each G_n above $\delta + \omega \cdot n_0$.

Cf. Figure 2(c).

Proof. By induction on $1 \leq h \leq n_0$, each $\eta_n(\alpha)$ is determined for $n \geq n_0 - h$, $\delta + \omega \cdot h \leq \alpha < \delta + \omega \cdot (h + 1)$. By induction on limit ordinals $\delta + \omega \cdot n_0 < \delta' < \omega_1$, each $\eta_n(\alpha)$ is determined for $\delta + \omega \cdot n_0 < \alpha < \delta'$. $\square$

Note: For all $n_0 \in \omega$ and $f : \omega \to \omega$, there are $p \in \tilde{P} * P_\omega$ that determine all $\eta_n(m)$ for $n < n_0$ or $m < f(n)$, cf. Figure 2(b). (The reason is that $\tilde{P} * P_{n_0}$ does not add new reals, as we will see in the next lemma, and that each Q_n-condition can be modified at finitely many places.) However, these conditions are *not* dense. (For exactly the same reason: There is a condition p_0 stating that $(\eta_n(0))_{n\in\omega}$ codes $(\eta_n{\restriction}\omega)_{n\in\omega}$, via a simple bijection from $\omega\times\omega$ to ω. Clearly no $p' \leq p_0$ can determine all $\eta_n(0)$.)

Lemma 3.9. *$\tilde{P} * P_n$ forces that Q_n is proper and does not add a new ω-sequence of ordinals.*

Proof. Work in $V' = V[G_n]$.
Let $N^* \prec H(\chi^*)$ be a countable elementary submodel containing G_0, η_{n-1} and $p_0 \in Q_n$. Set $\delta = N^* \cap \omega_1$. Let $(D_i)_{i\in\omega}$ list all dense subsets of Q_n that are in N^*, $D_0 = Q_n$.
We will show that there is a $q \leq p_0$ such that $\mathrm{ht}(q) = \delta$, and such that q is not only N^*-generic, but actually stronger than some $p_i \in D_i \cap N^*$ for every $i \in \omega$.
Pick (in V') a sequence $(N_i)_{i\in\omega}$ and a $\chi \in N^*$ such that:

- $N_i \in N^*$.
- $N_i \prec H(\chi)$ is countable.
- N_0 contains η_{n-1} and p_0, N_{i+1} contains N_i and D_{i+1}.

Set $\beta_i = N_i \cap \omega_1$. So $\sup_{i\in\omega}(\beta_i) = \delta$.
Fix (in V') any $\eta^* \in Q_n$ of length δ. In particular η^* and η_{n-1} cohere at $\delta + m$ for all m. Set $u_i = \{\alpha \in \nu_{\delta,n-1,m} : m < i, \beta_i \leq \alpha < \beta_{i+1}\}$. Each u_i is finite. We will construct q such that $q \supseteq \eta^*{\restriction}u_i$ for all $i \geq 1$. This guarantees that q and η_{n-1} cohere at $\delta + m$ for all $m \in \omega$.
Assume that $p_i \in N_i \cap D_i$ is already defined. We extend it to $p_{i+1} \in N_{i+1} \cap D_{i+1}$: The finite sequence $\eta^*{\restriction}u_{i+1}$ is in N_{i+1}, so we can (in N_{i+1}) extend p_i first to some $p' \supseteq \eta^*{\restriction}u_{i+1}$ in Q_n. Then extend p' to $p_{i+1} \in D_{i+1} \cap N_{i+1}$.
Set $q = \bigcup_{i\in\omega} p_i$. q is in Q_n: For $\alpha < \delta$, q and η_{n-1} cohere at $\alpha + m$: Let i be such that $\alpha < \beta_i$. Then q extends p_{i+1} which coheres with η_{n-1}. For all i, q extends $p_i \in D_i \cap N^*$. If $\underset{\sim}{f} \in N^*$ is a name for a function from ω to the ordinals, then the value of $\underset{\sim}{f}(n)$ is determined in some dense set $D_{i(n)}$ an therefore by q. $\square$

As an immediate consequence we get (cf. Figure 2(d)):

Corollary 3.10. *The conditions $(\tilde{p}, p_0, \ldots, p_{n-1})$ of the following form are dense in $\tilde{P} * P_n$: $\mathrm{ht}(\tilde{p}) = \delta + \omega \cdot (n-1)$ for some δ, and in V there is a sequence $(p'_0, \ldots, p'_{n-1})$ such that $p'_i \in 2^{\delta+\omega\cdot(n-1-i)}$ and p_i is the standard name for p'_i.*

4. A dense subset

Definition 4.1.
- A variable is one of the symbols $x_{n,m}$ for $n, m \in \omega$. We will interpret it as variable for the value of some $\eta_n(\zeta)$, i.e. 0 or 1. For better readability, we will also use $y_{n,m}$ or $z_{n,m}$.
- A term t has the form $t = f(v_0, \dots, v_{l-1})$ for some $0 \leq l < \omega$, variables $v_0, \dots, v_{l-1}$ and a function $f : 2^l \to 2$.
- An assignment a is a mapping from the variables to 2. Assignments naturally extends to all terms, and we write $t \circ a$ to denote the result of applying a to the term (or variable) t. We sometime also write $x_{n,m} := t$ instead of $x_{n,m} \circ a = t$.
- A term-matrix $\bar{r}$ is a sequence of terms $(r_{n,\alpha})_{n\in\omega,\alpha<\mathrm{ht}(\bar{r})}$ for $\mathrm{ht}(\bar{r}) \in \omega_1$.
- Let $\bar{r}$ be a term-matrix, and let $\bar{\eta} = (\eta_i)_{n\in\omega}$ be a sequence of partial functions from ω_1 to 2. Then r is compatible with $\bar{\eta}$, if there is an assignment a such that $r_{n,\alpha} \circ a = \eta_n(\alpha)$ for all $\alpha \in \mathrm{dom}(\eta_n) \cap \mathrm{ht}(\bar{r})$.

In $V[G_0]$, a term matrix $\bar{r}$ can be interpreted as a promise that the generic sequence $(\eta_i)_{n\in\omega}$ is compatible with $\bar{r}$. Of course such a promise can be inconsistent, e.g. if each $r_{n,\alpha}$ is (the constant) 0 and $\mathrm{ht}(\bar{r}) \geq \omega$.

Definition 4.2. $R = \bigcup_{\delta<\omega_1} R_{\delta+\omega}$. $R_{\delta+\omega}$ consists of pairs $(\tilde{p}, \bar{r})$ such that:
- $\tilde{p} \in \tilde{P}$, $\mathrm{ht}(\tilde{p}) = \delta + 1$ (or equivalently $\delta + \omega$).
- $\bar{r}$ is a term matrix with $\mathrm{ht}(\bar{r}) = \delta + \omega$.
- $r_{n,\delta+m}$ is the variable $x_{n,m}$.
- If $\alpha < \delta$, then $r_{n,\alpha}$ only depends on $x_{l,k}$ with $l < n$.
- For every $m, n \in \omega$ and $\alpha \leq \delta$ limit there is a $k_0 < \omega$ such that for all assignments a, $r_{n,\alpha+m} \circ a$ and $(r_{n+1,\zeta} \circ a)_{\zeta<\alpha}$ cohere at $\alpha + m$ above k_0.

For any sequence $\bar{\eta}$ with $\mathrm{dom}(\eta_n) \supseteq \delta+\omega$, there is at most one assignment a^c that can witness the compatibility of $\bar{\eta}$ and $\bar{r}$, the assignment that sets $x_{n,m} := \eta_n(\delta+m)$. We call a^c the canonical assignment.

Elements of R can be interpreted as (consistent) statements about the generic sequence:

Definition 4.3. We define a function $i : R \to \mathrm{ro}(\tilde{P} * P_\omega)$ by mapping $(\tilde{p}, \bar{r})$ to the (nonzero) truth value of the following statement: $\tilde{p} \in G_0$, and $\bar{r}$ is compatible with the generic sequence $\bar{\eta}$. Moreover, the truth value remains nonzero if we additionally assign specific values for finitely many of the variables $x_{n,m}$.

We can even interpret $(\tilde{p}, \bar{r})$ directly as element in $\tilde{P} * P_\omega$:

Lemma 4.4. *$i(\tilde{p}, \bar{r})$ is the condition $(\tilde{p}, p(0), p(1), \dots) \in \tilde{P} * P_\omega$, where $p(n)$ is the $P * P_n$-name $(r_{n,\alpha} \circ \underset{\sim}{a}^c)_{\alpha\in\delta}$. $\underset{\sim}{a}^c$ is the $P * P_n$-name for the canonical assignment $x_{l,m} := \eta_s(\delta + m)$ for $l > n, m \in \omega$.*

This shows in particular that $i(\tilde{p}, \bar{r})$ is nonzero. If we additionally fix (in V) some $f : \omega \to \omega$ and arbitrary values for $x_{n,m}$ $(m < f(n))$, the name $p(n)$ just has to be modified by adding these values on top.

If $\alpha < \delta$, $r_{n,\alpha}$ can be calculated from finitely many $r_{l,\delta+m}$ with $l < n$ (since $r_{n,\alpha}$ is a term using variables $x_{l,m}$, $l < m$, and $r_{l,\delta+m} = x_{l,m}$). We can also calculate values in the other direction, cf. Figure 3(a):

Lemma 4.5. *If $(\tilde{p}, \bar{r}) \in R_{\delta+\omega}$ and $\beta < \delta$, then every $r_{n,\alpha}$ with $\beta + \omega \leq \alpha < \delta + \omega$ can be determined by finitely many $r_{l,\zeta}$ with $l > n, \beta < \zeta < \beta + \omega$.*
In particular $x_{n,m}$ (i.e. $r_{n,\delta+m}$) can be determined by finitely many such $r_{l,\zeta}$.

More precisely: There is a sequence $(l_i, \zeta_i)_{i<k}$ such that $l_i > n$, $\beta < \zeta < \beta+\omega$ and for all assignments a, b the following holds: If $r_{n,\alpha} \circ a \neq r_{n,\alpha} \circ b$, then $(r_{l_i,\zeta_i} \circ a)_{i<k} \neq (r_{l_i,\zeta_i} \circ b)_{i<k}$.

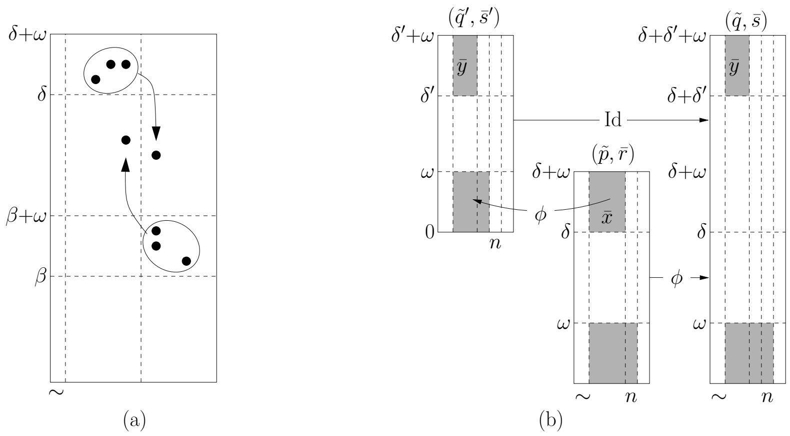


FIGURE 3. Elements of R. (a) Dependence in both directions, according to Definition 4.2 and Lemma 4.5. (b) Conditions can be stacked to get stronger conditions. If on the other hand $(\tilde{q},\bar{s})$ is stronger than $(\tilde{p},\bar{r})$, then it can be split accordingly.

Proof. By induction on α: Assume $\alpha=\beta+\omega+m$. $r_{n,\alpha}$ and $(r_{n+1,\zeta})_{\zeta<\beta+\omega}$ have to cohere above some k_0, so we can use $f_{\beta+\omega,n,m}$ to get $r_{n,\alpha}$. In more detail: Let ζ_i be the i-th element of $\nu_{\beta+\omega,n,m}$. There is some $k\geq k_0$ such that $\zeta_{j_{\beta+\omega,n,m}(k)}>\beta$. Let a be an assignment. Then $r_{n,\alpha}\circ a$ has to be the same as $f_{\beta+\omega,n,m,k}$ applied to the sequence $(r_{n+1,\zeta_i}\circ a)_{j_{\beta+\omega,n,m}(k)\leq i<j_{\beta+\omega,n,m}(k+1)}$.

Now assume that the statement is true for all $\alpha<\nu$, ν limit. If $\alpha=\nu+m$, then $r_{n,\alpha}$ again is determined by the values of certain $r_{l,\zeta}$ with $l>n,\beta+\omega\leq\zeta<\nu$, each of which in turn is determined (by induction) by finitely many $r_{l',\zeta'}$ with $\beta<\zeta'<\beta+\omega$. □

We now define the order on R.

Definition 4.6. A substitution ϕ maps variables to terms. Substitutions naturally extend to all terms, and we write $t\circ\phi$ to denote the result of applying ϕ to the term (or variable) t. We sometime also write $x_{n,m}:=t$ instead of $x_{n,m}\circ\phi=t$.

To improve readability, we sometimes use different variable symbols than $x_{n,m}$. For example we write the term $s(\bar{x})$ with the variables $x_{n,m}$, and the substitution ϕ assigns $x_{n,m}:=t_{n,m}(\bar{y})$. Then $s\circ\phi$ is a term in $\bar{y}$.

Definition 4.7. Let $(\tilde{p},\bar{r})\in R_{\delta+\omega}$ and $(\tilde{q},\bar{s})\in R_{\delta'+\omega}$. $(\tilde{q},\bar{s})$ is stronger than $(\tilde{p},\bar{r})$, if $\tilde{q}\leq\tilde{p}$ (in particular $\delta'\geq\delta$), and $r_{n,\alpha}\circ\phi=s_{n,\alpha}$ for all $\alpha<\delta$ and for the substitution ϕ defined by $x_{n,m}:=s_{n,\delta+m}$.

Note that our terms are functions, not syntactical objects, i.e. $r_{n,\alpha}\circ\phi=s_{n,\alpha}$ is equivalent to: $r_{n,\alpha}\circ\phi\circ a=s_{n,\alpha}\circ a$ for all assignments a.

If $(\tilde{q},\bar{s})$ is stronger than $(\tilde{p},\bar{r})$ and $\delta'=\delta$, then $(\tilde{q},\bar{s})=(\tilde{p},\bar{r})$.

If $(\tilde{p},\bar{r})\in R_{\delta+\omega}$ (written with the variables $\bar{x}$) and $(\tilde{q}',\bar{s}')\in R_{\delta'+\omega}$ (written with $\bar{y}$), then we can "stack" $(\tilde{q}',\bar{s}')$ on top of $(\tilde{p},\bar{r})$ — overlapping at $[\delta,\delta+\omega[$ — to get a condition $(\tilde{q},\bar{s})\in R_{\delta+\delta'+\omega}$ stronger than $(\tilde{p},\bar{r})$, cf. Figure 3(b).

More precisely:

- $\tilde{q}\restriction(\delta+1)=\tilde{p}$.
- $\tilde{q}(\alpha)$ for $\alpha\geq\delta+\omega$ is defined the following way: $\nu^q_{\delta+\alpha,n,m}=\{\delta+\beta:\ \beta\in\nu^{q'}_{\alpha,n,m}\}$, $j^q_{\delta+\alpha,n,m}=j^{q'}_{\alpha,n,m}$, $f^q_{\delta+\alpha,n,m,k}=f^{q'}_{\alpha,n,m,k}$.
- $s_{n,\delta+\alpha}=s'_{n,\alpha}$.

- If $\alpha < \delta$, then $s_{n,\alpha} = r_{n,\alpha} \circ \phi$ for the substitution ϕ defined by $x_{n,m} := s'_{n,m}$.

On the other hand, if $(\tilde{q}, \bar{s}) \in R_{\delta+\delta'+\omega}$ is stronger than $(\tilde{p}, \bar{r}) \in R_{\delta+\omega}$, then we can "split" $(\tilde{q}, \bar{s})$ into $(\tilde{p}, \bar{r})$ and $(\tilde{q}', \bar{s}') \in R_{\delta'+\omega}$ such that $(\tilde{q}, \bar{s})$ is the result of stacking $(\tilde{q}', \bar{s}')$ on top of $(\tilde{p}, \bar{r})$.

Of course we generally cannot split a condition at any level, i.e. if $(\tilde{q}, \bar{s}) \in R_{\delta''+\omega}$ and $\delta'' = \delta + \delta'$, then we generally cannot write $(\tilde{q}, \bar{s})$ as some $(\tilde{q}', \bar{s}') \in R_{\delta'+\omega}$ stacked on top of some $(\tilde{p}, \bar{r})$.

We now show that R is a dense subset of $\tilde{P} * P_\omega$:

Lemma 4.8. (1) *$i : R \to \tilde{P} * P_\omega$ is injective.*
(2) *$i(\tilde{q}, \bar{s}) \leq i(\tilde{p}, \bar{r})$ iff $\tilde{q} \leq \tilde{s}$ is stronger than $(\tilde{p}, \bar{r})$.*
(3) *i is dense (and preserves $\perp$).*
(4) *In particular, R can be interpreted as a dense subset or $\tilde{P} * P_\omega$.*

Proof. Assume that $(\tilde{p}, \bar{r}) \in R_{\delta_1}$, $(\tilde{q}, \bar{s}) \in R_{\delta_2}$, $\delta_1 \leq \delta_2$, and that $\tilde{q} \leq \tilde{p}$ in $\tilde{P}$.

(1) If $\delta_1 < \delta_2$, then $i(\tilde{p}, \bar{r}) \neq i(\tilde{q}, \bar{s})$. If $\delta_1 = \delta_2$ and $(\tilde{q}, \bar{s}) \neq (\tilde{p}, \bar{r})$, then $(\tilde{q}, \bar{s})$ is not stronger than $(\tilde{p}, \bar{r})$, so $i(\tilde{p}, \bar{r}) \not\leq i(\tilde{q}, \bar{s})$.

(2) It is clear that i preserves $\leq$. i preserves $\not\leq$: If $(\tilde{q}, \bar{s})$ is not stronger than $(\tilde{p}, \bar{r})$, then $(r_{n,\alpha} \circ \phi) \neq s_{n,\alpha}$ for some n, α, i.e. there is a finite partial assignment a such that $(r_{n,\alpha} \circ \phi \circ a) \neq s_{n,\alpha} \circ a$. $(\tilde{q}, \bar{s})$ together with a is consistent, i.e. there is a forcing extension such that $\bar{\eta}$ is compatible with $\bar{s}$ but not with $\bar{r}$.

(3) Fix a $(\tilde{p}, p(0), p(1), \dots) \in \tilde{P} * P_\omega$ and a countable $N^* \prec H(\chi^*)$ containing $(\tilde{p}, p(0), p(1), \dots)$. Set $\delta = N^* \cap \omega_1$. We will show that there is a $(\tilde{q}, \bar{s}) \in R_{\delta+\omega}$ such that $i(\tilde{q}, \bar{s}) \leq (\tilde{p}, p(0), p(1), \dots)$.

First extend $\tilde{p}$ to an $\tilde{p}'$ such that $\mathrm{ht}(\tilde{p}') = \delta$ and such that for every dense subset D of $\tilde{P}$ in N^* there is an $a \in D \cap N^*$ weaker that $\tilde{p}'$ (this is possible since $\tilde{P}$ is σ-closed). In particular, $\tilde{p}'$ is N^*-generic, and if $A \subseteq \tilde{P}$ is a maximal antichain in N^*, then $\tilde{p}'$ decides the $a \in A$ that will be in the generic filter (and $a \in N^*$). Further extend $\tilde{p}'$ to $\tilde{q}$ by adding some arbitrary value at δ. So $\mathrm{ht}(\tilde{q}) = \delta + 1$ or equivalently $\delta + \omega$.

We are looking for an $\bar{r}$ such that $(\tilde{q}, \bar{r}) \in R_{\delta+\omega}$ is such that $i(\tilde{q}, \bar{r}) \leq (\tilde{p}, p(0), \dots)$. $r_{n,\delta+m}$ has to be $x_{n,m}$.

Now we construct by induction on $n \geq 1$ the sequence $(r_{n-1,\alpha})_{\alpha<\delta}$ such that $(\tilde{q}, \bar{r}\restriction n)$ is N-generic and stronger than $(\tilde{p}, p(0), \dots, p(n-1))$, and decides every maximal antichain in N by finite case distinction. In more detail: For every maximal antichain $A \subseteq \tilde{P} * P_n$ in N^* there is a sequence $(a^A_0, \dots a^A_{l^A})$ of elements of $a^A_k \in A \cap N^*$ and a sequence $(t^A_0, \dots, t^A_{l^A})$ of terms using only variables $x_{i,j}$ with $i < n$ such that $(\tilde{q}, \bar{r}\restriction n)$ forces the following:[7]

- G_n is N^*-generic.
- $(\tilde{p}, p(0), \dots, p(n-1)) \in G_n$.
- There is exactly one $k < l^A$ such that $t^A_k \circ a^c = 1$ (using the canonical assignment a^c), and $a^A_k \in G_n$ for this k.

For $n = 1$, we have to define $(r_{0,\alpha})_{\alpha<\delta}$. Let $(\underset{\sim}{D}_i)_{i\in\omega}$ enumerate all $\tilde{P}$-names in N^* for dense subsets of Q_0. $p_0 := p(0)$ is decided by a maximal antichain in N^*. (Since $\tilde{P}$ is σ-closed, it does not add any new 0-1-sequences of countable length.) Therefore p_0 is decided by $\tilde{q}$. Given $p_i \in N^*$, there is a $\tilde{P}$-name $\tilde{p}$ in N^* for an element of $\underset{\sim}{D}_{i+1}$ such that $p_i \in Q_0$ implies $\tilde{p} \leq p_i$. Again, $\tilde{p}$ is decided by $\tilde{q}$ to be some sequence p_{i+1}. We set $r_{0,\alpha}$ to be the term with constant value $p_i(\alpha)$ for sufficiently large i. So $\tilde{q}$ forces that that $r(0)$ is $N^*[G_0]$-generic, i.e. $(\tilde{q}, r(0))$ is N^*-generic, and forces that $(\tilde{p}, p(0)) \in G_1$. Let $A \in N^*$ be a maximal antichains of

[7]More precisely: In $V[G_n]$, if G_0 contains $\tilde{q}$ and $\eta_0, \dots, \eta_{n-1}$ is compatible with $(r_{l,\alpha})_{l<n,\alpha<\delta+\omega}$, then the following holds.

$\tilde{P} * P_1$. We can assume without loss of generality that elements of A have the form of Corollary 3.10.

$$\underset{\sim}{D} = \{x \le a(0) : (\tilde{a}, a(0)) \in A, \tilde{a} \in G_0\}$$

is a $\tilde{P}$-name for an open dense set, i.e. $\underset{\sim}{D} = \underset{\sim}{D}_i$ for some i. p_i is forced by $\tilde{q}$ to be in $D_i \cap N^*$, and $A_0 = \{\tilde{a} \in \tilde{P} : (\tilde{a}, a(0)) \in A, a(0) \subseteq p_i\} \in N^*$ is an antichain in $\tilde{P}$. We know that $\tilde{q}$ forces $A_0 \cap G_0 \neq \emptyset$. So we know (in V) the element $a \in A \cap N^*$ such that $(\tilde{q}, \bar{r}(0))$ forces that $G_1 \cap A = \{a\}$.

Now we deal with the case $n+1$. We already have $(\tilde{q}, \bar{r}\restriction n)$ and have to find $r_{n,\alpha}$. Let $(\underset{\sim}{D}_i)_{i\in\omega}$ enumerate all $\tilde{P} * P_n$-names in N^* for dense subsets of Q_n, $\underset{\sim}{D}_0 = Q_n$.

First we fix (in V) a term-sequence $(t^*_\alpha)_{\alpha\in\bigcup_{m\in\omega}\nu_{\delta,n-1,m}}$ such that:

- If $\alpha \in \nu_{\delta,n-1,m}$, then t^*_α only depends on $x_{n-1,m}$.
- For all m, $\bar{t}^*$ and $(r_{n-1,\alpha})_{\delta\le\alpha<\delta+\omega}$ (i.e. the block $(x_{n-1,m})_{m\in\omega}$) cohere at $\delta + m$ above $k_0 = 0$.

(I.e. for all assignments a and $m \in \omega$, $x_{n-1,m} \circ a$ and $\bar{t}^* \circ a$ cohere at $\delta + m$ above 0.)

We construct in V by induction on $i \in \omega$:

- $\beta_i < \omega_1$.
- A finite set v_i of variables $x_{l,j}$ with $l < n$, containing every $x_{n-1,j}$ for $j \le i$. If $i > 0$, then $v_i \supseteq v_{i-1}$.
- For every assignment a of v_i a 0-1-sequence p^a_i in N^*. If $i > 0$, then
  - $\mathrm{ht}(p^a_i) \ge \beta_{i-1}$,
  - $p^a_i \supseteq p^{a'}_{i-1}$ (where a' is the restriction of a to v_{i-1}),
  - if $m < i$, $\mathrm{ht}(p^{a'}_{i-1}) \le \alpha < \mathrm{ht}(p^a_i)$, and $\alpha \in \nu_{\delta,n-1,m}$, then $p^a_i(\alpha) = t^*_\alpha \circ a$.

Such that for every assignment a of v_i the following is forced by $(\tilde{q}, \bar{r}\restriction n)\&a$:[8]

- p^a_i extends $p(n)$.
- $p^a_i \in D_i$ (i.e. in particular, $p^a_i \in Q_i$).

Set $\beta_0 = 0$. $p(n)$ is decided by an antichain in $\tilde{P} * P_n$, i.e. below $(\tilde{q}, \bar{r}\restriction n)$ it depends on a finite set v_0 of variables. For every assignment a, let p^a_0 be the sequence forced to be $p(0)$ by $(\tilde{q}, \bar{r}\restriction n)\&a$.

Given β_i, v_i and p^a_i, find in N^* an name $\underset{\sim}{N}$ for a countable $\underset{\sim}{N} \prec H^{V[G_n]}(\chi)$ containing $\underset{\sim}{D}_{i+1}$, β_i and p^a_i for all a. $\underset{\sim}{N} \cap \omega_1$ can be decided by an antichain, i.e. has only finitely many possibilities, and in particular there is a β_{i+1} such that $(\tilde{q}, \bar{r}\restriction n)$ forces that $\beta_{i+1} > \underset{\sim}{N} \cap \omega_1$.

Set $v'_{i+1} = v_i \cup \{x_{n-1,0}, \dots, x_{n-1,i}\}$. b ranges over the (finitely many) assignments of v'_{i+1}, and $a(b)$ denotes the restriction of b to v_i. Set

$$u = \{\alpha \in \nu_{\delta,n-1,m} : m \le i, \alpha < \beta_{i+1}\}.$$

For every b, set $s^b = (t^* \circ b)\restriction u$. This is a welldefined (and finite) 0-1-sequence, since every variable in t^*_α is element of v'_{i+1} for $\alpha \in u$.

Fix b. The following is forced (in N^*) by the empty condition:

> There is a $q \in 2^{\mathrm{ht}(q)} \cap \underset{\sim}{N}$ such that $\mathrm{ht}(q) > \beta_i$, $q \supseteq p^{a(b)}_i$, and such that $p^{a(b)}_i \in Q_n$ implies the following: $q \in \underset{\sim}{D}_{i+1}$, and if $\alpha \in u \cap \mathrm{ht}(q)$ and $\alpha \ge \mathrm{ht}(p^{a(b)}_i)$, then $q(\alpha) = s^b(\alpha)$.

(Proof: Assuming that $p^{a(b)}_i \in Q_n$, find in $\underset{\sim}{N}$ a $q' \supseteq p^b_i \cup [s^b\restriction(\omega_1 \cap \underset{\sim}{N} \setminus \mathrm{ht}(p^{a(b)}_i))]$ of height at least β_i, then extend it (still in $\underset{\sim}{N}$) into some condition $q \in \underset{\sim}{D}_{i+1}$.)

So we can pick (in N^*) a name $\underset{\sim}{q}^b$ for this q, and decide $\underset{\sim}{q}^b$ by an antichain, i.e. by a finite set v^b of variables. Set $v_{i+1} = v'_{i+1} \cup \bigcup_b v^b$. Let c be an assignment of v_{i+1}, b

[8] I.e.: In $V[G_n]$, if G_0 contains $\tilde{q}$ and $\eta_0, \dots, \eta_{n-1}$ is compatible with $(r_{l,\alpha})_{l<n,\alpha<\delta+\omega}$, and if for every $x_{l,k} \in v_i$, $\eta_{l,\delta+k} = x_{n,m} \circ a$, then the following holds.

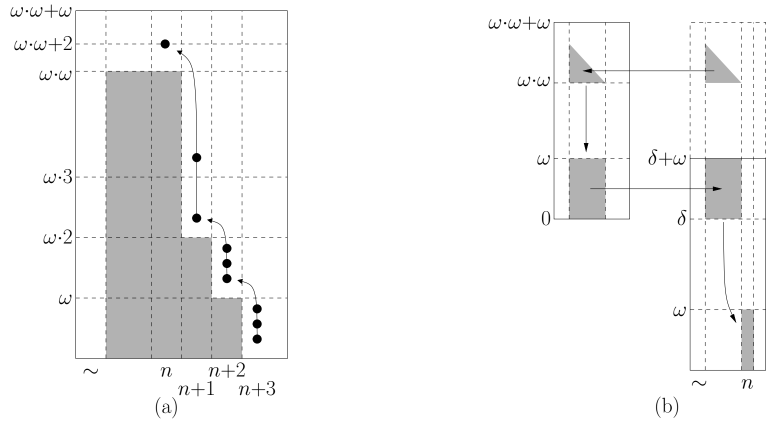


FIGURE 4. A useful example. (a) $y_{n,2}$ only appears in $r^{\Delta}_{l,k}$ if $l > n+2$. (b) After stacking the example on top of a condition, $r_{n,m}$ also only depends on $y_{l,k}$ with $l + k < n$.

its restriction to v'_{i+1}. Set p^c_{i+1} to be the sequence $\underset{\sim}{q}^b$ as decided by $(\tilde{q}, \bar{r}{\restriction}n)\&c$. Since $(\tilde{q}, \bar{r}{\restriction}n)\&b$ forces that $p^b_i \in Q_n$, it follows that $(\tilde{q}, \bar{r}{\restriction}n)\&c$ forces that $p^c_{i+1} \in \underset{\sim}{D}_{n+1}$. Since $(\tilde{q}, \bar{r}{\restriction}n)$ forces that $\beta_{i+1} > \underset{\sim}{N} \cap \omega_1$, it follows that $\mathrm{ht}(p^c_{i+1}) < \beta_{i+1}$, i.e. that $t^*_\alpha \circ c = p^c_{i+1}(\alpha)$ whenever $\mathrm{ht}(p^b_i) \le \alpha \in u \cap \mathrm{ht}(p^c_{i+1})$.

Now we define by induction on i the terms $r_{n,\alpha}$ for all $\beta_i \le \alpha < \beta_{i+1}$: $r_{n,\alpha}$ is the term using variables in v_{i+2} such that for all v_{i+2}-assignments b, $r_{n,\alpha} \circ b = p^b_{i+2}(\alpha)$.

For each m and $\alpha \le \delta$, $r(n)$ and $r(n-1)$ cohere at $\alpha + m$ above some fixed $k_0(m,n)$: If $\alpha = \delta$, assume that $\beta_{i+1} \le \alpha$, $i > m$, and that $\alpha \in \nu_{\delta,n-1,m}$. Then by the construction above, $t^*_\alpha \circ b = r_{n,\alpha} \circ b$ for all assignments b. If $\alpha < \delta$, pick an $i > 0$ such that $\beta_{i-1} > \alpha$. For all assignments a of v_i, $(\tilde{q}, \bar{r}{\restriction}n)\&a$ forces that $\underset{\sim}{p}^a_i$ and η_{n-1} coheres at $\alpha + m$ above some $\underset{\sim}{k}^a$. Since $\underset{\sim}{k}^a$ can be decided by a maximal antichain, there are only finitely many possibilities for $\underset{\sim}{k}^a$. There are finitely many a, so there is an upper bound k_0.

Let $A \in N$ be a maximal antichain of $\tilde{P} * P_{n+1}$. We can assume without loss of generality that elements of A have the form of Corollary 3.10. Let $\underset{\sim}{D}$ be the $\tilde{P} * P_n$-name for the following open dense subset of Q_n: $\{q \le a(n) : a \in A, a{\restriction}n \in G_n\}$. $\underset{\sim}{D}$ appears as $\underset{\sim}{D}_i$ in the list of dense sets in N^*. Fixing an assignment a of v_i, we get $p^a_i \in D_i[G_n] \cap N[G_n]$ in V. $\{a{\restriction}n : a \in A, a(n) \subseteq p^a_i\}$ is a $\tilde{P} * P_n$-antichain, and can be decided by finite case distinction. □

5. An example

Let us now give a typical example of a $p \in R$ (for later use, we now call the variables $y_{n,m}$). See Figure 4(a).

For this section, we fix an injective function from $\omega^{<\omega}$ to ω, $(a_1, \dots, a_l) \mapsto \ulcorner a_1, \dots, a_l \urcorner$ with coinfinite range.

Example 5.1. We define the following $(\tilde{p}^{\Delta}, \bar{r}^{\Delta}) \in R_{\omega\cdot\omega+\omega}$. Set $F(n_0, m_0) = m_0$. n,m,n_0,m_0,k,d,l range over ω.

- Each $j_{\alpha,n,m} : \omega \to \omega$ and each $f_{\alpha,n,m,k} : 2 \to 2$ is the identity function.
- $\nu_{\omega\cdot\omega,n_0,m_0} = \{\omega \cdot k + \ulcorner n_0, m_0 \urcorner : k > F(n_0, m_0) - 1\}$.
  If $\alpha \in \nu_{\omega\cdot\omega,n_0,m_0}$, then $r^{\Delta}_{n+1,\alpha} = y_{n_0,m_0}$.
- If m is the code of the sequence $(n_0, m_0, a_2, \dots, a_l)$ of length $l + 1 \le 2$, and $n > n_0$, $n - n_0 > F(n_0, m_0) - l$, then

$\nu_{\omega\cdot(k+1),n,m} = \{\omega\cdot k + \ulcorner n_0,\dots,a_l,j\urcorner : j\in\omega\}$, and
if $\alpha\in\nu_{\omega\cdot(k+1),n,m}$, then $r^{\Delta}_{n+1,\alpha} = y_{n_0,m_0}$.

- For other k,m, define $\nu_{\omega\cdot k+1,n,m}$ arbitrarily, and if $r^{\Delta}_{n,\alpha}$ is not yet defined set it 0.

In this example, each $r^{\Delta}_{n,\alpha}$ is either a y_{n_0,m_0} or 0. If $r^{\Delta}_{n,m} = y_{n_0,m_0}$, then $n > m_0+n_0$. Given n, $r^{\Delta}_{n,m}=0$ for infinitely many m.

Given any $(\tilde p,\bar r)\in R$, we can stack $\tilde p^{\Delta},\bar r^{\Delta}$ on top of $(\tilde p,\bar r)$, cf. Figure 4(b). This shows the following:

Definition 5.2. R' consists of $(\tilde p,\bar r)\in R$ such that $r_{n,m}$ only depends on $x_{l,k}$ with $l+k<n$.

Corollary 5.3. *R' is a dense subset of R.*

We now generalize this example:

Lemma 5.4. *Assume that $(r_{n,m})_{n,m\in\omega}$ is such that:*

- *$r_{n,m}$ is a term depending on variables $z_{i,j}$ with $i<n$.*
- *For all (n_0,m_0) there are infinitely many n such that for all k there are l and $k<m'_0<\dots<m'_{l-1}$ such that z_{n_0,m_0} is determined by $(r_{n,m'_j})_{j<l}$.*
- *For all n there are infinitely many m such that $r_{n,m}$ is constant (e.g. 0).*

Then there is a $(\tilde p^{\Delta},\bar r^{\Delta})\in R_{\omega\cdot\omega+\omega}$ such that $r^{\Delta}_{n,m}=r_{n,m}$ for all $n,m\in\omega$.

As usual, "z_{n_0,m_0} is determined by $(r_{n,m'_j})_{j<l}$" means: if a,b are assignments such that $z_{n_0,m_0}\circ a\neq z_{n_0,m_0}\circ b$, then $(r_{n,m'_j}\circ a)_{j<l}\neq(r_{n,m'_j}\circ b)_{j<l}$.

Proof. Define $\operatorname{index}(n_0,m_0)\subseteq\omega\times\omega$ the following way: $(n,0)\in\operatorname{index}(n_0,m_0)$ iff for all k there are l and $k<m'_0<\dots<m'_{l-1}$ such that z_{n_0,m_0} depends on $(r_{n,m'_j})_{j<l}$. So for all (n_0,m_0) there are infinitely many n such that $(n,0)\in\operatorname{index}(n_0,m_0)$. Such an n must be bigger than n_0, since $r_{n,m}$ does not depend on z_{n_0,m_0} for $n\le n_0$. $(n,k+1)\in\operatorname{index}(n_0,m_0)$ iff $(n+1,k)\in\operatorname{index}(n_0,m_0)$. So $(n,k)\in\operatorname{index}(n_0,m_0)$ iff $(n+k,0)\in\operatorname{index}(n_0,m_0)$.

Set $V_n=\{z_{n_0,m_0} : (n,1)\in\operatorname{index}(n_0,m_0)\}$. Enumerate V_n and the symbol "constant" with infinite repetitions as $v_0,v_1,\dots$. Set $l_0=0$, and l_{i+1} such that $(r_{n+1,m})_{l_i\le i<l_{i+1}}$ either determines the variable v_i, or contains some constant term in case that v_i is the symbol "constant". For $z_{n_0,m_0}\in V_n$, define u_{n,m,n_0,m_0} as a union of infinitely many of the intervals $[l_i,l_{i+1}-1]$ with $v_i=z_{n_0,m_0}$, such that u_{n,m,n_0,m_0} and u_{n,m',n_0,m_0} are disjoint if $m\neq m'$. Similarly, let $u_{n,m,\text{constant}}$ be an infinite subset of $\bigcup_{v_i=\text{constant}}[l_i,l_{i-1}]$ such that $l\in u_{n,m,\text{constant}}$ implies that $r_{n+1,l}$ is a constant term, and such that the sets $u_{n,m,\text{constant}}$ are disjoint for different m.

We now define $(\tilde p^{\Delta},\bar r^{\Delta})$:

- If $m,n\in\omega$, then $r^{\Delta}_{n,\omega\cdot\omega+m}=z_{n,m}$. (This is required of elements of R.)
- If $m,n\in\omega$, then $r^{\Delta}_{n,m}=r_{n,m}$. (That is what we want in the lemma.)
- If $\alpha>\omega$, then $j_{\alpha,n,m}:\omega\to\omega$ and $f_{\alpha,n,m,k}:2\to 2$ are the identity functions.
- $\nu_{\omega\cdot\omega,n,m}=\{\omega\cdot j+\ulcorner n,m\urcorner : j>0,(n+1,j)\in\operatorname{index}(n,m)\}$.
  If $\alpha\in\nu_{\omega\cdot\omega,n,m}$, then $r^{\Delta}_{n+1,\alpha}=z_{n,m}$.
- Assume that $1<k<\omega$, $m=\ulcorner n_0,m_0,a_2,\dots,a_l\urcorner$ is code of a sequence of length $l+1$ for some $l\ge 1$ and $n_0,m_0,a_2,\dots,a_{l+1}\in\omega$, $n=n_0+l$, and $(n,k)\in\operatorname{index}(n_0,m_0)$. Then we define
  $\nu_{\omega\cdot k,n,m}=\{\omega\cdot(k-1)+\ulcorner n_0,\dots,a_l,j\urcorner : j\in\omega\}$, and
  $r^{\Delta}_{n+1,\alpha}=z_{n_0,m_0}$ for $\alpha\in\nu_{\omega\cdot k,n,m}$.

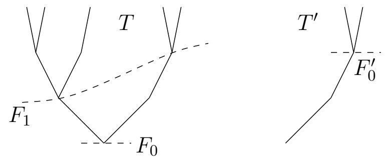


FIGURE 5. F_n is the front of n-th splitting nodes.

- Assume the same with $k = 1$. Then we define $\nu_{\omega,n,m} = u_{n,m,n_0,m_0}$, and we set $j_{\omega,n,m}$ and $f_{\omega,n,m,j}$ so that they calculate z_{n_0,m_0}.
- If $k > 1$, n and m are not covered by the previous cases, pick $\nu_{\omega\cdot k,n,m}$ between $\omega \cdot (k-1)$ and $\omega \cdot k$ disjoint from the image of $\ulcorner \cdot \urcorner$ (make sure to get disjoint sets for different m).
  For $\alpha \in \nu_{\omega\cdot k,n,m}$, set $r^{\Delta}_{n+1,\alpha} = 0$.
- If $k = 1$, n and m are not covered by the previous cases, set $\nu_{\omega,n,m} = u_{n,m,\text{constant}}$. Let $j_{\omega,n,m}$ be the identity function. If the l-th element of $\nu_{\omega,n,m}$ is ζ and $r_{n+1,\zeta}$ is the constant term 0, then let $f_{\omega,n,m,l}$ be the identity function, otherwise (i.e. if $r_{n+1,\zeta}$ is the constant term 1), let $f_{\omega,n,m,l}$ be the function $x + 1 \bmod 2$.
- Set $r^{\Delta}_{n,\alpha} = 0$ for all $r^{\Delta}_{n,\alpha}$ not defined previously. $\square$

Corollary 5.5. *Let $A \subseteq \omega \times \omega$ be such that $(\exists^\infty n)(\exists^\infty m)(n,m) \notin A$. Then there is a condition in $\tilde{P} * P_\omega$ that determines $(\eta_n(m))_{(n,m)\in A}$.*

Note however that these conditions are *not* dense (cf. Lemma 3.8 and the note following it).

6. Finding the Sacks real

Consider a Sacks tree T (i.e. $T \subseteq 2^{<\omega}$ is a perfect tree). Let F_n be the (finite) front of the n-th splitting nodes, cf. Figure 5. T consists of sequences $(t_0, t_1, \dots, t_n)$. We can interpret t_n to be a term, depending on some variables $v_0, v_1, \dots.$ v_n that tells us whether to choose the right or left branch at F_n.

Note that we can calculate the value an assignment a gives v_n if we know a sufficiently long sequence $(t_0 \circ a, t_1 \circ a, \dots, t_l \circ a)$ (l has to be larger than the height of every node in F_n). In the example, we get the following sequence of terms:

$$t_0 = v_0, \quad t_1 = \begin{cases} v_1 & \text{if } v_0 = 0 \\ 1 & \text{otherwise,} \end{cases} \quad t_2 = \begin{cases} v_2 & \text{if } v_0 = 0 \text{ and } v_1 = 0 \\ 1 & \text{if } v_0 = 0 \text{ and } v_1 = 1 \\ v_1 & \text{otherwise,} \end{cases}$$

and v_0 can be calculated from t_0, and v_1 from (t_0, t_1, t_2).

If $T' \subseteq T$ is a perfect subtree, then the fronts F'_n grow, and the variables $(v'_0, \dots, v'_l)$ have at least as much information as $(v_0, \dots, v_l)$: There is a substitution ϕ that sets $v_i := s_i(v'_0, \dots, v'_i)$ such that $t_n = t'_n \circ \phi$.

In the example of Figure 5, we get $v_0 := 1$, and $v_1 := v'_0$.

We will see that the set S^* of such term-sequences is equivalent to Sacks forcing:

Definition 6.1. S^* consists of sequences of terms $(t_i)_{i\in\omega}$ using the variables v_j ($j \in \omega$) such that

- t_i depends only on v_j with $j \le i$, and
- each v_j is determined by some $t_0, \dots, t_{l-1}$.

$\bar{t}(\bar{w})$ is stronger than $\bar{s}(\bar{v})$, if $\bar{t} = \bar{s} \circ \phi$ for some substitution ϕ such that $v_i \circ \phi$ only depends on w_j with $j \le i$.

As usual, "v_j is determined by some $t_0, \dots, t_{l-1}$" means: If a, b are assignments such that $v_j \circ a \neq v_j \circ b$, then $(t_0 \circ a, \dots, t_{l-1} \circ a) \neq (t_0 \circ b, \dots, t_{l-1} \circ b)$.

We can interpret a $\bar{t} \in S^*$ as continuous function from 2^ω to 2^ω, and map $\bar{t}$ to its image:

Lemma 6.2. *Let Ψ map $\bar{t} \in S^*$ to $\{(\bar{t} \circ a)\restriction n : n \in \omega, a \text{ assignment}\}$. Then Ψ is a surjective complete embedding from S^* into Sacks forcing.*

Proof. $\Psi(\bar{t})$ is a perfect tree: Pick any $s = (\bar{t} \circ a)\restriction n \in \Psi(\bar{t})$. $t_0, \dots t_{n-1}$ use a finite set A of variables. Pick $v_j \notin A$. Then v_j is determined by $t_0, \dots t_{l-1}$ for some l. Pick assignments b, c extending $a\restriction A$ such that $v_j \circ b \neq v_j \circ c$. Then $(\bar{t} \circ b)\restriction l \neq (\bar{t} \circ c)\restriction l$ both extend s.

It is clear that Ψ is surjective and preserves $\leq$.

Ψ preserves $\perp$: Assume that $\Psi(\bar{t})$ and $\Psi(\bar{s})$ both contain the perfect tree T. By thinning out T, we can assume the following: If l is the length of a node in F_n, then $t_1(\bar{v}), \dots, t_l(\bar{v})$ determines v_n, and the same for $\bar{s}$. Define $\bar{r}(\bar{w}) \in S^*$ by letting w_i indicate whether we choose the right or left branch at F_n. So $w_0 \dots w_{n-1}$ determine a node in F_n, and therefore sufficiently many $t_1, \dots, t_{l-1}$ to determine v_n. This defines a substitution ϕ witnessing that $\bar{r}$ is stronger than $\bar{t}$. The same applies to $\bar{s}$.

□

We now define an $\omega \times \omega$-version of S^*:

Note that $\tau : \omega \times \omega \to \omega$ defined by $(n, m) \mapsto n + \frac{1}{2}(n + m)(n + m + 1)$ is a bijection. So $(i, j) \lhd (n, m)$ defined by $\tau(i, j) \leq \tau(n, m)$ is a linear order of order type ω.

Definition 6.3. Define $(i, j) \lhd (n, m)$

$$i + \frac{1}{2}(i + j)(i + j + 1) \leq n + \frac{1}{2}(n + m)(n + m + 1).$$

Q_* is the set of term-matrices $\bar{t} = (t_{n,m})_{n,m \in \omega}$ such that

- $t_{n,m}$ only depends on $x_{i,j}$ with $i + j < n$, and
- each $x_{n,m}$ is determined by finitely many $t_{i,j}$.

$\bar{t}(\bar{y})$ is stronger (in Q_*) than $\bar{s}(\bar{x})$, if there is a substitution ϕ such that $\bar{t} = \bar{s} \circ \phi$ and such that $x_{n,m} \circ \phi$ depends only on $y_{i,j}$ with $(i, j) \lhd (n, m)$.

Lemma 6.4. *Q_* forces that there is an S^*-generic object.*

Proof. Using τ, we can identify v_l with $x_{n,m}$ and an ω-term-sequence with an $\omega \times \omega$-term-matrix. In particular, we interpret S^* as a set of $\omega \times \omega$-term-matrices.

We will show that Q^* is an open subset of S^* (the order is the same by definition): If $\bar{t} \in Q^*$, then $\bar{t} \in S^*$, since $i + j < n$ implies $(i, j) \lhd (n, m)$. If $\bar{s} \in Q_*$, and $\bar{t} \leq \bar{s}$ in S^*, then $\bar{t} \in Q_*$: Let ϕ witnessed $\bar{t}(\bar{y}) \leq \bar{s}(\bar{x})$. So $t_{n,m}(\bar{y}) = s_{n,m} \circ \phi(\bar{y})$ can be calculated from some $x_{i,j}$ with $i + j < n$ and therefore from some $y_{l,k}$ with $l + k < n$ as well. □

Note that Q_* is not a dense subset of (this representation of) S^*: If we set $t_{0,\tau(n_0,m_0)} = x_{n_0,m_0}$, and $t_{n,m} = 0$ for $n > 0$, then $\bar{t}$ is in S^* but no stronger condition can be in Q_*. Nevertheless, Q_* and S^* (and therefore Sacks forcing) are equivalent: Sacks forcing is homogeneous, and Q_* forces that $\underset{\sim}{s}$ is a Sacks real and that the Q_*-generic filter is determined by $\underset{\sim}{s}$.

For now on, we only work with R' and Q_*, and show that R' forces that there is a Q_*-generic object. Recall that R' is a dense subset of R (Corollary 5.3) and therefore of $\tilde{P} * P_\omega$ (Lemma 4.8). So this proves the first part of Theorem 1.

We will need the following simple fact: If ϕ witnesses that $\bar{t}(\bar{y})$ is stronger than $\bar{s}(\bar{x})$, then every $y_{n,m}$ is determined by finitely many $x_{i,j}$ (i.e. $y_{i,j} \circ \phi$), or equivalently: there is an l such that $(x_{i,j})_{i+j<l}$ determines $y_{n,m}$.

Again, we mean the following: If a, b are assignments of $\bar{y}$ such that $y_{n,m} \circ a \neq y_{n,m} \circ b$, then $(x_{i,j} \circ \phi \circ a)_{i+j<l} \neq (x_{i,j} \circ \phi \circ b)_{i+j<l}$.

(Proof: $y_{n,m}$ is determined by $(t_{i,j})_{i+j<l}$, each $t_{i,j} = s_{i,j} \circ \phi$ is a term using finitely many $x_{i,j}$, so the union of these $x_{i,j}$ determines $y_{n,m}$.)

Lemma 6.5. *Let G be R'-generic over V, and let $\sigma : R' \to Q_*$ map $\bar{r}$ to $\bar{r} \restriction (\omega \times \omega)$. Then*

$$G^* = \{\bar{r} \in Q_* : \exists (\tilde{q}, \bar{s}) \in G : \sigma(\bar{s}) \leq \bar{r}\}$$

is a Q_*-generic filter over V.

Proof. $\sigma(\bar{r})$ is in Q_* because of Lemma 4.5.

It is sufficient to show: Assume $(\tilde{p}, \bar{r}) \in R'$.

(1) There is a $(\tilde{q}, \bar{s}) \leq (\tilde{p}, \bar{r})$ such that $\sigma(\bar{s}) \leq r'$ for all $(\tilde{p}', \bar{r}') \geq (\tilde{p}, \bar{r})$.
(2) There is a $\bar{t} \in Q_*$ stronger than $\sigma(\bar{r})$ such that for all $\bar{t}'$ stronger than $\bar{t}$ there is a $(\tilde{q}, \bar{s}) \leq (\tilde{p}, \bar{r})$ with $\sigma(\bar{s}) \leq t'$.

Then the lemma follows: Assume that $(\tilde{p}, \bar{r}) \in R'$ forces that $(\tilde{p}', \bar{r}')$ and $(\tilde{p}'', \bar{r}'')$ are in G but $\sigma(\bar{r}')$ and $\sigma(\bar{r}'')$ are incompatible in Q_*. Without loss of generality, we can assume that $(\tilde{p}, \bar{r})$ is stronger than $(\tilde{p}', \bar{r}')$ and $(\tilde{p}'', \bar{r}'')$, Extend $(\tilde{p}, \bar{r})$ to $(\tilde{q}, \bar{s})$ according to (1). Then $(\tilde{q}, \bar{s})$ forces that $\sigma(\bar{s})$ is stronger than both $\sigma(\bar{r}')$ and $\sigma(\bar{r}'')$, a contradiction.

Assume that $(\tilde{p}, \bar{r})$ forces that G^* does not meet the open dense set $D \subseteq Q_*$. Pick $\bar{t} \leq \sigma(\bar{r})$ according to (2), and $\bar{t}' \leq \bar{t}$ in D. Then $(\tilde{q}, \bar{s})$ as in (2) forces that $\sigma(\bar{s}) \in G^* \cap D$, a contradiction.

So it remains to show (1) and (2).

(1): Stack the matrix of Example 5.1 on top of $(\tilde{p}, \bar{r})$ to get $(\tilde{q}, \bar{s}) \in R'$. So $(\tilde{q}, \bar{s})$ is stronger than $(\tilde{p}, \bar{r})$, witnessed by the substitution ϕ^Δ defined by $x_{n,m} := r^\Delta_{n,m}(\bar{y})$. Assume $(\tilde{p}', \bar{r}') \lneq (\tilde{p}, \bar{r})$ in R', witnessed be ϕ defined by $x'_{n,m} := r_{n,\delta'+m}(\bar{x})$. Then $x'_{n,m}$ depends on $x_{i,j}$ with $i < n$, and therefore on $y_{l,k}$ with $l + k < i < n$.

(2) Define a substitution ϕ such that $x_{n,m}$ is mapped to either 0 or some $y_{l,k}$ with $l+k+1 < n$, and such that for each (l, k) with $l+k+1 < n$ (and for 0) there are infinitely many m such that $x_{n,m} := y_{l,k}$ (or $x_{n,m} := 0$, respectively).

More explicitly: Let s^n be the finite sequence $(0, y_{0,0}, y_{0,1}, y_{1,1}, \dots)$ of 0 and all $y_{i,j}$ with $i + j < n$. The length of s^n is $\text{period}_n = 1 + \frac{1}{2}n(n + 1)$. Define ϕ by

$$x_{n,m} := s^n(k) \text{ for } k = m \bmod \text{period}_n.$$

Set $\bar{t}(\bar{y}) = \sigma(\bar{s}) \circ \varphi$. So $\bar{t} \leq \sigma(\bar{s})$ in Q_*. Let $\bar{t}'(\bar{z})$ be stronger in Q_* than $\bar{t}(\bar{y})$, witnessed (or: defined) by the substitution ψ.

Note that $(x_{n,m} \circ \varphi \circ \psi)_{n,m\in\omega}$ satisfies the requirements for Lemma 5.4. So the lemma gives us a $(\tilde{p}^\Delta, \bar{r}^\Delta)$ in $R'_{\omega\cdot\omega+\omega}$ (in the variables $z_{i,j}$) such that $r^\Delta_{n,m} = \varphi \circ \psi$ for $n, m \in \omega$.

Now we can stack $(\tilde{p}^\Delta, \bar{r}^\Delta)$ on top of $(\tilde{p}, \bar{r})$ and get some $(\tilde{q}, \bar{s}) \leq (\tilde{p}, \bar{r})$ such that $\sigma(\bar{s}) = \bar{t}'$. ☐

It might look tempting to assume $\diamondsuit$ to construct the coding sequences $\bar{\nu}, \bar{j}, \bar{f}$ instead of using the preparatory forcing. (We just have to "guess" correctly sufficiently often for the proofs to work.) However, this is not possible: Otherwise, Sacks forcing would be *equivalent* to P_ω (since P_ω adds a Sacks real s which in turn determines the P_ω-generic filter G_ω). But Sacks reals are minimal, and the Q_0 generic $\eta_0 \in 2^{\omega_1}$ is not in the ground model V. Therefore $V[s] = V[\eta_0]$, a contradiction.

In particular R does add a Sacks real s, but R is not equivalent to Sacks forcing, and s does not determine the R-generic object. However, every new ω-sequence is already added by s:

Lemma 6.6. *If G is R-generic over V, and if $r \in V[G]$ is an ω-sequence of ordinals, then $r \in V[s]$. Here we set $s = (\bar{\eta}_n \restriction \omega)_{n \in \omega}$.*

Proof. If $\tilde{q} \in G_0$ has height δ, then s together with $\tilde{q}$ determines G up to height δ (just as in Lemma 3.8). So if $N \prec H(\chi)$ and $\tilde{q} \in G_0$ has height $N \cap \omega_1$, then s together with $\tilde{q}$ determines whether $r \in G$ for any $r \in R \cap N$.

Assume towards a contradiction that $p \in R$ forces that $\underset{\sim}{f}$ is an omega sequence of ordinals not added by s. Choose an $N \prec H(\chi)$ containing $p, \underset{\sim}{f}$, and an N-generic $q \leq p$. Each $\underset{\sim}{f}(n)$ is decided by some maximal antichain $A \in N$. But for each $a \in A \cap N$, s together with $\tilde{q}$ determines whether a is in G. In particular, $\underset{\sim}{f}[G] \in V[s]$. □

This proves the second part of Theorem 1: Since R forces that there is some Sacks real over V and since Sacks forcing is homogeneous, R can be factored as Sacks composed with some P'. Since the sacks real already adds all new ω-sequences, P' is NNR.

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